Black hole solutions of dimensionally reduced Einstein-Gauss-Bonnet gravity

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(Received 30 August 2006; published 11 December 2006)

We study the phase space of the spherically symmetric solutions of the system obtained from the dimensional reduction of the six-dimensional Einstein-Gauss-Bonnet action. We show that all the physically significant solutions are either asymptotically flat or asymptotically anti–de Sitter.

DOI: 10.1103/PhysRevD.74.124008 PACS numbers: 04.70.Bw, 04.50.+h, 11.25.Mj, 97.60.Lf

I. INTRODUCTION

The possibility that higher-derivative corrections should be added to the Einstein-Hilbert (EH) Lagrangian of general relativity in order to obtain a better behaved theory has often been considered. Among the various possibilities, a prominent role is played by the so-called Gauss-Bonnet (GB) terms.

GB Lagrangians were introduced in [1] as the only possible generalization of the EH Lagrangian in higher dimensions that gives rise to field equations which are second order in the metric, linear in the second derivatives, and divergence free. Another important property is that GB corrections do not introduce any new propagating degrees of freedom in the spectrum of gravity [2]. However, since they vanish in lower dimensions unless nonminimally coupled to scalar fields, they are mainly useful in the context of higher-dimensional gravity, and especially Kaluza-Klein theories [3–5]. It must however be mentioned that GB contributions also appear in the low-energy limit of string theories, [2,6], and may also play an important role in the context of the braneworld scenario [7].

The introduction of GB terms in the action of Kaluza-Klein theories allows spontaneous compactification of higher-dimensional models without the need of introducing external fields. For example, GB models admit ground states in the form of the direct product of two maximally symmetric spaces [4]. They also have interesting applications in higher-dimensional cosmology [5].

In order to explore further the physical implications of the dimensional reduction of higher-dimensional models of gravity including GB corrections, it is interesting to study the existence of black hole solutions of the dimensionally reduced theory, with compact internal space. In the case of pure Einstein gravity, this investigation was performed in [8], where it was shown that the only solution of physical interest is the four-dimensional Schwarzschild metric with flat internal space. In the GB case, one may expect the existence of a greater variety of solutions, and, in particular, also black holes with anti–de Sitter asymptotics.

Some black hole solutions of the Einstein-GB field equations are already known in different physical situations, as spherical symmetry in higher dimensions [9] or GB-scalar coupling in four dimensions [10,11]. In these cases it results a modification with respect to the Einstein case of the short-distance behavior of the solutions near the singularities, but also asymptotic or global properties of the black hole may be altered.

In this paper, our aim is to classify all solutions of the Einstein-GB system taking the form of a direct product of a four-dimensional spherically symmetric black hole with a maximally symmetric internal space. Since a general discussion would be too involved, we shall limit ourselves to the case of six dimensions, where the only relevant GB correction is quadratic in the curvature and has the form $\mathcal{S} = \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^2$.

A powerful technique for investigating this topic is the study of the phase space of the solutions of the field equations. This method has been used for example in the Einstein case [8]. In particular, the classification of the critical points of the dynamical system placed at infinity of the phase space allows one to deduce the asymptotics of all possible solutions. However, as mentioned above, when a GB term is added to the action, the field equations are still second order, and linear in the second derivatives, but no longer quadratic in the first derivatives. This fact gives rise to several technical problems. In particular, the potential of the dynamical system is no longer polynomial, but presents poles for some values of the variables [11].

Not all the solutions of the dynamical system are physically relevant. We require that they are regular everywhere, except possibly for coordinate singularities associated to horizons, and spherically symmetric. Moreover, the size of the internal space (or equivalently the Kaluza-Klein scalar field) must go to a constant at spatial infinity, in order to avoid decompactification. The result of our investigation is that the solutions fulfilling these requirements are either asymptotically flat or asymptotically anti–de Sitter, discarding the possibility of more exotic behavior.

Our starting point is the $(n + 4)$-dimensional action

$$I = \int \sqrt{-g}d^{(n+4)}x(\mathcal{R}^{(n+4)} + \alpha\mathcal{S}^{(n+4)}), \qquad (1.1)$$

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where $\mathcal{R}^{(n+4)}$ is the curvature scalar and $\mathcal{S}^{(n+4)}$ the quadratic GB term of the manifold and α is a coupling constant of dimension $[L]^2$.

We want to perform a dimensional reduction which casts the metric in the form of a direct product of a four-dimensional manifold with an n-dimensional space of constant curvature, whose size is parametrized by a scalar field ϕ. In general, contrary to the Einstein case, it is not possible to find an ansatz for the metric of the Einstein-GB system that completely disentangles the scalar field ϕ from the curvature in the dimensionally reduced action, except when the internal space is flat. Therefore we maintain the usual ansatz

$$ds^2_{(n+4)} = e^{-n\phi} ds^2_{(4)} + e^{2\phi} g^{(n)}_{ab} dx^a dx^b, \tag{1.2}$$

where $ds^2_{(4)}$ is the line element of the four-dimensional spacetime and $g^{(n)}_{ab}$ is the metric of the n-dimensional maximally symmetric internal space, with $\mathcal{R}^{(n)}_{ab} = \lambda_i g^{(n)}_{ab}$. The action is dimensionally reduced to

$$I = \int \sqrt{-g} d^4x \Big[(1 + 2\alpha\lambda_i e^{-2\phi})\mathcal{R}^{(4)} + \alpha e^{n\phi} \mathcal{S}^{(4)} + 4n\alpha e^{n\phi} \mathcal{G}^{(4)}_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi + \left(\frac{n(n+2)}{2} - (n^2 - 2n - 12)\alpha\lambda_i e^{-2\phi}\right)(\nabla\phi)^2 - \frac{n(n+2)(n^2+n-3)}{3}\alpha e^{n\phi}(\nabla\phi)^4 + \lambda_i e^{-(n+2)\phi} + (n-2)(n-3)\alpha\lambda_i^2 e^{-(n+4)\phi} \Big]. \tag{1.3}$$

The dimensionally reduced action contains the Einstein and the GB terms nonminimally coupled to a scalar field, and a standard kinetic term and a potential for the scalar field. In addition, one has a nonstandard quartic correction to the kinetic term and a coupling between the Einstein tensor $\mathcal{G}_{\mu\nu}$ and derivatives of the scalar field. Of course, both these terms yield second order field equations. The action (1.3) displays some similarity with the string effective action studied in [11], but contains additional terms.

In the following discussion, it is important to fix the possible ground states for the model. These are taken to be the direct product of a four-dimensional and an n-dimensional maximally symmetric space, i.e. $\mathcal{R}^{(4)}_{\mu\nu\rho\sigma} = \Lambda_e (g^{(4)}_{\mu\rho} g^{(4)}_{\nu\sigma} - g^{(4)}_{\mu\sigma} g^{(4)}_{\nu\rho})$, $\mathcal{R}^{(n)}_{\mu\nu\rho\sigma} = \Lambda_i (g^{(n)}_{\mu\rho} g^{(n)}_{\nu\sigma} - g^{(n)}_{\mu\sigma} g^{(n)}_{\nu\rho})$. Substituting this ansatz into the field equations derived from (1.1), one obtains

$$\begin{gathered} \alpha[(n-1)(n-2)(n-3)(n-4)\Lambda_i^2 + 24\Lambda_e^2 + 24(n-1)(n-2)\Lambda_e\Lambda_i] + (n-1)(n-2)\Lambda_i + 12\Lambda_e = 0, \\ \alpha[n(n-1)(n-2)(n-3)\Lambda_i^2 + 12n(n-1)\Lambda_i\Lambda_e] + n(n-1)\Lambda_i + 6\Lambda_e = 0. \end{gathered} \tag{1.4}$$

The system always admits the solution $\Lambda_e = \Lambda_i = 0$, as in the Einstein case, but one can also obtain solutions with nonvanishing curvature, namely, de Sitter or anti–de Sitter.[1] Consequently, black hole solutions of (1.1) may have anti–de Sitter behavior at spatial infinity. In the following we shall concentrate on the case $n = 2$. Equation (1.4) then admits a solution $\Lambda_e = -\frac{1}{2\alpha}$, $\Lambda_i = -\frac{3}{10\alpha}$, i.e. AdS $\times H^2$ for $\alpha > 0$, or dS $\times S^2$ for $\alpha < 0$.

II. THE DYNAMICAL SYSTEM

Let us consider the case $n = 2$. For the four-dimensional metric we adopt the ansatz [8]

$$ds^2_{(4)} = -e^{2\nu} dt^2 + \sigma^{-2} e^{4\zeta - 2\nu} d\xi^2 + e^{2\zeta - 2\nu} g_{ij} dx^i dx^j, \tag{2.1}$$

where ν, ζ and σ as well as ϕ are functions of ξ and g_{ab} is the metric of a two-dimensional maximally symmetric space, with $\mathcal{R}_{ij} = \lambda_e g_{ij}$. The ansatz (2.1) enforces radial symmetry. Of course, the solutions of physical interest are the spherically symmetric ones, i.e. those with $\lambda_e > 0$.

It is then convenient to define new variables

$$\chi = 2\zeta - \nu - \phi, \qquad \eta = 2\zeta - \nu - 2\phi \tag{2.2}$$

Substituting the ansatz (1.2) and (2.1) into the action, performing some integrations by parts, and factoring out the internal space, the action can be cast in the form

[1] We are only interested in black hole with asymptotic regions, so we shall not consider the de Sitter case further.

$$I = -8\pi \int d^4x \Big\{ \sigma[6\chi'^2 + 3\zeta'^2 + 3\eta'^2 - 8\chi'\zeta' - 8\chi'\eta' + 4\zeta'\eta'] - \frac{1}{\sigma}(\lambda_e e^{2\zeta} + \lambda_i e^{2\eta}) + 4\alpha e^{-2\chi}\Big[\sigma(\eta' - \chi')(4\zeta' + 3\eta' - 5\chi')\lambda_e e^{2\zeta} + \sigma(\zeta' - \chi')(3\zeta' + 4\eta' - 5\chi')\lambda_i e^{2\eta} - \sigma^3(\zeta' - \chi')(\eta' - \chi')(11\chi'^2 + 4\zeta'^2 + 4\eta'^2 + 7\zeta'\eta' - 13\chi'\zeta' - 13\chi'\eta') - \lambda_e\lambda_i \frac{e^{2(\zeta+\eta)}}{\sigma}\Big]\Big\}. \tag{2.3}$$

As usual, the action (2.3) does not contain derivatives of σ, which acts therefore as a Lagrangian multiplier enforcing the Hamiltonian constraint. Another relevant property of (2.3) is that, in spite of the presence of the higher-derivative GB term, it contains only first derivatives of the fields, although up to the fourth power, and therefore gives rise to second order field equations. A further interesting property is that, due to our choice of variables, the action is invariant under the interchange of ζ and η.

One can now vary (2.3) and then write the field equations in first order form in terms of the new variables,

$$W = \chi', \qquad X = \zeta', \qquad Y = \eta', \qquad U = e^{\chi}, \qquad Z = e^{\zeta}, \qquad V = e^{\eta}, \tag{2.4}$$

which satisfy

$$U' = WU, \qquad Z' = XZ, \qquad V' = YV. \tag{2.5}$$

Varying with respect to σ and then choosing the gauge $\sigma = 1$, one obtains the Hamiltonian constraint

$$E \equiv P^2 + \lambda_e Z^2 + \lambda_i V^2 + \frac{4\alpha}{U^2}[\lambda_e\lambda_i Z^2V^2 + \lambda_e Z^2(Y - W)A + \lambda_i V^2(X - W)B - 3(X - W)(Y - W)C^2] = 0, \tag{2.6}$$

where

$$P^2 = 6W^2 + 3X^2 + 3Y^2 - 8WX - 8WY + 4XY, \qquad C^2 = 11W^2 + 4X^2 + 4Y^2 + 7XY - 13WX - 13WY,$$

$$A = 4X + 3Y - 5W, \qquad B = 3X + 4Y - 5W.$$

Variation with respect to χ, ζ and η gives rise to the other field equations

$$2X' + 2Y' - 3W' + \Big\{\frac{2\alpha}{U^2}[\lambda_e Z^2(2X + 4Y - 5W) + \lambda_i V^2(4X + 2Y - 5W) + 22W^3 - 2X^3 - 2Y^3 - 36W^2X - 36W^2Y - 12X^2Y - 12Y^2X + 17WX^2 + 17WY^2 + 44XYW]\Big\}' = \frac{2\alpha}{U^2}[-\lambda_e\lambda_i Z^2V^2 + \lambda_e Z^2(Y - W)A + \lambda_i V^2(X - W)B - (X - W)(Y - W)C^2], \tag{2.7}$$

$$X' + 2Y' - 2W' + \Big\{\frac{4\alpha}{U^2}[\lambda_e Z^2(2X + 2Y - 3W) - (X - W)(10W^2 + 2X^2 + 5Y^2 + 6XY - 9WX - 14WY - \lambda_i V^2)]\Big\}' = \lambda_e Z^2 + \frac{4\alpha}{U^2}[\lambda_i V^2(X - W)B - (X - W)(Y - W)C^2], \tag{2.8}$$

$$2X' + Y' - 2W' + \Big\{\frac{4\alpha}{U^2}[\lambda_i V^2(2X + 2Y - 3W) - (Y - W)(10W^2 + 5X^2 + 2Y^2 + 6XY - 14WX - 9WY - \lambda_e Z^2)]\Big\}' = \lambda_i V^2 + \frac{4\alpha}{U^2}[\lambda_e Z^2(Y - W)A - (X - W)(Y - W)C^2]. \tag{2.9}$$

In the variables (2.4), the problem takes the form of a six-dimensional dynamical system, subject to a constraint. Notice that the function E defined in (2.6) is a constant of the motion of the system (2.5), (2.7), (2.8), and (2.9), whose value vanishes by virtue of the Hamiltonian constraint. Since the system is obviously symmetric for $V \to -V$, $Z \to -Z$, $U \to -U$, we shall only consider positive values of these variables.

The Einstein limit

In the Einstein limit, $\alpha = 0$, one recovers the results of [8]. We summarize them in terms of the variables introduced above: when $\alpha = 0$, the dynamical system reduces to Eqs. (2.5) and

$$2X' + 2Y' - 3W' = 0, \qquad X' + 2Y' - 2W' = \lambda_e Z^2,$$
$$2X' + Y' - 2W' = \lambda_i V^2, \tag{2.10}$$

subject to the constraint

$$E = P^2 + \lambda_e Z^2 + \lambda_i V^2 = 0. \tag{2.11}$$

The physical trajectories lie on the four-dimensional hyperplane $E = 0$. Moreover, the system is independent of the variable U, and one may restrict the analysis to $U = 0$. It is evident that $2X + 2Y - 3W$ is a constant of the motion for the system (2.10) and therefore one of the variables, say W, could be eliminated, but we keep it for comparison with the GB case.

The structure of the phase space can be studied by classifying the critical points. These are the points where the trajectories of the solutions start or end, and their position is determined by the condition that the derivatives of all the phase space variables vanish there. The location of the critical points is therefore related to the behavior of the metric at infinity or near the horizon (or singularity), while the discussion of the linearized equations near the critical points permits to deduce which trajectories start or end at a specific point [8].

In particular, the critical points at finite distance in phase space correspond to the small radius limit of the solutions [8]. They lie on the surface $Z_0 = V_0 = P_0 = 0$, but only points with $X_0 = Y_0 = W_0$ correspond to regular horizons, while the others give rise to naked singularities. The eigenvalues of the linearized equations around the critical points are 0, with multiplicity 3, X_0, Y_0 and W_0

Since we are interested in solutions with asymptotic regions, we are led to study the phase space at infinity, which corresponds to the large radius limit of the solutions [8]. This can be investigated defining new variables

$$t = \frac{1}{W}, \qquad x = \frac{X}{W}, \qquad y = \frac{Y}{W},$$
$$u = \frac{U}{W}, \qquad z = \frac{Z}{W}, \qquad v = \frac{V}{W}. \tag{2.12}$$

In terms of these variables, the field equations at infinity are then obtained for $t \to 0$, and read

$$\dot{t} = -2(v^2 + z^2)t, \qquad \dot{x} = z^2 + 2v^2 - 2(v^2 + z^2)x$$
$$\dot{y} = 2z^2 + v^2 - 2(v^2 + z^2)y, \qquad \dot{u} = (1 - 2v^2 - 2z^2)u,$$
$$\dot{z} = (x - 2v^2 - 2z^2)z, \qquad \dot{v} = (y - 2v^2 - 2z^2)v,$$

where a dot denotes $td/d\xi$.

The critical points at infinity are found at $t_0 = 0$ and

a) $\lambda_i v_0^2 = \lambda_e z_0^2 = 0$, $x = x_0$, $y = y_0$, with $3x_0^2 + 3y_0^2 + 4x_0y_0 - 8x_0 - 8y_0 + 6 = 0$.
b) $\lambda_i v_0^2 = 0$, $\lambda_e z_0^2 = 1/4$, $x_0 = 1/2$, $y_0 = 1$.
c) $\lambda_e z_0^2 = 0$, $\lambda_i v_0^2 = 1/4$, $x_0 = 1$, $y_0 = 1/2$.
d) $\lambda_i v_0^2 = \lambda_e z_0^2 = 3/16$, $x_0 = y_0 = 3/4$.

Points a) are the endpoints of the hypersurface $V = Z = 0$, points b) of the hypersurface $V = 0$, points c) of the hypersurface $Z = 0$. Exact solutions with endpoints b), c) and d) are discussed in the appendix.

The eigenvalues of the linearized equations around the critical points with their degeneracy in parentheses are:
a) 0(3), 1, x_0, y_0.
b,c) -1, $-\frac{1}{2}(3)$, $\frac{1}{2}(2)$.
d) $-\frac{3}{2}$, $-\frac{3}{4}(2)$, $\frac{1}{4}$, $-\frac{3\pm\sqrt{15}}{8}$.

The asymptotic behavior of the solutions can be deduced from the location of the critical points at infinity [8]. Excluding points a) that do not correspond to physical trajectories, one has, in terms of a radial variable r:
b) $ds^2 \sim -dt^2 + dr^2 + r^2 d\Omega_+^2$, $e^{2\phi} \sim$ const
c) $ds^2 \sim -r^2 dt^2 + r^2 dr^2 + r^2 d\Omega_0^2$, $e^{2\phi} \sim r^2$.
d) $ds^2 \sim -r dt^2 + dr^2 + r^2 d\Omega_+^2$, $e^{2\phi} \sim r$.

Here, we have denoted with $d\Omega_+^2$ the metric of a unitary 2-sphere, and with $d\Omega_0^2$ that of a flat 2-plane. The solutions ending at points b) are asymptotically flat, while the others have more exotic behavior.

The phase space portrait is the following: solutions with regular horizons start at $Z_0 = V_0 = 0$, $X_0 = Y_0 = W_0 = a$, for some value of the parameter a, and end at points b) if $\lambda_i = 0$, or d) if $\lambda_i > 0$. These last solutions, however, decompactify for $r \to \infty$, since $e^{2\phi}$ diverges in such limit. Also cylindrical solutions with $\lambda_e = 0$ exist, which end at points c). The only spherically symmetric solutions with constant scalar field at infinity are therefore asymptotically flat.

III. THE GAUSS-BONNET PHASE SPACE

As discussed in Sec. I, in the GB case Eqs. (1.4) admit the ground state solution $\Lambda_e = -1/2\alpha$, $\Lambda_i = -3/10\alpha$ in addition to flat space, and therefore black holes with anti–de Sitter asymptotic behavior may be expected if $\alpha > 0$. The phase space of the system can be studied by the same methods used in the Einstein case. As usual in the presence of Gauss-Bonnet terms, some care must be taken because of the poles at $U = 0$. The limit $U \to 0$ must be therefore taken, when necessary, at the end of the calculations.

Equations (2.7), (2.8), and (2.9) must be solved for the variables X', Y' and W' in order to put the system in its canonical form. One can then find the critical points at finite distance by requiring the vanishing of the derivatives of the fields. As in the Einstein case, they lie on the hypersurface $U_0 = Z_0 = V_0 = 0$. However, in the GB system, the other variables must satisfy the constraint $W_0 = X_0 = Y_0$, or $X_0 = \frac{4\pm\sqrt{5}}{5} W_0$, $Y_0 = \frac{4\mp\sqrt{5}}{5} W_0$. Only the first case corresponds to regular horizons. In that case the eigenvalues of the linearized equations near the critical points are identical to those found in the Einstein limit.

TABLE I. Location of the critical points at infinity.

	x_0	y_0	u_0^2/α	$\lambda_e z_0^2$	$\lambda_i v_0^2$
a)	1	1	0	0	0
b)	1/2	1	0	1/4	0
c)	1	1/2	0	0	1/4
e)	2/3	1	2/9	0	−1/15
f)	1	2/3	2/9	−1/15	0
g)	1	1	2	−1	−1
h)	4/5	4/5	6/25	0	0
i)	2/3	1	0	1/3	0
l)	1	2/3	0	0	1/3
m$_\pm$)	$\frac{4\pm\sqrt{5}}{5}$	$\frac{4\mp\sqrt{5}}{5}$	0	0	0

TABLE II. The eigenvalues of the linearized equations near the critical points at infinity and the nature of the trajectories attracted.

	Eigenvalues (*with degeneracy*)	Trajectories attracted
a)	0(3), 1(3)	
b)	$-\frac{1}{2}(3),\ -1,\ \frac{1}{2}(2)$	$\lambda_e > 0,\ \lambda_i = 0$
c)	$-\frac{1}{2}(3),\ -1,\ \frac{1}{2}(2)$	$\lambda_e = 0,\ \lambda_i > 0$
e)	$-1(2),\ -2,\ -\frac{1}{3},\ -\frac{1\pm\sqrt{11/3}}{2}$	any λ_e, $\lambda_i < 0$
f)	$-1(2),\ -2,\ -\frac{1}{3},\ -\frac{1\pm\sqrt{11/3}}{2}$	$\lambda_e < 0$, any λ_i
g)	$-1,\ -2(2),\ 1,\ -\frac{1\pm i\sqrt{5/3}}{2}$	$\lambda_e < 0,\ \lambda_i < 0$
h)	$-1(3),\ -2,\ -\frac{1}{5}(2)$	any λ_e, λ_i
i)	$-\frac{2}{3},\ -\frac{1}{3},\ -1,\ \frac{1}{3}(3)$	$\lambda_e > 0,\ \lambda_i = 0$
l)	$-\frac{2}{3},\ -\frac{1}{3},\ -1,\ \frac{1}{3}(3)$	$\lambda_e = 0,\ \lambda_i < 0$
m$_+$)	$-\frac{2}{3}(2),\ 0,\ \frac{1}{3},\ \frac{2\pm3\sqrt{5}}{15}$	$\lambda_e > 0,\ \lambda_i = 0$
m$_-$)	$-\frac{2}{3}(2),\ 0,\ \frac{1}{3},\ \frac{2\pm3\sqrt{5}}{15}$	$\lambda_e = 0,\ \lambda_i < 0$

The critical points at infinity are obtained by writing the dynamical systems in terms of the variables (2.12) and requiring the vanishing of their derivatives as $t \to 0$. They are listed in Table I.

In case a), the curve of the previous section reduces to a single point. Furthermore, the points d) have disappeared, except in the limit $\alpha \to 0$ (i.e. $t \to \infty$). The points with $u_0 = 0$ are attained by taking the limit $u_0 \to 0$ at the end of the calculation. It is also evident that points e)-h) exist only if $\alpha > 0$.

It is interesting to notice that the location of the critical points at infinity is very similar to that of the pure Einstein system with a cosmological constant, investigated in [12], except for the presence of the new points i), l) and m$_\pm$).

It seems therefore that one of the main ingredients in fixing the structure of the phase space at infinity, and hence the asymptotic behavior of the solutions, is the relative dimension of the terms in the action. For a detailed discussion see [13].

From the eigenvalues and the eigenvectors of the linearized equations, one can deduce the nature of the trajectories attracted by the various critical points at infinity. The eigenvalues of the linearized equations near the critical points and their degeneracy are listed in Table II, together with the nature of the trajectories attracted, for $W > 0$.

The points a) do not attract any trajectory from finite distance. Moreover, the points i), l) and m$_\pm$) attract only trajectories on the hypersurface $U = 0$, corresponding to the limit $\alpha \to \infty$. These are therefore solutions of the pure Gauss-Bonnet theory, without the presence of the Einstein-Hilbert term.

The critical points b)–c) generalize those found in the Einstein case, and have the same asymptotic behavior. For what concerns the other points,

e) $ds^2 \sim -r^2dt^2 + r^{-2}dr^2 + r^2d\Omega_+^2$, $e^{2\phi} \sim$ const.
f) $ds^2 \sim -r^4dt^2 + dr^2 + r^2d\Omega_-^2$, $e^{2\phi} \sim r^2$.
g) $ds^2 \sim -r^2dt^2 + r^{-2}dr^2 + d\Omega_-^2$, $e^{2\phi} \sim$ const.
h) $ds^2 \sim -r^2dt^2 + r^{-1}dr^2 + r^2d\Omega_0^2$, $e^{2\phi} \sim r$.
i) $ds^2 \sim -r^2dt^2 + dr^2 + r^2d\Omega_+^2$, $e^{2\phi} \sim$ const.
l) $ds^2 \sim -r^4dt^2 + r^2dr^2 + r^2d\Omega_0^2$, $e^{2\phi} \sim r^2$.
$m_\pm$) $ds^2 \sim -r^{2\pm\sqrt{5}}dt^2 + r^{-(2\pm\sqrt{5})}dr^2 + r^2d\Omega_0^2$, $e^{2\phi} \sim r^{1\mp\sqrt{5}}$.

Here $d\Omega_-^2$ denotes the metric of a two-dimensional space H^2 with constant negative curvature.

Of particular interest are the solutions that end at the critical point e), which arise for positive α. These asymptote to the exact ground state solution $\mathrm{AdS}^4 \times H^2$, cited previously, that in the present coordinates takes the form

$$ds^2 = -\left(\frac{r^2}{2\alpha} + 1\right)dt^2 + \left(\frac{r^2}{2\alpha} + 1\right)^{-1}dr^2 + r^2d\Omega_+^2,$$
$$e^{2\phi} = \frac{10\alpha}{3}. \tag{3.1}$$

Also interesting is the solution g), that asymptotes the exact solution $\mathrm{AdS}^2 \times H^2 \times H^2$. Its four-dimensional section is analogous to a Bertotti-Robinson metric. The other solutions have less common behavior.

The results of the study of the phase space of the dynamical system can be summarized as follows: solutions with regular horizon start at the points $U = V = Z = 0$, $X = Y = W$ and terminate at one of the critical points listed in Table I, depending on the values of α, λ_e and λ_i (see Table II). These trajectories exhaust all the possible black hole solutions of the system, and their asymptotic behavior depends solely on their endpoints at infinity of phase space. The various possibilities that can arise are discussed above.

As explained in the introduction, the relevant solutions from the Kaluza-Klein point of view are those with $\lambda_e > 0$ and $e^{2\phi}$ asymptotically constant. From the previous discussion, it is evident that the only solutions satisfying these requirements are the asymptotically flat (Schwarzschild-like) solutions with flat internal space ending at the critical points b) and the asymptotically anti–de Sitter solutions with internal space of negative curvature ending at the critical points e).[2] We can therefore deduce that the only physically relevant black hole solutions of the dimensionally reduced six-dimensional Einstein-GB system with asymptotic regions are either asymptotically flat or asymptotically anti–de Sitter.

IV. CONCLUSIONS

Higher-dimensional models of gravity naturally admit Gauss-Bonnet terms in the Lagrangian. Our study has shown that the compactification of the simplest model admits black hole solutions displaying a variety of asymptotic behaviors. However, all physically reasonable solutions (i.e. spherically symmetric and with internal space of finite size) have either flat or anti–de Sitter asymptotics, and internal space of vanishing or negative curvature, respectively. This behavior actually reproduces that of the possible ground states.

It turns out that the phase space of the model is quite similar to that of pure Einstein gravity with a cosmological term [12]. It would be interesting therefore to consider the effect of adding a cosmological constant to our model. This topic is currently under study [13].

ACKNOWLEDGMENTS

I wish to thank Maurizio Melis for valuable comments.

APPENDIX A

In the pure Einstein case, exact solutions corresponding to the vanishing of λ_i or λ_e where obtained in [8] for generic spacetime dimensions. In our six-dimensional setting, the solutions take the following form: for $\lambda_i = 0$ one obtains of course the Schwarzschild metric with constant scalar field,

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2 d\Omega_+^2,$$
$$e^{2\phi} = \text{const}$$

For $\lambda_e = 0$, one has instead a solution of the form

$$ds^2 = -r^2(1 - 2M/r)dt^2 + r^2(1 - 2M/r)^{-1}dr^2 + r^2 d\Omega_0^2,$$
$$e^{2\phi} = r^2.$$

In our case, a family of exact solutions with asymptotic behavior d) can also be obtained, if one makes the ansatz $\eta = \zeta$. In fact, in this case, the field equations reduce to

$$4\zeta'' - 3\chi'' = 0, \qquad 3\zeta'' - 2\chi'' = e^{2\zeta},$$

subject to the constraint $6\chi'^2 + 10\zeta'^2 - 16\chi'\zeta' + 2e^{2\zeta} = 0$. Integrating the first equation, one obtains $\chi' = 4(\zeta' - c)/3$, and hence $e^{\chi} = Ae^{4(\zeta - c\xi)/3}$ for constant A and c. Substituting in the second equation, one obtains

$$\zeta'' = 3e^{2\zeta},$$

which is solved by

$$e^{\zeta} = \frac{2ae^{a\xi}}{\sqrt{3}(1 - e^{2a\xi})}.$$

Regular black hole solutions satisfying the constraint are obtained for $c = a/4$. For $\eta = \zeta$, the metric functions of (2.1) are related to our variables by

$$e^{\nu} = e^{3\zeta - 2\chi}, \qquad e^{\phi} = e^{\chi - \zeta}.$$

Using these relations with $A = 1$, defining $r = \int e^{2\zeta} d\xi$, $r_0 = 2a/3$, and substituting in (2.1), one finally obtains

$$ds^2 = -\frac{r - r_0}{r^{1/3}}dt^2 + \frac{r^{1/3}}{r - r_0}dr^2 + r^{4/3}d\Omega^2,$$
$$e^{2\phi} = r^{2/3},$$

or, in different coordinates,

$$ds^2 = -R\left(1 - \frac{r_0}{R^{3/2}}\right)dt^2 + \frac{27}{4}\left(1 - \frac{r_0}{R^{3/2}}\right)^{-1}dR^2 + R^2 d\Omega^2,$$
$$e^{2\phi} = R.$$

A more familiar expression can be obtained by writing the metric in its six-dimensional form:

$$ds^2 = -\left(1 - \frac{r_0}{\hat{r}^3}\right)dt^2 + \left(1 - \frac{r_0}{\hat{r}^3}\right)^{-1}d\hat{r}^2 + \frac{\hat{r}^2}{3}(d\Omega_i^2 + d\Omega_e^2).$$

This is a variant of the well known six-dimensional Tangherlini metric, where the 4-sphere is replaced by the direct product $S^2 \times S^2$.

APPENDIX B

It may be interesting to write down some special exact solutions of the Einstein-GB system corresponding to the possible asymptotic behavior associated with the different critical points at infinity. The properties of these solutions are more transparent in their six-dimensional form, in the Schwarzschild-like gauge

$$ds^2 = -e^{2\lambda}dt^2 + e^{-2\lambda}dr^2 + e^{2\rho}d\Omega_e^2 + e^{2\sigma}d\Omega_i^2.$$

In this gauge is evident the presence of a symmetry for the interchange of ρ and σ, that follows from the specific compactification considered. This entails a duality between points b), c) and e), f).

[2]As mentioned above, the solutions i), l) and $m_\pm$) arise only in the limit of vanishing Einstein-Hilbert contribution and can therefore be neglected.

b) $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_+^2 + d\Omega_0^2$.

c) $ds^2 = -dt^2 + dr^2 + d\Omega_0^2 + r^2 d\Omega_+^2$.

e) $ds^2 = -(\frac{r^2}{2\alpha} + 1)dt^2 + (\frac{r^2}{2\alpha} + 1)^{-1}dr^2 + r^2 d\Omega_+^2 + \frac{10\alpha}{3} d\Omega_-^2$, or $ds^2 = -\frac{r^2}{2\alpha}dt^2 + \frac{2\alpha}{r^2}dr^2 + r^2 d\Omega_0^2 + \frac{10\alpha}{3} d\Omega_-^2$.

f) $ds^2 = -(\frac{r^2}{2\alpha} + 1)dt^2 + (\frac{r^2}{2\alpha} + 1)^{-1}dr^2 + \frac{10\alpha}{3} d\Omega_-^2 + r^2 d\Omega_+^2$, or $ds^2 = -\frac{r^2}{2\alpha}dt^2 + \frac{2\alpha}{r^2}dr^2 + \frac{10\alpha}{3} d\Omega_-^2 + r^2 d\Omega_0^2$.

g) $ds^2 = -(\frac{r^2}{2\alpha} - m)dt^2 + (\frac{r^2}{2\alpha} - m)^{-1}dr^2 + 2\alpha d\Omega_-^2 + 2\alpha d\Omega_-^2$.

h) $ds^2 = -\frac{r^2}{6\alpha}dt^2 + \frac{6\alpha}{r^2}dr^2 + r^2 d\Omega_0^2 + r^2 d\Omega_0^2$.

The parameter m is an arbitrary constant.

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