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Conformally invariant wave equations and massless fields in de Sitter spacetime

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Conformally invariant wave equations in de Sitter space, for scalar and vector fields, are introduced in the present paper. Solutions of their wave equations and the related two-point functions, in the ambient space notation, have been calculated. The Hilbert space structure and the field operator, in terms of coordinate independent de Sitter plane waves, have been defined. The construction of the paper is based on the analyticity in the complexified pseudo-Riemannian manifold, presented first by Bros *et al.* Minkowskian limits of these functions are analyzed. The relation between the ambient space notation and the intrinsic coordinates is then studied in the final stage.

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I. INTRODUCTION

Recent astrophysical data indicate that our universe might currently be in a de Sitter (dS) phase. Quantum field theory in dS space-time has evolved as an exceedingly important subject, studied by many authors in the course of the past decade. The importance of dS space has been primarily ignited by the study of the inflationary model of the universe and the quantum gravity. The importance of the ''massless'' spin-2 field in the de Sitter space (dS linear quantum gravity) is due to the fact that it plays the central role in quantum gravity and quantum cosmology. Massless field equations in de Sitter space, similar to the flat space counterparts, have the conformal invariance properties. The massless field equations with $s \geq 1$ are also gauge invariant. In this paper, the conformally invariant aspects of the massless scalar field and the massless spin-1 field (vector field) in dS space are studied. This formulation establishes the base for conformally invariant wave equation of massless spin-2 field.

Bros *et al.* [1,2] presented the quantum field theory (QFT) of the scalar field in dS space that closely mimics the QFT in Minkowski space. They have introduced a new version of the Fourier-Bros transformation on the hyperboloid [3], which allows us to completely characterize the Hilbert space of a ''one-particle'' state and the corresponding irreducible unitary representations of the de Sitter group. In this construction, correlation functions are boundary values of analytical functions. It should be noted that the analyticity condition is only preserved in the case of Euclidean vacuum. In a series of papers, we generalized the Bros construction to the quantization of the various spin free fields in dS space [4]. Here we have applied the Bros construction to the conformally invariant massless scalar and vector fields in dS space.

The massive and massless conformally coupled scalar fields, respectively, correspond to the principal and complementary series representation of the de Sitter group [2]. The massive and massless vector fields in dS space have been associated with the principal series and the lowest representation in the vector discrete series representation of the dS group, respectively [5,6]. These representations have the physical meaning in the null curvature limit. The massless vector field, however, with the divergencelessness condition, is singular [5]. This type of singularity is actually due to the divergencelessness condition needed to associate this field with a specific unitary irreducible representation (UIR) of the dS group. To solve this problem, the divergencelessness condition must be dropped. The field equation is then gauge invariant [6]. Hence, the vector field is associated with an indecomposable representation of the dS group. By fixing the gauge, this field can be quantized. In this case, emergence of states with negative or null norms necessitates indefinite metric quantization [6]. In order to eliminate these unphysical states, certain conditions must be imposed on the field operators and on the vacuum state, similar to the pattern of Minkowskian space theories [7]. Physical states propagate on the light cone and correspond to the vector massless Poincaré field in the null curvature limit. It has been proven that the use of an indefinite metric is unavoidable if one insists on the preservation of causality (locality) and covariance in gauge quantum field theories [8]. The generalization of the Wightman axioms to the QFT in de Sitter space, for scalar, spinor, and vector fields, has been studied by Bros, Gazeau, and others [2,4–6].

The free massless de Sitter vector field in the flat coordinate system has been studied previously [9]. This covers only one-half of the dS hyperboloid. In 1986, Allen calculated the massless vector two-point functions in terms of a maximally symmetric bitensor. His simple choice of gauge broke the conformal invariance and led to the appearance of logarithmic singularity [10].

In Sec. II, we have briefly recalled the main result of the previous paper [6], i.e. the gauge invariant dS vector field equation in terms of the Casimir operator. The six-cone formalism is presented in Sec. III. Following the description of de Sitter coordinates, the projection techniques have

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been introduced. Conformally invariant wave equations have been obtained in the next stage. Section IV is devoted to the solutions of the field equations in terms of a de Sitter plane wave and a polarization vector $\mathcal{E}_\alpha$. Because of the presence of a multivalued phase factor and the presence of a singularity, these solutions are not globally defined. Extending these solutions to the complex dS space has allowed us to circumvent these problems altogether [2]. The two-point functions are calculated in Sec. V. The "Hilbert" space structure and the field operators, in terms of coordinate independent dS plane waves, have been defined in this section. The null curvature limit of the two-point functions and the relation between the ambient space notation and the intrinsic coordinates are studied in the next stage. Finally, a brief conclusion and an outlook have been given in Sec. VI.

II. DE SITTER FIELD EQUATIONS

The de Sitter space-time can be defined by the one-sheeted four-dimensional hyperboloid:

$$X_H = \{x \in \mathbb{R}^5: x^2 = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2}\},\quad \alpha, \beta = 0, 1, 2, 3, 4, \tag{2.1}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, -1)$. The de Sitter metric is

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta|_{x^2=-H^{-2}} = g^{\rm dS}_{\mu\nu}dX^\mu dX^\nu,\quad \nu, \mu = 0, 1, 2, 3, \tag{2.2}$$

where X^μ are the 4 space-time coordinates in the dS hyperboloid and x^α are the 5-dimensional coordinates in the ambient space notation. For simplicity one can put $H = 1$. The wave equation for massless conformally coupled scalar field is

$$(\Box_H + 2)\phi = 0, \tag{2.3}$$

where $\Box_H$ is the Laplace-Beltrami operator on dS space. In the ambient space notation, the wave equation is written in the following form [2]:

$$(Q^{(0)} - 2)\phi = 0, \tag{2.4}$$

where $Q^{(0)}$ is the second order scalar Casimir operator of de Sitter group $SO_0(1, 4)$. The covariant derivative of a tensor field, $T_{\alpha_1\ldots\alpha_n}$, in the ambient space notation is

$$\nabla_\beta T_{\alpha_1\ldots\alpha_n} = \bar\partial_\beta T_{\alpha_1\ldots\alpha_n} - \sum_{i=1}^n x_{\alpha_i}T_{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+1}\ldots\alpha_n}, \tag{2.5}$$

where $\bar\partial_\alpha = \theta_{\alpha\beta}\partial^\beta = \partial_\alpha + x_\alpha(x\cdot\partial)$ and $\theta_{\alpha\beta}$ is the transverse projector ($\theta_{\alpha\beta} = \eta_{\alpha\beta} + x_\alpha x_\beta$). In terms of the covariant derivative, the second order scalar Casimir operator is $Q^{(0)} = -\bar\partial^2$. The wave equation for massless vector fields $A_\mu(X)$ propagating on dS space gives [10]

$$(\Box_H + 3)A_\mu(X) - \nabla_\mu\nabla\cdot A = 0. \tag{2.6}$$

This field equation is invariant under the gauge transformation $A_\mu \to A^{\rm gt}_\mu = A_\mu + \nabla_\mu\phi_g$, where ϕ_g is an arbitrary scalar field. The gauge-fixed wave equation is [10]

$$(\Box_H + 3)A_\mu(X) - c\nabla_\mu\nabla\cdot A = 0, \tag{2.7}$$

where c is a gauge-fixing parameter. It is an arbitrary positive real number.

In order to simplify the relation between the field and the representation of the dS group, we have adopted the vector field notation $K_\alpha(x)$ in ambient space notation. Pursuing this notation, the solutions of the field equations are easily written in terms of scalar fields. Consequently, a gauge transformation has vividly appeared. The 4-vector field $A_\mu(X)$ is locally determined by the five-vector field $K_\alpha(x)$ through the relation

$$A_\mu(X) = \frac{\partial x^\alpha}{\partial X^\mu}K_\alpha(x(X)). \tag{2.8}$$

Using Eq. (2.8) and the transversality condition ($x\cdot K = 0$), we have

$$\nabla_\rho\nabla_\mu A_\nu = \frac{\partial x^\gamma}{\partial X^\rho}\frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}[\bar\partial_\gamma(\bar\partial_\alpha K_\beta - x_\beta K_\alpha) - x_\alpha(\bar\partial_\gamma K_\beta - x_\beta K_\gamma) - x_\beta(\bar\partial_\alpha K_\gamma - x_\gamma K_\alpha)]. \tag{2.9}$$

Using the above equations, the field equation (2.6), in the ambient space notation, gives [5,6]

$$((\bar\partial)^2 + 2)K(x) - 2x\bar\partial\cdot K(x) - \bar\partial\partial\cdot K = 0. \tag{2.10}$$

In terms of the second order vector Casimir operator $Q^{(1)}$, one obtains [5,6]:

$$Q^{(1)}K(x) + D_1\partial\cdot K = 0, \qquad x\cdot K = 0, \tag{2.11}$$

where $D_1 = \bar\partial$. The Casimir operator $Q_1^{(1)}$ is defined by

$$Q_1^{(1)} = -\tfrac{1}{2}L^{\alpha\beta}L_{\alpha\beta} = -\tfrac{1}{2}(M^{\alpha\beta} + S^{\alpha\beta})(M_{\alpha\beta} + S_{\alpha\beta}),$$

where $M_{\alpha\beta} = -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha) = -i(x_\alpha\bar\partial_\beta - x_\beta\bar\partial_\alpha)$ and the action of the spin generator $S_{\alpha\beta}$ is defined by

$$S_{\alpha\beta}K_\gamma = -i(\eta_{\alpha\gamma}K_\beta - \eta_{\beta\gamma}K_\alpha).$$

The field equation is gauge invariant, i.e.

$$K \to K^{gt} = K + D_1\phi_g \Rightarrow Q^{(1)}K^{gt}(x) + D_1\partial\cdot K^{gt} = 0, \tag{2.12}$$

where ϕ_g is an arbitrary scalar field. The gauge-fixed wave equation in the ambient space notation is

$$Q^{(1)}K(x) + cD_1\partial \cdot K = 0. \tag{2.13}$$

This could be directly obtained from (2.7).

If we consider the physical subspace of solutions with $\partial \cdot K = 0$, we have

$$Q^{(1)}K(x) = 0 = (Q^{(0)} - 2)K(x), \tag{2.14}$$

which is similar to Eq. (2.4). This field can be associated with the UIR's $\Pi^{\pm}_{1,1}$ of the dS group. For obtaining the solution of the equation (2.14) following the massive field case, the principal series parameter (ν) must be replaced by $\pm\frac{i}{2}$ [6]. Replacement of the principal series parameter by the discrete series results in appearance of singularities in vector field solution. This singularity is actually due to the divergencelessness condition needed to associate this field with a specific UIR of the dS group [6]. In Sec. IV, we calculate the general solutions of the field equations for the different values of c. In the next section we have obtained the specific value of c, which makes the wave equation conformally invariant.

III. CONFORMALLY INVARIANT WAVE EQUATIONS

In the Minkowski space, the massless field equations are conformally invariant. For every massless representation of the Poincaré group, there exists only one corresponding representation in the conformal group [11,12]. In the de Sitter space, for the vector field, only two representations in the discrete series ($\Pi^{\pm}_{1,1}$) have a Minkowskian interpretation. The signs $\pm$ correspond to two types of helicity for the massless vector field. The representation $\Pi^{+}_{1,1}$ has a unique extension to a direct sum of two UIR's $\mathcal{C}(2;1,0)$ and $\mathcal{C}(-2;1,0)$ of the conformal group $SO_0(2,4)$. Note that $\mathcal{C}(2;1,0)$ and $\mathcal{C}(-2;1,0)$ correspond to positive and negative energy representation in the conformal group, respectively [11,12]. The concept of energy cannot be defined in de Sitter space. The latter restricts to the vector massless Poincaré UIR's $P^{>}(0,1)$ and $P^{<}(0,1)$ with positive and negative energies, respectively. The following diagrams illustrate these connections:

$$\Pi^{+}_{1,1} \hookrightarrow \begin{array}{c}\mathcal{C}(2,1,0)\\ \oplus\\ \mathcal{C}(-2,1,0)\end{array} \overset{H=0}{\longrightarrow} \begin{array}{c}\mathcal{C}(2,1,0)\\ \oplus\\ \mathcal{C}(-2,1,0)\end{array} \begin{array}{c}\hookleftarrow\\ \\ \hookleftarrow\end{array} \begin{array}{c}\mathcal{P}^{>}(0,1)\\ \oplus\\ \mathcal{P}^{<}(0,1),\end{array}$$

$$\Pi^{-}_{1,1} \hookrightarrow \begin{array}{c}\mathcal{C}(2,0,1)\\ \oplus\\ \mathcal{C}(-2,0,1)\end{array} \overset{H=0}{\longrightarrow} \begin{array}{c}\mathcal{C}(2,0,1)\\ \oplus\\ \mathcal{C}(-2,0,1)\end{array} \begin{array}{c}\hookleftarrow\\ \\ \hookleftarrow\end{array} \begin{array}{c}\mathcal{P}^{>}(0,-1)\\ \oplus\\ \mathcal{P}^{<}(0,-1),\end{array}$$

where the arrows $\hookrightarrow$ designate unique extension. $\mathcal{P}^{\gtrless}(0,-1)$ are the massless Poincaré UIR with positive and negative energies and negative helicity. In this section, the conformal invariance of massless scalar and vector fields in de Sitter space is studied. Conformally invariant wave equations are best obtained by the use of Dirac's null cone in $\mathbb{R}^6$, followed by the projection of the equations to the de Sitter space [13].

A. Dirac's six-cone formalism

Dirac's six-cone is a 5-dimensional supersurface

$$\begin{aligned} u^2 &= \eta_{ab}u^a u^b = 0,\\ \eta_{ab} &= \mathrm{diag}(1,-1,-1,-1,-1,1), \end{aligned} \tag{3.1}$$

in $\mathbb{R}^6$, where $a, b = 0, 1, 2, 3, 4, 5$. An operator $\hat{A}$ which acts on the field ϕ, over $\mathbb{R}^6$, is said to be intrinsic if [14–16]

$$\hat{A}u^2\phi = u^2\hat{A}'\phi, \qquad \text{for any } \phi. \tag{3.2}$$

The following are examples of the intrinsic operators:

(1) Fifteen generators of the conformal group $SO_0(2,4)$,

$$M_{ab} = i(u_a\partial_b - u_b\partial_a).$$

(2) The conformal-degree operator N_5

$$N_5 \equiv u^a\partial_a.$$

(3) The intrinsic gradient

$$\mathrm{Grad}_a \equiv u_a\partial_b\partial^b - (2N_5 + 4)\partial_a.$$

(4) The powers of d'Alembertian

$$(\partial_a\partial^a)^p,$$

which acts intrinsically on fields of conformal degree $(p-2)$.

The following conformally invariant set of equations, on the cone, has been used most commonly:

$$\begin{cases} (\partial_a\partial^a)^p\Psi &= 0,\\ N_5\Psi &= (p-2)\Psi, \end{cases} \tag{3.3}$$

where Ψ is a tensor field of a definite rank and of a definite symmetry. The other conformally invariant conditions can be added to the above system in order to restrict the space of the solutions. The following conditions are introduced to achieve the above goal:

(1) transversality

$$u_a\Psi^{ab\ldots} = 0,$$

(2) divergencelessness

$$\mathrm{Grad}_a \Psi^{ab\ldots} = 0,$$

(3) tracelessness

$$\Psi^{a}_{ab\ldots} = 0.$$

B. Projective of the six-cone

In order to project the coordinates on the cone $u^2 = 0$, to the 4 + 1 de Sitter space we chose the following relation:

$$\begin{cases} x^\alpha &= (u^5)^{-1} u^\alpha, \\ x^5 &= u^5. \end{cases} \tag{3.4}$$

Note that x^5 becomes superfluous when we deal with the projective cone. Various intrinsic operators introduced in the previous section now read as

(1) the ten $\mathrm{SO}_0(1,4)$ generators

$$M_{\alpha\beta} = i(x_\alpha \partial_\beta - x_\beta \partial_\alpha), \tag{3.5}$$

(2) the conformal-degree operator (N_5)

$$N_5 = x_5 \frac{\partial}{\partial x_5}, \tag{3.6}$$

(3) the conformal gradient (Grad_α) [16]

$$\mathrm{Grad}_\alpha = -x_5^{-1}\{x_\alpha[Q^{(0)} - N_5(N_5 - 1)] + 2\bar{\partial}_\alpha(N_5 + 1)\}, \tag{3.7}$$

(4) the powers of d'Alembertian $(\partial_a \partial^a)^p$, which act intrinsically on the field of conformal degree $(p - 2)$,

$$(\partial_a \partial^a)^p = -x_5^{-2p} \prod_{j=1}^{p} [Q^{(0)} + (j+1)(j-2)]. \tag{3.8}$$

C. Conformally invariant equations

For scalar field, the simplest conformally invariant system is obtained from (3.3) with $p = 1$,

$$\begin{cases} (\partial_a \partial^a)\Psi &= 0, \\ N_5 \Psi &= -\Psi, \end{cases} \tag{3.9}$$

where Ψ is a scalar field on the cone. We introduce the scalar de Sitter field by $\phi = x_5 \Psi$. This obeys the conformally invariant equation derived from (3.8) and (3.9):

$$(Q^{(0)} - 2)\phi = 0, \tag{3.10}$$

which is a massless conformally coupled scalar field in de Sitter space [Eq. (2.4)].

Similarly, the conformally invariant system for vector field is obtained from (3.3) with $p = 1$. In this case Ψ is a tensor of rank one, and it is assumed to be the solution of

$$\begin{cases} (\partial_a \partial^a)\Psi_a &= 0, \\ N_5 \Psi_a &= -\Psi_a. \end{cases} \tag{3.11}$$

We classify the 6 degrees of freedom of the vector fields on the cone by

$$K_\alpha = x_5(\Psi_\alpha + x_\alpha x \cdot \Psi), \qquad \phi_1 = x_5 \Psi_5, \qquad \phi_2 = x_5 x \cdot \Psi, \tag{3.12}$$

where K_α is a vector field on de Sitter space ($x \cdot K = 0$). Using the equations (3.8) and (3.11), these fields are proved to obey the following conformal system of equations:

$$Q^{(1)} K_\alpha + \tfrac{2}{3} D_{1\alpha} \bar{\partial} \cdot K + \tfrac{1}{6} D_{1\alpha} Q^{(0)} \bar{\partial} \cdot K = 0, \tag{3.13}$$

$$(Q^{(0)} - 2)\phi_1 = 0, \tag{3.14}$$

$$\phi_2 = \frac{x_5}{12}(Q^{(0)} + 4)\bar{\partial} \cdot K. \tag{3.15}$$

The divergence of (3.13) leads to

$$Q^{(0)}(Q^{(0)} - 2)\bar{\partial} \cdot K = 0. \tag{3.16}$$

Adding the conformal invariance condition

$$u^a \Psi_a = x^5(x \cdot \Psi + \Psi_5) = 0 \tag{3.17}$$

to the above relations, we obtain the following conformal systems:

$$Q^{(1)} K_\alpha + D_{1\alpha} \bar{\partial} \cdot K = 0, \tag{3.18}$$

$$(Q^{(0)} - 2)\partial \cdot K = 0, \tag{3.19}$$

which correspond to the gauge fixing $c = 1$ in (2.13). This is not a fully gauge invariant case, since condition (3.19) restricts the gauge field space. Nonetheless it preserves the null-cone propagation of the solutions. Under the gauge transformation $K^{\mathrm{gt}} = K + D_1 \phi_g$, the equation (3.19) requires the scalar field ϕ_g to satisfy the following equation (this is indeed not an arbitrary scalar field):

$$Q^{(0)}(Q^{(0)} - 2)\phi_g = 0. \tag{3.20}$$

The general solutions of the field equations are calculated in the next section.

IV. CONFORMALLY INVARIANT SOLUTIONS

A general solution of Eq. (2.13) can be written in terms of two scalar fields ϕ_1 and ϕ_2:

$$K_\alpha = \bar{Z}_\alpha \phi_1 + D_{1\alpha} \phi_2, \tag{4.1}$$

where Z is a constant five-vector and $\bar{Z}_\alpha = \theta_{\alpha\beta} Z^\beta$. The vector field solution was obtained in terms of a ''massless'' conformally coupled scalar field ϕ [6],

$$K^c = \left(\bar{Z} - \frac{c}{2(1-c)} D_1[x \cdot Z + Z \cdot \bar{\partial}] + \frac{2-3c}{1-c} D_1[Q^{(0)}]^{-1} x \cdot Z\right)\phi, \qquad c \neq 1. \tag{4.2}$$

If we use the scalar dS plane wave for massless conformally coupled scalar field [2], i.e.

$$\phi = (x \cdot \xi)^\sigma, \qquad \sigma = -1, -2,$$

dS vector-plane wave (4.2) could not be properly defined since its last term is singular,

$$[Q^{(0)}]^{-1} x \cdot Z (x \cdot \xi)^\sigma = \frac{-1}{(\sigma+1)(\sigma+4)} x \cdot Z (x \cdot \xi)^\sigma, \qquad \sigma = -1.$$

The five-vector ξ lies on the null cone $\mathcal{C} = \{\xi \in \mathbb{R}^5; \xi^2 = 0\}$. For $c = \frac{2}{3}$, however, it can be properly defined [6]:

$$K_\alpha(x) = \left[\bar{Z}_\alpha + (\sigma - 1)\frac{Z \cdot \xi}{(x \cdot \xi)^2}\bar{\xi}_\alpha + (\sigma+1)\frac{Z \cdot x}{x \cdot \xi}\bar{\xi}_\alpha\right] \times (x \cdot \xi)^\sigma, \tag{4.3}$$

where $\bar{\xi}_\alpha = \theta_{\alpha\beta}\xi^\beta$. It is clear that this method could not be used for the conformally invariant case, i.e. $c = 1$.

An alternative pattern is presented here for other values of c corresponding to the conformally invariant case ($c = 1$) and the physical state ($\partial \cdot K = 0$ or $c = 0$). The general solution of the field equation can be written in terms of a generalized polarization five-vector and a dS plane wave

$$K_\alpha(x) = \mathcal{E}_\alpha(x, \xi, Z, \sigma)(x \cdot \xi)^\sigma.$$

Using the pattern of Eq. (4.3), we introduce two constant parameters b_1 and b_2 to define the following polarization vector:

$$\mathcal{E}_\alpha = \left[\bar{Z}_\alpha + b_1 \frac{Z \cdot \xi}{(x \cdot \xi)^2}\bar{\xi}_\alpha + b_2 \frac{Z \cdot x}{x \cdot \xi}\bar{\xi}_\alpha\right].$$

This vector satisfies the condition $x \cdot \mathcal{E} = 0$. Imposing the condition $\partial \cdot K = 0$ to meet the criterion of a physical state, i.e. divergenceless condition, it is shown that the two constant parameters are

$$b_1 = \frac{1-\sigma}{2+\sigma}, \qquad b_2 = -1.$$

In this case, the divergence of the polarization five-vector is

$$\partial \cdot \mathcal{E} = -\sigma \frac{\bar{\xi} \cdot \mathcal{E}}{x \cdot \xi} = \frac{-3\sigma}{2+\sigma}\frac{Z \cdot \xi}{x \cdot \xi}.$$

The choice of $Z \cdot \xi = 0$, which results in $\partial \cdot \mathcal{E} = 0$, is a suitable restriction on the arbitrary five-vector Z^α, since it renders a simplified solution and in the null curvature limit it embarks on the Minkowskian solution of the two-point function. By the use of this condition ($Z \cdot \xi = 0$), the degrees of freedom for the arbitrary five-vector field Z^α reduce to 4. The generalized polarization vector then becomes

$$\mathcal{E}^d_\alpha = \left[\bar{Z}_\alpha - \frac{Z \cdot x}{x \cdot \xi}\bar{\xi}_\alpha\right] = \left[Z_\alpha - \frac{Z \cdot x}{x \cdot \xi}\xi_\alpha\right].$$

This polarization vector satisfies the interesting relation:

$$Q^{(0)}\mathcal{E}^d_\alpha \phi = \mathcal{E}^d_\alpha Q^{(0)}\phi.$$

If the vector field K does not satisfy the divergenceless condition, by the choice of $Z \cdot \xi = 0$, it takes the following form:

$$K_\alpha(x) = \mathcal{E}_\alpha(x, \xi, Z, \sigma)(x \cdot \xi)^\sigma,$$
$$\mathcal{E}_\alpha = \left[\bar{Z}_\alpha + a\frac{Z \cdot x}{x \cdot \xi}\bar{\xi}_\alpha\right] = \mathcal{E}^d_\alpha + (1+a)\frac{Z \cdot x}{x \cdot \xi}\bar{\xi}_\alpha,$$

where a is an arbitrary constant parameter. This parameter depends on the gauge parameter c and homogenous degree σ. If K satisfies the vector field equation (2.13), the arbitrary constant parameter a is fixed at once. The divergence of the vector field, through the condition $Z \cdot \xi = 0$, is

$$\partial \cdot K(x) = (1+a)(\sigma+4) x \cdot Z (x \cdot \xi)^\sigma.$$

This equation is itself a scalar field with the homogenous degree ($\sigma + 1$)

$$Q^{(0)}\partial \cdot K(x) = -(\sigma+1)(\sigma+4)\partial \cdot K(x).$$

Implementation of K in the wave equation (2.13) together with the identity,

$$Q^{(0)} x \cdot Z (x \cdot \xi)^\sigma x^\alpha = -(\sigma+2)(\sigma+5) x \cdot Z (x \cdot \xi)^\sigma x^\alpha - 2\left[Z^\alpha + \sigma\frac{Z \cdot x}{x \cdot \xi}\xi^\alpha\right](x \cdot \xi)^\sigma,$$

results in the following system of equations:

$$\begin{cases} a[\sigma(\sigma+3)+2] - (1+a)\sigma(-2+c\sigma+4c) = 0, & \text{(I)} \\ \sigma(\sigma+3)+2-(1+a)(-2+c\sigma+4c) = 0, & \text{(II)} \\ (1+a)(\sigma+1)(\sigma+4)(1-c) = 0. & \text{(III)} \end{cases} \tag{4.4}$$

All other values of a and σ can be categorized in terms of various choices of the gauge parameter c. In the present chapter, three values of c ($c = 0, 1, \frac{2}{3}$) have been studied.

For $c = \frac{2}{3}$, the solutions of the system of the equations (4.4) are

$$\begin{cases} \sigma = -1, & a = \text{arbitrary}, \\ \sigma = -2, & a = -1. \end{cases}$$

For the case $\sigma = -1$, value $a = 0$ leads to the previous solution Eq. (4.3). In this gauge, the solution of the wave equation becomes

$$K_\alpha = \left(\bar{Z}_\alpha + (\sigma+1)\frac{Z\cdot x}{x\cdot\xi}\bar{\xi}_\alpha\right)(x\cdot\xi)^\sigma = \left(\bar{Z}_\alpha + \frac{\sigma+1}{\sigma}Z\cdot x\bar{\partial}_\alpha\right)(x\cdot\xi)^\sigma. \quad (4.5)$$

For $c=0$, the two solutions of the system of the equations (4.4) are

$$\begin{cases}\sigma=-1, & a=-1,\\ \sigma=-2, & a=-1.\end{cases}$$

In this gauge the solution of the wave equation becomes

$$K_\alpha = \left(\bar{Z}_\alpha - \frac{Z\cdot x}{x\cdot\xi}\bar{\xi}_\alpha\right)(x\cdot\xi)^\sigma = \left(\bar{Z}_\alpha - \frac{1}{\sigma}Z\cdot x\bar{\partial}_\alpha\right)(x\cdot\xi)^\sigma, \quad (4.6)$$

which is clearly divergenceless. This field can be associated with the UIR's $\Pi^{\pm}_{1,1}$ of the dS group and corresponds to the physical state.

For $c=1$, the solutions of the system of the equations (4.4) are

$$\begin{cases}\sigma=0, & a=0,\\ \sigma=-1, & a=-1,\\ \sigma=-2, & a=\text{arbitrary},\\ \sigma=-3, & a=-3.\end{cases} \quad (4.7)$$

In this gauge, fixing a to be -2 while $\sigma=-2$, the solution results in

$$K_\alpha = \left(\bar{Z}_\alpha + \sigma\frac{Z\cdot x}{x\cdot\xi}\bar{\xi}_\alpha\right)(x\cdot\xi)^\sigma = (\bar{Z}_\alpha + Z\cdot x\bar{\partial}_\alpha)(x\cdot\xi)^\sigma.$$

Equation (3.19) restricts the solutions to the values $\sigma=-2,-3$, which are the conformally invariant solutions.

It is more suitable to represent entire solutions of the field equation in the following form:

$$K_\alpha = \left(\bar{Z}_\alpha + a(c,\sigma)\frac{Z\cdot x}{x\cdot\xi}\bar{\xi}_\alpha\right)(x\cdot\xi)^\sigma = \left(\bar{Z}_\alpha + \frac{a(c,\sigma)}{\sigma}Z\cdot x\bar{\partial}_\alpha\right)(x\cdot\xi)^\sigma. \quad (4.8)$$

In contrast to the Minkowskian case, the generalized polarization vector $\mathcal{E}_\alpha(x,\xi,Z,c,\sigma)$ is a function of the spacetime x. These solutions, however, are problematic as well. In contrast to the "massive" field case in de Sitter space, the two solutions are not complex conjugate of each other. We shall return to this point when we construct the quantum field in the forthcoming chapter. There also appears an arbitrary constant five-vector Z (with one constraint $Z\cdot\xi=0$) in the solution of the field equation. This is reminiscent of the problem of the vacuum state in the curved space. For simplicity, we impose the condition that the solution in the limit $H=0$ must be exactly the Minkowskian solution. This condition in the massive scalar, spinor, and vector cases, results in the choice of Euclidian vacuum. The limit $H=0$ for the "massless" conformally coupled scalar field, however, cannot be defined in this notation [2]. In order to obtain the proper behavior of the massless conformally coupled scalar field in the limit $H=0$, we must use a system of bounded global coordinates (X^μ, $\mu=0,1,2,3$) well suited to describe a compactified version of dS space, namely $S^3\times S^1$ (Lie sphere) [6]. This mode defines the Euclidian vacuum [17]. The above procedure, however, cannot be used for the massless vector field, since the polarization vector which depends on the de Sitter plane wave could not be defined in the null curvature limit. It is important to note that the two-point function of the conformally coupled scalar field, obtained by the two different methods are one and the same. Proper choice of vacuum could also be achieved by imposing the condition that in the null curvature limit, the two-point function takes the form of its Minkowskian counterpart. By imposing the following conditions:

(i) setting the vector two-point function to have a maximally symmetric form of bivectors in the ambient space notation and,

(ii) its exact equivalence with the Minkowskian counterpart in the null curvature limit,

the constant vector Z and the normalization of the vector field are fixed,

$$Z^\lambda\cdot Z^{\lambda'} = \eta^{\lambda\lambda'}, \qquad \sum_{\lambda=0}^{3} Z^\lambda_\alpha Z^\lambda_\beta = -\eta_{\alpha\beta}. \quad (4.9)$$

λ takes four values for different polarizations. Henceforth the polarization vector can be defined as

$$\mathcal{E}(x,\xi,Z,c,\sigma) = \left(\bar{Z}^\lambda + a(c,\sigma)\frac{Z^\lambda\cdot x}{x\cdot\xi}\bar{\xi}\right) \equiv \mathcal{E}^\lambda(x,\xi,c,\sigma). \quad (4.10)$$

This polarization vector satisfies the following relation:

$$\mathcal{E}^\lambda(x,\xi,c,\sigma)\cdot\bar{\xi} = (a+1)(Z^\lambda\cdot x)(x\cdot\xi). \quad (4.11)$$

It can be shown that, by the use of Eq. (4.9), the properties of the dS polarization vector are very similar to the Minkowskian case:

$$\sum_{\lambda=0}^{3}\mathcal{E}^\lambda_\alpha(x,\xi,c,\sigma_1)\mathcal{E}^\lambda_{\alpha'}(x',\xi,c,\sigma_2) = -\left(\theta_\alpha\cdot\theta'_{\alpha'} + a_2\frac{\theta_\alpha\cdot x'}{x'\cdot\xi}\bar{\xi}'_{\alpha'} + a_1\frac{\theta'_{\alpha'}\cdot x}{x\cdot\xi}\bar{\xi}_\alpha + a_2a_1\frac{x'\cdot x}{(x\cdot\xi)(x'\cdot\xi)}\bar{\xi}_\alpha\bar{\xi}'_{\alpha'}\right). \quad (4.12)$$

Because of the presence of a singularity on the three-dimensional lightlike manifold, the dS vector-plane wave solutions (4.8) are not globally defined [2]. For a complete determination of the solutions (4.8), one may consider the solutions in the complex de Sitter space-time $X_H^{(c)}$. The complex de Sitter space-time is defined as

$$X_H^{(c)} = \{z = x + iy \in \mathbb{C}^5; \eta_{\alpha\beta} z^\alpha z^\beta = (z^0)^2 - \vec{z}\cdot\vec{z} - (z^4)^2 = -1\} = \{(x, y) \in \mathbb{R}^5 \times \mathbb{R}^5; x^2 - y^2 = -1, x \cdot y = 0\}. \tag{4.13}$$

Let $T^\pm = \mathbb{R}^5 + iV^\pm$ be the forward and backward tubes in $\mathbb{C}^5$. The domain V^+ (respectively V^-) stems from the causal structure on X_H:

$$V^\pm = \{x \in \mathbb{R}^5; x^0 <> \sqrt{\|\vec{x}\|^2 + (x^4)^2}\}. \tag{4.14}$$

Respective intersections with $X_H^{(c)}$ are

$$\mathcal{T}^\pm = T^\pm \cap X_H^{(c)}, \tag{4.15}$$

which will be called forward and backward tubes of the complex dS space $X_H^{(c)}$. Finally, we define the "tuboid" on $X_H^{(c)} \times X_H^{(c)}$ by

$$\mathcal{T}_{12} = \{(z, z'); z \in \mathcal{T}^+, z' \in \mathcal{T}^-\}. \tag{4.16}$$

Details are given in [2]. When z varies in $\mathcal{T}^+$ (or $\mathcal{T}^-$) and ξ lies on the positive cone $\mathcal{C}^+$,

$$\xi \in \mathcal{C}^+ = \{\xi \in \mathcal{C}; \xi^0 > 0\},$$

the plane wave solutions are globally defined, since the imaginary part of $(z\cdot\xi)$ has a fixed sign. The phase is chosen such that

$$\text{boundary value of } (z\cdot\xi)^\sigma|_{x\cdot\xi>0} > 0. \tag{4.17}$$

Therefore we have

$$K_{\sigma,c}^{\xi,\lambda}(z) = \mathcal{E}^{(\lambda)}(z, \xi, \sigma, c)(z\cdot\xi)^\sigma, \tag{4.18}$$

in which $z \in X_H^{(c)}$ and $\xi \in \mathcal{C}^+$.

V. THE TWO-POINT FUNCTION

The two-point function of the massless conformally coupled scalar field is studied first, in this section. The field operator and the vacuum states are defined properly to result in this two-point function. The massless vector two-point function is then calculated. The vector field operator and the vacuum state are defined to suit the above two-point function in the next stage. The null curvature limits of the two-point functions are then discussed. The relations between the ambient space notation and the intrinsic coordinates are studied in the final stage.

A. Scalar two-point function

The Wightman two-point function for a conformally coupled scalar field is [2]

$$\begin{aligned}\mathcal{W}_0(x, x') &= c_0 \int_T d\mu_T(\xi)[(x\cdot\xi)_+^{-1} + e^{i\pi}(x\cdot\xi)_-^{-1}] \\ &\quad\times [(x'\cdot\xi)_+^{-2} + e^{-2i\pi}(x'\cdot\xi)_-^{-2}] \\ &= bv W_0(z, z') = bv C_0 P_{-1}^{(5)}(z\cdot z'),\end{aligned} \tag{5.1}$$

where $C_0 = \frac{\Gamma(2)\Gamma(1)}{2^4\pi^2} = 2\pi^2 c_0$ and

$$W_0(z, z') = \frac{1}{8\pi^2}\frac{-1}{1 - \mathcal{Z}(z, z')}, \qquad \mathcal{Z}(z, z') = -z\cdot z'. \tag{5.2}$$

The function $P_\sigma^{(5)}$ is the generalized Legendre function of the first kind given by the following integral representation (valid for $\cos\theta \in \mathbb{C}\backslash]-\infty, -1[$) [2]:

$$\begin{aligned}P_\sigma^{(5)}(\cos\theta) &= \frac{4}{\pi}(\sin\theta)^{-2}\int_0^\theta \cos\left[\left(\sigma + \frac{3}{2}\right)\tau\right] \\ &\quad\times \sqrt{2(\cos\tau - \cos\theta)}d\tau.\end{aligned} \tag{5.3}$$

This has the interesting property of $P_\sigma^{(5)} = P_{-3-\sigma}^{(5)}$. By determining the boundary values of the equation (5.2), we obtain [4,18]

$$\begin{aligned}\mathcal{W}_0(x, x') = \frac{-1}{8\pi^2}\Big[&\frac{1}{1 - \mathcal{Z}(x, x')} \\ &- i\pi\epsilon(x^0 - x'^0)\delta(1 - \mathcal{Z}(x, x'))\Big].\end{aligned} \tag{5.4}$$

In the theorem 4.2 of [2], it has been shown that this two-point function satisfies the following conditions: (i) positivity, (ii) locality, (iii) covariance, and (iv) normal analyticity. A de Sitter free field can be defined at this stage.

Using the superposition principle and two solutions of the scalar field equation, globally defined in the complex de Sitter space, the general solution of the scalar field is thoroughly defined:

$$\phi(z) = \int_T \{a(\xi, \sigma_1)(z\cdot\xi)^{\sigma_1} + b(\xi, \sigma_2)(z\cdot\xi)^{\sigma_2}\}d\mu_T(\xi), \tag{5.5}$$

where T denotes an orbital basis of $\mathcal{C}^+$ and $\sigma_1 = -1$, $\sigma_2 = -3 - \sigma_1 = -2$. $d\mu_T(\xi)$ is an invariant measure on $\mathcal{C}^+$ [2]. The boundary value of this equation is the scalar field, defined globally in the de Sitter space

$$\phi(x) = \int_T \{a(\xi, \sigma_1)[(x\cdot\xi)_+^{\sigma_1} + e^{-i\pi\sigma_1}(x\cdot\xi)_-^{\sigma_1}] + b(\xi, \sigma_2)[(x\cdot\xi)_+^{\sigma_2} + e^{i\pi\sigma_2}(x\cdot\xi)_-^{\sigma_2}]\} d\mu_T(\xi), \tag{5.6}$$

where [19]

$$(x\cdot\xi)_+ = \begin{cases} 0 & \text{for } x\cdot\xi \le 0 \\ (x\cdot\xi) & \text{for } x\cdot\xi > 0. \end{cases}$$

Since the measure satisfies $d\mu_T(l\xi) = l^3 d\mu_T(\xi)$, a and b must satisfy the homogeneity condition

$$a(l\xi, \sigma) = l^{-\sigma-3}a(\xi), \qquad b(l\xi, \sigma) = l^{-\sigma-3}b(\xi, \sigma).$$

Implementation of these conditions results in the integral representation (5.6) that is independent of the orbital basis T [2].

As far as irreducible unitary representations of the de Sitter group are concerned, the two solutions of the wave equation $[(x\cdot\xi)^{\sigma_1}$ and $(x\cdot\xi)^{\sigma_2=-3-\sigma_1}]$ are equivalent to one another. Naturally, the solutions are a complex conjugate of each other ($\sigma_1^* = \sigma_2$) for the principal series representation, since the homogeneity degree of functions (σ) is complex in this case. In the case of the complementary series representation, however, the homogeneity degree is real and as a result the two solutions, in spite of the equivalence of their corresponding representation (σ_1, $\sigma_2 = -3-\sigma_1$), are not complex conjugates of each other.

Now we define the conformally scalar field operator, which results in the above two-point function

$$\phi(x) = \int_T \{a(\xi, \sigma_1)[(x\cdot\xi)_+^{\sigma_1} + e^{-i\pi\sigma_1}(x\cdot\xi)_-^{\sigma_1}] + a^\dagger(\xi, \sigma_2)[(x\cdot\xi)_+^{\sigma_2} + e^{i\pi\sigma_2}(x\cdot\xi)_-^{\sigma_2}]\} d\mu_T(\xi). \tag{5.7}$$

The vacuum state is defined as follows:

$$a(\xi, \sigma)|\Omega\rangle = 0,$$

which is fixed by imposing the condition that, in the null curvature limit, the Wightman two-point function becomes exactly the same as its Minkowskian counterpart. This vacuum, $|\Omega\rangle$, is equivalent to the Euclidean vacuum. "One particle" states are

$$a^\dagger(\xi, \sigma)|\Omega\rangle = |\xi, \sigma\rangle. \tag{5.8}$$

The field operator (5.7) gives the above two-point function (5.1)

$$\mathcal{W}_0(x, x') = \langle\Omega \mid \phi(x)\phi(x') \mid \Omega\rangle.$$

For the hyperbolic-type submanifold, T_4, the measure is $d\mu_{T_4}(\xi) = d^3\vec{\xi}/\xi_0$ and the canonical commutation relations are represented by

$$[a(\xi, \sigma), a^\dagger(\xi', \sigma')] = \sqrt{c_0}\,\delta_{\sigma,-\sigma'-3}\xi^0\delta^3(\vec{\xi} - \vec{\xi}'). \tag{5.9}$$

B. Vector two-point function

The general vector two-point function is calculated explicitly at this stage in the ambient space notation. The vector two-point function, which is invariant under the conformal group, is then calculated and the Minkowskian limits are discussed. Finally, the relation between the ambient space notation and the intrinsic coordinates are determined.

Similar to the field solution (4.1), the vector two-point function $\mathcal{W}_{\alpha\alpha'}(x, x')$, which is a solution of the wave equation (2.13), can be found simply in terms of scalar Wightman two-point functions,

$$\mathcal{W}_{\alpha\alpha'}(x, x') = \langle\Omega, K_\alpha(x)K_{\alpha'}(x')\Omega\rangle = \theta_\alpha\cdot\theta'_{\alpha'}\mathcal{W}_1(x, x') + \bar\partial_\alpha\bar\partial'_{\alpha'}\mathcal{W}_2(x, x'), \tag{5.10}$$

where $\bar\partial_\alpha\bar\partial'_{\alpha'} = \bar\partial'_{\alpha'}\bar\partial_\alpha$. The vector two-point function was obtained in terms of a massless conformally coupled scalar two-point function $\mathcal{W}_0(x, x')$ [6],

$$\mathcal{W}_{\alpha\alpha'}(x, x') = \theta_\alpha\cdot\theta'_{\alpha'}\mathcal{W}_0(x, x') - \frac{c}{2(1-c)}\bar\partial_\alpha[\bar\partial\cdot\theta'_{\alpha'} + x\cdot\theta'_{\alpha'}]\mathcal{W}_0 + \frac{2-3c}{1-c}\bar\partial_\alpha[Q^{(0)}]^{-1}x\cdot\theta'_{\alpha'}\mathcal{W}_0 \equiv D_{\alpha\alpha'}(x, x', c)\mathcal{W}_0, \qquad c \neq 1. \tag{5.11}$$

We can write this equation in the following form:

$$\mathcal{W}^c_{\alpha\alpha'} = \mathcal{W}^{2/3}_{\alpha\alpha'} + \frac{\frac{2}{3}-c}{(1-c)}\bar\partial_\alpha[Q^{(0)}]^{-1}\partial\cdot\mathcal{W}^{2/3}_{\alpha'}, \tag{5.12}$$

where

$$\partial\cdot\mathcal{W}^{2/3}_{\alpha'} = 3(x\cdot\theta'_{\alpha'} + [\bar\partial\cdot\theta'_{\alpha'} + x\cdot\theta'_{\alpha'}])\mathcal{W}_0,$$

$$Q^{(0)}\partial\cdot\mathcal{W}^{2/3}_{\alpha'} = 0.$$

These two-point functions can only be defined properly for the gauge $c = \frac{2}{3}$ since the term $[Q^{(0)}]^{-1}\partial\cdot\mathcal{W}^{2/3}_{\alpha'}$ becomes singular.

Now, we consider the case $c = 1$, i.e. the conformally invariant two-point function, and $c = 0$, i.e. the physical part of the two-point function. The massless vector two-point function, which satisfies the field equation, is obtained as the boundary value of the analytic two-point function $W_{\alpha\alpha'}(z, z')$:

$$W_{\alpha\alpha'}(z, z') = c_s\int_T\sum_\lambda \mathcal{E}^\lambda_\alpha(z, \xi, c, \sigma_1)\mathcal{E}^\lambda_{\alpha'}(z', \xi, c, \sigma_2) \times (z\cdot\xi)^{\sigma_1}(z'\cdot\xi)^{\sigma_2}d\mu_T(\xi), \tag{5.13}$$

where $\sigma_1 + \sigma_2 = -3$. With the help of Eq. (4.10), the vector two-point function is easily expanded in terms of the analytic scalar two-point function $W_s(z, z')$:

$$W_{\alpha\alpha'}(z,z') = c_s \int_T \sum_\lambda \left(\bar{Z}^\lambda_\alpha + a_1 \frac{Z^\lambda \cdot z}{z\cdot\xi}\bar{\xi}_\alpha\right)\left(\bar{Z}'^\lambda_{\alpha'} + a_2 \frac{Z^\lambda \cdot z'}{z'\cdot\xi}\bar{\xi}_{\alpha'}\right)(z\cdot\xi)^{\sigma_1}(z'\cdot\xi)^{\sigma_2} d\mu_T(\xi)$$
$$= \sum_\lambda \left(\bar{Z}^\lambda_\alpha + \frac{a_1}{\sigma_1} Z^\lambda \cdot z\bar{\partial}_\alpha\right)\left(\bar{Z}'^\lambda_{\alpha'} + \frac{a_2}{\sigma_2} Z^\lambda \cdot z'\bar{\partial}'_{\alpha'}\right) c_s \int_T (z\cdot\xi)^{\sigma_1}(z'\cdot\xi)^{\sigma_2} d\mu_T(\xi)$$
$$= \sum_\lambda \left(\bar{Z}^\lambda_\alpha + \frac{a_1}{\sigma_1} Z^\lambda \cdot z\bar{\partial}_\alpha\right)\left(\bar{Z}'^\lambda_{\alpha'} + \frac{a_2}{\sigma_2} Z^\lambda \cdot z'\bar{\partial}'_{\alpha'}\right) W_s(z,z'). \tag{5.14}$$

We define the arbitrary constant tensor T of rank-2 as

$$T_{\beta\gamma} \equiv \sum_\lambda Z^\lambda_\beta Z^\lambda_\gamma.$$

The equation (5.14) in terms of this arbitrary tensor can be written in the following form:

$$W_{\alpha\alpha'}(z,z') = T^{\beta\gamma}\left(\theta_{\alpha\beta}\theta'_{\alpha'\gamma} + \frac{a_1}{\sigma_1}\theta'_{\alpha'\gamma} z_\beta \bar{\partial}_\alpha + \frac{a_2}{\sigma_2}\theta_{\alpha\beta} z'_\gamma \bar{\partial}'_{\alpha'} + \frac{a_1 a_2}{\sigma_1\sigma_2} z_\beta z'_\gamma \bar{\partial}_\alpha \bar{\partial}'_{\alpha'}\right) c_s \int_T (z\cdot\xi)^{\sigma_1}(z'\cdot\xi)^{\sigma_2} d\mu_T(\xi)$$
$$\equiv D_{\alpha\alpha'}(z,z',c,\sigma) W_s(z,z'), \tag{5.15}$$

where $T_{\beta\gamma}$ and c_s are arbitrary constants and $W_s(z,z')$ is the scalar two-point function.

In the previous paper [20], we showed that a maximally symmetric bivector in the ambient space notation has the following form:

$$M_{\alpha\alpha'}(z,z') = \theta_\alpha \cdot \theta'_{\alpha'} f(Z) + (\theta_\alpha \cdot z')(\theta'_{\alpha'} \cdot z) g(Z). \tag{5.16}$$

By imposing the following conditions:

(i) setting the vector two-point function to have a maximally symmetric form of bivectors in the ambient space notation and,

(ii) its exact equivalence with the Minkowskian counterpart in the null curvature limit,

the constant tensor T and the normalization constant c_s are fixed:

$$Z^\lambda \cdot Z^{\lambda'} = \eta^{\lambda\lambda'}, \qquad \sum_{\lambda=0}^{3} Z^\lambda_\alpha Z^\lambda_\beta = -\eta_{\alpha\beta},$$
$$c_s = e^{i\pi(\sigma+(3/2))}\frac{\Gamma(-\sigma)\Gamma(3+\sigma)}{2^5\pi^4},$$

where $\sigma_1 \equiv \sigma$. Note that the choice of normalization constant corresponds to the Euclidean vacuum. $W_s(z,z')$ can be written as a hypergeometric function (see [2]):

$$W_s(z,z') = C_s\, {}_2F_1\left(-\sigma, 3+\sigma; 2; \frac{1+Z}{2}\right) = C_s P^5_\sigma(-Z)$$
$$\text{with } C_s = \frac{\Gamma(-\sigma)\Gamma(3+\sigma)}{2^4\pi^2}.$$

The two-point function, as the boundary value of the analytic two-point functions, can be attained explicitly in terms of the following class of integral representation:

$$\mathcal{W}_{\alpha\alpha'}(x,x') = c_s \int_T d\mu_T(\xi)[(x\cdot\xi)^\sigma_+ + e^{-i\pi\sigma}(x\cdot\xi)^\sigma_-]$$
$$\times[(x'\cdot\xi)^{-3-\sigma}_+ + e^{i\pi(\sigma+3)}(x'\cdot\xi)^{-3-\sigma}_-]$$
$$\times \sum_{\lambda=0}^{3} \mathcal{E}^\lambda_\alpha(x,\xi,c,\sigma)\mathcal{E}^\lambda_{\alpha'}(x',\xi,c,-3-\sigma). \tag{5.17}$$

This relation defines the two-point function in terms of global plane waves on X_H. Its explicit form is

$$\mathcal{W}_{\alpha\alpha'}(x,x') = \theta_\alpha \cdot \theta'_{\alpha'}\left(1 - \frac{a_1 a_2}{\sigma_1\sigma_2} Z\frac{d}{dZ}\right)\mathcal{W}_s(Z)$$
$$+ (\theta_\alpha \cdot x')(\theta'_{\alpha'} \cdot x)\left(\frac{a_1 a_2}{\sigma_1\sigma_2} Z\frac{d^2}{dZ^2} - \left(\frac{a_1}{\sigma_1} + \frac{a_2}{\sigma_2}\right)\frac{d}{dZ}\right)\mathcal{W}_s(Z), \tag{5.18}$$

where we have used the identity $\bar{\partial}_\alpha \mathcal{W}_s(Z) = -(\theta_\alpha \cdot x')\frac{d}{dZ}\mathcal{W}_s(Z)$. For $c \neq 1$, $\mathcal{W}_s$ is a conformally coupled scalar two-point function and for $c = \frac{2}{3}$, the previous result is obtained [6].

The vector two-point functions could be constructed by dS plane wave functions $(x\cdot\xi)^\sigma$ and $(x\cdot\xi)^{-3-\sigma}$ that are directly related to irreducible scalar representation of de Sitter group, i.e. discrete, complementary, and principal series representations. The two plane waves are equivalent as far as irreducible representation is concerned. In the case $c = 1$, four sets of solutions are obtained, all related to different values of σ [Eq. (4.7)]. Two solutions ($\sigma = -1, -2$) are related to the conformally coupled scalar field. The other solutions ($\sigma = -3, 0$) are related to the minimally coupled scalar field. In the above formalism, Eq. (5.18), two different vector two-point functions can be defined, which are not covariant under the conformal transformation. A conformally covariant vector two-point function must satisfy Eq. (3.19). This equation restricts the solutions of the wave equation corresponding to the values $\sigma = -2, -3$, which are not equivalent as far as irreducible representation is concerned. In this case, the integral representation (5.15) cannot be properly defined, i.e. it depends on the orbits of integration.

To obtain a conformally covariant vector two-point function, the two-point functions in the form of (5.10) can be utilized to satisfy the two equations (3.18) and (3.19) simultaneously. After some algebra, we obtained $\mathcal{W}_1$, which satisfies

$$\frac{d^2}{dZ^2}\mathcal{W}_1 = 0, \quad \text{or } \mathcal{W}_1 = C_1 + C_2 Z.$$

In order to obtain a regular solution in the large Z domain, we impose the condition $C_2 = 0$. This results in the following equation for $\mathcal{W}_2$:

$$Q^{(0)}(Q^{(0)} - 2)\mathcal{W}_2 = 24 C_1 Z,$$

where

$$Q^{(0)} = (1 - Z^2)\frac{d^2}{dZ^2} - 4Z\frac{d}{dZ}.$$

These solutions are simultaneously covariant under the conformal group transformation as well as the de Sitter group. These solutions are associated with a reducible representation of the de Sitter group. By imposing the condition that the vector two-point function should propagate on the light cone, we obtained

$$C_1 = 0, \qquad \mathcal{W}_2 = \mathcal{W}_0,$$

$$\mathcal{W}_{\alpha\alpha'}(x, x') = \left(-\theta_\alpha \cdot \theta'_{\alpha'}\frac{d}{dZ} + (\theta_\alpha \cdot x')(\theta'_{\alpha'} \cdot x)\frac{d^2}{dZ^2}\right) \times \mathcal{W}_0 = D_{\alpha\alpha'}\mathcal{W}_0, \tag{5.19}$$

where $\mathcal{W}_0$ is a conformally scalar two-point function.

The vector field commutation relation is

$$\begin{aligned} iG_{\alpha\alpha'}(x, x') &= \mathcal{W}_{\alpha\alpha'}(x, x') - \mathcal{W}^*_{\alpha\alpha'}(x, x') \\ &= [K_\alpha(x), K_{\alpha'}(x')] = D_{\alpha\alpha'}[\phi(x), \phi(x')] \\ &= D_{\alpha\alpha'} iG(x, x'), \end{aligned} \tag{5.20}$$

where $iG(x, x')$ is the commutation relation of the conformally coupled scalar field [2,4,18],

$$i[\phi(x), \phi(x')] = \frac{H^2}{4}\epsilon(x^0, x'^0)\delta(Z(x, x') - 1), \tag{5.21}$$

where $Z(x, x') = -H^2 x \cdot x' = 1 + \frac{H^2}{2}(x - x')^2$ and

$$\epsilon(x^0, x'^0) = \begin{cases} 1 & x^0 > x'^0 \\ 0 & x^0 = x'^0 \\ -1 & x^0 < x'^0. \end{cases} \tag{5.22}$$

We obtain

$$[K_\alpha(x), K_{\alpha'}(x')] = \frac{H^2}{4i} D_{\alpha\alpha'}\epsilon(x^0, x'^0)\delta(Z(x, x') - 1). \tag{5.23}$$

This field propagates on the light cone ($Z = 1$) and the logarithmic singularity does not appear. Similar to the scalar field [2], the retarded and advance propagators are defined, respectively, by

$$G^{\text{ret}}_{\alpha\alpha'}(x, x') = -\theta(x^0 - x'^0)G_{\alpha\alpha'}(x, x'), \tag{5.24}$$

$$\begin{aligned} G^{\text{adv}}_{\alpha\alpha'}(x, x') &= \theta(x'^0 - x^0)G_{\alpha\alpha'}(x, x') \\ &= G^{\text{ret}}_{\alpha\alpha'}(x, x') + G_{\alpha\alpha'}(x, x'). \end{aligned} \tag{5.25}$$

The "Feynman propagator" is also defined by

$$\begin{aligned} iG^{(F)}_{\alpha\alpha'}(x, x') &= \langle \Omega, TK_\alpha(x)K_{\alpha'}(x')\Omega\rangle \\ &= \theta(x^0 - x'^0)\mathcal{W}_{\alpha\alpha'}(x, x') \\ &\quad + \theta(x'^0 - x^0)\mathcal{W}_{\alpha'\alpha}(x', x). \end{aligned} \tag{5.26}$$

Using the Wightman two-point function that satisfies the conditions: (i) positivity, (ii) locality, (iii) covariance, (iv) normal analyticity, (v) transversality, and (vi) divergencelessness, we have already constructed the covariant quantum massive vector field in dS space [5]. In the previous paper [6], it has been shown that, for the massless vector field, we do not have necessarily the divergenceless condition and as a result we cannot associate this field with a UIR's of the dS group. To maintain the covariant condition in field quantization, we must use an indecomposable representation of dS group. In this case, however, we do not have the positivity condition and there appear unphysical negative and null norm states which are considered as the longitudinal "photons" and the scalar photons [6]. In order to remove the above problems it is necessary to impose some new conditions similar to Minkowskian vector field. Following this procedure, positivity and divergenceless conditions are simultaneously avoided. In contrast to the massive vector field, it is not evident that two-point function could be used for construction of massless quantum vector field.

However, in order to obtain the above two-point functions, the vector field operators are defined as

$$\begin{aligned} K_\alpha(x) = \int_T \sum_\lambda \{ & a(\xi, \sigma_1, \lambda)\mathcal{E}^{(\lambda)}_\alpha(x, \xi, c, \sigma_1)[(x\cdot\xi)^{\sigma_1}_+ \\ & + e^{-i\pi\sigma_1}(x\cdot\xi)^{\sigma_1}_-] + a^\dagger(\xi, \sigma_2, \lambda)\mathcal{E}^{(\lambda)}_\alpha(x, \xi, c, \sigma_2) \\ & \times [(x\cdot\xi)^{\sigma_2}_+ + e^{i\pi\sigma_2}(x\cdot\xi)^{\sigma_2}_-]\} d\mu_T(\xi). \end{aligned} \tag{5.27}$$

The vacuum state is defined as follows:

$$a(\xi, \sigma, \lambda)|\Omega\rangle = 0.$$

This vacuum, $|\Omega\rangle$, is equivalent to the Euclidean vacuum. "One particle" states are

$$a^{\dagger}(\xi, \sigma, \lambda)|\Omega\rangle = |\xi, \sigma, \lambda\rangle. \tag{5.28}$$

For the hyperbolic type submanifold T_4, the measure is $d\mu_{T_4}(\xi) = d^3\vec{\xi}/\xi_0$ and the canonical commutation relations are

$$[a(\xi, \sigma, \lambda), a^{\dagger}(\xi', \sigma', \lambda')] = -\sqrt{c_s}\delta_{\sigma,-\sigma'-3}\eta^{\lambda\lambda'}\xi^0\delta^3(\vec{\xi}-\vec{\xi}'). \tag{5.29}$$

It is important to note that the null curvature limit of this vector field operator is not defined, however, it does exist for the vector two-point function.

The Minkowskian limit is now established for the above problem. First, the two-point function of the massless conformally coupled scalar field is considered. In contrast to the field operator, where the null curvature limit ($H = 0$) exists only in the intrinsic notation, the two-point function [Eq. (5.4)] has a Minkowskian limit in both notations (intrinsic notations [18] and ambient space notations [4]). The limit $H = 0$ of this equation is

$$\lim_{H=0}\mathcal{W}_0(x, x') = \mathcal{W}^{(M)}(X, X') = \frac{-1}{8\pi^2}\left[\frac{1}{\mu} + i\pi\epsilon(t-t')\delta(\mu)\right],$$
$$2\mu = (X - X')^2. \tag{5.30}$$

This is exactly the two-point function for a massless scalar field in Minkowski space. For the null curvature limit, the vector two-point function of the Minkowski space is obtained in the same gauge $c = 0$,

$$\lim_{H=0}\mathcal{W}_{\alpha\alpha'}(x, x') = \eta_{\mu\nu}\mathcal{W}^{(M)}(X, X') = \mathcal{W}^{(M)}_{\mu\nu}(X, X').$$

Finally, let us write the intrinsic expression of the two-point functions. In the previous paper [20], we presented the following relations between the ambient space and intrinsic notations:

$$Q_{ab'} \equiv \frac{\partial x^{\alpha}}{\partial X^a}\frac{\partial x'^{\beta'}}{\partial X'^{b'}}\mathcal{W}_{\alpha\beta'}(x, x'),$$

where

$$\frac{\partial x^{\alpha}}{\partial X^a}\frac{\partial x'^{\beta'}}{\partial X'^{b'}}\theta_{\alpha}\cdot\theta'_{\beta'} = g_{ab'} + (\mathcal{Z}-1)n_a n_{b'},$$
$$\frac{\partial x^{\alpha}}{\partial X^a}\frac{\partial x'^{\beta'}}{\partial X'^{b'}}\frac{H^2(\theta'_{\beta'}\cdot x)(\theta_{\alpha}\cdot x')}{1-\mathcal{Z}^2} = n_a n_{b'}.$$

n_a, $n_{b'}$, and $g_{ab'}$ are defined in terms of the geodesic distance $\mu(x, x')$, which is the distance along the geodesic connecting the points x and x'. Note that $\mu(x, x')$ can be defined by a unique analytic extension even when no geodesic connects x and x'. In this sense, these fundamental tensors form a complete set. They can be obtained by differentiating the geodesic distance:

$$n_a = \nabla_a\mu(x, x'), \qquad n_{a'} = \nabla_{a'}\mu(x, x')$$

and the parallel propagator

$$g_{ab'} = \sqrt{1-\mathcal{Z}^2}\nabla_a n_{b'} + n_a n_{b'}.$$

The geodesic distance is implicitly defined [2] for $\mathcal{Z} = -H^2 x\cdot x'$ by

$$\mathcal{Z} = \cosh(\mu H) \quad \text{for } x \text{ and } x' \text{ timelike separated,}$$
$$\mathcal{Z} = \cos(\mu H) \quad \text{for } x \text{ and } x' \text{ spacelike separated such that } |x\cdot x'| < H^{-2}.$$

Therefore, the two-point function in the intrinsic coordinates is

$$Q_{aa'} = (g_{aa'} + (\mathcal{Z}-1)n_a n_{a'})\left(1 - \frac{a_1 a_2}{\sigma_1\sigma_2}\mathcal{Z}\frac{d}{d\mathcal{Z}}\right)\mathcal{W}_s(\mathcal{Z}) + (1-\mathcal{Z}^2)n_a n_{a'}\left(\frac{a_1 a_2}{\sigma_1\sigma_2}\mathcal{Z}\frac{d^2}{d\mathcal{Z}^2} - \left(\frac{a_1}{\sigma_1} + \frac{a_2}{\sigma_2}\right)\frac{d}{d\mathcal{Z}}\right)\mathcal{W}_s(\mathcal{Z}). \tag{5.31}$$

This equation can be written in the following form:

$$Q_{aa'} = g_{aa'}\left(1 - \frac{a_1 a_2}{\sigma_1\sigma_2}\mathcal{Z}\frac{d}{d\mathcal{Z}}\right)\mathcal{W}_s(\mathcal{Z}) + n_a n_{a'}\Bigg((\mathcal{Z}-1)\left(1 - \frac{a_1 a_2}{\sigma_1\sigma_2}\mathcal{Z}\frac{d}{d\mathcal{Z}}\right) + (1-\mathcal{Z}^2)\left(\frac{a_1 a_2}{\sigma_1\sigma_2}\mathcal{Z}\frac{d^2}{d\mathcal{Z}^2} - \left(\frac{a_1}{\sigma_1} + \frac{a_2}{\sigma_2}\right)\frac{d}{d\mathcal{Z}}\right)\Bigg)\mathcal{W}_s(\mathcal{Z}). \tag{5.32}$$

If we have the two-point function in the intrinsic coordinate

$$Q_{aa'}(X, X') = g_{aa'}\alpha(\mathcal{Z}) + n_a n_{a'}\beta(\mathcal{Z}), \tag{5.33}$$

where $\alpha(\mathcal{Z})$ and $\beta(\mathcal{Z})$ are introduced by Eq. (4.22) in [10], the two-point function in the ambient space notation can be obtained:

$$\mathcal{W}_{\alpha\beta'}(x, x') = \left[\theta_{\alpha}\cdot\theta'_{\beta'} + \frac{H^2(\theta'_{\beta'}\cdot x)(\theta_{\alpha}\cdot x')}{1+\mathcal{Z}}\right]\alpha(\mathcal{Z}) + \left[\frac{H^2(\theta'_{\beta'}\cdot x)(\theta_{\alpha}\cdot x')}{1-\mathcal{Z}^2}\right]\beta(\mathcal{Z}), \tag{5.34}$$

$$\mathcal{W}_{\alpha\beta'}(x,x') = \theta_\alpha \cdot \theta'_{\beta'}\alpha(\mathcal{Z}) + H^2(\theta'_{\beta'}\cdot x)(\theta_\alpha\cdot x') \times \left(\frac{\alpha(\mathcal{Z})}{1+\mathcal{Z}} + \frac{\beta(\mathcal{Z})}{1-\mathcal{Z}^2}\right). \quad (5.35)$$

VI. CONCLUSION

In a series of papers we have shown that the quantization of various free fields in de Sitter space has a similar pattern of field quantization as in the Minkowski space. The establishment of the above similarity between the two spaces is based on the analyticity in the complexified pseudo-Riemannian manifold. The dS plane wave solution and the Fourier-Bros transformation in the dS space play an essential role in the above construction. In this paper, the conformally invariant wave equations in de Sitter space, for scalar and vector fields, are introduced and their solutions and their related two-point functions have been calculated. We have defined the covariant vector field operator and the ''particle states'' in the ambient space notation. It is important to note that, although the null curvature limit of this vector field operator is not defined, the limit of the two-point function does exist.

The irreducible unitary representations of the de Sitter group, which are associated with rank-2 massless tensor fields, have nonambiguous extensions to the conformal group SO(4, 2). On the other hand, the irreducible unitary representations of conformal group are precisely the unique extension of the massless Poincaré group representations with helicity ± 2. In the quantization process, due to the zero mode problem of the Laplace-Beltrami operator, de Sitter covariance is broken. To avoid this problem, a new method was presented for quantization of the massless minimally coupled scalar field in dS space-time [4,21,22]. Using this method for linear gravity, the two-point function is covariant and free of any infrared divergence [23,24]. In the forthcoming paper, we shall generalize this construction to the traceless rank-2 massless tensor field (conformal linear quantum gravity in dS space).

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