

**$SO(10)$  grand unification model with  $S_4$  flavor symmetry**

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We present a supersymmetric grand unification model based on  $SO(10)$  group with  $S_4$  flavor symmetry. In this model, the fermion masses are from Yukawa couplings involving  $\mathbf{10}$  and  $\overline{\mathbf{126}}$  Higgs multiplets and the flavor structures of mass matrices of both quarks and leptons are determined by spontaneously broken  $S_4$ . This model fits all of the masses and mixing angles of the quarks and leptons. For the most general  $CP$ -violation scenario, this model gives  $\sin\theta_{13}$  a wide range of values from zero to the current bound with the most probable values 0.02–0.09. With certain assumptions where leptonic phases have same  $CP$ -violation source as CKM phase, one gets a narrower range 0.03–0.06 for  $\sin\theta_{13}$  with the most probable values 0.04–0.08. This model gives leptonic Dirac  $CP$  phase the most probable values 2–4 radians in the general  $CP$ -violation case.

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**I. INTRODUCTION**

The discovery of no-zero neutrino masses and lepton mixings have raised hope to understand the mystery of flavor structures of quarks and leptons in a unified way [1]. Although many similarities between leptons and quarks make such unification plausible, the mixing pattern in the lepton sector is very different from that in the quark sector. However, there are now many grand unification models based on  $SO(10)$  gauge group that can give small quark mixings and large lepton mixings along with all their masses with few assumptions [1].

Another interesting possibility is that there may exist horizontal underlying flavor symmetry. This is favored by leptonic mixing pattern with near maximal atmospheric mixing angle and the vanishing  $\theta_{13}$ . A permutation symmetry between  $\mu$  neutrino and  $\tau$  neutrino in the flavor basis has been proposed in recent years [2,3]. Even though there is no apparent evidence of such symmetry in charged lepton and quark sector, it has been shown that the unified description of quarks and leptons with this symmetry is possible [4]. Applications of higher permutation group  $S_3$ ,  $S_4$  and  $A_4$  to flavor symmetry also have been discussed in the literature [5–7]. Other discrete groups such as dihedral group  $D_4$  and  $D_5$  have been studied [8,9].

In this paper, we focus on the group  $S_4 \times SO(10)$ .  $S_4$  has certain good features to be a flavor symmetry. First, it has three dimensional irreducible representation to accommodate the three generations of fermions naturally. Note that this is different from  $S_3$  because the largest irreducible representation of  $S_3$  has dimension two and therefore we have to treat one family of fermions different from other two. Second, it can be embedded into continuous group  $SU(3)$  or  $SO(3)$  [10]. As we will show below,  $S_4$  symmetry also gives degenerate spectrum of the right-handed neutrinos naturally, which has some interesting consequences for the neutrino phenomenology. For example, in this case, one can use the resonant enhancement of leptogenesis for (quasi)degenerate right-handed neutrinos to generate

enough baryon asymmetry [11]. With the degenerate heavy right-handed neutrinos, the low energy neutrino flavor structure is determined by Dirac mass matrix at the seesaw scale completely, which makes it easier to reconstruct high energy physics from low energy observables. Some work has been done in this direction. In Ref. [12], Lee and Mohapatra constructed a  $S_4 \times SO(10)$  model, which naturally gives quasidegenerate spectrum of neutrinos masses with small solar angle, which already has been ruled out by large mixing angle MSW solution to the solar neutrino problem. In principle radioactive corrections may amplify the solar angle and keep the other two angles unchanged, but generally this needs extreme fine-tuning of parameters at the seesaw scale to realize it. On the other hand, in a recent paper [10] by Hagedorn, Linder, and Mohapatra, a low energy scale nonsupersymmetric model is presented based on  $S_4$  flavor symmetry, which can accommodate current neutrino data. Our goal is to see if we can embed the model of Ref. [10] into a SUSY GUT framework without running into the small solar angle problem of Ref. [12]. In this letter, we address this question and find that we can build a realistic model based on  $S_4 \times SO(10)$  with the proper choice of the parameter space.

In this model, all the quarks and leptons of one generation are unified into a  $\mathbf{16}$  spinor representation of  $SO(10)$  and the Yukawa coupling structures of three generations are determined by  $S_4$ . We use  $\mathbf{10}$  and  $\overline{\mathbf{126}}$  representations of  $SO(10)$  for Yukawa couplings to account for all the fermions masses and mixing angles [13,14]. Even though in the most general  $CP$ -violation case this model has 18 complex parameters, it is not obvious whether it can accommodate all observed masses and mixing angles because of constraints from  $S_4$  flavor symmetry and the correlations between quarks and leptons indicated by  $SO(10)$  unification. For instance, with the particle assignment of  $S_4$  in this model, the heavy right-handed neutrino mass matrix is proportional to an identity matrix, and the Dirac mass matrix of neutrino determines the mixing

among light neutrinos completely. The general mechanism to generate the lepton sector mixing independently from the quark sector by right-handed neutrinos does not work in this model. On the other hand, one may argue that since the total number of parameters is much larger than that of observables, this model may lose predicability even if it can fit all the observables. We find this not to be the case. It turns out that half of complex phases can be rotated away by choices of basis and redefinitions of the right-handed fields of charged leptons and down-type quarks. For the most general  $CP$ -violation case, this model gives wide range of  $\sin\theta_{13}$  from zero to current bound with the most probable values 0.02–0.09. The most probable values of leptonic  $CP$  phase are 2–4 radians. With certain assumptions where the leptonic phases have same  $CP$ -violation source as CKM phase, one gets narrower predicted range 0.03–0.09 for  $\sin\theta_{13}$  with the most probable values 0.04–0.08.

Some issues about Higgs sector still need to be addressed. As we have six  $\mathbf{10}$ s and three  $\overline{\mathbf{126}}$ s, without analyzing the  $S_4 \times SO(10)$  invariant Higgs potential, whether or not we can get the desired vacuum configuration still remains an open question. We do not concern with doublet-doublet splitting and doublet-triplet splitting problems in this paper. With such rich Higgs fields, we assume they can be realized in some way. And another fact we should be careful is that generally the discrete flavor symmetry can enhance the accidental global symmetry of Higgs potential and lead to unwanted massless Nambu-Goldstone bosons. There are ways found in the literature to avoid it. One can introduce gauge singlet Higgs fields whose couplings are invariant under discrete symmetry but break the global symmetry [15], or introduce soft terms which break discrete symmetry and global symmetry [16].

The paper is organized as follows: in Sec. II, we present an  $SO(10)$  model with  $S_4$  flavor symmetry and present the mass matrices of quarks and leptons; in Sec. III, we present a detailed numerical analysis including  $CP$  violation in quark and lepton sector. We end with conclusions and remarks in Sec. IV.

## II. SUSY $SO(10)$ MODEL WITH $S_4$ FLAVOR SYMMETRY

The group  $S_4$  is the permutation group of the four distinct objects, which has 24 distinct elements. It has five conjugate classes and contains five irreducible representations  $\mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{3}$  and  $\mathbf{3}'$ . Our assignment of fermions and Higgs multiplets to  $S_4 \times SO(10)$  are shown in Table I.

In this model, we assign three generations of  $\mathbf{16}$  to  $\mathbf{3}'$  irreducible representation of  $S_4$ , because  $\mathbf{3}'$  can be identified with the fundamental representation of continuous group  $SO(3)$  or  $SU(3)$  [10,17]. In Higgs sector, because of  $\mathbf{3}' \times \mathbf{3}' = \mathbf{1} + \mathbf{2} + \mathbf{3} + \mathbf{3}'$ , to make Yukawa coupling  $S_4$  invariant, Higgs fields can not belong to  $\mathbf{1}'$ .  $\mathbf{1}$  is necessary for phenomenological reason, otherwise all of the mass matrices would be traceless. To get symmetric mass matrices which is required by group structure of  $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$  or  $\mathbf{16} \cdot \mathbf{16} \cdot \overline{\mathbf{126}}$ , Higgs should not belong to  $\mathbf{3}'$ . We include both  $\mathbf{2}$  and  $\mathbf{3}$  to get realistic mass and mixing of quark and lepton. One might think six  $\mathbf{10}$  Higgs fields transforming as  $\mathbf{1} + \mathbf{2} + \mathbf{3}$  under  $S_4$  are enough. But there are two reasons why we also need  $\overline{\mathbf{126}}$ , one is to give right-handed neutrinos heavy masses and the other is to fix the bad mass relation between quark sector and lepton sector indicated by  $\mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$ . In this sense, our choice of Higgs fields is minimal.

The breaking of  $SO(10)$  to Standard Model (SM) can be realized in many ways. In this model, we choose  $\mathbf{210}$  Higgs field, which is  $\mathbf{1}$  under  $S_4$  transformation, to break  $SO(10)$  to  $SU(2)_L \times SU(2)_R \times SU(4)_C$  ( $G_{224}$ ) while keep the  $S_4$  symmetry. We choose  $(\mathbf{1}, \mathbf{3}, \mathbf{10})$  components of only  $\overline{\Delta}_0$  (the numbers denote representation under the  $G_{224}$ ) to get vev  $v_R$  that breaks  $G_{224}$  down to the SM and gives heavy masses to right-handed neutrinos. With this breaking pattern,  $S_4$  symmetry is kept down to the electroweak scale.

To see what this model implies for fermion masses, let us first explain how the MSSM doublets emerge. Besides the  $SU(2)_L$  Higgs doublets from submultiplets  $(\mathbf{2}, \mathbf{2}, \mathbf{1})$  and  $(\mathbf{2}, \mathbf{2}, \mathbf{15})$  contained in  $\mathbf{10}$  and  $\overline{\mathbf{126}}$  respectively, we also have Higgs doublets contained in  $(\mathbf{2}, \mathbf{2}, \mathbf{10}) \oplus (\mathbf{2}, \mathbf{2}, \overline{\mathbf{10}})$  from  $\mathbf{210}$ . Furthermore, to obtain anomaly-free theory, we need to introduce three  $\mathbf{126}$ , which we denote by  $\Delta$ , that also contain Higgs doublets. Altogether, we have 14 pairs of Higgs doublets:  $\phi_u = (H_{iu}, \overline{\Delta}_{ju}, \Delta_{ju}, \Phi_{u1}, \Phi_{u2})$ ,  $\phi_d = (H_{id}, \overline{\Delta}_{jd}, \Delta_{jd}, \Phi_{d1}, \Phi_{d2})$ , where  $i = 0, \dots, 5$  and  $j = 0, \dots, 2$ . As noted, six pairs from  $H$ s, three pairs from  $\overline{\Delta}$ s, three pairs from  $\Delta$ s and two pairs from  $\Phi$ . We can write Higgs doublet mass matrix as  $\phi_u M_H \phi_u^T$ .  $M_H$  can be diagonalized by  $X M_H Y^T$ , which  $X$  and  $Y$  are unitarity matrices acting on  $\phi_u$  and  $\phi_d$  respectively. At the GUT scale, by some doublet-triplet and doublet-doublet splitting mechanisms, we assume only one pair of linear combinations of  $X_{\alpha\beta}^* \phi_{u\beta}$  and  $Y_{\alpha\beta}^* \phi_{d\beta}$ , say  $X_{1\beta}^* \phi_{u\beta}$  and  $Y_{1\beta}^* \phi_{d\beta}$ , has masses of order of the weak scale and all others are kept super heavy near GUT scale, which generally can be

TABLE I. Transformation property of fermions and Higgs multiplets under  $S_4 \times SO(10)$

| Fermions                                 | Higgs Bosons                             |   |   |   |   |   |
|--|--|---|---|---|---|---|
| $\Psi_a, a = 1, 2, 3$                    | $\Phi$                                   | $\overline{\Delta}_0$                               | $\overline{\Delta}_{1,2}$                           | $H_0$                                   | $H_{1,2}$                               | $H_{3,4,5}$                             |
| $\{\mathbf{3}'\} \times \{\mathbf{16}\}$ | $\{\mathbf{1}\} \times \{\mathbf{210}\}$ | $\{\mathbf{1}\} \times \{\overline{\mathbf{126}}\}$ | $\{\mathbf{2}\} \times \{\overline{\mathbf{126}}\}$ | $\{\mathbf{1}\} \times \{\mathbf{10}\}$ | $\{\mathbf{2}\} \times \{\mathbf{10}\}$ | $\{\mathbf{3}\} \times \{\mathbf{10}\}$ |

realized by one fine-tuning of the parameters in the Higgs mass matrix. The MSSM Higgs doublets are given by this lightest pair:  $H_u^{\text{MSSM}} = X_{1\beta}^* \phi_{u\beta}$  and  $H_d^{\text{MSSM}} = Y_{1\beta}^* \phi_{d\beta}$ . Since we focus on the structures of Yukawa couplings, we do not discuss the details of the splitting mechanisms that lead to the above results.

With Higgs fields and fermions listed in Table I, we can write down  $S4 \times SO(10)$  invariant Yukawa coupling as [18]

$$W_{\text{Yukawa}} = (\Psi_1 \Psi_1 + \Psi_2 \Psi_2 + \Psi_3 \Psi_3)(h_0 H_0 + f_0 \bar{\Delta}_0) + \frac{1}{\sqrt{2}}(\Psi_2 \Psi_2 - \Psi_3 \Psi_3)(h_1 H_1 + f_2 \bar{\Delta}_1) + \frac{1}{\sqrt{6}}(-2\Psi_1 \Psi_1 + \Psi_2 \Psi_2 + \Psi_3 \Psi_3)(h_1 H_2 + f_2 \bar{\Delta}_2) + h_3[(\Psi_2 \Psi_3 + \Psi_3 \Psi_2)H_3 + (\Psi_1 \Psi_3 + \Psi_3 \Psi_1)H_4 + (\Psi_1 \Psi_2 + \Psi_2 \Psi_1)H_5]. \quad (1)$$

After electroweak symmetry breaking,  $(2, 2, 1)$  of  $H_i (i = 0, \dots, 5)$  component acquires vevs (denoted by  $\langle H_i \rangle^u$  and  $\langle H_i \rangle^d$ ). And  $(2, 2, 15)$  submultiplet of  $\bar{\Delta}_j (j = 0, \dots, 2)$  also get induced vevs. Their vevs are denoted by  $\langle \bar{\Delta}_j \rangle^u$  and  $\langle \bar{\Delta}_j \rangle^d (j = 0, 1, 2)$ .

The mass matrices for the quarks and the leptons have following sum rules:

$$M_u = M_u^{(10)} + M_u^{(126)}, \quad (2)$$

$$M_d = M_d^{(10)} + M_d^{(126)}, \quad (3)$$

$$M_\nu^D = M_u^{(10)} - 3M_u^{(126)}, \quad (4)$$

$$M_l = M_d^{(10)} - 3M_d^{(126)}, \quad (5)$$

$$M_\nu = -M_\nu^{DT} M_\nu^D / f_0 v_R, \quad (6)$$

where

$$M_u^{(10)} = \begin{bmatrix} a_0 - 2a_2 & a_5 & a_4 \\ a_5 & a_0 + a_1 + a_2 & a_3 \\ a_4 & a_3 & a_0 - a_1 + a_2 \end{bmatrix}, \quad (7)$$

$$M_d^{(10)} = \begin{bmatrix} b_0 - 2b_2 & b_5 & b_4 \\ b_5 & b_0 + b_1 + b_2 & b_3 \\ b_4 & b_3 & b_0 - b_1 + b_2 \end{bmatrix}, \quad (8)$$

$$M_u^{(126)} = \begin{bmatrix} d_0 - 2d_2 & 0 & 0 \\ 0 & d_0 + d_1 + d_2 & 0 \\ 0 & 0 & d_0 - d_1 + d_2 \end{bmatrix}, \quad (9)$$

$$M_d^{(126)} = \begin{bmatrix} e_0 - 2e_2 & 0 & 0 \\ 0 & e_0 + e_1 + e_2 & 0 \\ 0 & 0 & e_0 - e_1 + e_2 \end{bmatrix}, \quad (10)$$

and where  $a_i$  and  $b_i$  are products of the type  $h\langle H_i \rangle^u$  and  $h\langle H_i \rangle^d$  respectively. Similarly, we use  $d_j$  and  $e_j$  to denote products of the type  $f\langle \bar{\Delta}_j \rangle^u$  and  $f\langle \bar{\Delta}_j \rangle^d$  respectively. The MSSM vevs are given by  $v_u = X_{1\beta}^* \langle \phi_{u\beta} \rangle$  and  $v_d = Y_{1\beta}^* \langle \phi_{d\beta} \rangle$ , where we use  $v_u$  and  $v_d$  to denote vevs of  $H_u^{\text{MSSM}}$  and  $H_d^{\text{MSSM}}$  respectively. The Yukawa couplings and vevs of Higgs fields in general are complex, and there are 18 complex parameters. We choose a basis in which the down-quark mass matrix is diagonalized and set  $b_3 = 0$ ,  $b_4 = 0$ , and  $b_5 = 0$ . Note this is our main difference with Ref. [12], where they choose a basis in which up-quark mass matrix is diagonal and set off-diagonal entries of  $M_u$  to zeros, which leads to small solar mixing angle. In the basis we choose, the charged lepton mass matrix is also diagonalized. Therefore, the phases of  $b_0, b_1, b_2, e_0, e_1$ , and  $e_2$  can be rotated away by redefining 3 right-handed down-type quarks fields and three right-handed charged leptons. We treat  $b_0, b_1, b_2, e_0, e_1$ , and  $e_2$  as real parameters in later analysis, and they can be determined by the masses of down-quark and charged lepton completely.

Because the mass matrix of down-quark sector is diagonalized and  $M_u$  is symmetric, one can have

$$M_u = V_{\text{CKM}}^T \hat{M}_u V_{\text{CKM}}, \quad (11)$$

where  $\hat{M}_u \equiv \text{diag}(m_u, m_c, m_b)$ . By fitting mass matrix of up-quark in Eq. (11), parameters  $a_3, a_4, a_5$  can be determined. In addition, we get three conditions among the parameters  $a_0, a_1, a_2, d_0, d_1$ , and  $d_2$ . Therefore, there are three complex parameters left to be determined by masses and mixings of neutrino sector. Without loss of generality, we choose  $d_0, d_1$ , and  $d_2$  to be determined by fitting of neutrino sector. And Dirac neutrino mass matrix can be written conveniently as

$$M_\nu^D = V_{\text{CKM}}^T \hat{M}_u V_{\text{CKM}} - 4m_t \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \quad (12)$$

with

$$x \equiv \frac{1}{m_t}(d_0 - 2d_2), \quad y \equiv \frac{1}{m_t}(d_0 + d_1 + d_2), \quad (13)$$

$$z \equiv \frac{1}{m_t}(d_0 - d_1 + d_2).$$

Because we know nothing about leptonic phases, in principle, there is no constraint on the phases of  $d_0, d_1$ , and  $d_2$ .

To see how this model can give a large atmospheric mixing angle, we give an approximate analysis first. Using first order Wolfenstein parameterization [19] for the quark mixing,  $V_{\text{CKM}}^T \hat{M}_u V_{\text{CKM}}$  can be written as

$$m_t \begin{bmatrix} \lambda^6 + A^2 \lambda^6 (1 - i\eta - \rho) & \cdot & \dots \\ -\lambda^5 - A^2 \lambda^5 (1 - i\eta - \rho) & \lambda^4 + A^2 \lambda^4 & \dots \\ A\lambda^3 (1 - i\eta - \rho) & -A\lambda^2 & 1 \end{bmatrix} \quad (14)$$

where we use  $m_c/m_t \simeq \lambda^4$  and  $m_u/m_t \simeq \lambda^8$ . Therefore, to get near maximal mixing of  $\theta_{23}$ ,  $y$  and  $z$  should satisfy

$$\lambda^4(1 + A) - 4y \simeq 1 - 4z. \quad (15)$$

### III. DETAILED NUMERICAL ANALYSIS

To see if the model is phenomenologically acceptable, we first fit the masses of the charged leptons and down-type quarks using the mass values of leptons and quarks at the GUT scale with  $\tan\beta = 10$  [20] given in Ref. [21]:

| input observable | $\tan\beta = 10$                  |
|------------------|-----------------------------------|
| $m_u$ (MeV)      | 0.7238 $^{+0.1365}_{-0.1467}$     |
| $m_c$ (MeV)      | 210.3273 $^{+19.0036}_{-21.2264}$ |
| $m_t$ (GeV)      | 82.4333 $^{+30.2676}_{-14.7686}$  |
| $m_d$ (MeV)      | 1.5036 $^{+0.4235}_{-0.2304}$     |
| $m_s$ (MeV)      | 29.9454 $^{+4.3001}_{-4.5444}$    |
| $m_b$ (GeV)      | 1.0636 $^{+0.1414}_{-0.0865}$     |
| $m_e$ (MeV)      | 0.3585 $^{+0.0003}_{-0.0003}$     |
| $m_\mu$ (MeV)    | 75.6715 $^{+0.0578}_{-0.0501}$    |
| $m_\tau$ (GeV)   | 1.2922 $^{+0.0013}_{-0.0012}$     |

We use standard parametrization form for the  $V_{\text{CKM}}$  and take the following values at the scale Mz [22]:  $\sin\theta_{q12} = 0.2272$ ,  $\sin\theta_{q13} = 0.00382$ ,  $\sin\theta_{q23} = 0.04178$  and the  $CP$  phase  $\delta_q = \frac{\pi}{3}$ , where we use subscript  $q$  to distinguish them from the lepton section mixing angles. And we use RGE running factor  $\eta = 0.8853$ . At the GUT scale, we have the  $V_{\text{CKM}}$

$$\begin{bmatrix} 0.973841 & 0.227198 & 0.00169092 - 0.00292876i \\ -0.227079 - 0.000134603i & 0.97298 - 0.000031403i & 0.0369876 \\ 0.00675837 - 0.00284968i & -0.0364044 - 0.000664834i & 0.99912 \end{bmatrix} \quad (16)$$

#### A. Quark and charged lepton sector

Using the central values of charged lepton and down-quark masses at GUT scale,  $b_0, b_1, b_2, e_0, e_1$  and  $e_2$  are solved from Eq. (3) and (5) (in Mev)

$$\begin{aligned} b_0 &= 387.756, & b_1 &= -539.649, & b_2 &= 193.27, \\ e_0 &= -22.7734, & e_1 &= 22.8717, & e_2 &= -11.5298. \end{aligned} \quad (17)$$

For up-quark sector, by solving Eq. (2) and (11), we get values of  $a_3, a_4, a_5$ , and three conditions for  $a_0, a_1, a_2, d_0, d_1, d_2$  (in Mev):

$$\begin{aligned} a_3 &= -2990.72 - i54.757, \\ a_4 &= 554.859 - i234.705, \\ a_5 &= -66.748 + i8.155, \\ a_0 - 2a_2 + d_0 - 2d_2 &= 14.628 - i3.162, \\ a_0 + a_1 + a_2 + d_0 + d_1 + d_2 &= 308.363 + i3.977, \\ a_0 - a_1 + a_2 + d_0 - d_1 + d_2 &= 82288.5 - i7.169 \times 10^{-6}. \end{aligned} \quad (18)$$

We can see that accommodation of hierarchical structure of

fermions masses is realized by adjusting the parameters,  $S_4$  flavor symmetry itself does not provide hints on it [23].

#### B. Neutrino sector

In this model, the light neutrino mass matrix is given by type-I seesaw [24]. The mass matrix of right-handed neutrinos is proportional to an identity matrix due to the  $S_4$  quantum number assignment, therefore the Dirac mass matrix  $M^D$  determines the lepton sector mixing because the charged lepton mass matrix is diagonalized.

$$M_\nu = -\frac{1}{f_0 v_R} M_\nu^{DT} M_\nu^D. \quad (19)$$

This model gives hierarchical neutrino mass spectrum naturally. One can choose  $f_0 \sim 1$  and  $v_R \sim 10^{14}$  GeV, so the mass of the heaviest light neutrino is around  $10^{-2} - 10^{-1}$  eV.

The fit of neutrino sector are found by scanning whole parameter space spanned by  $x, y$  and  $z$  under the constrain of the current experiment requirements.

We choose the standard parametrization for the lepton sector mixing:

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \cdot \text{diag}(e^{-i\varphi_1/2}, e^{-i\varphi_2/2}, 1) \quad (20)$$

with  $c_{ij} \equiv \cos\theta_{ij}$  and  $s_{ij} \equiv \sin\theta_{ij}$ .  $\delta$  is the Dirac phase and  $\varphi_1, \varphi_2$  are Majorana phases of neutrinos. These phases have range from 0 to  $2\pi$ .

We take  $3\sigma$  experiment bound [25]:

$$\begin{aligned} 0.24 \leq \sin^2\theta_{12} \leq 0.40 & \quad 0.34 \leq \sin^2\theta_{23} \leq 0.68 \\ \sin^2\theta_{13} \leq 0.040 & \quad 0.024 \leq \Delta m_{\odot}^2/\Delta m_{\text{ATM}}^2 \leq 0.040. \end{aligned} \quad (21)$$

As mentioned earlier  $x, y$ , and  $z$  generally are complex numbers. For the most general  $CP$ -violation case, we treat the phases of  $x, y$ , and  $z$  as random input numbers with range  $0 - 2\pi$ . The results are shown in Fig. 1. In this case,  $\sin\theta_{13}$  has wide range from zero to the current bound with the most probable values  $0.02-0.09$  as shown in Fig. 1(a). Figure 1(b) shows the correlation between  $\sin\theta_{23}$  and  $\sin\theta_{13}$ . Figure 1(c) is the value distribution of Dirac  $CP$ -violation phase in the lepton sector. The allowed range of  $\delta$  is quite large from 0 to  $2\pi$  radians with the most probable values  $2-4$  radians. Two Majorana phases  $\varphi_1$  and  $\varphi_2$  have wide range from 0 to  $2\pi$  as shown in Fig. 1(d), which is expected.

Now we consider an interesting special case where  $x, y$ , and  $z$  are all real. Note the complexity of  $f_0\nu_R$  only

contributes an overall phase to the light neutrino mass matrix, which can be rotated away. Therefore, in this case leptonic  $CP$ -violation phases have same source as CKM phase.

The allowed range  $0.03-0.09$  for  $\sin\theta_{13}$  is narrower compared to the general case, and the most probable range is  $0.04-0.08$  as shown in Fig. 2(a). Unlike Fig. 1(b) and 2(b) exhibits an interesting correlation between  $\sin\theta_{23}$  and  $\sin\theta_{13}$ . If we take the central value of  $\theta_{23} = \frac{\pi}{4}$ , we can get two much narrower ranges for  $\sin\theta_{13}$ . One is  $0.055-0.06$ , and the other is  $0.070-0.075$ . The values of  $\delta$  are  $2.8-3$  radians, and  $6.0-6.1$  with small possibility as shown in Fig. 2(c). Figure 2(d) shows the allowed values of two Majorana phases. Note this parameter region is just left-up corner of Fig. 1(d) for the most general case. The most probable value ranges for  $\varphi_1$  and  $\varphi_2$  are  $0.02-0.15$  radians and  $6.19-6.25$  radians, respectively.

For illustration, we give a typical example of fit for this case. We take

$$x = 0.0139726, \quad y = 0.025914, \quad z = 0.273173 \quad (22)$$

and solve  $d_0, d_1, d_2, a_0, a_1, a_2$  from Eq. (13) and (18) (in

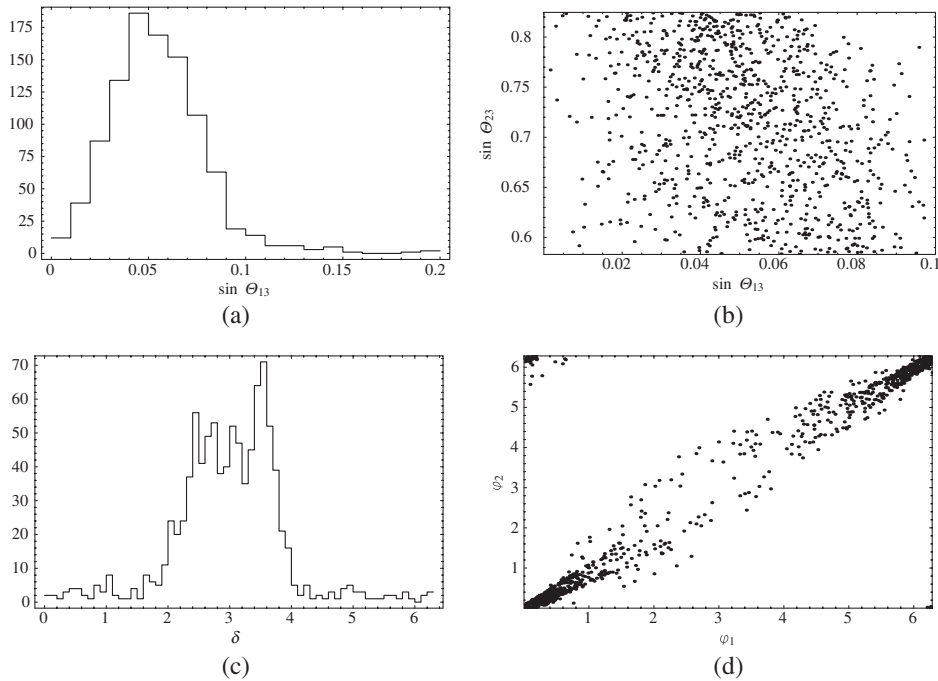


FIG. 1. Numerical analysis for the most general case where  $x, y$ , and  $z$  are complex consistent with current experimental bound Eq. (21). (a) Value distribution of  $\sin\theta_{13}$ . (b) Correlation between  $\sin\theta_{23}$  and  $\sin\theta_{13}$ . (c) Value distribution of leptonic Dirac  $CP$ -violation phase. (d) Scatter plot of two Majorana  $CP$ -violation phases  $\varphi_1$  and  $\varphi_2$ .

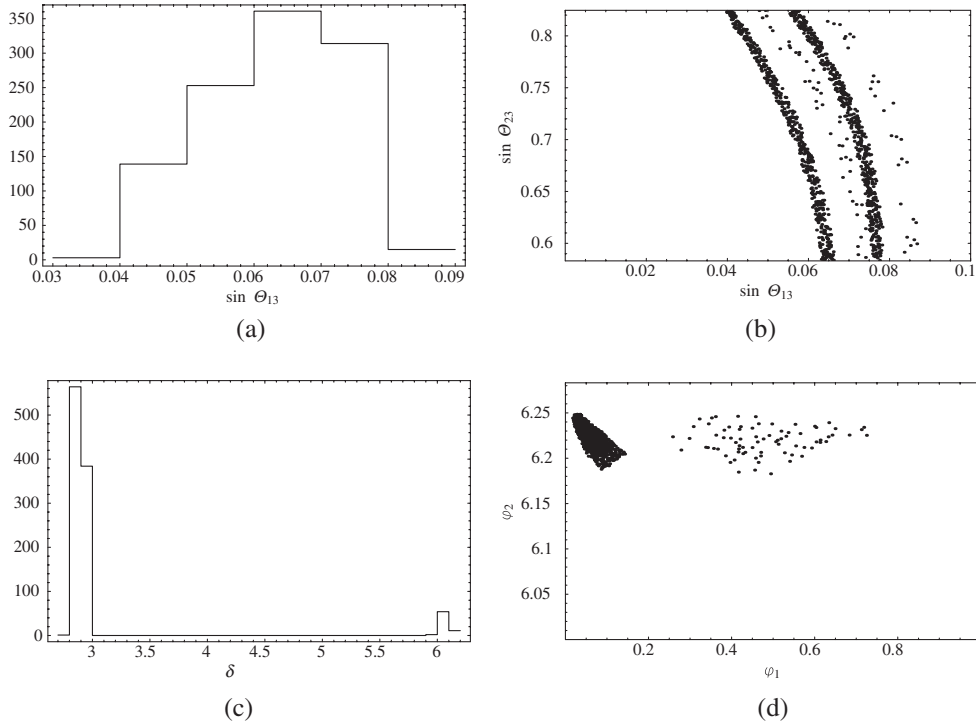


FIG. 2. Numerical analysis for case where  $x, y, z$ , are real consistent with current experimental bound Eq. (21). (a) Value distribution of  $\sin \theta_{13}$ . (b) Correlation between  $\sin \theta_{23}$  and  $\sin \theta_{13}$ . (c) Value distribution of leptonic Dirac  $CP$ -violation phase. (d) Scatter plot of two Majorana  $CP$ -violation phases  $\varphi_1$  and  $\varphi_2$ .

Mev)

$$\begin{aligned}
 d_0 &= 8602.18, \\
 d_1 &= -10191.2, \\
 d_2 &= 3725.19, \\
 a_0 &= 18935 + i0.271681, \\
 a_1 &= -30798.9 + i1.9887, \\
 a_2 &= 10036.1 + i1.71701.
 \end{aligned} \tag{23}$$

With these parameters values as input, one then obtains for the neutrino parameters

$$\begin{aligned}
 \sin \theta_{12} &\simeq 0.53, \\
 \sin \theta_{23} &\simeq 0.73 \\
 \sin \theta_{13} &\simeq 0.054, \\
 \Delta m_{\odot}^2 / \Delta m_{\text{ATM}}^2 &\simeq 0.031.
 \end{aligned} \tag{24}$$

And light neutrino masses are  $m_1 = 0.00774$  eV,  $m_2 = 0.0118$  eV,  $m_3 = 0.051$  eV, which are normalized by  $\Delta m_{31}^2 = 2.6 \times 10^{-3}$  eV. The Dirac phase appearing in MNS matrix is  $\delta = 2.84$  radians. And two Majorana phases are (in radians):  $\varphi_1 = 0.093$ ,  $\varphi_2 = 6.21$ . The Jarlskog invariant [26] has the value  $J_{cp} = 1.80 \times 10^{-3}$ . One can evaluate the effective neutrino mass for the neutrinoless double beta decays process to be

$$\left| \sum U_{ei}^2 m_{\nu i} \right| \simeq 0.009 \text{ eV}.$$

#### IV. SUMMARY AND CONCLUSION

In summary, we build a supersymmetric  $SO(10)$  model with  $S_4$  flavor symmetry. The three dimensional irreducible representation of  $S_4$  group unify three generations of fermions horizontally.  $\mathbf{10}$  and  $\mathbf{126}$  Higgs fields have been used to give the Yukawa couplings and generate all the masses and mixings of quarks and leptons. This model accommodates all observables including CKM  $CP$ -Violation phase. We studied the prediction of this model in the neutrino sector. For the most general  $CP$ -violation case, this model gives the most probable values 0.02–0.09 for  $\sin \theta_{13}$ . In a special case where leptonic phases have same  $CP$ -violation source as CKM phase, one gets narrower range 0.03–0.09 for  $\sin \theta_{13}$  with the most probable values 0.04–0.08.

In the model we present here, the masses of light neutrinos purely come from the type-I seesaw [24]. Generally, one also can include the contribution from type-II seesaw [27], which can generate a scenario with degenerate neutrino mass spectrum naturally because of the  $S_4$  symmetry if the type-II seesaw dominates the contribution to the light neutrinos masses. It is interesting to study the mixing pattern and its radioactive stability. We leave this possibility for future work.

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