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Power law in a gauge-invariant cut-off regularization

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We study one-loop quantum corrections of a compactified Abelian $5d$ gauge field theory. We use a cut-off regularization procedure which respects the symmetries of the model, i.e. gauge invariance, exhibits the expected powerlike divergences and therefore allows the derivation of power-law behavior of the effective $4d$ gauge coupling in a coherent manner.

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I. INTRODUCTION

Since a decade, it has been realized that large and/or universal extra-dimensions could be included in models beyond the standard model (SM) without being in conflict with experimental data [1–3]. The SM can then be thought as a low-energy limit of a higher-dimensional theory. This kind of scenario offers new insights to challenging problems, like gauge hierarchy problem [2,4], supersymmetry breaking [5], electroweak symmetry breaking [6], dark matter [7], etc.

Higher-dimensional gauge field theories are nonrenormalizable and by dimensional analysis, powerlike divergences will appear in the loop integrations. A well-known feature of these theories is the powerlike quantum corrections of effective four-dimensional gauge couplings [8]. For the purpose of computing loop corrections, it is of fundamental importance to respect the symmetries of the theory. On the one hand, dimensional regularization [9] is well known to preserve gauge invariance but hides powerlike divergences. Extensions to extra-dimensional theories are given in [10]. On the other hand, a naive proper-time cut-off regularization provides the expected powerlike divergences but breaks gauge symmetry (see for example [11]). Other approaches, preserving the symmetries of the theory are possible, for example, Pauli-Villars regularization (see [12] for an application to extra-dimensions).

In this paper we study a compactified five-dimensional Abelian gauge theory with a cut-off regularization which preserves gauge invariance and exhibits powerlike divergences (details and applications to other subjects are given in [13]). We present briefly the model in Sec. II and introduce the regularization to study Ward-identities in Sec. III. We then focus on the calculation of the vacuum polarization function, Sec. IV, to deduce the power-law behavior of the effective four-dimensional gauge coupling with respect to the cut-off, Sec. V. We end with a discussion on the results.

II. A BRIEF PRESENTATION OF THE MODEL

The action of quantum electrodynamics (QED) in 5 dimensions, or a generic $5d$ Abelian gauge theory, is

$$\mathcal{S} = \int d^5x\Big(-\tfrac{1}{4}F^{MN}F_{MN} + \bar{\Psi}(i\gamma^M D_M - m_e)\Psi\Big) \quad (1)$$

with the capital indices $M, N = (0, 1, 2, 3, 5) = (\mu, 5)$. The gamma matrices are $\gamma^M = (\gamma^\mu, i\gamma^5)$ and the five-dimensional covariant derivative is defined by

$$D_M = (\partial_\mu - i\tilde{e}A_\mu, \partial_5 - i\tilde{e}A_5).$$

The compactification on a circle S^1 implies the following Fourier decomposition of the fields:

$$\Psi(x^M) = \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{+\infty}\psi^{(n)}(x^\mu)e^{inx^5/R} \quad (2)$$

$$A_\mu(x^M) = \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{+\infty}A_\mu^{(n)}(x^\mu)e^{inx^5/R} \quad (3)$$

$$A_5(x^M) = \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{+\infty}A_5^{(n)}(x^\mu)e^{inx^5/R}. \quad (4)$$

The four-dimensional fields $\psi^{(n)}$, $A_\mu^{(n)}$, $A_5^{(n)}$ are the Kaluza-Klein (KK) excitations (or modes) of the original five-dimensional fields Ψ, A_μ and A_5 respectively. Other compactifications are possible and indeed widely studied in the literature, depending on the precise field content and masses one wants to obtain in the four-dimensional theory. In the following we shall consider S^1, but the method can be easily applied to other cases (if one introduces an additional Z_2 symmetry, for example).

Performing the integration over the extra-coordinate x^5 in Eq. (1) leads to the effective $4d$ action:

$$\begin{aligned}
\mathcal{S}_4 = &\sum_{n=-\infty}^{+\infty} \int d^4x - \frac{1}{4} F^{\mu\nu(-n)} F^{(n)}_{\mu\nu} \\
&+ \sum_{n=-\infty}^{+\infty} \int d^4x \bar{\psi}^{(-n)} (i\partial\!\!\!/ - M_n) \psi^{(n)} \\
&+ \sum_{n,m=-\infty}^{+\infty} \int d^4x e \bar{\psi}^{(-n)} \gamma^\mu A^{(n-m)}_\mu \psi^{(m)} \\
&+ \sum_{n,m=-\infty}^{+\infty} \int d^4x i e \bar{\psi}^{(-n)} \gamma^5 A^{(n-m)}_5 \psi^{(m)} \\
&+ \sum_{n=-\infty}^{+\infty} \int d^4x - \frac{1}{2} \partial_\mu A^{(-n)}_5 \partial^\mu A^{(n)}_5 \\
&+ \sum_{n=-\infty}^{+\infty} \int d^4x \frac{-in}{R} A^{\mu(-n)} \partial_\mu A^{(n)}_5 \\
&+ \sum_{n=-\infty}^{+\infty} \int d^4x - \frac{1}{2} \frac{n^2}{R^2} A^{(-n)}_\mu A^{\mu(n)} \\
&+ \int d^4x \mathcal{L}_{\text{gauge-fixing}}. \qquad (5)
\end{aligned}$$

The effective four-dimensional gauge coupling and the mass matrices of the fermions are respectively

$$e = \frac{\tilde{e}}{\sqrt{2\pi R}} \quad \text{and} \quad M_n = m_e + i\gamma^5 n/R.$$

The $4d$ sub-Lagrangian with zero-mode bosons and n fermions is invariant under the following $4d$ $U(1)$ gauge transformations:

$$\begin{aligned}
A^{(0)}_\mu \to A^{(0)}_\mu + \frac{1}{e}\partial_\mu \theta(x^\mu) \qquad A^{(0)}_5 \to A^{(0)}_5 \\
\psi^{(n)} \to e^{i\theta(x^\mu)} \psi^{(n)}.
\end{aligned} \qquad (6)$$

This sector is the relevant one for the purpose of this paper. It will be shown that the $U(1)$ gauge symmetry is preserved in the regularization procedure. The gauge-fixing term of this sector is taken to be the usual Stückelberg Lagrangian:

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{\lambda}{2} (\partial^\mu A^{(0)}_\mu)^2. \qquad (7)$$

We will limit our study to renormalization in the effective four-dimensional theory obtained after compactification. For a comparison of the one-loop renormalization of the full extradimensional theory with the four-dimensional effective one see [14].

III. REGULARIZATION AND WARD IDENTITIES

Since we want to generate explicitly the high-energy dependence of the correlations functions at one-loop in our extra-dimensional model, we have to choose a cut-off regularization procedure that fulfills the necessity of preserving the $U(1)$ gauge symmetry. Such procedure has been developed in detail in [13]. The strategy is to deduce all the integrals encountered during the regularization procedure from only one single integral. More precisely, starting from

$$I(\alpha, \beta) \equiv \int \frac{d^dk}{i(2\pi)^d} \frac{1}{[\alpha k^2 - \beta m^2]} \qquad (8)$$

we can deduce all the integrals of the general form (obtained after partial traces on gamma matrices and introduction of Feynman parameters):

$$\int \frac{d^dk}{i(2\pi)^d} \frac{k^a}{[k^2 - m^2]^b} \qquad (9)$$

by derivations of (8) with respect to α and β.

The starting integral can be computed with the Schwinger proper-time method (for example, see [11]), and reads

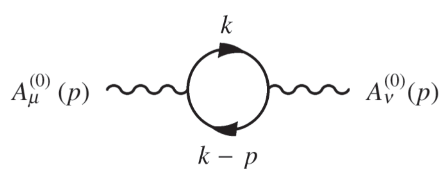
$$I(1,1)_{\text{div}} = -\frac{1}{4\pi^2}(\Lambda^2 - m^2 \log\Lambda^2) \qquad (10)$$

where Λ is the cut-off. The identification of the cut-off with a physical mass scale and the running with respect to the scale are discussed in detail in [15].

It is worthwhile to mention that a key-point for the consistency of the method is to take a special care of the dimension d, which has be taken equal to 4 (resp. 2) for logarithmic (resp. quadratic) terms (for further details, see [13]).

The vacuum polarization function of the $4d$ photon, $A^{(0)}_\mu$, is the infinite sum of vacuum polarization in which massive Kaluza-Klein excitations run in the loop

$$i\Pi^{\mu\nu}(p) = -e^2 \sum_{n=-\infty}^{+\infty} \int \frac{d^4k}{(2\pi)^4} \text{Tr}\left[\gamma^\mu \frac{1}{k\!\!\!/ - m_n} \gamma^\nu \frac{1}{k\!\!\!/ - p\!\!\!/ - m_n}\right]. \qquad (11)$$



Neglecting for the moment the sum over the Kaluza-Klein modes, Eq. (11) is then formally equivalent to the standard 4-dimensional QED for a fermion of mass m_n. Thus, the polarization tensor reads

$$i\Pi^{\mu\nu}(p) = -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{N^{\mu\nu}}{(k^2 - m_n^2)((k-p)^2 - m_n^2)} \qquad (12)$$

with

$$\begin{aligned}
N^{\mu\nu} = &k^\mu(k^\nu - p^\nu) + k^\nu(k^\mu - p^\mu) - g^{\mu\nu} k(k-p) \\
&+ g^{\mu\nu} m_n^2.
\end{aligned}$$

After the regularization procedure, the divergent part of $\Pi^{\mu\nu}(p)$ is

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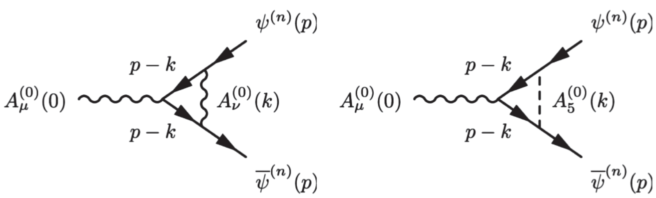
$$\Pi^{\mu\nu}_{\rm div}(p) = -\frac{e^2}{12\pi^2}(p^2 g^{\mu\nu} - p^\mu p^\nu)\ln\left(\frac{\Lambda^2}{m_n^2}\right). \quad (13)$$

As it should be, $\Pi^{\mu\nu}_{\rm div}(p)$ is transverse (independently of the value of the Kaluza-Klein number n in the loop). Moreover, it behaves logarithmically, in total agreement with what is expected for gauge theories.

A complementary test of our regularization procedure is the fact that the Ward identity between the three-point function $\Gamma_\mu(p, p)$ and the nth "electron" self-energy $\Sigma(p)$ is satisfied. This test can also be worked out in standard 4d QED using our regularization procedure. In our effective model, one has to check that the two above quantities, depicted diagrammatically below (we consider here the diagrams for the zero-mode photon $A_\mu^{(0)}$ stemming from the effective four-dimensional action we have described in Sec. II), satisfy the following relation:

$$\Gamma_\mu(p, p) = -\frac{\partial}{\partial p^\mu}\Sigma(p) \quad (14)$$

where the contributions to $\Gamma_\mu(p, p)$ can be decomposed into two parts $\Gamma^{(1)}_\mu(p, p)$ and $\Gamma^{(2)}_\mu(p, p)$, respectively, represented by



Likewise, the contributions $\Sigma^{(1)}(p)$ and $\Sigma^{(2)}(p)$ to the nth mode electron self-energy are

Following our regularization procedure, it is straightforward to obtain

$$\begin{aligned} -i\Sigma^{(1)}(p) &= \frac{i(-ie)^2}{16\pi^2}\log\Lambda^2\left(\left(\frac{1}{\lambda}+3\right)M_n - \frac{1}{\lambda}\not{p}\right) \\ -i\Sigma^{(2)}(p) &= \frac{i(ie^2)}{32\pi^2}\log\Lambda^2(2M_n - \not{p}) \\ -ie\Gamma^{(1)}_\mu(p, p) &= \frac{-(-ie)^3}{16\pi^2\lambda}\gamma_\mu\log\Lambda^2 \\ -ie\Gamma^{(2)}_\mu(p, p) &= \frac{-(-ie)^3}{32\pi^2}\gamma_\mu\log\Lambda^2 \end{aligned} \quad (15)$$

where λ is the parameter of the Stückelberg gauge fixing term, Eq. (7). We see explicitly that (14) is satisfied, the gauge-invariance of the sector with the zero-mode $A_\mu^{(0)}$ is preserved.

IV. VACUUM POLARIZATION

The aim of this section is to apply our regularization procedure to the vacuum polarization function of $A_\mu^{(0)}$ in the effective theory obtained by compactifying on S^1 the original 5-dimensional QED Lagrangian. Taking into account the sum over the Kaluza-Klein modes, one obtains

$$\Pi^{\mu\nu}_{\rm div}(p) = -\sum_{n=-\infty}^{\infty}\frac{e^2}{12\pi^2}(p^2 g^{\mu\nu} - p^\mu p^\nu) \times \ln\left(\frac{\Lambda^2}{m_e^2 + n^2/R^2}\right). \quad (16)$$

Our strategy is then to separate the standard 4-dimensional part ($n = 0$) from the extra dimension contributions ($n \neq 0$). Then, we have

$$\ln\left(\prod_{n=-\infty}^{\infty}\frac{\Lambda^2}{m_e^2 + n^2/R^2}\right) = \ln\left(\frac{\Lambda^2}{m_e^2}\right) + 2\ln\left(\prod_{n=1}^{\infty}\frac{\Lambda^2}{m_e^2 + n^2/R^2}\right). \quad (17)$$

Some approximations can be done in order to simplify the calculation of the previous expression (for an alternative analytical approach see the appendix of [16]):

(1) $m_e^2 \ll 1/R^2$ This can be justified by the fact that we assume the first Kaluza-Klein resonance to be far above the fundamental mass, which is the case in phenomenological applications [3].

(2) $n_{\rm max} = \Lambda R \gg 1$ In the spirit of a Wilsonian effective theory [17], Λ is the typical scale that enables to select the relevant degrees of freedom present in the theory : thus the sum over n is truncated at some value $n_{\rm max} \simeq \Lambda R$. Moreover since we are interested in the high-energy behavior of the theory, we will make the assumption $n_{\rm max} \gg 1$ in the following. The limits of the truncation of KK sums have been discussed by Ghilencea [18]. It has been shown that, when one performs the infinite KK sums, higher derivative operators of higher dimension are generated as one-loop counterterms describing a nondecoupling effect of the very massive KK states. However, this effect appears only in six dimensions or when one sums over 2 KK numbers, which is not our case.

Using these two approximations, we can then rewrite the second term of Eq. (17) as

$$\ln\left(\prod_{n=1}^{n_{\rm max}}\frac{\Lambda^2 R^2}{n^2}\right) = \ln\left(\frac{(\Lambda^2 R^2)^{\Lambda R}}{(\Lambda R)!^2}\right) \sim 2\Lambda R - \ln\Lambda R \quad (18)$$

where the Stirling formula has been used in the last step.

Finally, the total contribution for the divergent part of the vacuum polarization function reads

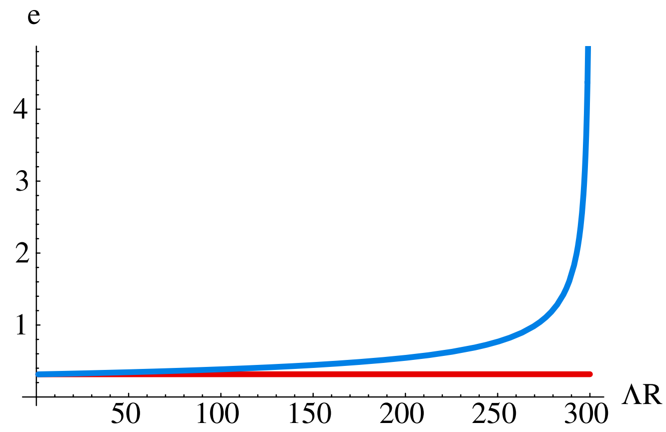


FIG. 1 (color online). Cut-off dependence of e in the model discussed (upper curve) and in the standard $4d$ QED (lower curve), as a function of ΛR. The radius is arbitrarily chosen at $(100\text{ GeV})^{-1}$. The Landau pole is at $\Lambda = 300R^{-1}$.

$$\Pi^{\mu\nu}_{\text{div}}(p) = -\frac{e^2}{12\pi^2}(p^2 g^{\mu\nu} - p^\mu p^\nu)(4\Lambda R). \tag{19}$$

In agreement with the results obtained in the literature [8], the divergence is linear in the cut-off. We see by the preceding manipulations that the power-law is the result of the sum of the individual logarithmic contributions.

V. RENORMALIZATION OF THE 4*d* EFFECTIVE COUPLING CONSTANT

In an Abelian theory, due to Ward identities, the beta function of the $4d$ effective coupling e can be calculated directly:

$$\beta_e = -\frac{e}{2}\Lambda\frac{\partial\Pi_\Lambda}{\partial\Lambda} \tag{20}$$

where Π_Λ is the divergent part of scalar vacuum polarization function defined by

$$\Pi_\Lambda = \frac{1}{3}g_{\mu\nu}\Pi^{\mu\nu}_{\text{div}} = \frac{-e^2}{3\pi^2}\Lambda R. \tag{21}$$

We obtain

$$\beta_e = \frac{e^3}{6\pi^2}\Lambda R \tag{22}$$

which gives the following asymptotic behavior of $e(\Lambda)$ with respect to the cut-off, between the scale $\mu = R^{-1}$ and $\mu = \Lambda$:

$$e(\Lambda) = \left(\frac{e(R^{-1})^2}{1 - \frac{e(R^{-1})^2}{3\pi^2}(\Lambda R - 1)}\right)^{1/2}. \tag{23}$$

This expression shows the expected power-law running of e, Fig. 1. It admits a Landau pole at $\Lambda_L = (1 + \frac{3\pi^2}{e(R^{-1})^2})R^{-1}$.

Equivalently, we can write

$$\alpha_e^{-1}(\Lambda) = \alpha_e^{-1}(R^{-1}) - \frac{b}{2\pi}X(\Lambda R - 1) \tag{24}$$

with $\alpha_e = \frac{e^2}{4\pi}$, $b = 4/3$ and $X = 2$. We found the same result for $\alpha^{-1}(\Lambda)$ as in [8] but with the fundamental difference that we do not need any final rescaling of the cut-off.

VI. DISCUSSION AND CONCLUSIONS

Regularization dependence of quantum corrections in higher-dimensional field theory has been extensively discussed. It is of fundamental importance for all phenomenological applications such as the study of divergences in a given model (gauge couplings, yukawas etc). Indeed, at the end of the regularization process, one would like to identify the cut-off Λ with the mass scale M at which our effective theory breaks down. However, the identification cannot be done with the knowledge of the effective theory only. One needs a matching with the theory taking place at M to fix the coefficient of the powerlike quantum corrections. For an example of this type of calculations see [19].

Without any UV completion of the higher-dimensional model, one can fix the coefficient by an *external* requirement. In a sense, the choice of a given regularization procedure (and the definition of the cut-off) is a feature of the model. For example, the authors of [20], asked the gauge couplings in the $4d$ effective theory to recover the result of the noncompactified theory in the limit of large radius. In the calculation of [8], the cut-off was redefined in order to recover the asymptotic result of including KK states at their thresholds, between $\mu = \frac{n}{R}$ and $\mu = \frac{n+1}{R}$ and so on. As in all standard cut-off calculations, gauge invariance was explicitly broken through the appearance of a $\Lambda g_{\mu\nu}$ term with no compensating $p_\mu p_\nu$ in the vacuum polarization $\Pi_{\mu\nu}$.

In our calculation, we included all states of KK number $|n| \leq \Lambda R$ at once, assuming the decoupling of the heavy states and without introducing any *ad-hoc* redefinition of the cut-off. Moreover, we explicitly checked in Sec. III that our calculation of loop integrals does not break gauge-invariance. The main advantage of the method proposed in this paper is the possibility to keep the physical insight of the cut-off procedure together with symmetry conservation.

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