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Particle motion around magnetized black holes: Preston-Poisson space-time

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We analyze the motion of massless and massive particles around black holes immersed in an asymptotically uniform magnetic field and surrounded by some mechanical structure, which provides the magnetic field. The space-time is described by the Preston-Poisson metric, which is the generalization of the well-known Ernst metric with a new parameter, tidal force, characterizing the surrounding structure. The Hamilton-Jacobi equations allow the separation of variables in the equatorial plane. The presence of a tidal force from the surroundings considerably changes the parameters of the test particle motion: it increases the radius of circular orbits of particles and increases the binding energy of massive particles going from a given circular orbit to the innermost stable orbit near the black hole. In addition, it increases the distance of the minimal approach, time delay, and bending angle for a ray of light propagating near the black hole.

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I. INTRODUCTION

Black holes in the centers of galaxies are immersed in a strong magnetic field due to charged matter surrounding them. The strong magnetic field in the center of galaxies is stipulated by toroidal currents around galactic black holes [1]. Therefore an exact solution of Einstein-Maxwell equations describing a black hole immersed in an asymptotically uniform magnetic field, known as the Ernst solution [2], was of considerable interest [3]. The light and particle motion around the Ernst-Schwarzschild black hole was analyzed in a few papers [4,5]. In particular, in [4] it was shown that the Hamilton-Jacobi equation allows the separation of variables in the equatorial plane, where the motion of neutral and charged particles were analyzed. In [5] the motion of neutral particles were considered for a more general situation of the electromagnetized Kerr background. There it was shown that the release of binding energy is considerably increased because of the presence of the electromagnetic field, and the binding energy for circular orbits was calculated. Yet, in a more realistic situation, the strong magnetic field in the central region near the black hole is created by some surrounding matter, such as accretion disk or an active galactic nuclei. This surrounding structure exerts strong gravitational tidal force on particles moving near black holes, so that the magnetic influence of the structure might be even much smaller than its gravitational influence. Therefore a more physical situation should include into consideration the corrections to the black hole metric due to that structure. Fortunately, recently Preston and Poisson [6] have found such a corrected metric. This is the solution to the perturbative Einstein-Maxwell equations depending on three parameters: the black hole mass M, the magnetic field B, and a new parameter K, which characterize the above surrounding structure. The solution is very accurate for $r^2B^2 \ll M/a \ll 1$, and $r^2K \ll 1$, where r is the distance form the black hole and a is the length scale of the mechanical structure. Indeed, a comparison with the exact Ernst solution shows that next order corrections are of order B^4 and are very small.

In the present paper we generalize the analysis of works [4,5] and study the motion of test particles near black holes immersed in an asymptotically uniform magnetic field and some gravitating surrounding structure, which provides the magnetic field. The paper is organized as follows: In Sec. II we reduce the Preston-Poisson metric to the Ernst-like form, by going over to a new coordinate. Then in Sec. III we consider the Hamilton-Jacobi equation in the equatorial plane, and use it for the analysis of massless particles. There the lens effects for the Preston-Poisson metric are considered. The motion of massive particles is described in Sec. IV, where the binding energy for particles on circular orbits are calculated.

II. PRESTON-POISSON METRIC

Following [6], let us consider a model consisting of a nonrotating black hole immersed in a uniform magnetic field, and a large mechanical structure, such as a giant solenoid, producing the magnetic field of strength B. The structure has a mass M' and its linear extension is $\sim a$. In order to have a magnetic field which is uniform for $r \gg M$, one chooses

$$A^{\alpha} = \frac{1}{2}B\phi^{\alpha}, \tag{1}$$

where ϕ^{α} is the rotational Killing vector of the unperturbed Schwarzschild metric.

The metric which describes the space-time of the above model, written in light-cone gauge, is [6]

$$ds^2 = -g_{vv}dv^2 + 2dvdr + g_{v\theta}dvd\theta + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \tag{2}$$

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where

$$g_{vv} = -(1 - (2M/r)) - \tfrac{1}{9}B^2 r(3r - 8M) - (\tfrac{1}{9}B^2(3r^2 - 14Mr + 18M^2) + K(r - 2M)^2)(3\cos^2\theta - 1), \quad (3)$$

$$g_{v\theta} = (\tfrac{2}{3}B^2(r - 3M) - 2K)r^2 \sin\theta\cos\theta, \quad (4)$$

$$g_{\theta\theta} = r^2 + (-\tfrac{1}{3}B^2r^2 + B^2M^2 + K(r^2 - 2M^2))r^2\sin^2\theta, \quad (5)$$

$$g_{\phi\phi} = r^2\sin^2\theta + B^2r^2((-\tfrac{1}{3}r^2 - M^2) - K(r^2 - 2M^2)r^2)\sin^4\theta. \quad (6)$$

The above metric is accurate through order (B^2, K), whenever $r^2B^2 \ll$ and $r^2K \ll 1$. The new parameter K characterizing the mechanical structure, containing the black hole, can be interpreted as a tidal gravity or Weyl curvature when $r \gg M$. Thus we have a three-parameter solution.

From now on, we shall consider motion in the equatorial plane $\theta = \pi/2$. Therefore, we shall put $\theta = \pi/2$ in formulas (1)–(5). Then, let us make the following coordinate transformations:

$$v = t + \bar{r} + 2M\ln|(\bar{r}/2M) - 1|, \quad (7)$$

$$r = \bar{r}(1 + (1/6)B^2r^2 + (1/3)(K - (1/2)B^2)(\bar{r} - 2M)\bar{r}^2 + O(B^4, K^2). \quad (8)$$

These transformations are the generalization of transformations (3.60–3.61) of [6] in the equatorial plane and they cast the metric (1) into the diagonal form

$$ds^2 = g_{tt}dt^2 + g_{\bar{r}\bar{r}}d\bar{r}^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \quad (9)$$

where the metric components (neglecting orders $O(B^4, K^2)$ and higher) are

$$g_{\bar{r}\bar{r}} = (1 - 2M/\bar{r})^{-1} + \frac{(4M - 3\bar{r})\bar{r}^2}{6M - 3\bar{r}}K - \frac{2M\bar{r}^2}{6M - 3\bar{r}}B^2, \quad (10)$$

$$g_{tt} = (1 - 2M/\bar{r}) + (1/3)(-8M^2 + 10M\bar{r} - 3\bar{r}^2)K + (2/3)M(2M - \bar{r})B^2, \quad (11)$$

$$g_{\theta\theta} = \bar{r}^2 + (1/3)\bar{r}^2(-6M^2 - 4M\bar{r} + 5\bar{r}^2)K + (1/3)\bar{r}^2(3M - \bar{r})(M + \bar{r})B^2, \quad (12)$$

$$g_{\phi\phi} = \bar{r}^2 - (1/3)\bar{r}^2(-6M^2 + 4M\bar{r} + 5\bar{r}^2)K - (1/3)\bar{r}^2(3M^2 - 2M\bar{r} + \bar{r}^2)B^2. \quad (13)$$

This form of the metric does not have nondiagonal components in the equatorial plane and is much simpler for the consideration of motion of test particles. To be exact, nondiogonal components are of order (B^4, K^2) and higher, and therefore can be safely neglected.

III. MOTION OF MASSLESS PARTICLES

From now on we shall write r instead of $\bar{r}$. The four-momentum is

$$p_\mu = g_{\mu\nu}\frac{dx^\mu}{ds}, \quad (14)$$

where s is an invariant affine parameter. The Hamiltonian has the form

$$H = \tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu. \quad (15)$$

The action can be represented in the form:

$$S = -\mu s - Et + L\phi + S_r(r) + S_\theta(\theta), \quad (16)$$

where E and L are the particle‘s energy and angular momentum, respectively.

Then, the Hamilton-Jacobi equations for null geodesics read

$$\frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = -\frac{\partial S}{\partial s} = \mu^2. \quad (17)$$

It is evident that the equations of motion allow the separation of variables in the equatorial plane $\theta = \pi/2$. The first integrals of motion are

$$\mu g_{rr}\frac{dr}{ds} = \pm\sqrt{-\frac{g_{rr}}{g_{tt}}(E^2 - U_{\rm eff}^2)}, \quad (18)$$

$$p_\phi = \mu g_{\phi\phi}\frac{d\phi}{ds} = L, \quad (19)$$

$$p_t = \mu g_{tt}\frac{dt}{ds} = -E, \quad (20)$$

$$U_{\rm eff}^2 = -g_{tt}\mu^2\left(1 + \frac{L^2}{\mu^2 g_{\phi\phi}}\right). \quad (21)$$

The trajectory and propagation equations take the form

$$\left(\frac{dr}{dt}\right)^2 = -\frac{g_{tt}}{g_{rr}}\left(1 + \frac{g_{tt}}{g_{\phi\phi}}\left(\frac{p_\phi}{p_t}\right)^2 - g_{tt}\frac{m^2}{p_t^2}\right), \quad (22)$$

$$\left(\frac{dr}{d\phi}\right)^2 = -\frac{g_{\phi\phi}}{g_{rr}}\left(1 + \frac{g_{\phi\phi}}{g_{tt}}\left(\frac{p_t}{p_\phi}\right)^2 - g_{\phi\phi}\frac{m^2}{p_\phi^2}\right). \quad (23)$$

For massless particles, from the above Eqs. (22) and (23), one can see that propagation and trajectory equations contain only the ratio $b = L/E$, which is called the impact parameter. The qualitative description of the motion of massless particles can be made by considering the effective potential of the motion (21, where $\mu = 0$). The equation for the radii of circular orbits can be found from the condition $dU_{\rm eff}/dr = 0$:

$$g_{tt}g_{\phi\phi},_r = g_{\phi\phi}g_{tt},_r. \quad (24)$$

This gives the algebraic equation for r:

$$-6(B^2 - 2K)M^3 - 2r + 10(B^2 - 2K)M^2 r + (4/3)(B^2 + K)r^3 + M(6 - 6B^2 r^2 + 4Kr^2) = 0. \quad (25)$$

When $B^2 \ll M$ and $K \ll M$, we have, that above some critical region of values of B and K, there are two null circular orbits with radii

$$r_1 = (3M + 12M^3 K + (3M^3 + 120M^5 K)B^2, \quad (26)$$

$$r_2 = \frac{2K(\sqrt{6} - 6\sqrt{K}M)(1 + 2KM^2) - B^2(\sqrt{6} + \sqrt{K}M(-15 + 13\sqrt{6K}M + 6KM^2))}{4(K)^{3/2}}. \quad (27)$$

When $B = K = 0$, r_1 takes its Schwarzschild value $3M$. Unfortunately, we cannot find accurate values for the critical region (B_{cr}, K_{cr}), because the values we get are quite large, and, for instance, for $K = B^2/2$, it is about $0.189M$, which is on the boundary of applicability of the approximate metric under consideration. The physical situation corresponds to some tidal force K, which is larger than its pure Ernst value $B^2/2$. Therefore, we shall further consider the new parameter h, which is given by the relation:

$$K = \frac{B^2}{2} + h.$$

Now, let us consider the effect of tidal gravitational attraction of the surrounding structure upon such lens effects as a light bending angle and time delay. For this, let us follow the approach of [7].

If we know the distance of minimal approach $r_{\rm min}$ with great accuracy, we can perform integrations for finding the bending angle:

$$\alpha = \phi_s - \phi_o = -\int_{r_s}^{r_{\rm min}} \frac{d\phi}{dr} dr + \int_{r_{\rm min}}^{r_o} \frac{d\phi}{dr} dr - \pi. \quad (28)$$

Here r_o is the radial coordinate of an observer and r_s is the radial coordinate of the source.

In a similar fashion one can find the time delay, which is the difference between the light travel time for the actual ray, and the travel time for the ray the light would have taken in the Minkowskian space-time:

$$t_s - t_o = -\int_{r_s}^{r_{\rm min}} \frac{dt}{dr} dr + \int_{r_{\rm min}}^{r_o} \frac{dt}{dr} dr - \frac{d_{s-o}}{\cos\mathcal{B}}. \quad (29)$$

Here the term $\frac{d_{s-o}}{\cos\mathcal{B}}$ represents the propagation time for a ray of light, if the black hole is absent. The distance of minimal approach is the corresponding root of the equation $dr/dt = 0$. For pure Schwarzschild black hole it would be the largest root, yet in our case it the second largest root of the equation:

$$r^3(-3 + B^2(3M^2 - 2Mr + r^2) + K(-6M^2 + 4Mr + r^2)) - (2M - r)(3 + 2B^2 Mr + Kr(-4M + 3r)b^2 = 0. \quad (30)$$

Looking numerically for the solution of Eq. (30) one can see that the tidal force K pulls out the radius of the minimal approach further from the black hole. We also can see from Table I, that the presence of the mechanical structure leads to the increasing of the banding angle and time delay near the black hole.

TABLE I. Bending angle α and propagation time τ for Ernst-Schwarzschild space-times (in geometrical units, $M = 1$) for $b = 6$. "Observer" and "source" are supposed to be situated not far from the black hole in order to estimate the influence of a magnetic field in the central region of the black hole: $r_o = r_s = 20$, $b = 6$.

B	h	$\alpha + \pi$	$t_s - t_o + d_{s-o}/\cos\mathcal{B}$
0	0	4.252 334	56.845 54
0	5×10^{-4}	4.406 231	57.252 24
0	10^{-3}	4.577 176	57.720 44
5×10^{-4}	0	4.252 388	56.845 67
5×10^{-4}	5×10^{-4}	4.406 293	57.252 40
5×10^{-4}	10^{-3}	4.577 247	57.720 63

IV. MOTION OF MASSIVE PARTICLES

The effective potential for the massive neutral particles (21) is shown in Figs. 1 and 2 for zero and nonzero angular momentum L of the particle.

From the above figures one can see that the effective potential can have the form of the barrier or of a monotonically increasing function.

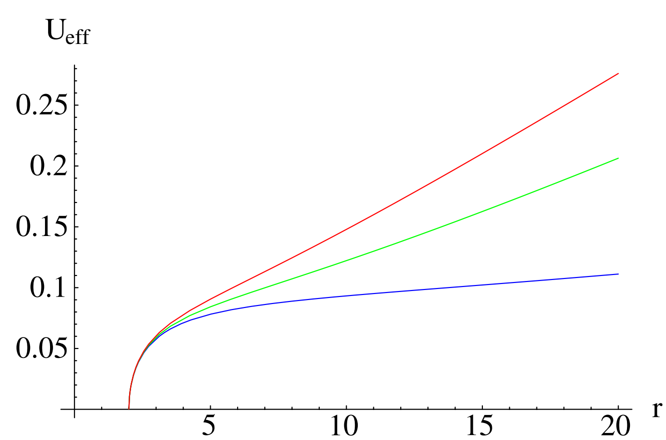

FIG. 1 (color online). Effective potential for neutral particles: $M = 1$, $B = 0.001$, $\mu = 0.1$, $L = 0$. $K = 0.001$ (lower), $K = 0.01$ (middle), $K = 0.02$ (upper).

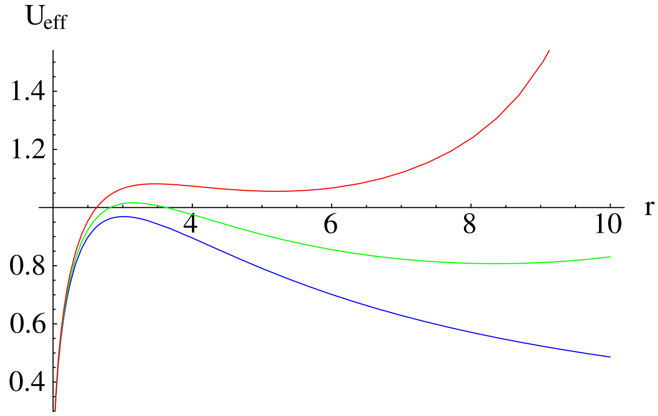

FIG. 2 (color online). Effective potential for neutral particles: $M = 1$, $B = 0.001$, $\mu = 0.1$, $L = 5$. $K = 0.001$ (lower), $K = 0.01$ (middle), $K = 0.02$ (upper).

For circular orbits the equation $V_{\text{eff},r} = 0$ gives

$$-L^2(g_{tt},_r g_{\phi\phi} - g_{\phi\phi} g_{tt},_r) + \mu^2 g_{tt},_r g_{\phi\phi}^2 = 0. \quad (31)$$

If one solves (27) for L, and uses it in the equation $dr/dt = 0$, or, equivalently, in

$$U_{\text{eff}}^2 = E^2, \quad (32)$$

one obtains a rather cumbersome system of the equation for determination of the parameters of orbits of massive particles.

The values L/μ and E/μ as functions of the radius of circular orbits are presented in Figs. 3 and 4. They are found there from accurate Eqs. (31) and (32). From Figs. 3 and 4, one can see that the particle angular momentum and energy per units mass is monotonically growing as functions of the radius of circular orbit r_c, starting from some minimal value. This minimum value of the test particle angular momentum corresponds to the orbit with the innermost stable circular radius r_{ic}. The large K, the more radius

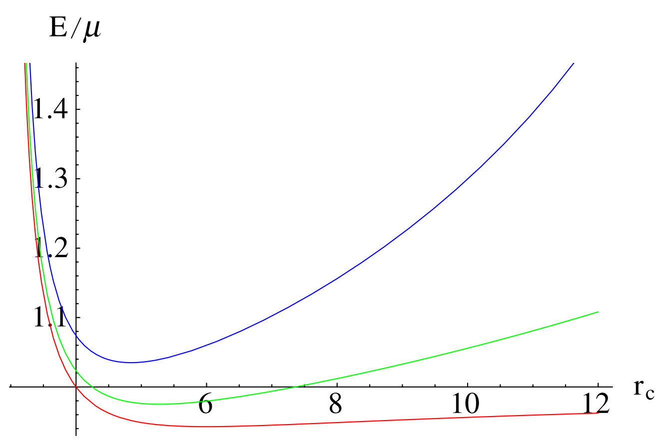

FIG. 3 (color online). E/μ as a function of the radius of the circular orbit r_c $M = 1$, $B = 0.0001$, $K = (B^2/2)$ (bottom), $K = (B^2/2) + 0.001$, $K = (B^2/2) + 0.003$ (top).

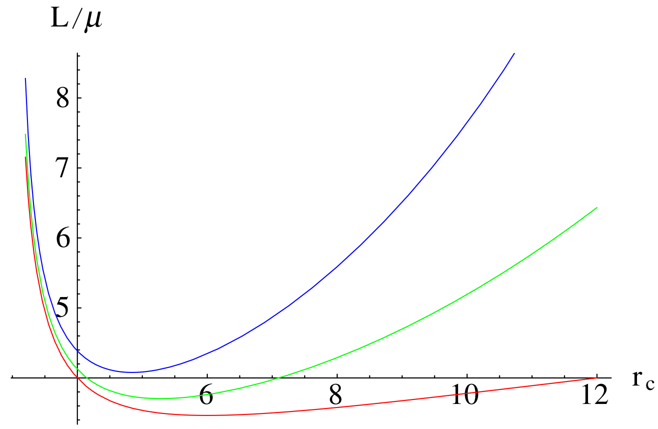

FIG. 4 (color online). L/μ as a function of the radius of the circular orbit r_c $M = 1$, $B = 0.0001$, $K = (B^2/2)$ (bottom), $K = (B^2/2) + 0.001$, $K = (B^2/2) + 0.003$ (top).

of the innermost stable circular orbit r_{ic} is pulled toward the black hole.

The binding energy is defined as the amount of energy that is released by the test particle going from a stable circular orbit r_c, to the innermost stable orbit of radius r_{ic}, i.e.

$$\text{Binding energy} = \frac{(E/\mu)_{r_c} - (E/\mu)_{r_{ic}}}{(E/\mu)_{r_c}}. \quad (33)$$

A test particle in an unstable circular orbit will fall into the black hole and the infall time is small compared to the radiative time, so that the particle energy will be brought to the black hole almost completely.

From Fig. 5 one can see two features: first, the binding energy is greater for a larger radius of circular orbit, and this dependence on the radius is strictly monotonic. Second, the larger the tidal force K, the larger the binding energy for a given radius of the circular orbit of the particle r_c. The last feature means that in the presence of the surrounding attracting structure a test particle, when going

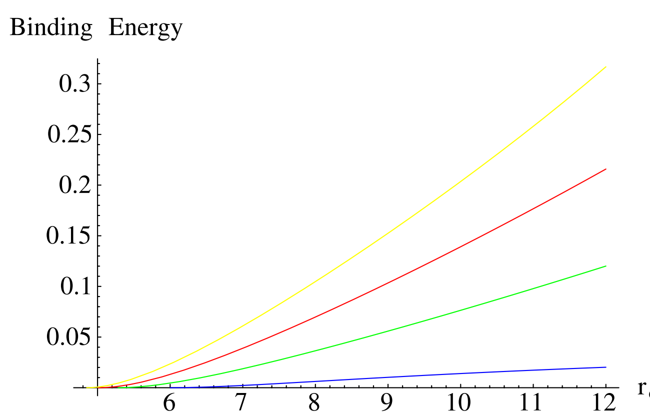

FIG. 5 (color online). Binding energy as a function of the radius of the circular orbit r_c $M = 1$, $B = 0.0001$, $K = (B^2/2)$ (bottom), $K = (B^2/2) + 0.001$, $K = (B^2/2) + 0.002$, $K = (B^2/2) + 0.003$ (top).

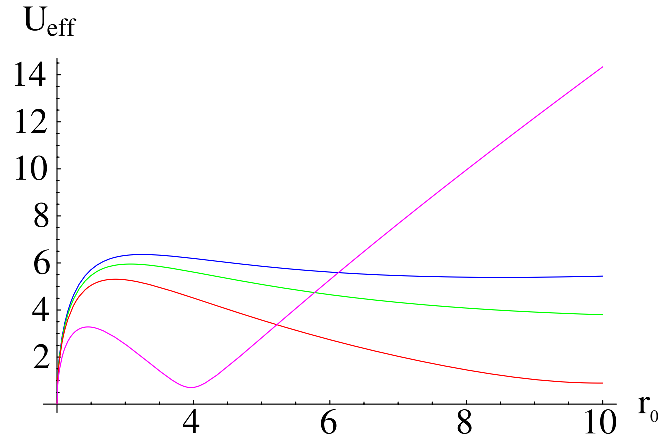

FIG. 6 (color online). Effective potential for charged particles: $e = 0.6$ (top), 0, -0.6, -3.8 (bottom), $h = 0$, $\mu M = 1$, $L = 30$.

from its stable orbit to the innermost stable one, releases more energy than it would release without the above structure.

Finally, let us give the explicit form of Eqs. (31) and (32) through the orders B^2, K,

$$\frac{E^2}{\mu^2} = \frac{1}{3}\left(\frac{r-2M}{r-3M}\right)^2\left(\frac{3}{r}(3M-r) + 2K(9M^2 - 10Mr + 3r^2) - M(9M-4r)B^2\right), \tag{34}$$

$$\frac{L^2}{\mu^2} = \frac{1}{3}\left(\frac{r}{r-3M}\right)^2(3r(r-3M) - 18M^4 + 6M^3r + 19M^2r^2 - 14Mr^3 + 3r^4 + M(9M^3 - 3M^2r - 2Mr^2 + r^3)B^2). \tag{35}$$

These expressions are much simpler than the exact Eqs. (31) and (32).

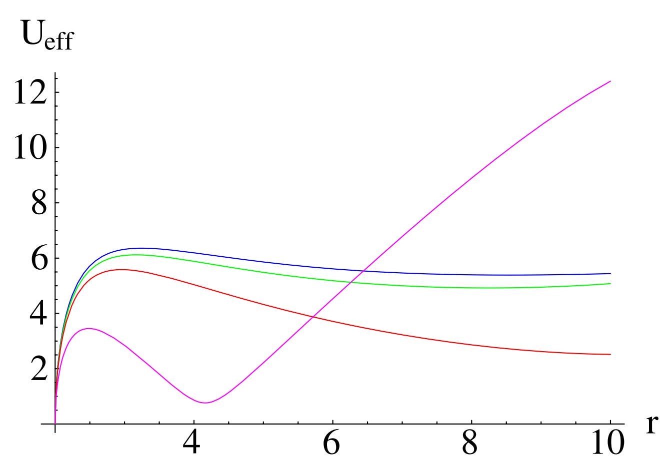

FIG. 7 (color online). Effective potential for charged particles: $e = 0.6$ (top), 0, -0.6, -3.8 (bottom), $h = 0.005$, $\mu M = 1$, $L = 30$.

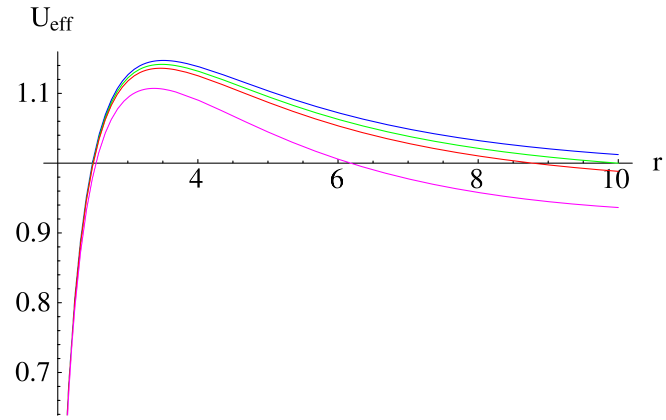

FIG. 8 (color online). Effective potential for charged particles: $eB = 0.006$ (top), 0, -0.006, -0.038 (bottom), $h = 0$, $\mu M = 1$, $L = 5$.

For massive charged particles in the vicinity of uncharged black holes, one should change the angular momentum of the particle L to the generalized momentum:

$$L \rightarrow L + g_{\phi\phi}\frac{eB}{2}. \tag{36}$$

The effective potentials for the case of charged particles are shown in Figs. 6–9. The situation is dependent on the sign of the charge, because the Lorentz force acting on the charged particles has opposite directions for positive and negative charges.

There are two reasons why we do not analyze the case of charged particles in detail: First, the magnetic field used for derivation of the considered metric is given only through the first order in B. Second, the effect of the strong magnetic field for charged particles is stipulated by the factor eB/μ (when $M = 1$), and is very large even for small $B \ll M$, because of the large ratio e/μ. Therefore, it is generally accepted, to neglect "geometric" influence on the propagation of charged particles, and to consider the more

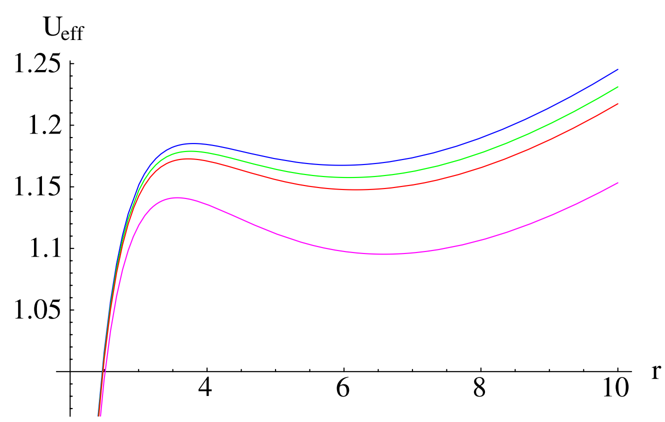

FIG. 9 (color online). Effective potential for charged particles: $eB = 0.006$ (top), 0, -0.006, -0.038 (bottom), $h = 0.005$, $\mu M = 1$, $L = 5$.

realistic decaying magnetic fields on the black hole background [8].

V. CONCLUSION

We have considered the motion of massless and massive test particles near black holes immersed in an asymptotically uniform magnetic field and some surrounding structure which provides this field. The tidal force from the surrounding structure has *considerable* influence on the parameters of the test particle motion. Let us enumerate them: (a) it pulls radius of the circular orbits off the black hole, (b) it increases the radius of the minimal approach for light, (c) it increases the time delay and bending angle for light, (d) it increases the energy and momentum (per unit mass) for a circular orbit of a given radius, (e) it increases the binding energy of massive particles, which releases when a particle goes from a given stable circular orbit to the innermost stable circular orbit, and (f) the radius of the innermost stable circular orbit is pulled closer to the black hole.

The used Preston-Poisson metric gives an excellent opportunity to investigate the motion of test particles in the vicinity of a supermassive "dirty" black hole, surrounded by some distribution of matter and the uniform magnetic field, and to approach, thereby, a more realistic situation than that given by the Ernst solution.

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