Various versions of analytic QCD and skeleton-motivated evaluation of observables

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We present skeleton-motivated evaluation of QCD observables. The approach can be applied in analytic versions of QCD in certain classes of renormalization schemes. We present two versions of analytic QCD which can be regarded as low-energy modifications of the "minimal" analytic QCD and which reproduce the measured value of the semihadronic τ decay ratio r_τ. Further, we describe an approach of calculating the higher-order analytic couplings $\mathcal{A}_k$ ($k = 2, 3, \ldots$) on the basis of logarithmic derivatives of the analytic coupling $\mathcal{A}_1(Q^2)$. This approach can be applied in any version of analytic QCD. We adjust the free parameters of the aforementioned two analytic models in such a way that the skeleton-motivated evaluation reproduces the correct known values of r_τ and of the Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$ at a given point (e.g., at $Q^2 = 2$ GeV^2). We then evaluate the low-energy behavior of the Adler function $d_v(Q^2)$ and the BjPSR $d_b(Q^2)$ in the aforementioned evaluation approach, in the three analytic versions of QCD. We compare with the results obtained in the minimal analytic QCD and with the evaluation approach of Milton *et al.* and Shirkov.

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I. INTRODUCTION

In perturbative QCD (pQCD), the coupling parameter $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ [where: $Q^2 = -q^2 = -(q^0)^2 + \mathbf{q}^2$] is obtained on the basis of the perturbative β-function which is a (truncated) polynomial of a. As a consequence, $a(Q^2)$ has Landau singularities in an infrared spacelike zone ($Q^2 > 0$), and therefore these singularities are unphysical. This problem was fully recognized and a solution found about ten years ago by Shirkov and Solovtsov [1]. The solution found was minimal in the sense that the analytization $a(Q^2) \mapsto \mathcal{A}_1(Q^2)$ was performed by removing the Landau-cut singularities, while keeping the singularities on the timelike axis unchanged. Further, completely analogous minimal analytization was performed for the higher powers $a^k \mapsto \mathcal{A}_k$ ($k \geq 2$) and this replacement was performed term-by-term in the simple truncated perturbation series (STPS—in powers of a) of observables by Milton, Solovtsov, Solovtsova, and Shirkov [2–4] ("analytic perturbation theory"—APT).[1] The resulting series have in general better convergence behavior and much less sensitivity under the variation of the renormalization scale (RScl) and scheme (RSch). We will call the analytic QCD model based on the aforementioned analytic coupling the "minimal analytic" (MA) model [$\mapsto \mathcal{A}_1^{(\mathrm{MA})}(Q^2)$], and the aforementioned evaluation approach (involving the truncated analytic series) the APT-evaluation approach.

The MA coupling $\mathcal{A}_1^{(\mathrm{MA})}(Q^2)$ contains just one adjustable parameter, the QCD scale Λ. Reproduction of the measured values of the higher energy QCD observables ($|q^2| > 10$ GeV^2) fixes the scale parameter to the value $\Lambda_{(n_f=5)} \approx 0.26$ GeV, corresponding to $\Lambda_{(n_f=3)} \approx 0.4$ GeV. However, then the well-measured value of the massless strangeless semihadronic τ-decay ratio $r_\tau(\triangle S = 0, m_q = 0) = 0.204 \pm 0.005$ [7–9] (cf. Appendix E) cannot be reproduced [4] unless large values of the u, d, and s quark masses are introduced ($m_q \approx 0.25$–0.45 GeV) [10] and the threshold effects become very important. One may want to avoid introduction of such large quark masses, by modifying the MA model at low energies while keeping the analyticity of $\mathcal{A}_1(Q^2)$ in the non-time-like region. In this work we introduce two somewhat different modifications $\Delta\mathcal{A}_1(Q^2)$ ($\mathcal{A}_1 \equiv \mathcal{A}_1^{(\mathrm{MA})} + \Delta\mathcal{A}_1$), both having powerlike behaviors. We construct in a systematic way the higher-order couplings $\mathcal{A}_k(Q^2)$ based on the logarithmic derivatives of $\mathcal{A}_1(Q^2)$. Further, we construct a skeleton-expansion-motivated algorithm of evaluation of QCD observables, which can be applied in any analytic version of QCD and in a large class of renormalization schemes. For such an evaluation, we have to know the first few coefficients of STPS and all the leading-β_0 coefficients of the full perturbation series. We believe that the inclusion in this evaluation of the light-by-light contributions, if they contribute, should be avoided. Such contributions have a different topological structure and their evaluation should be performed separately in most evaluation (resummation) methods—see, for example, Ref. [11]. Some of the main results of the present work were published by us in a summarized form in Ref. [12].

In Sec. II, we explain the main features of the analytic versions of QCD (anQCD), we present the known MA model, and propose two versions of modified MA—the models "M1" and "M2" [$\mapsto \mathcal{A}_1^{(\mathrm{M1})}(Q^2), \mathcal{A}_1^{(\mathrm{M2})}(Q^2)$]. In Sec. III, we introduce the higher-order couplings $\mathcal{A}_k(Q^2)$ ($k \geq 2$) in a way that can be applied in any version of

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[1] Analytization of noninteger powers in MA was performed and used in Refs. [5], representing a generalization of results of Ref. [6].

anQCD, by imposing on them specific natural behavior under the change of scale Q^2 and of RSch. In Sec. IV we then present an algorithm which allows us to evaluate any QCD observable in any version of anQCD, an algorithm motivated by the skeleton expansion. In Sec. V we fix the free parameters in the M1 and M2 anQCD couplings $\mathcal{A}_1^{(\mathrm{M1})}(Q^2)$ and $\mathcal{A}_1^{(\mathrm{M2})}(Q^2)$ in such a way that the aforementioned skeleton-motivated approach gives us the measured values of r_τ and of the Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$ at $Q^2 = 2\ \mathrm{GeV}^2$. We then present the resulting low-energy curves for the V-channel Adler function $d_v(Q^2)$ and of the BjPSR $d_b(Q^2)$ in the skeleton-motivated approach, in the anQCD versions MA, M1, M2. We investigate the RScl and RSch dependence of the numerical curves, and in the MA case we compare the results of $d_b(Q^2)$ obtained by our skeleton-motivated evaluation approach with those of the APT approach of Refs. [2–4]. Numerical calculations were performed using MATHEMATICA [13]. In Sec. VI we present our conclusions and prospects for further work in this direction. Appendix A contains details of the coefficients appearing in the evaluation method. In Appendix B we present another evaluation method that is even more closely related to the skeleton expansion. Appendix C contains a derivation of the leading skeleton (LS) characteristic function of the BjPSR, and relations between the spacelike and timelike formulations for the LS term. Appendix D is a compilation of expressions of some coefficients used in this work, and Appendix E describes an extraction of the experimental value of $r_\tau(\triangle S = 0, m_q = 0)$.

II. MINIMAL ANALYTIC QCD AND TWO EXTENSIONS OF IT

The perturbative QCD coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ in the spacelike region [Q^2 not in $(-\infty, 0)$] has the scale dependence governed by the renormalization group equation (RGE)

$$\frac{\partial a(\ln Q^2; \beta_2, \ldots)}{\partial \ln Q^2} = -\sum_{j=2}^{j_{\max}} \beta_{j-2} a^j(\ln Q^2; \beta_2, \ldots), \qquad (1)$$

where the first two coefficients $\beta_0 = (1/4)(11 - 2n_f/3)$ and $\beta_1 = (1/16)(102 - 38n_f/3)$ are scheme independent in mass-independent schemes, and the other coefficients β_j ($j \geq 2$) characterize the RSch. In practice, the above sum is truncated at a certain $j_{\max}$ where $j_{\max} - 1$ is the loop level. The perturbative RGE (1) has a standard iterative solution in the form

$$a(Q^2) = \sum_{k=1}^{\infty} \sum_{\ell=0}^{k-1} K_{k\ell} \frac{(\ln L)^\ell}{L^k}, \qquad (2)$$

where $L = \ln(Q^2/\Lambda^2)$ and $K_{k\ell}$ are constants depending on the β_j coefficients and on the choice of the scale Λ. If the conventional (“$\overline{\mathrm{MS}}$”) scale $\Lambda = \bar{\Lambda}$ [14,15] is used, the coefficients $K_{k\ell}$ are

$$K_{10} = 1/\beta_0; \qquad K_{20} = 0; \qquad K_{21} = -\beta_1/\beta_0^3;$$
$$K_{30} = -\beta_1^2/\beta_0^5 + \beta_2/\beta_0^4; \qquad (3)$$
$$K_{31} = -K_{32} = -\beta_1^2/\beta_0^5; \quad \ldots$$

Further coefficients $K_{k\ell}$, up to $k = 6$, are given in Appendix D. The coupling $a(Q^2)$, Eq. (2), has nonanalytic structure along the timelike axis $Q^2(\equiv -q^2) < 0$. In addition, it has singularities in the spacelike region $0 < Q^2 \leq \bar{\Lambda}^2$, which are formally the consequence of the (truncated) power expansion structure of the beta function on the right-hand side of Eq. (1). Application of the Cauchy theorem to function $a(Q^2)$ in the Q^2-plane gives then the following dispersion relation for a:

$$a(Q^2) = \frac{1}{\pi} \int_{\sigma=-\Lambda^2-\eta}^{\infty} \frac{d\sigma \rho_1^{(\mathrm{pt})}(\sigma)}{(\sigma + Q^2)}, \qquad (4)$$

where $\rho_1^{(\mathrm{pt})}(\sigma)$ is the (pQCD) discontinuity function of a along the cut axis in the Q^2-plane: $\rho_1^{(\mathrm{pt})}(\sigma) = \mathrm{Im}\, a(-\sigma - i\epsilon)$. In the integration, η is positive ($\eta \to +0$ can be taken), reflecting the fact that the corresponding contour integration path avoids entirely the singularities of $a(z)$ in the complex plane, including the singularity at $z \equiv -\sigma = \Lambda^2$ [cf. Eq. (2)].

By special relativity and causality, observables are analytic functions of the associated physical momentum squared $q^2 \equiv -Q^2$ in the Q^2-plane with the timelike axis ($Q^2 < 0$) excluded. Since QCD observables are functions of the invariant coupling $a(Q^2)$, both should have the same analyticity properties. The singularity sector $0 < Q^2 \leq \Lambda^2$ in $a(Q^2)$, Eqs. (2) and (4), is therefore nonphysical. The most straightforward rectification of this problem is to eliminate that sector from the dispersion relation (4) while keeping the pQCD discontinuity function $\rho^{(\mathrm{pt})}(\sigma; \beta_2, \ldots)$ unchanged on the timelike axis $\sigma > 0$ [1], thus leading to the specific “minimal analytic” (MA) coupling

$$\mathcal{A}_1^{(\mathrm{MA})}(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\sigma \rho_1^{(\mathrm{pt})}(\sigma)}{(\sigma + Q^2)}. \qquad (5)$$

In practice, truncated series (2) can be used to obtain the discontinuity function $\rho_1^{(\mathrm{pt})}(\sigma)$ and thus the coupling (5). Prescription (5) was investigated from calculational viewpoints in Refs. [16–18]. There exists a practical iterative solution [16,17] to RGE (1) based on the Lambert function [19]. This solution is an expansion of a different form than (2). When the number of terms in the Lambert-based expansion and in expansion (2) increases, the two solutions for $\mathcal{A}_1^{(\mathrm{MA})}$ converge to the exact numerical solution rapidly for all Q^2, but the Lambert-based expansion converges faster. When $k_{\max} \geq 4$ in (2), the corresponding solution $\mathcal{A}_1^{(\mathrm{MA})}(Q^2)$ differs in $\overline{\mathrm{MS}}$ RSch from the exact numerical solution by less than one per cent for all $Q^2 > 0$ [17]. In the present work, we will use expansion (2) with $k_{\max} = 5$ or 6.

Other types of analytization of a can be performed by focusing on the analyticity properties of the beta function [20,21], or by subtracting certain power correction terms $1/(Q^2)^n$ from the MA coupling $\mathcal{A}_1^{(\mathrm{MA})}$ [22]. For a review of various models, see Ref. [23].

In general, the discontinuity function can be different, and the analytic coupling must have the form

$$\mathcal{A}_1(Q^2) = \frac{1}{\pi}\int_{\sigma=0}^{\infty}\frac{d\sigma\rho_1(\sigma)}{(\sigma+Q^2)}, \tag{6}$$

where $\rho_1(\sigma) = \mathrm{Im}\mathcal{A}_1(-\sigma - i\epsilon)$. Relation (6) defines an analytic coupling in the Q^2-plane excluding the timelike semiaxis $-s = Q^2 < 0$. On this semiaxis, it is convenient to define the timelike coupling [24–26]

$$\mathfrak{A}_1(s) = \frac{i}{2\pi}\int_{-s+i\epsilon}^{-s-i\epsilon}\frac{d\sigma'}{\sigma'}\mathcal{A}_1(\sigma'). \tag{7}$$

The integration here is in the $Q^2 \equiv \sigma'$ plane avoiding the (timelike) cut $\sigma' < 0$. The relation between $\mathcal{A}_1(Q^2)$ and $\mathfrak{A}_1(s)$ is the same as the relation between the (vector channel) Adler function $D_V(Q^2)$ and its timelike analogue, the e^+e^- hadronic scattering cross section ratio $R_V(s)$. Therefore, while the leading QCD correction to $D_V(Q^2)$ in anQCD is $\mathcal{A}_1(Q^2)$ [the anQCD analogue of $a(Q^2)$], the leading QCD correction to $R_V(s)$ is $\mathfrak{A}_1(s)$. The following additional relations [3] hold between $\mathcal{A}_1$, $\mathfrak{A}_1$, and ρ_1 in any anQCD:

$$\mathfrak{A}_1(s) = \frac{1}{\pi}\int_s^{\infty}\frac{d\sigma}{\sigma}\rho_1(\sigma), \tag{8}$$

$$\mathcal{A}_1(Q^2) = Q^2\int_0^{\infty}\frac{ds\mathfrak{A}_1(s)}{(s+Q^2)^2}, \tag{9}$$

$$\frac{d}{d\ln\sigma}\mathfrak{A}_1(\sigma) = -\frac{1}{\pi}\rho_1(\sigma). \tag{10}$$

The MA coupling (5) contains only one free parameter, the value of the ($\overline{\mathrm{MS}}$) scale $\bar{\Lambda}$, which is not equal to the value of $\bar{\Lambda}$ in pQCD, but has to be adjusted so that the measured values of QCD observables be reproduced. By introducing and using a specific evaluation method within the MA QCD, the authors of Refs. [2–4] reproduced the measured values of the higher-energy QCD observables ($|q^2| > 10\ \mathrm{GeV}^2$) when the scale parameter $\bar{\Lambda}$ had the value $\bar{\Lambda}_{(n_f=5)} \approx 0.26$ GeV (where n_f is the number of active quark flavors). This corresponds to $\bar{\Lambda}_{(n_f=3)} \approx 0.4$ GeV. However, the measured value of the massless part of the semihadronic strangeless τ-decay ratio $r_\tau(\triangle S = 0, m_q = 0) = 0.204 \pm 0.005$ [7–9] [cf. Appendix E, Eq. (E6)] cannot be reproduced with such values of $\bar{\Lambda}$ [4] unless large masses of u, d, and s quarks are introduced ($m_q \approx$ 0.25–0.45 GeV) [10] and the mass threshold effects become central.

The above consideration motivates us to introduce low-energy modifications of the MA coupling. Modifications, although simple, introduce additional parameters which have to be fixed by requiring reproduction of the measured values of low-energy QCD observables, including of r_τ. One possible modification is inspired by the well measured [7,8] isovector hadronic spectral function $R_V(s)$. At low energies ($s < 1\ \mathrm{GeV}^2$), it is dominated by the ρ-resonance ($M_\rho = 0.776$ GeV) which, in the narrow width approximation, can be represented as a delta function $\delta(s - M_\rho^2)$ [27]. This is in the spirit of the vector meson dominance (VMD). If we assume that the s-dependence of the timelike quantity $R_V(s)$ is at least qualitatively described by the first order timelike coupling $\mathfrak{A}_1(s)$, Eq. (8), then the aforementioned deltalike structure should appear in it. This then leads to the following ansatz (model "M1"):

$$\mathfrak{A}_1^{(\mathrm{M1})}(s) = c_f\bar{M}_r^2\delta(s-\bar{M}_r^2) + k_0\Theta(\bar{M}_0^2 - s) + \Theta(s-\bar{M}_0^2)\mathfrak{A}_1^{(\mathrm{MA})}(s), \tag{11}$$

where c_f, k_0, $c_r = \bar{M}_r^2/\bar{\Lambda}^2$, $c_0 = \bar{M}_0^2/\bar{\Lambda}^2$ are four dimensionless parameters of the model; $\Theta(x)$ is the Heaviside step function (+ 1 for $x > 0$, zero otherwise). In this model, the MA behavior of $\mathfrak{A}_1(s)$ at low energies $s < \bar{M}_0^2$ has been replaced by a constant (k_0) plus a delta function (at $s = \bar{M}_r^2 < \bar{M}_0^2$). The more literal application of the VMD approach results in $k_0 = -1$ [28]. This is so because $R_V(s) = 1 + \mathfrak{A}_1(s) + \mathcal{O}(\mathfrak{A}_1^2)$, and $R_V(s) \to 0$ when $s \to 0$, implying $\mathfrak{A}_1(s) \to -1$. However, such a model appears to restrict the low-energy behavior of $\mathfrak{A}_1(s)$ and of $\mathcal{A}_1(Q^2)$ too severely, especially if we want to impose the condition of merging $\mathcal{A}_1(Q^2)$ of the model with $\mathcal{A}_1^{(\mathrm{MA})}(Q^2)$ at high Q^2. As a consequence, values of various unrelated low-energy observables, such as Adler function (or r_τ) and Bjorken polarized sum rule, cannot be reproduced simultaneously in such a model. Therefore, unlike the choice $k_0 = -1$ in Ref. [28], we keep here the constant k_0 in Eq. (11) free. Applying transformation (9) to expression (11) gives the spacelike analytic coupling of the model:

$$\mathcal{A}_1^{(\mathrm{M1})}(Q^2) = \mathcal{A}_1^{(\mathrm{MA})}(Q^2) + \Delta\mathcal{A}_1^{(\mathrm{M1})}(Q^2), \tag{12}$$

$$\Delta\mathcal{A}_1^{(\mathrm{M1})}(Q^2) = -\frac{1}{\pi}\int_{\sigma=0}^{\bar{M}_0^2}\frac{d\sigma\rho_1^{(\mathrm{pt})}(\sigma)}{(\sigma+Q^2)} + c_f\frac{\bar{M}_r^2Q^2}{(Q^2+\bar{M}_r^2)^2} - d_f\frac{\bar{M}_0^2}{(Q^2+\bar{M}_0^2)}, \tag{13}$$

where the constant d_f is

$$d_f \equiv -k_0 + \frac{1}{\pi}\int_{\bar{M}_0^2}^{\infty}\frac{d\sigma}{\sigma}\rho_1^{(\mathrm{pt})}(\sigma). \tag{14}$$

The coupling (12) and (13) can also be rewritten in a somewhat different, but equivalent, form:

$$\mathcal{A}_1^{(\mathrm{M1})}(Q^2) = c_f \frac{\bar{M}_r^2 Q^2}{(Q^2+\bar{M}_r^2)^2} + k_0 \frac{\bar{M}_0^2}{(Q^2+\bar{M}_0^2)} + \frac{Q^2}{(Q^2+\bar{M}_0^2)} \frac{1}{\pi} \times \int_{\sigma=\bar{M}_0^2}^{\infty} \frac{d\sigma \rho_1^{(\mathrm{pt})}(\sigma)(\sigma-\bar{M}_0^2)}{\sigma(\sigma+Q^2)}. \tag{15}$$

In general, this coupling differs from the MA coupling (5) by terms $\Delta\mathcal{A}_1^{(\mathrm{M1})} \sim \bar{\Lambda}^2/Q^2$. However, we will choose to require $\Delta\mathcal{A}_1^{(\mathrm{M1})} \sim \bar{\Lambda}^4/Q^4$, i.e., that M1 effectively merge into MA at higher energies, as we did in Ref. [28]. This condition eliminates one of the four new parameters, for example k_0:

$$k_0 = -\frac{c_r c_f}{c_0} + \frac{1}{\pi}\frac{1}{c_0\bar{\Lambda}^2}\int_0^{c_0\bar{\Lambda}^2} d\sigma \rho_1^{(\mathrm{pt})}(\sigma) + \frac{1}{\pi}\int_{c_0\bar{\Lambda}^2}^{\infty}\frac{d\sigma}{\sigma}\rho_1^{(\mathrm{pt})}(\sigma). \tag{16}$$

Since the presented version of M1 merges with MA at higher energies, the value of the scale parameter $\bar{\Lambda}$ remains practically unchanged, $\bar{\Lambda}_{(n_f=3)} = 0.4$ GeV, and the model contains only three dimensionless parameters c_f, c_r, and c_0.

Another, somewhat simpler, modification of the MA coupling consists in adding a constant value (c_v) in the low-energy region of the MA timelike coupling (model "M2"):

$$\mathfrak{A}_1^{(\mathrm{M2})}(s) = \mathfrak{A}_1^{(\mathrm{MA})}(s) + c_v \Theta(\bar{M}_p^2 - s), \tag{17}$$

$$\mathcal{A}_1^{(\mathrm{M2})}(Q^2) = \mathcal{A}_1^{(\mathrm{MA})}(Q^2) + c_v \frac{\bar{M}_p^2}{(Q^2+\bar{M}_p^2)}, \tag{18}$$

where c_v and $c_p = \bar{M}_p^2/\bar{\Lambda}^2$ are two dimensionless parameters of the model. For simplicity, we will assume that the scale parameter is unchanged: $\bar{\Lambda}_{(n_f=3)} = 0.4$ GeV. The resulting additional term $\propto 1/(Q^2+\bar{M}_p^2)$ in $\mathcal{A}_1(Q^2)$ can be interpreted, or motivated, as the leading powerlike modification ($\propto 1/Q^2$) of the MA coupling such that the condition $|\mathcal{A}_1(Q^2=0)| < \infty$ is preserved. The latter condition is regarded as desirable in our approach developed in Sec. IV, because the so-called leading-skeleton resummation of observables remains finite in such a model.

Model M1 was motivated by simulating roughly the ρ-resonance contribution in the *one-loop* expression for $R_V(s)$, via a VMD narrow width approximation ansatz in $\mathfrak{A}_1(s)$. However, this was only a motivation for the construction of an explicit form of $\mathfrak{A}_1(s)$ as the starting point of the model, and the higher-loop contributions $\mathfrak{A}_k(s)$ and $\mathcal{A}_k(Q^2)$ ($k \geq 2$) are then constructed on the basis of this $\mathfrak{A}_1(s)$ (see the next section). The approximation of the ρ-resonance is then expected to get worse at higher-loop level. Another possible approach, which we will not follow here, would be to refine (retroactively) $\mathfrak{A}_1(s)$ so that higher-loop evaluations of $R_V(s)$ give us a given specified approximation of the ρ-resonance at low energies. A similar approach could possibly be followed also in M2. In general, reproduction of the correct low-energy behavior of timelike observables such as $R_V(s)$ represents a difficult problem. In this work, we will follow a more modest approach—in Sec. V we will fix the free parameters of models M1 and M2 by requiring, at loop-level three or four, the reproduction of the central experimental values for the Bjorken polarized sum rule $d_b(Q^2)$ at two (in M1) or one (in M2) values of scale $Q(\geq 1$ GeV), and the reproduction of the measured value of $r_\tau(\triangle S = 0)$.

All the versions of anQCD presented here are infrared finite, i.e., the zero momentum limits $\mathcal{A}_1(0) = \mathfrak{A}_1(0)$ are finite.

III. ANALYTIZATION OF HIGHER POWERS OF THE COUPLING PARAMETER

In the previous section, a few of the possibilities of constructing the analytic version $\mathcal{A}_1(Q^2)$ of $a(Q^2)$ were presented. For evaluation of QCD observables, the analytic versions of higher powers $a^k(Q^2)$ are needed as well. For that, there is no unique way of constructing the correspondence $a^k \leftrightarrow \mathcal{A}_k$. In the MA QCD, one possibility is to apply the MA procedure (5) to each power of a [2]:

$$a^k(Q^2) \mapsto \mathcal{A}_k^{(\mathrm{MA})}(Q^2) = \frac{1}{\pi}\int_0^{\infty}\frac{d\sigma}{\sigma+Q^2}\rho_k^{(\mathrm{pt})}(\sigma) \quad (k=1,2,\ldots), \tag{19}$$

where $\rho_k^{(\mathrm{pt})} = \mathrm{Im}[a^k(-\sigma - i\epsilon)]$, and a is given, e.g., by Eq. (2). Other choices would be, e.g. $a^k \mapsto \mathcal{A}_1^k$, $\mathcal{A}_1^{k-2}\mathcal{A}_2$, etc. With construction (19), it was shown [16] that the RGE's governing the evolution of $\mathcal{A}_k$'s are identical to those governing the evolution of a^k's in pQCD when the replacements $a^j \mapsto \mathcal{A}_j^{(\mathrm{MA})}$ are made [cf. Eq. (1)]

$$\begin{aligned}\frac{\partial\mathcal{A}_k^{(\mathrm{MA})}(\mu^2)}{\partial\ln\mu^2} &= -k\sum_{j=2}^{j_{\max}}\beta_{j-2}\mathcal{A}_{j+k-1}^{(\mathrm{MA})}(\mu^2) \\ &= -k\beta_0\mathcal{A}_{k+1}^{(\mathrm{MA})}(\mu^2) - \cdots, \\ \frac{\partial^2\mathcal{A}_k^{(\mathrm{MA})}(\mu^2)}{\partial(\ln\mu^2)^2} &= k\sum_{j,\ell=2}^{j_{\max}}\beta_{j-2}\beta_{\ell-2}(\ell+k-1) \times \mathcal{A}_{j+\ell+k-2}^{(\mathrm{MA})}(\mu^2) \\ &= k(k+1)\beta_0^2\mathcal{A}_{k+2}^{(\mathrm{MA})}(\mu^2) + \cdots, \quad \text{etc.}\end{aligned} \tag{20}$$

The reason for this lies in the fact that a^k, and consequently $\rho_k^{(\mathrm{pt})}(\sigma)$, fulfill analogous RGE's. Further, the renormalization scheme (RSch) dependence in pQCD, i.e., dependence of a^k of β_j ($j \geq 2$), is known [29] (cf. also [30]), the same dependence holds for the discontinuity functions $\rho_k^{(\mathrm{pt})}(\sigma, \beta_2, \ldots)$ and thus for the MA couplings (19) the analogous dependence via $a^j \leftrightarrow \mathcal{A}_j^{(\mathrm{MA})}$ is obtained ($k = 1, 2, \ldots$):

$$\frac{\partial \mathcal{A}_k^{(\mathrm{MA})}(\mu^2)}{\partial \beta_2} = \frac{k}{\beta_0} \mathcal{A}_{k+2}^{(\mathrm{MA})}(\mu^2) + \frac{k\beta_2}{3\beta_0^2} \mathcal{A}_{k+4}^{(\mathrm{MA})}(\mu^2) + \mathcal{O}(\mathcal{A}_{k+5}^{(\mathrm{MA})}), \tag{21}$$

$$\frac{\partial \mathcal{A}_k^{(\mathrm{MA})}(\mu^2)}{\partial \beta_3} = \frac{k}{2\beta_0} \mathcal{A}_{k+3}^{(\mathrm{MA})}(\mu^2) - \frac{k\beta_1}{6\beta_0^2} \mathcal{A}_{k+4}^{(\mathrm{MA})}(\mu^2) + \mathcal{O}(\mathcal{A}_{k+5}^{(\mathrm{MA})}), \tag{22}$$

$$\frac{\partial \mathcal{A}_k^{(\mathrm{MA})}(\mu^2)}{\partial \beta_4} = \frac{k}{3\beta_0} \mathcal{A}_{k+4}^{(\mathrm{MA})}(\mu^2) + \mathcal{O}(\mathcal{A}_{k+5}^{(\mathrm{MA})}). \tag{23}$$

The RGE-type relations (20)–(23), valid in the MA QCD, imply the following important property: If the evaluation of a spacelike QCD observable quantity $\mathcal{D}(Q^2)$ is based on the analytization of STPS of that quantity according to the rule $a^k(\mu^2) \mapsto \mathcal{A}_k^{(\mathrm{MA})}(\mu^2)$ ($k \geq 1$), then the evaluated value of $\mathcal{D}(Q^2)$ has a dependence on RScl μ and on RSch (β_j, $j \geq 2$) which is suppressed systematically. The suppression gets stronger as the number of terms increases, just as in pQCD. The precision $\mathcal{O}(\mathcal{A}_n^{(\mathrm{MA})})$ corresponds in pQCD to the precision $\mathcal{O}(a^n)$.

Having the STPS with terms up to $\sim a^{n_{\max}}$ ($n_{\max} \equiv n_{\mathrm{m}}$), as well as its analytized analog

$$\mathcal{D}_{\mathrm{STPS}}^{(n_{\mathrm{m}})}(Q^2) = a(\mu^2; \beta_2, \ldots) + \sum_{n=2}^{n_{\mathrm{m}}} d_{n-1} a^n(\mu^2; \beta_2, \ldots), \tag{24}$$

$$\mathcal{D}_{\mathrm{an.}}^{(n_{\mathrm{m}})}(Q^2) = \mathcal{A}_1(\mu^2; \beta_2, \ldots) + \sum_{n=2}^{n_{\mathrm{m}}} d_{n-1} \mathcal{A}_n(\mu^2; \beta_2, \ldots), \tag{25}$$

it is then enough to include in the evolution rules (20)–(23) (for $k = 1$ only) terms of up to $\mathcal{A}_{n_{\mathrm{m}}}$ on the right-hand side (RHS). Then the analytized evaluated values $\mathcal{D}_{\mathrm{an.}}(Q^2)$ will have the RScl- and RSch-independence precision $\partial \mathcal{D}_{\mathrm{an.}}^{(n_{\mathrm{m}})}(Q^2)/\partial X \sim A_{n_{\mathrm{m}}+1}$ ($X = \ln \mu^2, \beta_j$) which has its perturbative analog $\partial \mathcal{D}_{\mathrm{STPS}}^{(n_{\mathrm{m}})}(Q^2)/\partial X \sim a^{n_{\mathrm{m}}+1}$.

In view of these considerations, we propose to maintain evolution relations (20) (for $k = 1$) for any version of anQCD, including models M1 and M2 of the previous section, truncating them as just mentioned:

$$\frac{\partial \mathcal{A}_1(\mu^2; \beta_2, \ldots)}{\partial \ln \mu^2} = -\beta_0 \mathcal{A}_2 - \cdots - \beta_{n_{\mathrm{m}}-2} \mathcal{A}_{n_{\mathrm{m}}},$$
$$\frac{\partial^2 \mathcal{A}_1(\mu^2; \beta_2, \ldots)}{\partial (\ln \mu^2)^2} = 2\beta_0^2 \mathcal{A}_3 + 5\beta_0\beta_1 \mathcal{A}_4 + \cdots + \kappa_{n_{\mathrm{m}}}^{(2)} \mathcal{A}_{n_{\mathrm{m}}}, \text{ etc.,} \tag{26}$$

where we have altogether $n_{\mathrm{m}} - 1$ equations, and $\kappa_n^{(\ell)}$ are the corresponding coefficients of the pQCD evolution equations. Equations (26) represent *definitions* of $\mathcal{A}_k$'s ($2 \leq k \leq n_{\mathrm{m}}$) via combinations of derivatives $\partial^n \mathcal{A}_1/\partial(\ln \mu^2)^n$.

On the other hand, evolution Eqs. (21)–(23) (for $k = 1$) for the change of RSch remain of the same form, but with aforementioned truncation

$$\frac{\partial \mathcal{A}_1(\mu^2; \beta_2, \ldots)}{\partial \beta_2} \approx \frac{1}{\beta_0} \mathcal{A}_3 + \frac{\beta_2}{3\beta_0^2} \mathcal{A}_5 + \cdots + k_{n_{\mathrm{m}}}^{(2)} \mathcal{A}_{n_{\mathrm{m}}},$$
$$\frac{\partial \mathcal{A}_1(\mu^2; \beta_2, \ldots)}{\partial \beta_3} \approx \frac{1}{2\beta_0} \mathcal{A}_4 - \frac{\beta_1}{6\beta_0^2} \mathcal{A}_5 + \cdots + k_{n_{\mathrm{m}}}^{(3)} \mathcal{A}_{n_{\mathrm{m}}}, \text{ etc.} \tag{27}$$

where we have altogether $n_{\mathrm{m}} - 2$ equations, and $k_n^{(\ell)}$ are the corresponding coefficients of the pQCD evolution equations. Equations (27) are, in contrast to Eqs. (26), not definitions, but in general approximations for the evolution under RSch changes. The RSch dependence of $\mathcal{A}_1(\mu^2)$ is treated in more detail later in this work.

On the basis of Eqs. (26) and (27), expressions for the (truncated) derivatives $\partial \mathcal{A}_k/\partial X$, for $k \geq 2$ ($X = \ln \mu^2, \beta_j$), can be obtained.

In our approach, the basic spacelike quantities are $\mathcal{A}_1(\mu^2)$ of a given anQCD model (e.g., MA, M1, M2) and its logarithmic derivatives

$$\tilde{\mathcal{A}}_n(\mu^2) \equiv \frac{(-1)^{n-1}}{\beta_0^{n-1}(n-1)!} \frac{\partial^{n-1} \mathcal{A}_1(\mu^2)}{\partial (\ln \mu^2)^{n-1}}, \quad (n = 1, 2, 3, \ldots), \tag{28}$$

whose pQCD analogs are

$$\tilde{a}_n(\mu^2) \equiv \frac{(-1)^{n-1}}{\beta_0^{n-1}(n-1)!} \frac{\partial^{n-1} a(\mu^2)}{\partial (\ln \mu^2)^{n-1}}, \quad (n = 1, 2, 3, \ldots). \tag{29}$$

The quantities $(\mathcal{A}_1(\mu^2), \tilde{\mathcal{A}}_2(\mu^2), \tilde{\mathcal{A}}_3(\mu^2), \ldots)$, all derived from $\mathcal{A}_1(\mu^2) \equiv \tilde{\mathcal{A}}_1(\mu^2)$, are known functions of the spacelike momenta μ in any chosen anQCD version in a given chosen RSch ($\beta_2, \beta_3, \ldots$). On the basis of these quantities and the (truncated) evolution equations (26), any higher-order quantity $\mathcal{A}_k(\mu^2)$ ($k \geq 2$) can be constructed, in the given RSch. Further, (truncated) Eqs. (26) and (27) then give us the values of $\tilde{\mathcal{A}}_k(\mu^2)$ and of $\mathcal{A}_k(\mu^2)$ ($k \geq 1$) in any other chosen RSch ($\beta_2', \beta_3', \ldots$). We emphasize that in this approach, the higher-order quantities $\mathcal{A}_k(\mu^2)$ ($k \geq 2$) are not as basic, they are defined via Eqs. (26) for convenience of having closer notational analogy with pQCD formulas (and $a^k \leftrightarrow \mathcal{A}_k$). In these definitions (26), as well as in β_j-running Eqs. (27), we could have kept one more term ($\sim \mathcal{A}_{n_m+1}$), in order to come closer to the exact analogy $\mathcal{A}_k = a^k + \text{NP}$ for $k \geq 2$, where NP stands for nonperturbative contributions (nonanalytic functions of a at $a = 0$).[2] However, this is not necessary, as argued below.

The basic analytization rule we adopt will thus be

$$\tilde{a}_n \mapsto \tilde{\mathcal{A}}_n \qquad (n = 1, 2, \ldots), \tag{30}$$

where $\tilde{\mathcal{A}}_n$ and $\tilde{a}_n$ are defined in Eqs. (28) and (29), respectively.

At loop level $n_{\max} \equiv n_{\mathrm{m}}$, and in a chosen "starting" RSch ($\beta_2, \beta_3, \ldots$), the truncation ("tr") of the RGE-running of the pQCD coupling $a(\mu^2)$ is in principle via Eq. (1) with $j_{\max} = n_{\max} + 1$ ($a = a_{\mathrm{tr}}$, $\tilde{a}_n = \tilde{a}_{n,\mathrm{tr}}$). The corresponding truncated $\tilde{\mathcal{A}}_n = \tilde{\mathcal{A}}_{n,\mathrm{tr}}$ are then

$$\tilde{\mathcal{A}}_n(\mu^2) = \tilde{a}_n + \text{NP} = \tilde{a}_n(\mu^2)_{(\infty)} + \text{NP} + \mathcal{O}(\beta_0^{n_{\mathrm{m}}-1} a^{n_{\mathrm{m}}+n}), \quad (n = 1, 2, \ldots), \tag{31}$$

and we assumed that we are in the class of the RSch's where $\beta_j \sim \beta_0^j$ in the large-β_0 limit. We recall that $\tilde{\mathcal{A}}_1 \equiv \mathcal{A}_1$ and $\tilde{a}_1 \equiv a$. The subscript (∞) in Eq. (31) means that this is the quantity obtained by not truncating RGE beta function (1), i.e., for $j_{\max} = \infty$ and keeping the same value of Λ in expansion (2) as in the case of the truncated beta function (i.e., $j_{\max} = n_{\max} + 1$). The second identity in Eq. (31) thus shows, as an additional reference, the magnitude of error committed due to the truncation of the beta function. Definitions (26) of $\mathcal{A}_n$'s then imply

$$\mathcal{A}_n(\mu^2) = a^n(\mu^2) + \text{NP} + \mathcal{O}(\beta_0^{n_{\mathrm{m}}-n} a^{n_{\mathrm{m}}+1}) \quad (n = 2, \ldots, n_{\mathrm{m}}). \tag{32}$$

Since the RGE-running (1) of a is truncated, we have $a^n = a^n_{(\infty)} + \mathcal{O}(\beta_0^{n_{\mathrm{m}}-1} a^{n_{\mathrm{m}}+n})$, and relations (32) remain unchanged when $a^n(\mu^2)$ there is replaced by $a^n_{(\infty)}(\mu^2)$.

The β_j-running Eqs. (27) are also truncated, i.e., the RHS's there have errors $\sim \mathcal{A}_{n_m+1}$, so that the changes of RSch entail additional errors. It can be verified that this effect, when going from a chosen "starting" RSch ($\beta_2, \beta_3, \ldots$) to another RSch ($\beta_2', \beta_3', \ldots$), modifies relations (31) to

$$\begin{aligned} \tilde{\mathcal{A}}_1(\equiv A_1(\mu^2)) &= a(\mu^2) + \mathcal{O}(\beta_0^{n_{\mathrm{m}}-2} a^{n_{\mathrm{m}}+1}) + \text{NP}, \\ \tilde{\mathcal{A}}_n(\mu^2) &= \tilde{a}_n + \mathcal{O}(\beta_0^{n_{\mathrm{m}}-2} a^{n_{\mathrm{m}}+n}) + \text{NP} \\ &(n = 2, \ldots, n_{\mathrm{m}}), \end{aligned} \tag{33}$$

while relations (32) do not get modified. We should keep in mind that there is yet another truncation involved, namely, in the solution (2) of RGE (1) the sum over index k has in the calculational practice finite number of terms. In our calculations, we will take there $k_{\max} = n_{\max} + 2$ ($= j_{\max} + 1$), which is so high that it does not affect "precision estimate" relations (32) and (33).

For example, at loop level 3 ($n_{\max} = 3$), where we include in RGE (1) term with $j_{\max} = 4$ (thus β_2), relations (26) are

$$\begin{aligned} \tilde{\mathcal{A}}_2(\mu^2) &= \mathcal{A}_2(\mu^2) + \frac{\beta_1}{\beta_0} A_3(\mu^2), \\ \tilde{\mathcal{A}}_3(\mu^2) &= \mathcal{A}_3(\mu^2), \end{aligned} \tag{34}$$

implying

$$\begin{aligned} \mathcal{A}_2(\mu^2) &= \tilde{\mathcal{A}}_2(\mu^2) - \frac{\beta_1}{\beta_0} \tilde{\mathcal{A}}_3(\mu^2), \\ \mathcal{A}_3(\mu^2) &= \tilde{\mathcal{A}}_3(\mu^2). \end{aligned} \tag{35}$$

The RSch (β_2) dependence is obtained from the truncated Eqs. (26) and (27)

$$\frac{\partial \tilde{\mathcal{A}}_j(\mu^2;\beta_2)}{\partial \beta_2} = \frac{1}{2\beta_0^3} \frac{\partial^2 \tilde{\mathcal{A}}_j(\mu^2;\beta_2)}{\partial (\ln \mu^2)^2} \left(\equiv \frac{1}{\beta_0} \tilde{\mathcal{A}}_3(\mu^2;\beta_2) \right) \quad (j = 1, 2, \ldots), \tag{36}$$

where $\tilde{\mathcal{A}}_1 \equiv \mathcal{A}_1$. These are second order approximate partial differential equations for $\mathcal{A}_1(\mu^2;\beta_2)$, $\tilde{\mathcal{A}}_2(\mu^2;\beta_2)$, $\tilde{\mathcal{A}}_3(\mu^2;\beta_2)$. Higher-order terms ($\sim \tilde{\mathcal{A}}_4$) are neglected on the right-hand side of the RSch evolution Eq. (36).

At loop level 4 ($n_{\max} = 4$), where we include in RGE (1) term with $j_{\max} = 5$ (thus β_3), relations analogous to Eq. (35) are

$$\begin{aligned} \mathcal{A}_2(\mu^2) &= \tilde{\mathcal{A}}_2(\mu^2) - \frac{\beta_1}{\beta_0} \tilde{\mathcal{A}}_3(\mu^2) \\ &\quad + \left(\frac{5}{2} \frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \tilde{\mathcal{A}}_4(\mu^2), \\ \mathcal{A}_3(\mu^2) &= \tilde{\mathcal{A}}_3(\mu^2) - \frac{5}{2} \frac{\beta_1}{\beta_0} \tilde{\mathcal{A}}_4(\mu^2), \\ \mathcal{A}_4(\mu^2) &= \tilde{\mathcal{A}}_4(\mu^2), \end{aligned} \tag{37}$$

while the changes of the RSch are governed by (approxi-

[2] $\mathcal{A}_k = a^k + \text{NP}$ holds exactly for the construction Eq. (19), i.e., the construction by Milton *et al.* [2–4] in MA.

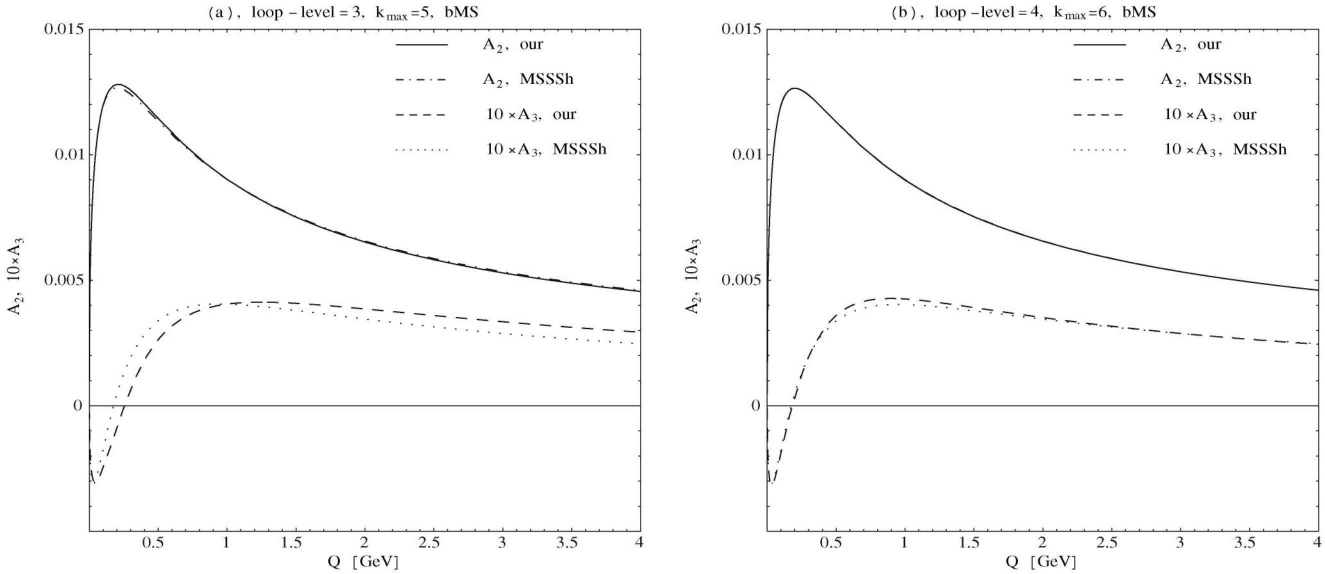

FIG. 1. The coupling parameters $\mathcal{A}_2(Q^2)$ and $\mathcal{A}_3(Q^2)$ in MA in $\overline{\rm MS}$ RSch, with $n_f = 3$ and $\bar{\Lambda}_{(n_f=3)} = 0.4$ GeV, calculated at (a) loop level = 3 (and $k_{\max} = 5$), and (b) loop level = 4 (and $k_{\max} = 6$). Presented are results of construction of Milton *et al.* (MSSSh) [2–4], and of our construction.

mate) relations

$$\frac{\partial \tilde{\mathcal{A}}_j(\mu^2)}{\partial \beta_2} = \left[\frac{1}{2!\beta_0^3}\frac{\partial^2}{\partial(\ln\mu^2)^2} + \frac{5}{3!2}\frac{\beta_1}{\beta_0^5} \times \frac{\partial^3}{\partial(\ln\mu^2)^3}\right]\tilde{\mathcal{A}}_j(\mu^2),$$
$$\frac{\partial \tilde{\mathcal{A}}_j(\mu^2)}{\partial \beta_3} = -\frac{1}{3!2\beta_0^4}\frac{\partial^3 \tilde{\mathcal{A}}_j(\mu^2)}{\partial(\ln\mu^2)^3} \qquad (j = 1, 2, \ldots). \tag{38}$$

Our approach is in a sense maximally truncating. Namely, the evolution under the changes of the RSch is truncated in such a way that $\partial \mathcal{D}^{(n_{\rm m})}_{\rm an.}(Q^2)/\partial\beta_j \sim \mathcal{A}_{n_{\rm m}+1}$. Further, our definition of $\mathcal{A}_k$'s ($k \geq 2$) via Eqs. (26) [cf. Eqs. (35) and (37)] involve short ("truncated") series which, however, still ensure the correct RScl dependence $\partial \mathcal{D}^{(n_{\rm m})}_{\rm an.}(Q^2)/\partial\mu^2 \sim \mathcal{A}_{n_{\rm m}+1}$. Furthermore, it may seem that, for loop level 3 ($n_{\max} = 3$), the RHS of the first of Eqs. (34) represents only two perturbative terms [$a^2 + (\beta_1/\beta_0)a^3$] plus nonperturbative terms (NP). However, since taking $j_{\max} = n_{\max} + 1 = 4$ in RGE (1) as the basis for calculation of $\mathcal{A}_1(\mu^2)$, it[3] is straightforward to show that the following holds:

$$(\tilde{\mathcal{A}}_2(\mu^2) =)\mathcal{A}_2(\mu^2) + \frac{\beta_1}{\beta_0}\mathcal{A}_3(\mu^2) = a^2(\mu^2) + \frac{\beta_1}{\beta_0}a^3(\mu^2) + \frac{\beta_2}{\beta_0}a^4(\mu^2) + \mathcal{O}(\beta_0^2 a^5) + \text{NP}. \tag{39}$$

The completely analogous result holds at loop level 4 ($n_{\max} = 4$ and $j_{\max} = 5$).

In the MA QCD, in the approach of Ref. [2], here Eq. (19) for $\mathcal{A}_k$, a truncation is performed only in expansion (2) for a [$\to \rho_1^{(\rm pt)}(\sigma)$, apparently with $k_{\max} = n_{\max}$], and then powers of this truncated a are used to define $\rho_k^{(\rm pt)}$ and thus $\mathcal{A}_k$ ($k \geq 2$). This implies that in the MA QCD our $\mathcal{A}_k$'s ($k = 2, \ldots$), on the one hand, and those of the approach of Milton, Solovtsov, Solovtsova, and Shirkov (MSSSh) [2–4], on the other hand, are not the same, although they must gradually merge when the loop level is increased. This is illustrated in Figs. 1 and 2, where the MA-coupling parameters $\mathcal{A}_2(Q^2)$ and $\mathcal{A}_3(Q^2)$ of both approaches are compared, for $n_f = 3$, at loop level ($= n_{\max}$) three and four, in $\overline{\rm MS}$ and in RSch A, respectively. The Adler (A) RSch is defined later in Eqs. (93) [cf. Eq. (94)]. For both $\mathcal{A}_2$ and $\mathcal{A}_3$, one can see a decrease in the absolute difference between our and MSSSh methods when going from loop level = 3 to 4, Fig. 1 in $\overline{\rm MS}$ RSch, and Fig. 2 in RSch A. The decrease can be understood as coming largely from the fact that the perturbative part of this difference is $\mathcal{O}(a^4)$ when loop level = 3, and $\mathcal{O}(a^5)$ when loop level = 4. Further, inspection of Figs. 1(a) and 2(a) reveals that the $\mathcal{A}_2$ curves practically merge already at loop level = 3 if RSch is $\overline{\rm MS}$, but less so

[3] When the anQCD is not MA, but rather M1 or M2, RGE (1) and the (truncated) expansion (2) still remain the basis for calculation of the MA part of $\mathcal{A}_1(\mu^2)$, the difference between $\mathcal{A}_1(\mu^2)$ and $\mathcal{A}_1^{(\rm MA)}(\mu^2)$ being purely nonperturbative, cf. Eqs. (12), (13), and (18).

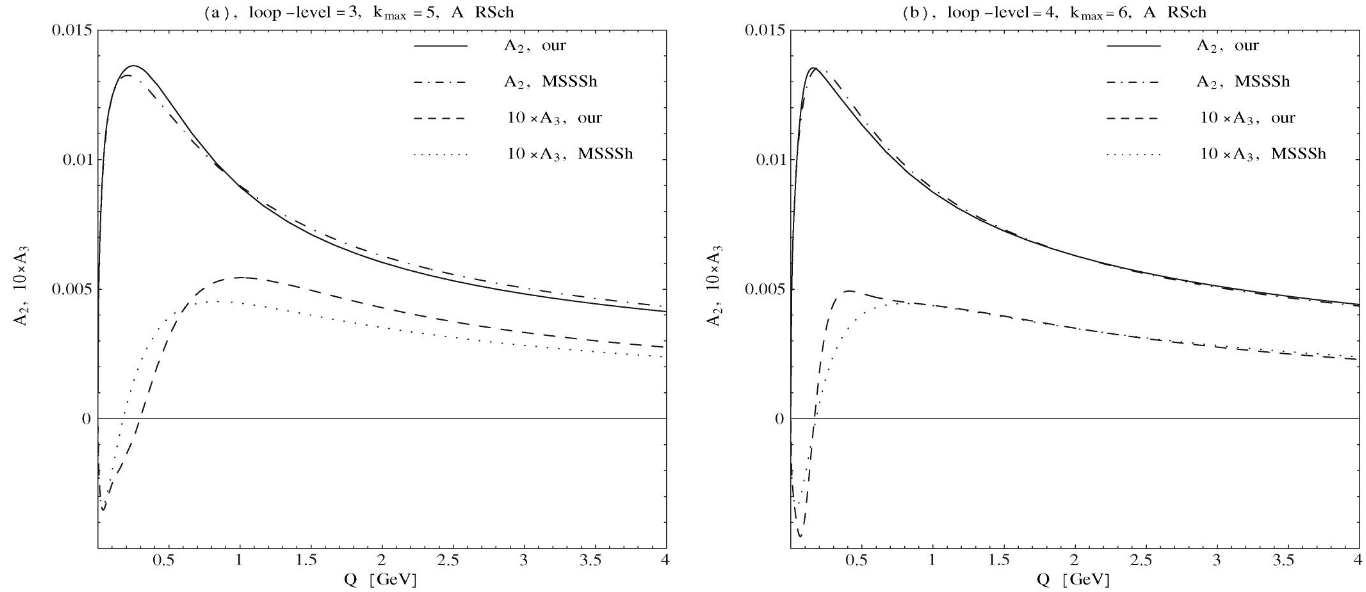

FIG. 2. Same as in Fig. 1, but now in RSch A, Eq. (93).

if RSch is A. An indication towards understanding this resides in the fact that the coefficient at a^4 of the difference between the two curves is proportional to $(2\beta_0\beta_2 - 5\beta_1^2)$, this being in $\overline{\rm MS}$ about one-fifth of the corresponding value in RSch A (when $n_f = 3$). In Fig. 3 the coupling parameters $\mathcal{A}_1(Q^2)$ and $\mathcal{A}_2(Q^2)$ of anQCD models M1, M2, and MA are presented as functions of the scale Q, for specific chosen fixed parameters of the models M1 and M2 (see Sec. V) and in the aforementioned specific RSch A. Note that we used $k_{\rm max} = n_{\rm max} + 2$ in the calculation of $\rho_1^{\rm (pt)}$ via Eq. (2) in all cases, i.e., also in the MSSSh cases. In Fig. 3, loop level $= 4$ and $k_{\rm max} = 6$ was taken (using our described approach). In Figs. 1–3, the basis for calculation was the $k_{\rm max}$-truncated series (2) in the corresponding RSch.

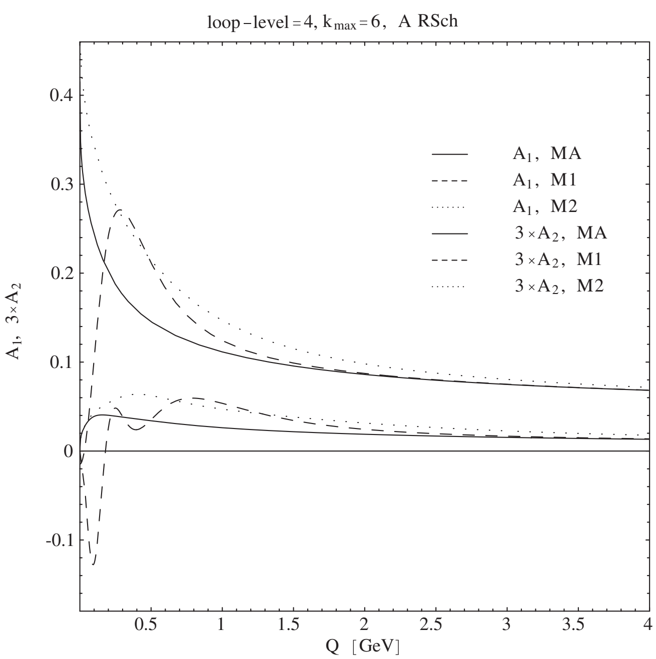

FIG. 3. Same as in Fig. 2, but now $\mathcal{A}_1$ and $\mathcal{A}_2$ for various models (M1, M2, and MA) with specific model parameters (see Sec. V): $c_0 = 2.94$, $c_r = 0.45$, $c_f = 1.08$ for M1; $c_v = 0.1$ and $c_p = 3.4$ for M2; $n_f = 3$ and $\bar{\Lambda}_{(n_f=3)} = 0.4$ GeV in all three models. The upper three curves are for $\mathcal{A}_1$, the lower three curves are for $3 \times \mathcal{A}_2$. All couplings are in RSch A, Eq. (93). $\mathcal{A}_2$ is constructed with our approach.

Even when already having anQCD coupling $\mathcal{A}_1(Q^2)$, there is no unique way to merge analyticity requirements with the perturbative results at higher orders, i.e., Eq. (32) for $\mathcal{A}_k(Q^2)$ ($k \geq 2$). The latter relations are ensured by our definitions of $\mathcal{A}_k(Q^2)$ for $k \geq 2$ via relations (26), but this is just one of the possibilities of addressing the problem. In MA the construction of $\mathcal{A}_1(Q^2)$ is very closely related to the perturbative solution $a(Q^2)$ via the dispersion relation (5). Therefore, it is very natural to keep that close analogy at higher orders, via dispersion relations (19). As a consequence, the RGE-type of relations (26) are fulfilled in MA [16]. For a general anQCD model, this approach does not apply. Deviations of $\mathcal{A}_1(Q^2)$ and $\mathfrak{A}_1(s)$ from their MA values imply that the discontinuity function $\rho_1(\sigma)$ deviates from its MA analog $\rho_1^{\rm (pt)}(\sigma) = {\rm Im}\, a(-\sigma - i\epsilon)$ at low values of σ, cf. Eqs. (10), (11), and (17). Therefore, there is no direct natural way of prescribing the low-σ behavior of the higher-order discontinuity functions $\rho_k(\sigma)$ appearing in the dispersion relations of the type of Eq. (19) for $\mathcal{A}_k$, i.e., prescribing their deviations from $\rho_k^{\rm (pt)}(\sigma) = {\rm Im} a^k(-\sigma - i\epsilon)$ for $k \geq 2$. We define $\mathcal{A}_k(Q^2)$ for $k \geq 2$ by forcing them to obey the truncated RGE-type relations (26). We emphasize that these relations define, in our approach, the couplings $\mathcal{A}_k(Q^2)$ for $k \geq 2$. Thus, we indirectly define the corresponding discontinuity functions ρ_k. This construction of $\mathcal{A}_k$'s is motivated also by the skeleton approach as discussed in Ref. [12]. Furthermore, as we will see later, this construction of $\mathcal{A}_k$'s allows us to suppress systematically the RScl and RSch dependence in the evaluated observables with the increasing order, because an RGE-type of analogy with pQCD is being preserved.

IV. SKELETON-MOTIVATED EXPANSION

Consider an observable $\mathcal{D}(Q^2)$ depending on a single spacelike physical scale $Q^2(\equiv -q^2) > 0$. Its perturbation expansion has the form

$$\mathcal{D}(Q^2)_{\rm pt} = a + d_1 a^2 + d_2 a^3 + \cdots, \qquad (40)$$

where $a = a(\mu^2; \beta_2, \ldots)$ is taken at a given RScl (μ) and RSch ($\beta_2, \ldots$). As mentioned before, we will take the convention $\Lambda = \bar{\Lambda}$, i.e., the $\overline{\rm MS}$ QCD scale as the reference scale for μ [cf. Eq. (2) and (3)]. Further, we will work in the following classes of RSch: each β_k ($k \geq 2$) is a polynomial in n_f of order k; equivalently, it is a polynomial in β_0:

$$\beta_k = \sum_{j=0}^{k} b_{kj}\beta_0^j, \qquad k = 2, 3, \ldots. \qquad (41)$$

The $\overline{\rm MS}$ clearly belongs to this class of schemes. In such schemes, the coefficients d_n of expansion (40) have the following specific form in terms of β_0, as can be deduced from the scheme independence of observable $\mathcal{D}(Q^2)$, e.g. by using relations of Ref. [29]:

$$d_1 = c_{11}^{(1)}\beta_0 + c_{10}^{(1)}, \qquad d_n = \sum_{k=-1}^{n} c_{nk}^{(1)}\beta_0^k, \qquad (42)$$

i.e., each d_n is a polynomial of order n in β_0 and includes in general, in addition, a term with the negative power $1/\beta_0$ (d_1 does not have it). In the $\overline{\rm MS}$ scheme, the negative powers do not occur.

We will now construct a separation of the series (40) into a sum of RScl-independent subseries

$$\mathcal{D}(Q^2)_{\rm pt} = \mathcal{D}^{(1)}(Q^2)_{\rm pt} + \sum_{n=2}^{\infty} k_n \mathcal{D}^{(n)}(Q^2)_{\rm pt}, \qquad (43)$$

with the following properties: (a) each dimensionless constant k_n is RScl independent; (b) each subseries $\mathcal{D}_{\rm pt}^{(n)}$ ($n \geq 1$) is RScl independent, and it is normalized so that $\mathcal{D}_{\rm pt}^{(n)} = a^n + \mathcal{O}(a^{n+1})$; (c) the subseries $\mathcal{D}^{(n)}(Q^2)_{\rm pt}$ contains all the leading-β_0 coefficients of the following "rest":

$$\frac{1}{k_n}[\mathcal{D}(Q^2)_{\rm pt} - \mathcal{D}^{(1)}(Q^2)_{\rm pt} - \cdots - k_{n-1}\mathcal{D}^{(n-1)}(Q^2)_{\rm pt}]. \qquad (44)$$

We will show that these conditions uniquely determine factors k_n and perturbation expansions of all $\mathcal{D}^{(n)}(Q^2)$. Further, we show in Appendix B that the above subseries, which always exist, would coincide with the expansions of the corresponding skeleton terms in the skeleton expansion of the observable if such an expansion existed in the considered RSch.

We consider first the leading-β_0 part of expansion (40):

$$\mathcal{D}_0^{(1)}(Q^2)_{\rm pt} = a + \sum_{j=2}^{\infty} a^j c_{jj}^{(1)}\beta_0^j. \qquad (45)$$

Under the change of RScl from μ^2 to μ_*^2, using the notation $L_* \equiv \ln(\mu_*^2/\mu^2)$, we have by RGE (1)

$$\begin{aligned} a &= a_* + \sum_{n=1}^{\infty} \tilde{a}_{*n+1}\beta_0^n L_*^n \\ &= a_* + a_*^2\beta_0 L_* + a_*^3(\beta_0^2 L_*^2 + \beta_1 L_*) \\ &\quad + a_*^4\left(\beta_0^3 L_*^3 + \frac{5}{2}\beta_0\beta_1 L_*^2 + \beta_2 L_*\right) + \cdots, \end{aligned} \qquad (46)$$

where $a \equiv a(\mu^2)$ and $a_* \equiv a(\mu_*^2)$. Inserting this into expansion (40) we obtain the transformation rules for the coefficients $c_{ij}^{(1)}$ (42) under the change of RScl. Specifically, for the diagonal coefficients the transformations are

$$c_{*kk}^{(1)} = \sum_{s=0}^{k} \binom{k}{s} L_*^s c_{k-s,k-s}^{(1)}, \qquad (47)$$

where we use the notations $c_{ij}^{(1)} \equiv c_{ij}^{(1)}(\mu^2)$ and $c_{*ij}^{(1)} \equiv c_{ij}^{(1)}(\mu_*^2)$ (and $c_{00}^{(1)} = 1$ by definition). Inserting expansion (46) into expansion (45) we obtain

$$\begin{aligned} \mathcal{D}_0^{(1)}(Q^2)_{\rm pt} &= a_* + a_*^2[\beta_0 c_{*11}^{(1)}] \\ &\quad + a_*^3[\beta_0^2 c_{*22}^{(1)} + \beta_1(c_{*11}^{(1)} - c_{11}^{(1)})] + \mathcal{O}(\beta_0^3 a^4). \end{aligned} \qquad (48)$$

This implies that the leading-β_0 series (45) does not maintain its form under the change of RScl, since a new term $a_*^3\beta_1(c_{*11}^{(1)} - c_{11}^{(1)})$ appears at $\sim a^3$. The RScl-"covariant" form, up to $\sim a^3$, is then

$$\begin{aligned} \mathcal{D}_1^{(1)}(Q^2)_{\rm pt} &= a + a^2[\beta_0 c_{11}^{(1)}] + a^3[\beta_0^2 c_{22}^{(1)} + \beta_1 c_{11}^{(1)}] \\ &\quad + a^4[\beta_0^3 c_{33}^{(1)}] + \mathcal{O}(\beta_0^4 a^5). \end{aligned} \qquad (49)$$

We now iteratively repeat the procedure: we insert expansion (46) into expansion (49) and, after some algebra and using relations (47), obtain

$$\begin{aligned} \mathcal{D}_1^{(1)}(Q^2)_{\rm pt} &= a_* + a_*^2[\beta_0 c_{*11}^{(1)}] + a_*^3[\beta_0^2 c_{*22}^{(1)} + \beta_1 c_{*11}^{(1)}] \\ &\quad + a_*^4[\beta_0^3 c_{*33}^{(1)} + \tfrac{5}{2}\beta_0\beta_1(c_{*22}^{(1)} - c_{22}^{(1)}) \\ &\quad + \beta_2(c_{*11}^{(1)} - c_{11}^{(1)})] + \mathcal{O}(\beta_0^4 a^5). \end{aligned} \qquad (50)$$

The new terms appearing at $\sim a_*^5$ here require the following restoration of the RScl "covariance" up to order $\sim a^5$:

$$\mathcal{D}^{(1)}(Q^2)_{\rm pt} = a + a^2[\beta_0 c_{11}^{(1)}] + a^3[\beta_0^2 c_{22}^{(1)} + \beta_1 c_{11}^{(1)}] + a^4\left[\beta_0^3 c_{33}^{(1)} + \frac{5}{2}\beta_0\beta_1 c_{22}^{(1)} + \beta_2 c_{11}^{(1)}\right] + \mathcal{O}(\beta_0^4 a^5). \tag{51}$$

This procedure can be continued to any required order. Expression (51) is now the *RScl-covariant* leading-β_0 part of the full perturbation expansion (40). This means that it keeps its form (51) under any change of RScl μ^2. Variations of $a = a(\mu^2)$ and of $c_{kk} = c_{kk}(\mu^2)$ under the RScl variation are governed by the RScl invariance of the entire observable $\mathcal{D}$ and of its perturbation expansion (40), as reflected by relations (46) and (47). The additional terms appearing in expansion (51), in comparison with the original leading-β_0 series (45), are subleading in β_0 and represent effects beyond one loop involving diagonal coefficients $c_{kk}^{(1)}$. As shown in Appendix B, the covariant leading-β_0 expansion (51) is the expansion of the leading skeleton (LS) term in an assumed skeleton expansion of the observable $\mathcal{D}$.

Now we subtract the LS expansion (51) from expansion (40), and the difference now involves only subleading-β_0 terms

$$[\mathcal{D}(Q^2)_{\rm pt} - \mathcal{D}^{(1)}(Q^2)_{\rm pt}] = k_2\left[a^2 + \sum_{n\geq 1} a^{n+2} d_n^{(2)}\right] \quad (k_2 = c_{10}^{(1)}), \tag{52}$$

where the coefficients $d_n^{(2)}$ have a structure similar to that of d_n's (42)

$$d_n^{(2)} = \sum_{k=-1}^{n} c_{nk}^{(2)} \beta_0^k \qquad (n = 1, 2, \ldots). \tag{53}$$

Coefficients $c_{ij}^{(2)}$ are related to the original coefficients $c_{ij}^{(1)}$ by relations

$$\begin{aligned} c_{10}^{(1)} c_{1j}^{(2)} &= c_{2j}^{(1)} - b_{1j} c_{11}^{(1)}, \qquad (j = 1, 0, -1),\\ c_{10}^{(1)} c_{2j}^{(2)} &= c_{3j}^{(1)} - \tfrac{5}{2} b_{1,j-1} c_{22}^{(1)} - b_{2j} c_{11}^{(1)}, \\ &(j = 2, 1, 0, -1), \end{aligned} \tag{54}$$

and coefficients b_{kj} are those of the expansion of β_k coefficients (41) in powers of β_0 (including the case $k = 1$). Specifically, we have $b_{k,-1} = 0$ $(k = 1, 2, \ldots)$. For $k = 1$, we have $b_{11} = 19/4$ and $b_{10} = -107/16$, both numbers being RSch independent. Now we repeat the previous construction, but now for the (canonically normalized) rest $(1/k_2)(\mathcal{D} - \mathcal{D}^{(1)})$ of Eq. (52) instead of $\mathcal{D}$ (40). Its RScl-covariant leading-β_0 part $\mathcal{D}^{(2)}$ then turns out to give

$$k_2\mathcal{D}^{(2)}(Q^2)_{\rm pt} = k_2\{a^2 + a^3[\beta_0 c_{11}^{(2)}] + a^4[\beta_0^2 c_{22}^{(2)} + \beta_1 c_{11}^{(2)}] + \mathcal{O}(\beta_0^3 a^5)\}. \tag{55}$$

Subtracting this from the rest (52), we obtain

$$[\mathcal{D}(Q^2)_{\rm pt} - \mathcal{D}^{(1)}(Q^2)_{\rm pt} - k_2\mathcal{D}^{(2)}(Q^2)_{\rm pt}] = k_3\left[a^3 + \sum_{n\geq 1} a^{n+3} d_n^{(3)}\right], \tag{56}$$

$$k_3 = c_{10}^{(1)}\left(c_{10}^{(2)} + \frac{1}{\beta_0} c_{1,-1}^{(2)}\right), \tag{57}$$

$$d_1^{(3)} = \beta_0(c_{21}^{(2)} - b_{11}c_{11}^{(2)})/c_{10}^{(2)} + k_4/k_3, \tag{58}$$

where k_4/k_3 is a number $\sim\beta_0^0$ which will be given explicitly below. The (RScl-covariant) leading-β_0 part $\mathcal{D}^{(3)}$ of the canonically normalized expression $(1/k_3)\times(\mathcal{D} - \mathcal{D}^{(1)} - k_2\mathcal{D}^{(2)})$ gives

$$k_3\mathcal{D}^{(3)}(Q^2)_{\rm pt} = k_3\{a^3 + a^4[\beta_0 c_{11}^{(3)}] + \mathcal{O}(\beta_0^2 a^5)\}, \tag{59}$$

$$c_{10}^{(2)} c_{11}^{(3)} = (c_{21}^{(2)} - b_{11} c_{11}^{(2)}). \tag{60}$$

Defining

$$\mathcal{D}^{(4)}(Q^2)_{\rm pt} = a^4 + \mathcal{O}(\beta_0 a^5), \tag{61}$$

and following the procedure pattern, we subtract expression (59) from expression (56) and obtain

$$\mathcal{D}(Q^2)_{\rm pt} = \mathcal{D}^{(1)}(Q^2)_{\rm pt} + k_2\mathcal{D}^{(2)}(Q^2)_{\rm pt} + k_3\mathcal{D}^{(3)}(Q^2)_{\rm pt} + k_4\mathcal{D}^{(4)}(Q^2)_{\rm pt} + \mathcal{O}(\beta_0^0 a^5), \tag{62}$$

where perturbation expansions for $\mathcal{D}^{(j)}$'s are given by (51), (55), (59), and (61); coefficients k_2 and k_3 are given by Eqs. (52) and (57); coefficients $c_{ij}^{(1)}$, $c_{ij}^{(2)}$, $c_{ij}^{(3)}$ are given by Eqs. (42), (54), and (60); and an explicit expression for the coefficient k_4 is

$$k_4 = c_{10}^{(1)}\left[c_{20}^{(2)} - b_{10}c_{11}^{(2)} - \frac{c_{1,-1}^{(2)}}{c_{10}^{(2)}}(c_{21}^{(2)} - b_{11}c_{11}^{(2)}) + \frac{1}{\beta_0} c_{2,-1}^{(2)}\right]. \tag{63}$$

It is straightforward to check that all the coefficients k_2, k_3, k_4 are RScl independent [as are the subseries $\mathcal{D}^{(j)}(Q^2)$]. Thus, identity (62), obtained by our construction, represents identity (43) to order $n = 4$. This construction can be continued to any order.

In practice, we know only all the leading-β_0 parts of the coefficients d_j of observable $\mathcal{D}(Q^2)$ Eq. (40), i.e., all the coefficients $c_{jj}^{(1)}$; and in addition, we usually know only one, two, or three full coefficients (d_1, d_2, and possibly d_3). This implies that the first term $\mathcal{D}^{(1)}$ on the RHS of identity (62) is known to all orders, while the other terms ($\mathcal{D}^{(2)}$, $\mathcal{D}^{(3)}$, and possibly $\mathcal{D}^{(4)}$) are known only in their truncated version. This means that the rest term in Eq. (62) is, in such a case, $\mathcal{O}(\beta_0^3 a^5)$, not $\mathcal{O}(\beta_0^0 a^5)$.

The perturbation expansion $\mathcal{D}_{\rm pt}^{(1)}$ of the "leading-skeleton" (LS) term can be written in a resummed form [31,32]

$$\mathcal{D}^{(1)}(Q^2)_{\rm pt} = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t) a(te^{\mathcal{C}} Q^2), \qquad (64)$$

where $F_{\mathcal{D}}^{\mathcal{E}}(t)$ is the LS-characteristic function[4] which often can be written in a closed explicit form [31]. In principle, $F_{\mathcal{D}}^{\mathcal{E}}(t)$ can be obtained for any spacelike observable whose leading-β_0 parts ($c_{kk}^{(1)}$) of all coefficients are known. The value of $\mathcal{C}$ in (64) depends on the value of the reference scale Λ used in the RGE running; in our convention, as mentioned before, we use $\Lambda = \bar{\Lambda}$ which corresponds to $\mathcal{C} = \bar{\mathcal{C}} \equiv -5/3$.

At this point, we will turn to the question of the RSch dependence of the (truncated) perturbation series (62). The RSch independence of the series (40) implies specific transformation rules of the expansion coefficients d_j under the change of β_j's ($j \geq 2$) [29]:

$$d_1 = \bar{d}_1, \; d_2 = \bar{d}_2 - \frac{1}{\beta_0}(\beta_2 - \bar{\beta}_2),$$
$$d_3 = \bar{d}_3 - 2\bar{d}_1 \frac{1}{\beta_0}(\beta_2 - \bar{\beta}_2) - \frac{1}{2\beta_0}(\beta_3 - \bar{\beta}_3), \ldots, \qquad (65)$$

where the bars denote the values with $\overline{\rm MS}$ RSch parameters $\beta_k = \bar{b}_k = \sum \bar{b}_{kj} \beta_0^j$, and unchanged RScl. This implies, in view of relations (42), (54), and (60), specific transformation rules for $c_{nk}^{(s)}$ coefficients. We will consider that the first term in skeleton-motivated expansion (62) has a known characteristic function, cf. Eq. (64), and that at most the first three nonleading coefficients of the perturbation expansion (40) of observable $\mathcal{D}$ are known: $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$—in $\overline{\rm MS}$ RSch and at RScl $\mu^2 = Q^2$. Since each term in expansion (62) is RScl independent, we can reexpand each $\mathcal{D}^{(j)}(Q^2)_{\rm pt}$ ($j \geq 2$) in powers of $a(Q_j^2)$, i.e., at different chosen RScl's Q_j, in a chosen common RSch ($\beta_2, \beta_3, \ldots$). The resulting subseries, however, will now be truncated since d_j's for $j \geq 4$ are not known. This leads to the following form of the skeleton-motivated expansion (62):

$$\mathcal{D}(Q^2)_{\rm pt} = \mathcal{D}(Q^2)_{\rm (TPS)} + \mathcal{O}(\beta_0^3 a^5), \qquad (66)$$

$$\mathcal{D}(Q^2)_{\rm (TPS)} = \mathcal{D}^{(1)}(Q^2) + t_2^{(2)} a^2(Q_2^2) + \sum_{j=2}^{3} t_3^{(j)} a^3(Q_j^2) + \sum_{j=2}^{4} t_4^{(j)} a^4(Q_j^2), \qquad (67)$$

[4] The superscript $\mathcal{E}$ means "Euclidean," since the scales involved (Q^2, $te^{\mathcal{C}}Q^2$) are spacelike.

where the coefficients $t_i^{(j)}$ depend on the scale ratios Q_j^2/Q^2 and the RSch parameters β_k (41), and are written explicitly in Appendix A in terms of the coefficients $\bar{c}_{ij}^{(1)}$, the latter comprising via Eq. (42) the coefficients $\bar{d}_n$ of the original perturbation series (40) in $\overline{\rm MS}$ RSch and at the RScl $\mu^2 = Q^2$.

We now turn to the question of analytization of the perturbation series (67), within a given anQCD model with known analytic couplings $\mathcal{A}_k$, Eqs. (6) and (28)–(37). For the first (LS) term, the natural analytization procedure is to replace the perturbative coupling $a(te^{\mathcal{C}}Q^2)$ by its anQCD counterpart $\mathcal{A}_1(te^{\mathcal{C}}Q^2)$:[5]

$$\mathcal{D}^{(1)}(Q^2)_{\rm an} \equiv \mathcal{D}^{\rm (LS)}(Q^2) = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t) \mathcal{A}_1(te^{\mathcal{C}} Q^2). \qquad (68)$$

In contrast to expression (64) which is an ill-defined integral due to the Landau singularities of a, expression (68) is a well-defined integral in any given anQCD [unless $\mathcal{A}_1(Q^2)$ diverges too strongly when $Q^2 \to 0$]. We can adopt the viewpoint that any anQCD model is defined: (a) by a specific expression for $\mathcal{A}_1(Q^2)$, *and* (b) by prescription (68) for calculation of the LS terms of any spacelike observable. The analytization of the other terms in Eq, (67), after the choice of an anQCD model, i.e., of $\mathcal{A}_1(Q^2)$, can be performed in different ways. For example, the replacements $a^k(Q_j^2) \mapsto \mathcal{A}_1^k(Q_j^2)$, $\mathcal{A}_1^{k-2}(Q_j^2)\mathcal{A}_2(Q_j^2)$, $\ldots$, $\mathcal{A}_k(Q_j^2)$ all appear equally natural at first, since the perturbative parts of these expressions are all the same to the order considered—cf. relations (32) and (33). However, construction of the higher order couplings $\mathcal{A}_k$ ($k \geq 2$) on the basis of the anQCD coupling $\mathcal{A}_1$, as presented in Sec. III, suggests that it is the replacement

$$[\tilde{a}_k(Q_j^2) \mapsto \tilde{\mathcal{A}}_k(Q_j^2) \Rightarrow] \, a^k(Q_j^2) \mapsto \mathcal{A}_k(Q_j^2), \qquad (k \geq 1) \qquad (69)$$

that appears to be the most natural from the point of view of the requirement of the RScl and RSch invariance of the observables. Namely, $\mathcal{A}_k(\mu^2; \beta_2, \ldots)$'s fulfill, to the order considered, the same evolution equations under the changes of the RScl and of RSch as $a^k(\mu^2; \beta_2, \ldots)$'s when the replacements (69) are performed everywhere. Further, the LS analytization (68) of the first term $\mathcal{D}_{\rm pt}^{(1)}$ of (67) is also equivalent to the term-by-term analytization

[5] A different approach to considering the perturbative LS term (64) was developed by the authors of Ref. [33]. They present a novel version of the leading-β_0 renormalon calculus, and consider that an OPE-term exists whose Q^2 dependence is the same as that of the renormalon ambiguity of the perturbative LS term and that the ambiguity cancels in the sum ("PT + NP"). This sum can be presented in the LS form (64) with the perturbative coupling $a(te^{\mathcal{C}}Q^2)$ there replaced by a modified (but nonanalytic) coupling with one parameter. Since they work in the OPE framework, the latter parameter is observable-dependent.

(69) of the perturbation expansion of $\mathcal{D}^{(1)}_{\rm pt}$, as is explicitly shown in Appendix B. The analytization (69) of the TPS (67), which results in the "truncated analytic series" (TAS),

$$\mathcal{D}(Q^2) = \mathcal{D}(Q^2)_{\rm (TAS)} + \mathcal{O}(\beta_0^3 \mathcal{A}_5), \tag{70}$$

$$\mathcal{D}(Q^2)_{\rm (TAS)} = \mathcal{D}^{\rm (LS)}(Q^2) + t_2^{(2)}\mathcal{A}_2(Q_2^2) + \sum_{j=2}^{3} t_3^{(j)}\mathcal{A}_3(Q_j^2) + \sum_{j=2}^{4} t_4^{(j)}\mathcal{A}_4(Q_j^2), \tag{71}$$

has, as a consequence, the suppression of the RScl and RSch dependence just as is known for the corresponding TPS in pQCD, but with $a^k \mapsto \mathcal{A}_k$:

$$\frac{\partial \mathcal{D}(Q^2)_{\rm (TAS)}}{\partial \ln Q_j^2} = \mathcal{O}(\beta_0^{5-j}\mathcal{A}_5) \qquad (j = 2, 3, 4), \tag{72}$$

$$\frac{\partial \mathcal{D}(Q^2)_{\rm (TAS)}}{\partial \beta_k} \leq \mathcal{O}(\beta_0^{3-k}\mathcal{A}_5) \qquad (k = 2, 3). \tag{73}$$

We are allowed, in principle, to vary in the TAS series (71) three different RScl's Q_j and 3 + 4 RSch parameters b_{2j} and b_{3j} appearing in β_2 and β_3. One may want to have, for given chosen RScl's Q_j, such a RSch that effectively only the first coefficient $t_2^{(2)}$ in the beyond-the-LS contribution is nonzero. This implies various conditions involving the other five $t_i^{(j)}$'s [Eqs. (A4)–(A8)]:

$$t_3^{(2)} = t_3^{(3)} = 0; \qquad \sum_{j=2}^{4} t_4^{(j)} = 0, \tag{74}$$

$$\Rightarrow \mathcal{D}(Q^2) = \mathcal{D}^{\rm (LS)}(Q^2) + t_2^{(2)}\mathcal{A}_2(Q_2^2) + \mathcal{O}(\beta_0^3\mathcal{A}_5). \tag{75}$$

Specifically, if we choose for all three $\mathcal{D}^{(j)}(Q^2;\mu^2 = Q_j^2)_{\rm (TAS)}$ $(j = 2, 3, 4)$ the same RScl

$$Q_2^2 = Q_3^2 = Q_4^2 = Q^2 \exp(\mathcal{C}), \tag{76}$$

the corresponding $\beta_k = b_{kj}\beta_0^j$ $(k = 2, 3)$ have the following $\delta b_{kj} \equiv b_{kj} - \bar{b}_{kj}$:

$$\delta b_{22} = \bar{c}_{10}^{(1)}(\bar{c}_{11}^{(2)} + 2\mathcal{C}), \tag{77}$$

$$\delta b_{21} = \bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)}, \qquad \delta b_{20} = 0, \tag{78}$$

$$\tfrac{1}{2}\delta b_{33} = \bar{c}_{10}^{(1)}\bar{c}_{22}^{(2)} + 3\bar{c}_{10}^{(1)}\mathcal{C}(\bar{c}_{11}^{(2)} + \mathcal{C}) - \delta b_{22}3(\bar{c}_{11}^{(1)} + \mathcal{C}), \tag{79}$$

$$\tfrac{1}{2}\delta b_{32} = \bar{c}_{10}^{(1)}\bar{c}_{21}^{(2)} + \mathcal{C}(3\bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)} + 2b_{11}\bar{c}_{10}^{(1)}) - \delta b_{22}2\bar{c}_{10}^{(1)} - \delta b_{21}3(\bar{c}_{11}^{(1)} + \mathcal{C}), \tag{80}$$

$$\tfrac{1}{2}\delta b_{31} = \bar{c}_{10}^{(1)}\bar{c}_{20}^{(2)} + \mathcal{C}2b_{10}\bar{c}_{10}^{(1)} - \delta b_{21}2\bar{c}_{10}^{(1)} - \delta b_{20}3(\bar{c}_{11}^{(1)} + \mathcal{C}), \tag{81}$$

$$\tfrac{1}{2}\delta b_{30} = -\delta b_{20}2\bar{c}_{10}^{(1)}(= 0). \tag{82}$$

Here, $\bar{c}_{ij}^{(k)} \equiv c_{ij}^{(k)}$ $(\mu^2 = Q^2; \overline{\rm MS})$. Results (77)–(82) are obtained by using explicit expressions (A4)–(A8) obtained in Appendix A, applying to them conditions (74) for the RScl choice (76). Specifically, result (77) is obtained by the requirement $t_3^{(2)} = 0$; results (78) from the requirement $t_3^{(3)} = 0$, being zero both the coefficient at β_0^0 and at $1/\beta_0$, respectively; results (79)–(82) are obtained from requirement $\sum t_4^{(j)} = 0$, being zero all the coefficients at the β_0 powers β_0^2, β_0^1, β_0^0, $1/\beta_0$, respectively.

Our evaluation method (71), with the choice of the scheme described above [Eqs. (75)–(77) and (80)–(82)], emphasizes in the beyond-the-LS parts the role of the analytic couplings $\mathcal{A}_k(\mu^2)$ $(k \geq 2)$ constructed in Sec. III from the couplings $\tilde{\mathcal{A}}_n(\mu^2)$, Eq. (28) [see Eqs. (37)]. The couplings $\mathcal{A}_k(\mu^2)$ $(k \geq 2)$ were constructed in such a way as to have, at perturbative level, their equivalence with $a^n(\mu^2)$. However, the construction in Sec. III strongly suggests that the couplings $\tilde{\mathcal{A}}_n(\mu^2)$ $(n \geq 2)$ are more basic since they are constructed as derivatives of $\mathcal{A}_1(\mu^2)$ which is the basic quantity in any anQCD model. Further, the skeleton-expansion arguments presented in Appendix B show that $\tilde{\mathcal{A}}_n(\mu^2)$ are the basic elements for the expansion of each term in the skeleton expansion. Therefore, a more natural choice for RSch (β_2, β_3) in the evaluation method (71), with RScl's (76), would be such that the resulting TAS expression is

$$\mathcal{D}(Q^2) = \mathcal{D}(Q^2)_{\rm (TAS)} + \mathcal{O}(\beta_0^3\tilde{\mathcal{A}}_5), \tag{83}$$

$$\mathcal{D}(Q^2)_{\rm (TAS)} = \mathcal{D}^{\rm (LS)}(Q^2) + \tilde{t}_2\tilde{\mathcal{A}}_2(Q^2 e^{\mathcal{C}}). \tag{84}$$

To obtain the β_k's $(k = 2, 3)$ necessary for this result, we first reexpress all $\mathcal{A}_k$'s $(k \geq 2)$ in TAS (71) in terms of $\tilde{\mathcal{A}}_n$'s, Eqs. (37). Keeping the RScl's according to (76), this implies that, in a general RSch (β_2, β_3) expression (71) can be reexpressed as

$$\mathcal{D}(Q^2)_{\rm (TAS)} = \mathcal{D}^{\rm (LS)}(Q^2) + \tilde{t}_2\tilde{\mathcal{A}}_2(Q^2 e^{\mathcal{C}}) + \tilde{t}_3\tilde{\mathcal{A}}_3(Q^2 e^{\mathcal{C}}) + \tilde{t}_4\tilde{\mathcal{A}}_4(Q^2 e^{\mathcal{C}}), \tag{85}$$

where the coefficients $\tilde{t}_i$ are certain combinations of $t_s^{(k)}$, and are written explicitly in Appendix A, Eqs. (A16)–(A21). Requiring the form (84), i.e.,

$$\tilde{t}_3 = \tilde{t}_4 = 0, \tag{86}$$

implies, by Eqs. (A16)–(A21), that the corresponding $\beta_k = b_{kj}\beta_0^j$ $(k = 2, 3)$ have the following $\delta b_{kj} \equiv b_{kj} - \bar{b}_{kj}$:

$$\delta b_{22} = \bar{c}_{10}^{(1)}(\bar{c}_{11}^{(2)} + 2\mathcal{C}), \tag{87}$$

$$\delta b_{21} = \bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)} - b_{11}\bar{c}_{10}^{(1)}, \qquad \delta b_{20} = -b_{10}\bar{c}_{10}^{(1)}, \tag{88}$$

$$\tfrac{1}{2}\delta b_{33} = \bar{c}_{10}^{(1)}\bar{c}_{22}^{(2)} + 3\bar{c}_{10}^{(1)}\mathcal{C}(\bar{c}_{11}^{(2)} + \mathcal{C}) - \delta b_{22}3(\bar{c}_{11}^{(1)} + \mathcal{C}), \tag{89}$$

$$\begin{aligned}\tfrac{1}{2}\delta b_{32} = {}& \bar{c}_{10}^{(1)}\bar{c}_{21}^{(2)} - \tfrac{5}{2}b_{11}\bar{c}_{10}^{(1)}\bar{c}_{11}^{(2)} - \bar{c}_{10}^{(1)}\bar{b}_{22} \\ & + 3\mathcal{C}\bar{c}_{10}^{(1)}(\bar{c}_{10}^{(2)} - b_{11}) + \delta b_{22}(-3\bar{c}_{10}^{(1)} + \tfrac{5}{2}b_{11}) \\ & - \delta b_{21}3(\bar{c}_{11}^{(1)} + \mathcal{C}),\end{aligned} \tag{90}$$

$$\begin{aligned}\tfrac{1}{2}\delta b_{31} = {}& \bar{c}_{10}^{(1)}\bar{c}_{20}^{(2)} - \tfrac{5}{2}b_{10}\bar{c}_{10}^{(1)}\bar{c}_{11}^{(2)} - \tfrac{5}{2}b_{11}\bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)} \\ & + \bar{c}_{10}^{(1)}(\tfrac{5}{2}b_{11}^2 - \bar{b}_{21} - 3b_{10}\mathcal{C}) + \delta b_{22}\tfrac{5}{2}b_{10} \\ & + \delta b_{21}(-3\bar{c}_{10}^{(1)} + \tfrac{5}{2}b_{11}) - \delta b_{20}3(\bar{c}_{11}^{(1)} + \mathcal{C}),\end{aligned} \tag{91}$$

$$\begin{aligned}\tfrac{1}{2}\delta b_{30} = {}& -\tfrac{5}{2}b_{10}\bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)} + 5b_{10}b_{11}\bar{c}_{10}^{(1)} - \bar{b}_{20}\bar{c}_{10}^{(1)} \\ & + \tfrac{5}{2}b_{10}\delta b_{21} + \delta b_{20}(-3\bar{c}_{10}^{(1)} + \tfrac{5}{2}b_{11}).\end{aligned} \tag{92}$$

In these expressions, $\bar{b}_{2j}$ are the coefficients b_{2j} in $\overline{\rm MS}$: $\bar{b}_{22} = 325/96$, $\bar{b}_{21} = 243/32$, $\bar{b}_{20} = -37\,117/1536$ (and $b_{11} = 19/4$, $b_{10} = -107/16$). We will apply, as a rule, our evaluation approach in the RSch (87)–(92), i.e., where the resulting formula is (83) and (84), and will use the RScl's (76) with $\mathcal{C} = \bar{\mathcal{C}} = -5/3$. The RSch evidently depends on the observable. Our starting point will be this RSch for the massless Adler function $\mathcal{D}(Q^2) = d_v(Q^2)$, where the STPS is known to a large degree of accuracy up to $\sim a^4$ (up to $\sim a^3$ it is known exactly)—we will call this RSch A ("A" for Adler).[6] If an observable is known in STPS only up to $\sim a^3$, only formulas (87) and (88) are to be applied, as $\tilde{t}_4$ is not known; in that case, in Eq. (83) the unknown rest term is $\mathcal{O}(\beta_0^2\tilde{\mathcal{A}}_4)$. For example, Bjorken polarized sum rule $d_b(Q^2)$ is such an observable.

In Appendix B, a different method of evaluation is presented, which would be an evaluation of the skeleton expansion itself if such an expansion existed in the considered RSch. The RSch dependence of that method is numerically stronger, which may be a reflection of the fact that this expansion, if it exists, is valid only in a specific ("skeleton") RSch that is hitherto unknown [34,35].

[6] The difference between this RSch A and the RSch A′ (77)–(82) for the Adler function is small. For example, for $n_f = 3$, the values are $\beta_2^{(\rm A)} = -18.92$, $\beta_2^{(\rm A')} = -18.59$; $\beta_3^{(\rm A)} = -33.84$, $\beta_3^{(\rm A')} = -32.72$. In Ref. [12], we used RSch A′ (77)–(82) [with RScl's (76) with $\mathcal{C} = \bar{\mathcal{C}} = -5/3$], and denoted there this approach as "v2".

V. NUMERICAL RESULTS

In this section, we take the position that the anQCD models M1 and M2, introduced in Sec. II, the form of $\mathcal{A}_1(Q^2)$ there, Eqs. (15) and (18), is achieved in the aforementioned "optimal" RSch (87)–(92) for the massless Adler function $d_v(Q^2)$—RSch A. We must keep in mind that models M1 and M2 change the form of $\mathcal{A}_1(Q^2)$ when the RSch $(\beta_2, \beta_3, \ldots)$ is changed.[7]

We will calculate numerically various low-energy QCD observables in the anQCD models MA, M1, and M2, with $n_f = 3$, by using the skeleton-motivated evaluation method presented in the previous section, Eq. (85). One such quantity is the massless Adler function $d_v(Q^2)$ whose pQCD expansion coefficients d_1 and d_2 (in $\overline{\rm MS}$ RSch and at RScl $\mu^2 = Q^2$) are known exactly [36,37], and d_3 has been estimated as a polynomial in n_f to a high degree of accuracy [38] (see Appendix D for explicit expressions of d_1, d_2, d_3). The normalization of d_v is taken according to Eq. (40) when $n_f = 3$. The additional light-by-light contributions [37] do not contribute when $n_f = 3$. Further, the LS characteristic function $F_v^{\mathcal{E}}(t)$ for $d_v(Q^2)$ was obtained in Ref. [31], and is given in Appendix C in Eqs. (C6) and (C7). Evaluation method (85) can thus be applied by including terms $\sim\tilde{\mathcal{A}}_4$ in the case of the massless Adler function (for a different approach to evaluating Adler function, see Ref. [39]). The optimal RSch for the massless Adler function $d_v(Q^2)$ is then obtained by requiring disappearance of $\sim\tilde{\mathcal{A}}_3$ and $\sim\tilde{\mathcal{A}}_4$ terms, Eq. (86), where we choose RScl according to (76) with $\mathcal{C} = \bar{\mathcal{C}} = -5/3$. We call this RSch Adler (A), and it can be obtained from $\overline{\rm MS}$ RSch by applying relations (87)–(92), resulting in

$$\begin{aligned}\beta_2^{(\rm A)} &= -23.6074 - 16.0248\beta_0 + 8.04784\beta_0^2, \\ \beta_3^{(\rm A)} &= 127.38 - 35.8577\beta_0 - 12.8734\beta_0^2 - 1.349\,26\beta_0^3.\end{aligned} \tag{93}$$

The values for $n_f = 3$ are $\beta_2 = -18.9211$ and $\beta_3 = -33.8404$ (in $\overline{\rm MS}$ RSch, at $n_f = 3$, the values are 10.0599 and 47.2281, respectively). In RSch A, the evaluated massless $d_v(Q^2)$ is thus

$$\begin{aligned}d_v(Q^2)_{\rm TAS} = {}& \int_0^\infty \frac{dt}{t} F_v^{\mathcal{E}}(t)\mathcal{A}_1(te^{\bar{\mathcal{C}}}Q^2; \beta_2^{(\rm A)}, \beta_3^{(\rm A)}) \\ & + \frac{1}{12}\tilde{\mathcal{A}}_2(e^{\bar{\mathcal{C}}}Q^2),\end{aligned} \tag{94}$$

[7] When β_j's $(j \geq 2)$ change, the change of $\mathcal{A}_1(Q^2)$ in general cannot be described just by running of the parameters of the model with β_j's, since new terms appear that depend on those parameters.

and the difference between the (massless) true $d_v(Q^2)$ and $d_v(Q^2)_{\rm TAS}$ is formally $\mathcal{O}(\beta_0^3 \tilde{\mathcal{A}}_5)$.

The ($V + A$-channel) semihadronic τ decay rate ratio r_τ is one of the best measured low-energy QCD quantities, its massless part for nonstrange hadron production has the value $r_\tau(\triangle S = 0, m_q = 0) = 0.204 \pm 0.005$ [8,9] [cf. Appendix E, Eq. (E6)]. The heavy quarks (c and b) do not contribute, since r_τ is a Minkowskian observable, and the τ particle cannot decay to charmed mesons because their masses are larger than m_τ.[8] Our evaluation approach for $r_\tau(\triangle S = 0, m_q = 0)$ uses the aforementioned evaluation (94) of the (massless) Adler function $d_v(Q^2)$ which is then inserted in the contour integral (C8). The LS part can then be written in the form (C9) with the timelike LS characteristic function (C10) and (C11). The beyond-the-LS (bLS) contribution is the contour integral

$$r_\tau(\triangle S = 0, m_q = 0)^{(\rm bLS)} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi (1 + e^{i\phi})^3 (1 - e^{i\phi}) \times \frac{1}{12} \tilde{\mathcal{A}}_2(e^{\bar{\mathcal{C}}} m_\tau^2 e^{i\phi}). \tag{95}$$

Yet another low-energy QCD observable that we will consider is the Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$. Its LS-characteristic function is obtained in Appendix C, on the basis of the known leading-β_0 coefficients [40] using the technique of Ref. [31]. The full perturbation coefficients d_1 and d_2 for the massless $d_b(Q^2)$, in $\overline{\rm MS}$ RSch and at RScl $\mu^2 = Q^2$, were obtained in Refs. [41] (see Appendix D for explicit expressions for d_1 and d_2). For the coefficient d_3, only the leading-n_f part ($\propto n_f^3$) is known exactly [40]; based on this, estimates of d_3 as a polynomial in β_0 were performed in Ref. [42] using naive non-Abelianization (NNA) $n_f \mapsto -6\beta_0$ [43]. For the evaluation of (the massless part of) $d_b(Q^2)$ we will not use estimates of the full d_3, i.e., we will use method (85) with terms up to $\tilde{t}_3 \tilde{\mathcal{A}}_3$ included, in any chosen RSch and with RScl's (76) with $\mathcal{C} = \bar{\mathcal{C}} = -5/3$. The formal difference between the evaluated and the true value is then $\mathcal{O}(\beta_0^2 \tilde{\mathcal{A}}_4)$. The experimental values of $d_b(Q^2)$ at low Q^2 are much less precise than those of $r_\tau(\triangle S = 0)$. At $Q^2 = 2$ and 1 GeV^2, they are $d_b(2\ {\rm GeV}^2) = 0.16 \pm 0.11$ and $d_b(1\ {\rm GeV}^2) = 0.17 \pm 0.07$ [44] (for an application, cf. Ref. [45]). The contributions of massive quarks (m_c, m_b) are $|\delta d_b(Q^2; m_q \neq 0)| < 10^{-3}$ for $Q^2 \leq 2\ {\rm GeV}^2$ [46], thus negligible. We recall that both d_v and d_b are massless observables which are normalized here according to the convention (40) for $n_f = 3$. Although the uncertainty of the measured values of $d_b(Q^2)$ is significantly lower at $Q^2 = 1\ {\rm GeV}^2$ than at $Q^2 = 2\ {\rm GeV}^2$, we will use both central values. We expect the theoretical predictions of our evaluations in general to be more reliable at higher momenta $Q^2 > 1\ {\rm GeV}^2$.

Now we will fix the parameters of models M1 and M2. Model M1 (11)–(16) has three independent parameters c_f, c_r, c_0 (and $\bar{\Lambda} = 0.4$ GeV as in MA). Requiring the reproduction of the aforementioned experimental central values $r_\tau(\triangle S = 0, m_q = 0) = 0.204$, $d_b(2\ {\rm GeV}^2) = 0.16$, and $d_b(1\ {\rm GeV}^2) = 0.17$, we obtain a solution for the three parameters, with the following values: $c_f = 1.08$, $c_r = 0.45$, $c_0 = 2.94$. We will use these parameter values in M1 (in RSch A). In general, the predicted values of observables do not change a lot when c_0 is varied in the regime ~ 1; they change more when c_r and/or c_f are varied. The experimental values of various higher-energy QCD observables $\mathcal{D}(Q^2)$, $\mathcal{R}(s)$ ($Q^2, s \gtrsim 10\ {\rm GeV}^2$) should be well reproduced in M1, because condition (16) ensures that M1 and MA merge at higher energies Q^2, $s \gg \bar{\Lambda}^2$, and it has been demonstrated that MA with $\bar{\Lambda}_{(n_f=3)} = 0.4$ GeV ($\Rightarrow \bar{\Lambda}_{(n_f=5)} = 0.26$ GeV) reproduces well those values [3]. We note that model MA (with $\bar{\Lambda} = 0.4$ GeV) predicts $r_\tau(\triangle S = 0, m_q = 0) \approx 0.14$, which is significantly too low.

Model M2 (17) and (18) has two free parameters c_v and c_p, both assumed to be ~ 1. Requiring reproduction of the central value of $r_\tau(\triangle S = 0, m_q = 0) = 0.204$, and requiring $|c_p|, |c_v| \geq 0.1$, it turns out that the model then always predicts values $d_b(2\ {\rm GeV}^2) > 0.19$. Requiring the minimal possible value $d_b(2\ {\rm GeV}^2) \approx 0.19$ gives us the parameter values $c_v = 0.1$ and $c_p = 3.4$. We will use these parameter values in M2 (in RSch A).

In Table I we present results of calculations of $r_\tau(\triangle S = 0, m_q = 0)$ and $d_b(Q^2 = 2\ {\rm GeV}^2)$ with our evaluation method (85), in the aforementioned RSch A (93) and (94) and at loop level = 4 and 3, in various anQCD models: M1, M2, and MA. When loop level = 4 (and $k_{\rm max} = 6$), we used in the calculation of $r_\tau(\triangle S = 0, m_q = 0)$ the estimated N^{3}LO perturbation coefficient d_3 of Ref. [38] for the Adler function (cf. Appendix D), as mentioned earlier. In the case of $d_b(Q^2 = 2\ {\rm GeV}^2)$, when loop level = 3 or 4, evaluation formula (85) was used in RSch A by inclusion of terms up to $\tilde{\mathcal{A}}_3$ only, as the N^{3}LO coefficient d_3 is not known there. We note that MA (with $\bar{\Lambda}_{(n_f=3)} = 0.4$ GeV), with light quark masses $m_u, m_d, m_s \ll \bar{\Lambda}$ ($m_u, m_d, m_s \approx 0$), does not reproduce the well-measured experimental value $r_\tau(\triangle S = 0, m_q = 0) = 0.204 \pm 0.005$, as already mentioned in the Introduction. This fact led us to suggest alternative versions of anQCD (e.g., M1, M2).

Now that the parameters of the presented anQCD models have been fixed, we can present various results of these models, evaluated with the method (85). In Fig. 4(a) we present curves for the massless Adler function $d_v(Q^2)$ (with $n_f = 3$) as functions of energy Q, in models M1,

[8]The contributions of heavy quarks in Euclidean observables $\mathcal{D}(Q^2)$, such as the Adler function, can be more important, even though $Q^2 < m_c^2$—see the discussion later in this section.

TABLE I. Results of evaluation of the semihadronic tau decay ratio $r_\tau(\triangle S = 0, m_q = 0)$ and of BjPSR $d_b(Q^2 = 2\mathrm{GeV}^2)$, in various anQCD models, using evaluation method (85) in RSch A (93). The basis for calculation of $\rho_1^{(\mathrm{pt})}(\sigma)$ is expansion (2) at loop level = 4 (i.e., when $\beta_3^{(\mathrm{A})}$ included) and with $k_{\max} = 6$. In parentheses are the results at loop level = 3 and $k_{\max} = 5$ (in that case, the d_3-term of the Adler function is not included). Presented are the results of the full evaluation (leading skeleton and beyond: LS + bLS), Eq. (85), and for $r_\tau(\triangle S = 0, m_q = 0)$ also the results of LS. The experimental values are $r_\tau(\triangle S = 0, m_q = 0) = 0.204 \pm 0.005$, $d_b(Q^2 = 2\ \mathrm{GeV}^2) = 0.16 \pm 0.11$, and $d_b(Q^2 = 1\ \mathrm{GeV}^2) = 0.17 \pm 0.07$. See the text for further details.

	$r_\tau(\triangle S = 0, m_q = 0)$	$r_\tau(\triangle S = 0, m_q = 0)$ [LS]	$d_b(Q^2 = 2\ \mathrm{GeV}^2)$	$d_b(Q^2 = 1\ \mathrm{GeV}^2)$
MA	0.141 (0.142)	0.139 (0.141)	0.137 (0.138)	0.155 (0.155)
M1	0.204 (0.205)	0.197 (0.198)	0.160 (0.161)	0.170 (0.171)
M2	0.204 (0.206)	0.203 (0.204)	0.189 (0.190)	0.219 (0.220)

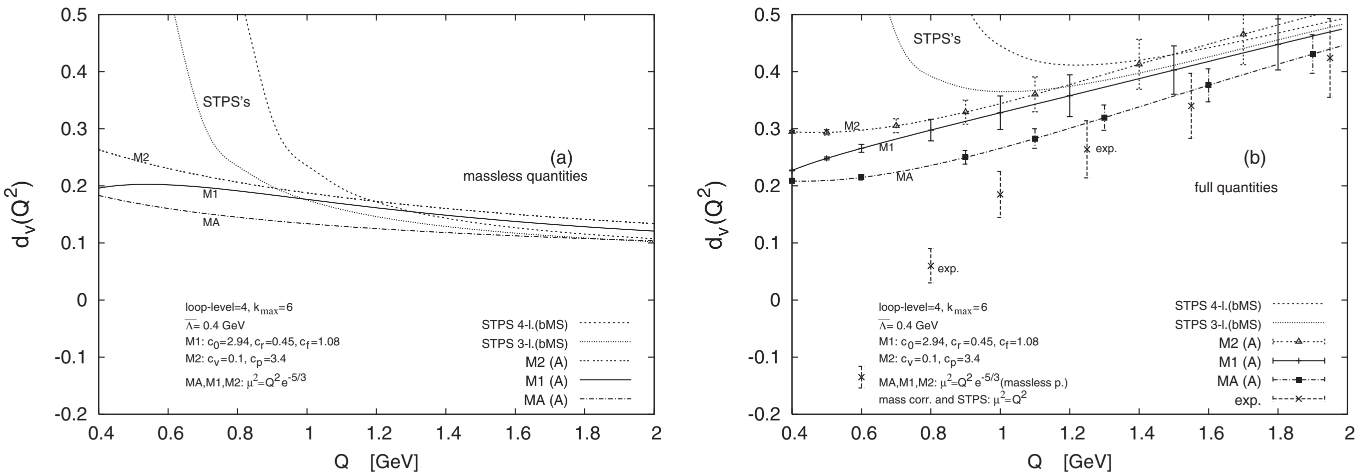

FIG. 4. Adler function as predicted by pQCD, and by our approach in several analytic QCD models (see the text): (a) the massless part ($n_f = 3$); (b) the full quantity, with the contribution of massive quarks included.

M2, and MA. The RSch used is RSch A (93) and (94). Loop level is 4, i.e., we include the value $\beta_3^{(\mathrm{A})}$ in our calculation for $\rho_1^{(\mathrm{pt})}$, with $k_{\max} = 6$ [cf. Eq. (2)], and use the estimated N^{3}LO perturbation coefficient d_3 of Ref. [38] (cf. Appendix D). The light-by-light contributions, which have a different topology of diagrams and should probably be resummed separately (cf. Ref. [11]), appear for the first time at $\sim a^3$ and are proportional to the square of the sum of the quark charges $(\sum Q_f)^2$ [37]. This sum is zero in the case $n_f = 3$ considered here. Figure 4(b) represents the results for the full Adler function, i.e., the V-channel heavy quark corrections $\delta d_v(Q^2; m_c, m_b)$ have been added there. For the calculation of the latter, we follow the procedure of Ref. [47], including the a^2-contributions [note that $d_v(Q^2) \equiv (1/2)D(Q^2) - 1$, where D is defined in [47]]. The first seven coefficients of the low-momentum Taylor expansion for the heavy quark a^2-contributions are calculated in Ref. [48]. Through a conformal mapping together with Padé improvement, as proposed in Ref. [49], an approximant is obtained. The approximant reproduces the low-momentum behavior and fits very well the large-momentum expansion [50] for this quantity up to energies $Q^2 \approx 16 m_q^2$ (see also Fig. 4 of Ref. [47]). Thus, this method can be safely used for the $q = c, b$ quarks in the energy range we are interested in.[9] In the heavy quark contributions, we simply replaced $a(Q^2)$ and $a^2(Q^2)$ by $\mathcal{A}_1(Q^2)$ and $\mathcal{A}_2(Q^2)$ (using $\Lambda = \bar{\Lambda} = 0.4$ GeV). The indicated $\pm$ uncertainties in the full Adler function curves are those c quark contributions which are proportional to $\mathcal{A}_2$. In Figs. 4(a) and 4(b) we included the STPS's [truncated forms of Eq. (40)] in $\overline{\mathrm{MS}}$ RSch and with RScl $\mu^2 = Q^2$. In Fig. 4(b) we included experimental values, for comparison. The experimental values of $d_v(Q^2)$ are taken from Ref. [47] where the integral expression for $d_v(Q^2)$ in terms of the e^+e^- QCD ratio $R_{e^+e^-}(s)$ is evaluated. All the values of $R_{e^+e^-}(s)$ are needed—from the two-pion thresh-

[9]Some contributions from heavy quarks are not considered here as we base our analysis on the expressions of Ref. [48]. The relevant diagrams are shown in Fig. 2 of Ref. [48]; the contributions with internal heavy and external light quarks are not included. These type of (a^2)-contributions have been obtained for the $R_{e^+e^-}(s)$ function in Refs. [51–53]. We checked that these contributions, when translated into the corresponding contributions for $d_v(Q^2)$ via the usual integral transformation relating R and d_v, result in a^2-contributions which are an order of magnitude smaller than the heavy quark a^2-contributions included in our curves.

old to infinity. The evaluation is based on the data compilation of Ref. [54]. The pQCD result for $R_{e^+e^-}(s)$ is used in the integral where it can be trusted, and data in the rest of the energy interval. Resonances are included separately. In Fig. 4(b) we can see that various anQCD models predict at low energies ($Q < 1.2$ GeV) values which are significantly closer to the experimental values than STPS's. Further, STPS's lose any predictability at $Q < 1.2$ GeV, mainly because of the vicinity of the unphysical Landau pole in the pQCD coupling $a(Q^2)$.

In Fig. 5(a), we present results of calculation of BjPSR $d_b(Q^2)$ at low energies in model M1, at loop level $= 3$ (and $k_{\max} = 5$), in two different RSch's: RSch A (93), and RSch B which is the "optimal" RSch for $d_b(Q^2)$, i.e., $\beta_2^{(\mathrm{B})}$ is obtained from the requirement $\tilde{t}_3 = 0$ for d_b, Eqs. (87) and (88)

$$\beta_2^{(\mathrm{B})} = -30.2949 - 10.4415\beta_0 + 7.445\,82\beta_0^2. \qquad (96)$$

At $n_f = 3$ we have $\beta_2^{(\mathrm{B})}(n_f = 3) = -16.0938$. The analytic couplings in RSch B are obtained from those in RSch A by applying the loop level $= 3$ RSch evolution Eqs. (36). In addition, we present in Fig. 5(a) results when the RScl in the beyond-the-LS terms (Q_2^2, Q_3^2) is increased from $Q^2 \exp(-5/3)$ to Q^2 (note that coefficients $\tilde{t}_2$ and $\tilde{t}_3$ then change accordingly). We see that at low energies $Q < 2$ GeV, the results in M1 change moderately but not insignificantly under the variation of RSch and RScl. For comparison, we included the curve obtained from the skeleton evaluation (B21) in RSch A [with $Q_2^2 = Q_3^2 = Q^2 \exp(-5/3)$], assuming that the skeleton expansion exists in RSch A (which is probably not true). We include the present experimental data, with the crosses representing the central values; the error bars extend in general over the entire depicted range of values, most of the experimental uncertainties are of the order of ± 0.1. The experimental data were deduced from Fig. 2 of Ref. [44], with the neutron decay parameter value $|g_A| = 0.211\,58 \pm 0.000\,48$ taken from [55]. The present experimental errors are too high to discriminate between various evaluation methods. In Fig. 5(b) we compare the results for of MA and M1. The RSch and RScl dependence of MA results remains very weak in all the shown region.

In Fig. 6(a) we present the same type of curves for M2 model. We see that the RSch and RScl dependence in M2 remains quite weak down to low energies. In Fig. 6(b) we compare the results of M2 and M1 models. Only the curves in RSch A are presented in Fig. 6(b).

Up until now, we applied the (skeleton-motivated) method (85) for the evaluation of QCD observables, in various anQCD models for $\mathcal{A}_1(\mu^2)$, with the higher-order couplings $\tilde{\mathcal{A}}_k$ ($k \geq 2$) constructed by Eqs. (28) in a certain RSch (usually RSch A) and equivalently the higher-order couplings $\mathcal{A}_k$ by Eqs. (35) [Eqs. (37) if loop level $= 4$]. There remains a question of how this method of evaluation compares with the APT evaluation approach of Milton *et al.* and Shirkov [2,3]. We recall that the APT approach was defined for the MA anQCD model, and it consists of using the available next-to-leading order (NLO) and N^{2}LO STPS of an observable (40) and replacing there $a^k(Q^2) \mapsto \mathcal{A}_k(Q^2)^{(\mathrm{MA})}$ ($k \geq 1$), where the higher-order MA couplings $\mathcal{A}_k(Q^2)^{(\mathrm{MA})}$ were constructed according to formula (19). In the N^{2}LO STPS case [e.g., for $d_b(Q^2)$], this reads

$$\mathcal{D}_{\mathrm{APT}}(Q^2) = \mathcal{A}_1(Q^2)^{(\mathrm{MA})} + d_1\mathcal{A}_2(Q^2)^{(\mathrm{MA})} + d_2\mathcal{A}_3(Q^2)^{(\mathrm{MA})}. \qquad (97)$$

The RSch is usually taken to be $\overline{\mathrm{MS}}$, but could in principle be any RSch. One of the differences between our and APT evaluation method here is the construction of the higher-order couplings $\mathcal{A}_k(Q^2)^{(\mathrm{MA})}$ of the model MA, where

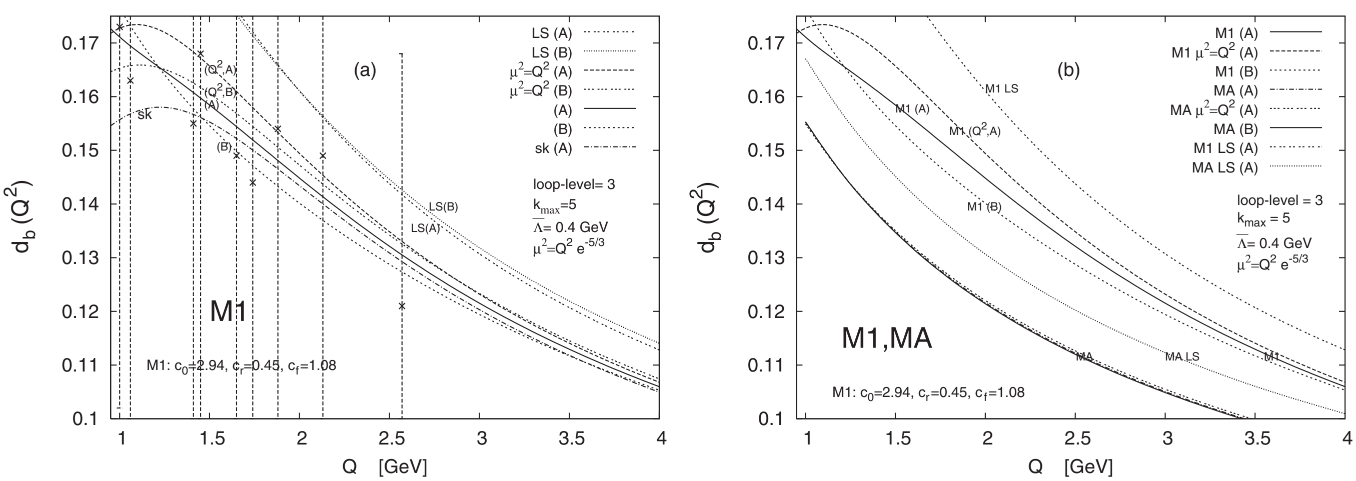

FIG. 5. Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$ in (a) model M1, and (b) comparison of M1 and MA; in various RSch's and at various RScl's. The vertical lines in (a) represent experimental data, with error bars in general covering the entire depicted range of values.

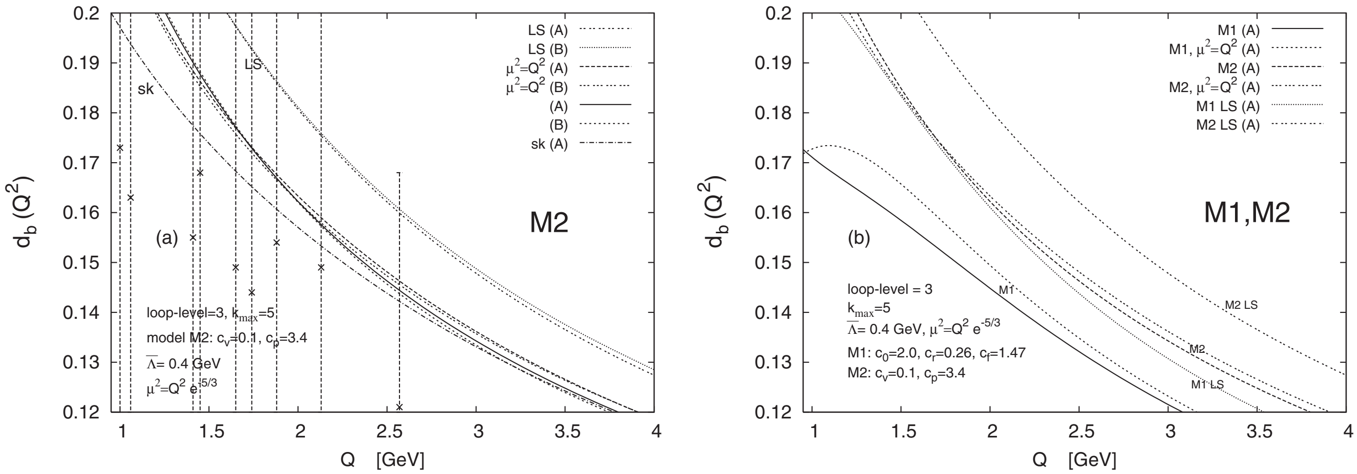

FIG. 6. BjPSR $d_b(Q^2)$ in (a) model M2, and (b) comparison of M2 and M1; at various RScl's (a),(b) and in various RSch's (a). The vertical lines in (a) represent the experimental data.

comparison with our construction has been made in Figs. 1 and 2 in Sec. III. Another difference is that our evaluation method (85) includes, in addition, the leading-β_0 contributions to all orders. We compare in Figs. 7(a) and 7(b) the results of our method (85) and the APT method (97) for BjPSR $d_b(Q^2)$, in MA model, in two different RSch's: A (93), and $\overline{\text{MS}}$ (relevant for β_2 coefficient only). In Figs. 7(a) and 7(b) the renormalization scale was taken to be $\mu^2 = Q^2 \exp(-5/3)$ and $\mu^2 = Q^2$, respectively, in both beyond-the-LS terms of our approach ($\propto \tilde{\mathcal{A}}_2, \tilde{\mathcal{A}}_3$), and in the APT approach. The RSch change from RSch A to $\overline{\text{MS}}$ was performed in our approach according to the loop level $= 3$ evolution equations (36), while in the APT approach the corresponding values of β_2 were inserted directly in (2) and thus in all $\rho_k^{(\text{pt})}$'s. For additional comparison, we included the skeleton evaluation (in RSch A), Eq. (B21). We see in Figs. 7 that in both our and APT evaluation approaches, in MA anQCD model, the RSch and RScl dependence of $d_b(Q^2)$ is very weak down to quite low energies. More detailed inspection reveals that our evaluation approach (85) gives for $d_b(Q^2)$ even somewhat less RScl-and RSch-dependent results than APT approach.

VI. SUMMARY AND CONCLUSIONS

In this work we suggested various models of analytic QCD (anQCD), i.e., models for construction of the anQCD coupling $\mathcal{A}_1(Q^2)$ which is an analytic analog of the perturbative QCD coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$. The main reason why we suggest alternatives to the minimal analytic (MA) model, i.e., to the coupling $\mathcal{A}_1^{(\text{MA})}(Q^2)$ of Shirkov and Solovtsov [1], is that it cannot correctly reproduce

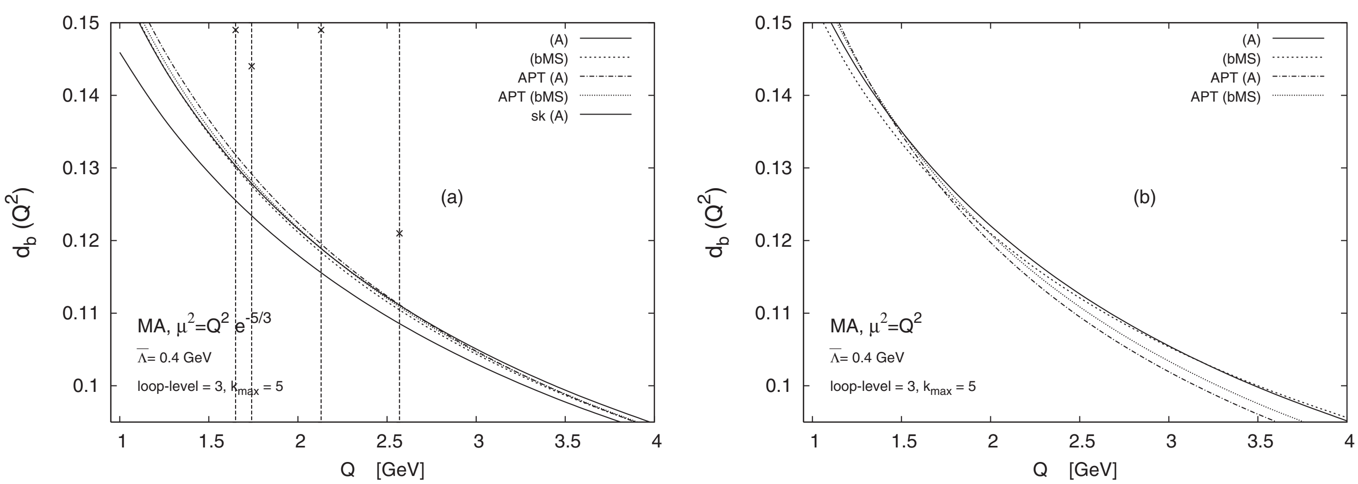

FIG. 7. BjPSR $d_b(Q^2)$ in the MA anQCD, with our evaluation method (85) and that of Milton *et al.* and Shirkov [2,3] (APT), in RSch A (93), and in $\overline{\text{MS}}$; the RScl is chosen to be (a) $\mu^2 = Q^2 \exp(-5/3)$, (b) $\mu^2 = Q^2$. The vertical lines in (a) represent the experimental data.

simultaneously various higher-energy QCD observables on the one hand and the low-energy observable r_τ (semihadronic τ decay rate ratio) on the other hand, unless large masses of u, d, and s quarks are introduced [4]. The described alternative models (M1 and M2) have $\mathcal{A}_1(Q^2)$ with additional dimensionless parameters in it, which can be adjusted in order to modify the behavior at low Q^2. Furthermore, we presented, for any anQCD model, an algorithm which allows construction of higher-order analytic couplings $\mathcal{A}_k(Q^2)$ ($k \geq 2$) which are the analytic analogs of $a^k(Q^2)$. In addition, we presented a method of evaluation of Euclidean QCD observables in anQCD models, a method which is (partly) motivated by the so-called skeleton-expansion structure but does not depend on the existence of such a skeleton expansion. The evaluation method sums up all the leading-β_0 contributions (LS: leading-skeleton) and adds those contributions beyond the LS which are known by the knowledge of a first few perturbation expansion coefficients of the considered observable. We tested this evaluation method, for three anQCD models, in the case of the Adler function, semihadronic τ decay ratio, and the Bjorken polarized sum rule (BjPSR) at low energies. The results show in general good stability under variation of the renormalization scale and scheme down to low energies $Q \sim 1$ GeV. We further carried out comparison of our evaluation method with that of Milton *et al.* (APT) [2–4], for the BjPSR, in the MA model where the latter method can be applied. The two methods give results which at low energies differ in general by only a few percent for this observable.

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APPENDIX A: RELEVANT COEFFICIENTS OF THE SKELETON-MOTIVATED EXPANSION

In this Appendix, we present explicit formulas for the coefficients $t_i^{(j)}$ appearing in the skeleton-motivated expansion (67), which is a slightly reorganized form of expansion (62). We consider the case when, in $\overline{\text{MS}}$ RSch ($\beta_k = \bar{\beta}_k = \sum \bar{b}_{kj}\beta_0^j$, $k \geq 2$) and at RScl $\mu^2 = Q^2$, the first three coefficients in expansion (40) are explicitly known ($\bar{d}_j$, $j = 1, 2, 3$), and all the leading-β_0 parts $\bar{c}_{nn}^{(1)}\beta_0^n$ of coefficients $\bar{d}_n$ ($n \geq 1$) in expansion (42) are known (we note that $\bar{c}_{n,-1}^{(1)} = 0$ in $\overline{\text{MS}}$). The RSch ($\beta_2, \beta_3, \ldots$) is chosen and common to all terms $\mathcal{D}^{(j)}$ ($j \geq 1$), and belongs to the class of RSch's of Eq. (41). The RScl's used in the resulting truncated versions of $\mathcal{D}^{(j)}(Q^2)$ ($j \geq 2$) are Q_j^2, they may be mutually different as each $\mathcal{D}^{(j)}(Q^2)$ (and k_j) is RScl independent. For the RSch and the RScl's we will use notations

$$\delta b_{kj} \equiv b_{kj} - \bar{b}_{kj}, \tag{A1}$$

$$Q_j^2 \equiv Q^2 \exp(\mathcal{C}_j). \tag{A2}$$

We then obtain for the coefficients $t_i^{(j)}$ of expansion (67) the following expressions, on the basis of relations (51)–(63), as well as (41), (42), and (65):

$$t_2^{(2)} = \bar{t}_2^{(2)} = \bar{c}_{10}^{(1)}, \tag{A3}$$

$$t_3^{(2)} = \bar{t}_3^{(2)} - \beta_0 \delta b_{22} + \beta_0 2\bar{c}_{10}^{(1)}\mathcal{C}_2, \tag{A4}$$

$$t_3^{(3)} = \bar{t}_3^{(3)} - \delta b_{21} - \frac{1}{\beta_0}\delta b_{20}, \tag{A5}$$

$$\begin{aligned} t_4^{(2)} = \bar{t}_4^{(2)} &+ (b_{11}\beta_0 + b_{10})(-\delta b_{22} + 2\bar{c}_{10}^{(1)}\mathcal{C}_2) \\ &+ \beta_0^2(-\delta b_{22}3\bar{c}_{11}^{(1)} - \tfrac{1}{2}\delta b_{33} + 3\bar{c}_{10}^{(1)}\bar{c}_{11}^{(2)}\mathcal{C}_2 - \delta b_{22}3\mathcal{C}_2 \\ &+ 3\bar{c}_{10}^{(1)}\mathcal{C}_2^2), \end{aligned} \tag{A6}$$

$$\begin{aligned} t_4^{(3)} = \bar{t}_4^{(3)} &+ \beta_0\Big(-\delta b_{22}2\bar{c}_{10}^{(1)} - \delta b_{21}3\bar{c}_{11}^{(1)} - \frac{1}{2}\delta b_{32} \\ &+ b_{11}\delta b_{22}\Big) - \delta b_{20}(\bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)} - \delta b_{21})^{-1}\Big(\bar{c}_{10}^{(1)}\bar{c}_{21}^{(2)} \\ &- \delta b_{22}2\bar{c}_{10}^{(1)} - \delta b_{21}3\bar{c}_{11}^{(1)} - \frac{1}{2}\delta b_{32} \\ &- b_{11}\bar{c}_{10}^{(1)}\bar{c}_{11}^{(2)} + b_{11}\delta b_{22}\Big) \\ &+ \beta_0\Big(\bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)} - \delta b_{21} - \frac{1}{\beta_0}\delta b_{20}\Big)3\mathcal{C}_3, \end{aligned} \tag{A7}$$

$$\begin{aligned} t_4^{(4)} = \bar{t}_4^{(4)} &+ \Big(-\delta b_{21}2\bar{c}_{10}^{(1)} - \delta b_{20}3\bar{c}_{11}^{(1)} - \frac{1}{2}\delta b_{31} + b_{10}\delta b_{22}\Big) \\ &+ \delta b_{20}(\bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)} - \delta b_{21})^{-1}\Big(\bar{c}_{10}^{(1)}\bar{c}_{21}^{(2)} - \delta b_{22}2\bar{c}_{10}^{(1)} \\ &- \delta b_{21}3\bar{c}_{11}^{(1)} - \frac{1}{2}\delta b_{32} - b_{11}\bar{c}_{10}^{(1)}\bar{c}_{11}^{(2)} + b_{11}\delta b_{22}\Big) \\ &+ \frac{1}{\beta_0}\Big(-\delta b_{20}2\bar{c}_{10}^{(1)} - \frac{1}{2}\delta b_{30}\Big). \end{aligned} \tag{A8}$$

Here, $\bar{t}_i^{(j)}$ are the values of the $t_i^{(j)}$ in $\overline{\text{MS}}$ RSch and with RScl $\mu^2 = Q^2$:

$$\bar{t}_2^{(2)} = \bar{c}_{10}^{(1)}, \tag{A9}$$

$$\bar{t}_3^{(2)} = \beta_0\bar{c}_{10}^{(1)}\bar{c}_{11}^{(2)}, \tag{A10}$$

$$\bar{t}_3^{(3)} = \bar{c}_{10}^{(1)}\bar{c}_{10}^{(2)}, \tag{A11}$$

$$\bar{t}_4^{(2)} = (b_{11}\beta_0 + b_{10})\bar{c}_{10}^{(1)}\bar{c}_{11}^{(2)} + \beta_0^2\bar{c}_{10}^{(1)}\bar{c}_{22}^{(2)}, \tag{A12}$$

$$\bar{t}_4^{(3)} = \beta_0 \bar{c}_{10}^{(1)} (\bar{c}_{21}^{(2)} - b_{11} \bar{c}_{11}^{(2)}), \tag{A13}$$

$$\bar{t}_4^{(4)} = \bar{c}_{10}^{(1)} (\bar{c}_{20}^{(2)} - b_{10} \bar{c}_{11}^{(2)}). \tag{A14}$$

Coefficients $\bar{c}_{ij}^{(2)}$ appearing in the above formulas can be obtained directly from coefficients $\bar{c}_{k\ell}^{(1)}$ by using relations (54) in $\overline{\rm MS}$ scheme (with RScl $\mu^2 = Q^2$):

$$\begin{aligned} \bar{c}_{10}^{(1)} \bar{c}_{1j}^{(2)} &= \bar{c}_{2j}^{(1)} - b_{1j} \bar{c}_{11}^{(1)} \qquad (j = 1, 0, -1), \\ \bar{c}_{10}^{(1)} \bar{c}_{2j}^{(2)} &= \bar{c}_{3j}^{(1)} - \tfrac{5}{2} b_{1,j-1} \bar{c}_{22}^{(1)} - b_{2j} \bar{c}_{11}^{(1)} \\ &(j = 2, 1, 0, -1). \end{aligned} \tag{A15}$$

Formulas (A9)–(A15), with notations (A1) and (A2), allow us to obtain all the coefficients $t_i^{(j)}$ of the skeleton-motivated expansion (67) in any chosen RSch and with chosen RScl's Q_j^2, if we know in $\overline{\rm MS}$ RSch at RScl $\mu^2 = Q^2$ all the leading-β_0 parts $\bar{c}_{nn}^{(1)} \beta_0^n$ of the expansion coefficients $\bar{d}_n = \sum_0^n \bar{c}_{nk}^{(1)} \beta_0^k$ of observable $\mathcal{D}(Q^2)$ Eq. (40), and we know exactly the full coefficients $\bar{d}_j$ for $j = 1, 2, 3$, i.e., we know $\bar{c}_{jk}^{(1)}$ for $j = 1, 2, 3$ and $k = 0, \ldots, j$. If, on the other hand, we do not know $\bar{d}_3$, the above formulas can be applied for $t_i^{(j)}$ for $i = 2, 3$ only.

When the beyond-the-LS contributions in our approach (71) are expressed in terms of $\tilde{\mathcal{A}}_k$'s, Eq. (85), with the RScl choice (76) $[\mathcal{C}_j = \mathcal{C}]$, coefficients $\tilde{t}_i$ can be expressed in terms of the above coefficients $t_s^{(k)}$ via relations (37) between $\mathcal{A}_k$'s and $\tilde{\mathcal{A}}_n$'s. After some straightforward algebra, we obtain

$$\tilde{t}_2 = \bar{\tilde{t}}_2 = \bar{c}_{10}^{(1)}, \tag{A16}$$

$$\tilde{t}_3 = \bar{\tilde{t}}_3 - \beta_0 \delta b_{22} + \beta_0 2 \bar{c}_{10}^{(1)} \mathcal{C} - \delta b_{21} - \frac{1}{\beta_0} \delta b_{20}, \tag{A17}$$

$$\begin{aligned} \tilde{t}_4 = \bar{\tilde{t}}_4 &+ \beta_0^2 \bigg[-\frac{1}{2} \delta b_{33} - \delta b_{22} 3 (\bar{c}_{11}^{(1)} + \mathcal{C}) + 3 \mathcal{C} \bar{c}_{10}^{(1)} (\bar{c}_{11}^{(2)} + \mathcal{C}) \bigg] + \beta_0 \bigg[-\frac{1}{2} \delta b_{32} + \delta b_{22} \bigg(-3 \bar{c}_{10}^{(1)} + \frac{5}{2} b_{11} \bigg) - \delta b_{21} 3 \bar{c}_{11}^{(1)} \\ &+ 3 \mathcal{C} \bar{c}_{10}^{(1)} (\bar{c}_{10}^{(2)} - b_{11}) \bigg] + \bigg[-\frac{1}{2} \delta b_{31} + \frac{5}{2} b_{10} \delta b_{22} + \delta b_{21} \bigg(-3 \bar{c}_{10}^{(1)} + \frac{5}{2} b_{11} \bigg) - 3 \delta b_{20} (\bar{c}_{11}^{(1)} + \mathcal{C}) - 3 b_{10} \bar{c}_{10}^{(1)} \mathcal{C} \bigg] \\ &+ \frac{1}{\beta_0} \bigg[-\frac{1}{2} \delta b_{30} + \frac{5}{2} b_{10} \delta b_{21} + \delta b_{20} \bigg(-3 \bar{c}_{10}^{(1)} + \frac{5}{2} b_{11} \bigg) \bigg] + \frac{1}{\beta_0^2} \frac{5}{2} b_{10} \delta b_{20}, \end{aligned} \tag{A18}$$

where $\bar{\tilde{t}}_i$ are the values of $\tilde{t}_i$ in $\overline{\rm MS}$ and with RScl $\mu^2 = Q^2$:

$$\bar{\tilde{t}}_2 = \bar{c}_{10}^{(1)}, \tag{A19}$$

$$\bar{\tilde{t}}_3 = \beta_0 \bar{c}_{10}^{(1)} \bar{c}_{11}^{(2)} + \bar{c}_{10}^{(1)} (\bar{c}_{10}^{(2)} - b_{11}) - \frac{1}{\beta_0} \bar{c}_{10}^{(1)} b_{10}, \tag{A20}$$

$$\begin{aligned} \bar{\tilde{t}}_4 = \beta_0^2 \bar{c}_{10}^{(1)} \bar{c}_{22}^{(2)} &+ \beta_0 \bar{c}_{10}^{(1)} \bigg(\bar{c}_{21}^{(2)} - \frac{5}{2} b_{11} \bar{c}_{11}^{(2)} - \bar{b}_{22} \bigg) + \bar{c}_{10}^{(1)} \bigg(\bar{c}_{20}^{(2)} - \frac{5}{2} b_{10} \bar{c}_{11}^{(2)} - \frac{5}{2} b_{11} \bar{c}_{10}^{(2)} + \frac{5}{2} b_{11}^2 - \bar{b}_{21} \bigg) \\ &+ \frac{1}{\beta_0} \bar{c}_{10}^{(1)} \bigg(5 b_{11} b_{10} - \frac{5}{2} b_{10} \bar{c}_{10}^{(2)} - \bar{b}_{20} \bigg) + \frac{1}{\beta_0^2} \frac{5}{2} \bar{c}_{10}^{(1)} b_{10}^2. \end{aligned} \tag{A21}$$

APPENDIX B: SKELETON EXPANSION

In this Appendix, we will construct an expression for evaluation of QCD spacelike observables $\mathcal{D}(Q^2)$ (for any anQCD model) which will be derived directly from the QCD skeleton expansion. Here we will take the position that such an expansion exists in the class of schemes with the QCD scale $\Lambda_{\mathcal{C}}^2 = \Lambda_0^2 \exp(\mathcal{C})$, where Λ_0 is the so-called V-scheme scale and $\mathcal{C}$ is an arbitrary n_f-independent constant, and with β_k of Eq. (41) where b_{kj} are arbitrary constants. In this context, choosing the $\overline{\rm MS}$ scale parameter $\mathcal{C} = \bar{\mathcal{C}} \equiv -5/3$ $(\Lambda = \bar{\Lambda})$ for scaling the RScl μ^2 represents no additional restriction. This expansion involves in the integrands the characteristic functions $F_{\mathcal{D}}^{\mathcal{E}}(t_1, \ldots, t_n)$, which are considered n_f-independent, and the (singular) pQCD coupling $a(\mu^2)$. We replace $a(\mu^2)$ by an anQCD coupling $\mathcal{A}_1(\mu^2)$ in the skeleton integrals which makes the integrals unambiguous:

$$\begin{aligned} \mathcal{D}(Q^2)_{\rm skel.} = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t) \mathcal{A}_1(t Q^2 e^{\bar{\mathcal{C}}}) &+ \sum_{n=2}^{\infty} s_{n-1}^{\mathcal{D}} \bigg[\prod_{j=1}^{n} \int_0^\infty \frac{dt_j}{t_j} \mathcal{A}_1(t_j Q^2 e^{\bar{\mathcal{C}}}) \bigg] \\ &\times F_{\mathcal{D}}^{\mathcal{E}}(t_1, \ldots, t_n) \end{aligned} \tag{B1}$$

$$= \mathcal{D}^{(\mathrm{LS})}(Q^2) + s_1^{\mathcal{D}} \mathcal{D}^{(\mathrm{NLS})}(Q^2) + s_2^{\mathcal{D}} \mathcal{D}^{(\mathrm{N^2LS})}(Q^2) + s_3^{\mathcal{D}} \mathcal{D}^{(\mathrm{N^3LS})}(Q^2) + \cdots. \tag{B2}$$

Here, $F_{\mathcal{D}}^{\mathcal{E}}(t_1, \ldots, t_n)$ are the characteristic functions and have the normalizations

$$\int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t) = 1, \qquad \int \frac{dt_1}{t_1} \frac{dt_2}{t_2} F_{\mathcal{D}}^{\mathcal{E}}(t_1, t_2) = 1, \ldots, \tag{B3}$$

implying for the perturbative parts

$$\mathcal{D}^{(\kappa)}(Q^2)_{\mathrm{pt}} = a^{n_\kappa}[1 + \mathcal{O}(a)], \tag{B4}$$

where $n_\kappa = 1$ for $\kappa = \mathrm{LS}$, $n_\kappa = 2$ for $\kappa = \mathrm{NLS}$, etc. The perturbative part of $\mathcal{A}_1(\mu^2)$ is $a(\mu^2)$ [$\mathcal{A}_1(\mu^2) = a(\mu^2) +$ NP, where NP involves nonanalytic in $a = 0$ functions of a, cf. Eq. (31)]. We will use RGE evolution series (46) for expansion of $a(te^{\bar{C}}Q^2)$ around $a(\mu^2) \equiv a$

$$\begin{aligned} a(te^{\bar{C}}Q^2) &= a + \sum_{n=1}^\infty \tilde{a}_{n+1} \beta_0^n (-\ln \mathcal{T})^n \\ &= a + a^2 \beta_0 (-\ln \mathcal{T}) \\ &\quad + a^3[\beta_0^2 \ln^2 \mathcal{T} - \beta_1 \ln \mathcal{T}] \\ &\quad + a^4\Big[-\beta_0^3 \ln^3 \mathcal{T} + \frac{5}{2}\beta_0 \beta_1 \ln^2 \mathcal{T} \\ &\quad - \beta_2 \ln \mathcal{T}\Big] + \cdots, \end{aligned} \tag{B5}$$

where $\mathcal{T} \equiv tQ^2 e^{\mathcal{C}}/\mu^2$, and $\tilde{a}_n$ are defined in Eq. (29). Using expansion (B5) in the leading-skeleton (LS) term in Eq. (B1), this term can be shown to have the following expansion for its perturbative part:

$$\begin{aligned} \mathcal{D}^{(\mathrm{LS})}(Q^2)_{\mathrm{pt}} &= a + a^2 \beta_0 \langle -\ln \mathcal{T} \rangle_{(1)} \\ &\quad + a^3[\beta_0^2 \langle (-\ln \mathcal{T})^2 \rangle_{(1)} + \beta_1 \langle -\ln \mathcal{T} \rangle_{(1)}] \\ &\quad + a^4\Big[\beta_0^3 \langle (-\ln \mathcal{T})^3 \rangle_{(1)} \\ &\quad + \frac{5}{2}\beta_1 \beta_0 \langle (-\ln \mathcal{T})^2 \rangle_{(1)} + \beta_2 \langle -\ln \mathcal{T} \rangle_{(1)}\Big] \\ &\quad + \mathcal{O}(\beta_0^4 a^5), \end{aligned} \tag{B6}$$

where we adhere to notations summarized in the following:

$$\mathcal{T} = \frac{tQ^2 e^{\bar{C}}}{\mu^2}, \qquad a \equiv a(\mu^2), \tag{B7}$$

$$\langle f(t_1, \ldots, t_n) \rangle_{(n)} \equiv \prod_{j=1}^n \int_0^\infty \frac{dt_j}{t_j} F_{\mathcal{D}}^{\mathcal{E}}(t_1, \ldots, t_n) f(t_1, \ldots, t_n). \tag{B8}$$

Requiring that the perturbative part of the LS term absorb all the leading-β_0 parts of $\mathcal{D}(Q^2)_{\mathrm{pt}}$ [see Eqs. (40)–(42)] implies that

$$\langle (-\ln \mathcal{T})^n \rangle_{(1)} = c_{nn}^{(1)} \qquad (n = 0, 1, 2, \ldots). \tag{B9}$$

This, in conjunction with expansion (B6), implies that $\mathcal{D}^{(\mathrm{LS})}(Q^2)_{\mathrm{pt}}$ is precisely $\mathcal{D}^{(1)}(Q^2)_{\mathrm{pt}}$ of construction in Sec. IV, Eq. (51), i.e., we really have for $\mathcal{D}^{(1)}(Q^2)_{\mathrm{pt}}$ the resummed form (64). Taylor expansion of $\mathcal{A}_1(te^{\bar{C}}Q^2)$ around Q^2 is completely analogous to expansion (B5):

$$\begin{aligned} \mathcal{A}_1(te^{\bar{C}}Q^2) &= \mathcal{A}_1 + \sum_{n=1}^\infty \tilde{\mathcal{A}}_{n+1} \beta_0^n (-\ln \mathcal{T})^n \\ &= \mathcal{A}_1 + \mathcal{A}_2 \beta_0 (-\ln \mathcal{T}) + \mathcal{A}_3[\beta_0^2 \ln^2 \mathcal{T} - \beta_1 \ln \mathcal{T}] + \mathcal{A}_4\Big[-\beta_0^3 \ln^3 \mathcal{T} + \frac{5}{2}\beta_0 \beta_1 \ln^2 \mathcal{T} - \beta_2 \ln \mathcal{T}\Big] + \cdots, \end{aligned} \tag{B10}$$

where $\tilde{\mathcal{A}}_k \equiv \tilde{\mathcal{A}}_k(\mu^2)$ and $\mathcal{A}_k \equiv \mathcal{A}_k(\mu^2)$. In the last identity we used the fact that $\tilde{\mathcal{A}}_n$'s appear on the left-hand side of RGE's (26), which are analogous to pQCD RGE's with $\tilde{a}_n$'s on the left-hand side [analogy valid up to terms $\mathcal{O}(\tilde{\mathcal{A}}_{n_\mathrm{m}})$ where n_m = loop level] when the correspondence $a^k \leftrightarrow \mathcal{A}_k$ is made. Equation (B10) implies for the (full analytic) LS term of the skeleton expansion (B1) a nonpower analytic expansion

$$\begin{aligned} \mathcal{D}^{(\mathrm{LS})}(Q^2) &\equiv \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t) \mathcal{A}(tQ^2 e^{\bar{C}}) = \mathcal{A}_1 + \sum_{n=1}^\infty \tilde{\mathcal{A}}_{n+1} \beta_0^n \langle (-\ln \mathcal{T})^n \rangle_{(1)} \\ &= \mathcal{A}_1 + \mathcal{A}_2 \beta_0 \langle -\ln \mathcal{T} \rangle_{(1)} + \mathcal{A}_3[\beta_0^2 \langle (-\ln \mathcal{T})^2 \rangle_{(1)} + \beta_1 \langle -\ln \mathcal{T} \rangle_{(1)}] \\ &\quad + \mathcal{A}_4\Big[\beta_0^3 \langle (-\ln \mathcal{T})^3 \rangle_{(1)} + \frac{5}{2}\beta_1 \beta_0 \langle (-\ln \mathcal{T})^2 \rangle_{(1)} + \beta_2 \langle -\ln \mathcal{T} \rangle_{(1)}\Big] + \mathcal{O}(\mathcal{A}_5), \end{aligned} \tag{B11}$$

which is just the analytized analog [according to the rule (69)] of perturbation expansion (51) and (B6).

Now we will investigate the beyond-the-LS contributions of the skeleton expansion (B1). In view of normalization conditions (B4), it follows immediately that

$$s_1^{\mathcal{D}} = c_{10}^{(1)}, \tag{B12}$$

which is just the coefficient k_2 (52) in the approach of Sec. IV. In analogy with the LS part, we now require that $\mathcal{D}^{(\mathrm{NLS})}(Q^2)_{\mathrm{pt}}$ be such as to absorb all the leading-β_0 parts of the difference $(1/s_1^{\mathcal{D}})[\mathcal{D}(Q^2) - \mathcal{D}^{(\mathrm{LS})}(Q^2)]_{\mathrm{pt}}$. In a completely analogous way as before, we can show that this is equivalent to

$$(-1)^n \langle \ln^n \mathcal{T}_1 + \ln^{n-1}\mathcal{T}_1 \ln \mathcal{T}_2 + \cdots + \ln^n \mathcal{T}_2 \rangle_{(2)} = c_{nn}^{(2)} \quad (n = 1, 2 \ldots), \tag{B13}$$

where $\mathcal{T}_j = t_j Q^2 e^{\bar{\mathcal{C}}}/\mu^2$, and coefficients $c_{nn}^{(2)}$ are defined in Eqs. (52)–(54). These coefficients are known if the perturbative coefficients d_j (40) are known. The (nonpower) expansion in $\mathcal{A}_k \equiv \mathcal{A}_k(\mu^2)$ of the next-to-leading skeleton (NLS) term is then

$$s_1^{\mathcal{D}} \mathcal{D}^{(\mathrm{NLS})}(Q^2) = c_{10}^{(1)} \{ \mathcal{A}_1^2 + \mathcal{A}_1 \mathcal{A}_2 \beta_0 c_{11}^{(2)} + \mathcal{A}_1 \mathcal{A}_3 [\beta_0^2 c_{22}^{(2)} + \beta_1 c_{11}^{(2)}] \tag{B14}$$

$$+ [\mathcal{A}_2^2 - \mathcal{A}_1 \mathcal{A}_3] \beta_0^2 \langle \ln \mathcal{T}_1 \ln \mathcal{T}_2 \rangle_{(2)} + \mathcal{O}(\mathcal{A}_1 \mathcal{A}_4, \mathcal{A}_2 \mathcal{A}_3, \ldots)\}. \tag{B15}$$

The last term in brackets has a coefficient $\propto \langle \ln \mathcal{T}_1 \ln \mathcal{T}_2 \rangle_{(2)}$ which cannot be obtained on the basis of the perturbative coefficients d_j (40). The perturbative part of this last term is zero. We know $c_{10}^{(1)}$ if we know the NLO coefficient d_1 of the perturbation expansion (40) of observable $\mathcal{D}(Q^2)$; for the knowledge of $c_{11}^{(2)}$ we need, in addition, the knowledge of d_2, and for $c_{22}^{(2)}$ the knowledge of d_3.

We now continue analogously one step further. In view of the normalization conditions (B4), it follows immediately

$$s_2^{\mathcal{D}} = c_{10}^{(1)} \left(c_{10}^{(2)} + \frac{1}{\beta_0} c_{1,-1}^{(2)} \right), \tag{B16}$$

which is identical to the coefficient k_3 (57) in the approach of Sec. IV. We require that the third ($\mathrm{N^2LS}$) term $s_2^{\mathcal{D}} \mathcal{D}^{(\mathrm{N^2LS})}(Q^2)$ in skeleton expansion (B2) satisfy the condition: $\mathcal{D}^{(\mathrm{N^2LS})}(Q^2)_{\mathrm{pt}}$ be such as to absorb all the leading-β_0 parts of the difference $(1/s_2^{\mathcal{D}})[\mathcal{D}(Q^2) - \mathcal{D}^{(\mathrm{LS})}(Q^2) - s_1^{\mathcal{D}} \mathcal{D}^{(\mathrm{NLS})}(Q^2)]_{\mathrm{pt}}$. This then implies

$$\langle -\ln \mathcal{T}_1 - \ln \mathcal{T}_2 - \ln \mathcal{T}_3 \rangle_{(3)} = c_{11}^{(3)}, \tag{B17}$$

where $c_{11}^{(3)}$ is given in Eq. (60); and similarly for higher terms ($c_{22}^{(3)}$, etc.). The (nonpower) expansion in $\mathcal{A}_k \equiv \mathcal{A}_k(\mu^2)$ of the $\mathrm{N^2LS}$ term is then

$$s_2^{\mathcal{D}} \mathcal{D}^{(\mathrm{N^2LS})}(Q^2) = s_2^{\mathcal{D}} \{ \mathcal{A}_1^3 + \mathcal{A}_1^2 \mathcal{A}_2 \beta_0 c_{11}^{(3)} + \mathcal{O}(\mathcal{A}_1^2 \mathcal{A}_3, \mathcal{A}_1 \mathcal{A}_2^2, \ldots)\}. \tag{B18}$$

We know the quantity $s_2^{\mathcal{D}}$ if we know the coefficients d_1 and d_2 in the perturbation expansion (40) of observable $\mathcal{D}(Q^2)$; for the knowledge of $c_{11}^{(3)}$ we need, in addition, the knowledge of d_3.

Normalization conditions (B4) now imply that the coefficient $s_3^{\mathcal{D}}$ of the $\mathrm{N^3LS}$ term in the skeleton expansion (B1) and (B2) is

$$s_3^{\mathcal{D}} = c_{10}^{(1)} \left[c_{20}^{(2)} - b_{10} c_{11}^{(2)} - \frac{c_{1,-1}^{(2)}}{c_{10}^{(2)}} (c_{21}^{(2)} - b_{11} c_{11}^{(2)}) + \frac{1}{\beta_0} c_{2,-1}^{(2)} \right], \tag{B19}$$

which is identical to the coefficient k_4 (63) in the approach of Sec. IV. The (nonpower) expansion in $\mathcal{A}_k \equiv \mathcal{A}_k(\mu^2)$ of the $\mathrm{N^3LS}$ term is then

$$s_3^{\mathcal{D}} \mathcal{D}^{(\mathrm{N^3LS})}(Q^2) = s_3^{\mathcal{D}} \mathcal{A}_1^4 + \mathcal{O}(\mathcal{A}_1^3 \mathcal{A}_2, \mathcal{A}_1^2 \mathcal{A}_3, \ldots). \tag{B20}$$

We know the quantity $s_3^{\mathcal{D}}$ if we know the coefficients d_1, d_2, and d_3 in the perturbation expansion (40) of observable $\mathcal{D}(Q^2)$.

Finally, we can combine the LS term (68), whose characteristic function is usually known, with all the beyond-the-LS terms written hitherto (B14)–(B20) which are known if d_1, d_2 and d_3 are known; since each of these terms is RScl independent, we can use in the most general case various (spacelike) RScl's $Q_j^2 = Q^2 \exp(\mathcal{C}_j)$ as Eq. (A2) ($j = 2, 3, 4$ for the NLS, $\mathrm{N^3LS}$ and $\mathrm{N^3LS}$ terms, respectively). This then results in

$$\begin{aligned} \mathcal{D} = {} & \mathcal{D}^{(\mathrm{LS})}(Q^2) + t_2^{(2)} [\mathcal{A}_1(Q_2^2)]^2 + \{ t_3^{(2)} \mathcal{A}_1(Q_2^2) \mathcal{A}_2(Q_2^2) + t_3^{(3)} [\mathcal{A}_1(Q_3^2)]^3 \} + \{ t_4^{(2)} \mathcal{A}_1(Q_2^2) \mathcal{A}_3(Q_2^2) \\ & + t_4^{(3)} [\mathcal{A}_1(Q_3^2)]^2 \mathcal{A}_2(Q_3^2) + t_4^{(4)} [\mathcal{A}_1(Q_4^2)]^4 \} + \{ [\mathcal{A}_2(Q_2^2)]^2 - \mathcal{A}_1(Q_2^2) \mathcal{A}_3(Q_2^2) \} c_{10}^{(1)} \beta_0^2 \langle \ln \mathcal{T}_1 \ln \mathcal{T}_2 \rangle_{(2)} \\ & + \mathcal{O}(\mathcal{A}_1^5, \mathcal{A}_1^3 \mathcal{A}_2, \ldots), \end{aligned} \tag{B21}$$

where the coefficients $t_{(j)}^{(i)}$ are precisely those given in Appendix A, Eqs. (A3)–(A14). Therefore, the evaluation method presented in the present Appendix, which is a representation of an assumed skeleton expansion (B1) and (B2), reduces to the evaluation method presented in Sec. IV when the following replacements are made:

$$[\mathcal{A}_1(\mu^2)]^{k_1} [\mathcal{A}_2(\mu^2)]^{k_2} \cdots [\mathcal{A}_s(\mu^2)]^{k_s} \mapsto \mathcal{A}_{k_1 + 2k_2 \cdots + s k_s}(\mu^2). \tag{B22}$$

In the present method, the coefficient at the last term in brackets in expression (B21) can be evaluated only if certain assumptions about the NLS characteristic function

$F_{\mathcal{D}}^{\mathcal{E}}(t_1, t_2)$ are made. For simplicity, we will make the factorization assumption

$$F_{\mathcal{D}}^{\mathcal{E}}(t_1, t_2) = w_{\mathcal{D}}^{\mathcal{E}}(t_1) w_{\mathcal{D}}^{\mathcal{E}}(t_2) \Rightarrow \langle \ln \mathcal{T}_1 \ln \mathcal{T}_2 \rangle_{(2)} = \tfrac{1}{4}(c_{11}^{(2)})^2, \tag{B23}$$

where the last identity is obtained on the basis of identity (B13) for $n = 1$.

Similarly as the skeleton-motivated evaluation method (70) and (71), the skeleton evaluation method (B21) can be performed in principle at any chosen RScl's Q_j and in any RSch of the class (41). This method was denoted as "v1" in Ref. [12]. However, skeleton method (B21) makes sense only if the skeleton expansion (B2) really exists. If the latter exists, it probably does so only in a specific (skeleton) scheme [34,35]. In contrast, the skeleton-motivated evaluation method (70) and (71) does not rely on the existence of the skeleton expansion.

APPENDIX C: LEADING-SKELETON CHARACTERISTIC FUNCTIONS IN THE SPACELIKE AND TIMELIKE FORM

In this Appendix we summarize the knowledge of the LS characteristic functions for the spacelike observables $\mathcal{D}(Q^2)$. In the spacelike formulation (68), which involves the spacelike coupling $\mathcal{A}_1$, the characteristic function can be obtained from the knowledge of the leading-β_0 coefficients $c_{nn}^{(1)}$—cf. Eqs. (40) and (42), following the formalism of Neubert [31].

For example, in the case of the Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$, the leading-β_0 coefficients were obtained in Ref. [40]. In $\overline{\rm MS}$ RSch and at RScl $\mu^2 = Q^2 \exp(\bar{\mathcal{C}})$ (we use $\Lambda = \bar{\Lambda}$ throughout, i.e., $\mathcal{C} = \bar{\mathcal{C}} \equiv -5/3$), they are

$$c_{nn}^{(1)} = n!\left[\frac{8}{9} + \frac{4}{9}(-1)^n - \frac{5}{18}\frac{1}{2^n} - \frac{1}{18}\frac{1}{2^n}(-1)^n\right] \quad (n = 0, 1, \ldots). \tag{C1}$$

This implies that the (leading-β_0) Borel transform is

$$\hat{S}_b(u; Q^2; \mu^2 = Q^2 e^{\bar{\mathcal{C}}}) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} c_{nn}^{(1)} u^n = \frac{1}{3}\frac{(3+u)}{(1-u^2)(1-u^2/4)}. \tag{C2}$$

The renormalon poles are at $u = \pm 1, \pm 2$. The LS characteristic function appearing in (68) is then obtained by the general formula

$$F_{\mathcal{D}}^{\mathcal{E}}(\tau) = \frac{1}{2\pi i}\int_{u_0 - i\infty}^{u_0 + i\infty} du \hat{S}_{\mathcal{D}}(u) \tau^u, \tag{C3}$$

where u_0 is any real number closer to the origin than the leading renormalon ($-1 < u_0 < 1$). We can choose $u_0 = 0$ and introduce a new integration variable $r = -iu$. The integral, with $\hat{S}(u)$ of Eq. (C2), then reduces to

$$F_b^{\mathcal{E}}(\tau) = \frac{2}{3\pi}\int_{-\infty}^{+\infty} dr e^{ir\ln\tau} \frac{(3+ir)}{(r+i)(r-i)(r+2i)(r-2i)}, \tag{C4}$$

which can be calculated by the use of the Cauchy theorem in the complex r-plane: when $\tau > 1$, we close the path with a large semicircle in the upper half plane; when $\tau < 1$, in the lower half plane. This gives us the result

$$F_b^{\mathcal{E}}(\tau) = \begin{Bmatrix} \frac{8}{9}\tau(1 - \frac{5}{8}\tau) & \tau \le 1 \\ \frac{4}{9\tau}(1 - \frac{1}{4\tau}) & \tau \ge 1 \end{Bmatrix}, \tag{C5}$$

which we already used in Ref. [12].

The LS characteristic function for the Adler function $d_v(Q^2)$ was obtained in Ref. [31], on the basis of the large-β_0 expansion of the Borel transform of d_v obtained in Refs. [56,57]

$$F_v^{\mathcal{E}}(t) = 2C_F t\left[\left(\frac{7}{4} - \ln t\right)t + (1+t)(\mathrm{PolyLog}_2(-t) + \ln t \ln(1+t))\right] \quad (t \le 1) \tag{C6}$$

$$= 2C_F\left[t(1+\ln t) + \left(\frac{3}{4} + \frac{1}{2}\ln t\right) + t(1+t)(\mathrm{PolyLog}_2(-1/t) - \ln t \ln(1+1/t))\right] \quad (t \ge 1), \tag{C7}$$

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$.

The semihadronic τ decay ratio r_τ is a timelike observable. The LS term of r_τ can be obtained from the LS term of the Adler function on the basis of the relation

$$r_\tau(\Delta S = 0, m_q = 0) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi (1 + e^{i\phi})^3 (1 - e^{i\phi}) \times d_v(Q^2 = m_\tau^2 e^{i\phi}). \tag{C8}$$

This implies for the LS term of $r_\tau(\Delta S = 0, m_q = 0)$

$$r_\tau(\Delta S = 0, m_q = 0)^{(\mathrm{LS})} = \int_0^\infty \frac{dt}{t} F_r^{\mathcal{M}}(t) \mathfrak{A}_1(t e^{\mathcal{C}} m_\tau^2), \tag{C9}$$

where $\mathfrak{A}_1$ is the timelike coupling appearing in Eqs. (7)–(10), and superscript $\mathcal{M}$ in the characteristic function means that it is Minkowskian (timelike). The latter was obtained in Ref. [32][10]

[10] We use a different normalization, so an additional factor of $t/4$ appears, in comparison to [32].

$$F_r^{\mathcal{M}}(t) = 4C_F t\Big[4 - \frac{73}{12}t - \frac{23}{24}t^2 - \frac{259}{432}t^3 - 2\,\mathrm{PolyLog}_3(-t) - 3\zeta(3) + \Big(\frac{17}{6}t + \frac{1}{3}t^2 + \mathrm{PolyLog}_2(-t)\Big)\ln t + \Big(\frac{3}{4}t^2 + \frac{1}{6}t^3\Big)\ln^2 t - \Big(\frac{11}{6} + 3t + \frac{3}{2}t^2 + \frac{1}{3}t^3\Big)(\ln t \ln(1+t) + \mathrm{PolyLog}_2(-t))\Big] \quad (t \le 1), \tag{C10}$$

$$F_r^{\mathcal{M}}(t) = 4C_F\Big[-\frac{575}{216}t + \frac{37}{48} - \frac{17}{12}t^2 - \frac{1}{3}t^3 + 2t\,\mathrm{PolyLog}_3(-1/t) - \Big(\frac{85}{36}t - \frac{1}{4} + \frac{4}{3}t^2 + \frac{1}{3}t^3 - t\,\mathrm{PolyLog}_2(-1/t)\Big)\ln t + \Big(\frac{11}{6}t + 3 + \frac{3}{2}t^3 + \frac{1}{3}t^4\Big)(\ln t \ln(1+1/t) - \mathrm{PolyLog}_2(-1/t))\Big] \quad (t \ge 1). \tag{C11}$$

Here, $\mathrm{PolyLog}_3$ is the polylogarithm function of nth order (using notation of [13]).

The LS part of any spacelike observable $\mathcal{D}(Q^2)$ can be written in two equivalent forms—the form involving the spacelike coupling $\mathcal{A}_1$, Eq. (68), and the form involving the timelike $\mathfrak{A}_1$ of Eqs. (7)–(10):

$$\mathcal{D}^{(\mathrm{LS})}(Q^2) = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t)\mathcal{A}_1(te^{\mathcal{C}}Q^2) = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{M}}(t)\mathfrak{A}_1(te^{\mathcal{C}}Q^2), \tag{C12}$$

where the superscript $\mathcal{M}$ stands for the "Minkowskian" (timelike) formulation, and the two characteristic functions are related via relations

$$F_{\mathcal{D}}^{\mathcal{M}}(t) = -\pi \frac{d}{d\ln t}\mathcal{F}_{\mathcal{D}}(t) = t\int_0^\infty \frac{dt'}{(t'+t)^2} F_{\mathcal{D}}^{\mathcal{E}}(t') \tag{C13}$$

$$F_{\mathcal{D}}^{\mathcal{E}}(t) = \mathrm{Im}\mathcal{F}_{\mathcal{D}}(-t - i\varepsilon) \quad \text{where} \quad \mathcal{F}_{\mathcal{D}}(t) \equiv \frac{1}{\pi}\int_0^\infty \frac{d\tau}{(\tau + t)} F_{\mathcal{D}}^{\mathcal{E}}(\tau). \tag{C14}$$

Identity (C14) is a direct consequence of the definition of $\mathcal{F}_{\mathcal{D}}$ there. On the other hand, relation (C12) is a direct consequence of identity (C13) and of the following identity in the complex σ-plane (where $\sigma = k^2$ is square of a four-vector):

$$\int_{\mathcal{C}_1+\mathcal{C}_2} \frac{d\sigma}{\sigma}\mathcal{A}_1(K^2 = -\sigma e^{\mathcal{C}})[\mathcal{F}_{\mathcal{D}}(\sigma/Q^2) - \mathcal{F}_{\mathcal{D}}(0)] = 0, \tag{C15}$$

where function $\mathcal{F}_{\mathcal{D}}$ is defined by identity (C14), and the path $\mathcal{C}_1 + \mathcal{C}_2$ is depicted in Fig. 8. In the σ-plane, the only singularities of the integrand in Eq. (C15) are the cut of $\mathcal{A}_1(-\sigma)$ along the positive semiaxis, and the cut of $[\mathcal{F}_{\mathcal{D}}(\sigma/Q^2) - \mathcal{F}_{\mathcal{D}}(0)]$ along the negative semiaxis. Identity (C15) thus follows from the Cauchy theorem.

When applying relation (C13) to the characteristic function (C5) of BjPSR, we obtain for the timelike characteristic function of that observable

$$F_b^{\mathcal{M}}(t) = t\Big[-\frac{10}{9} - \frac{1}{3t} - \frac{2}{9t^2} - \frac{2}{9}(5t+4)\ln t + \frac{2}{9}\Big(5t + 4 + \frac{2}{t^2} + \frac{1}{t^3}\Big)\ln(1+t)\Big], \tag{C16}$$

which agrees with the corresponding expression in Ref. [58] after identifying in their Eq. (4.57): $\dot{\mathcal{F}}_3(\epsilon, N = 1) \equiv (-3/2)F_b^{\mathcal{M}}(t)$, and $\epsilon \equiv t = \mu^2/Q^2$ (Ref. [58] uses apparently $\mathcal{C} = 0$).

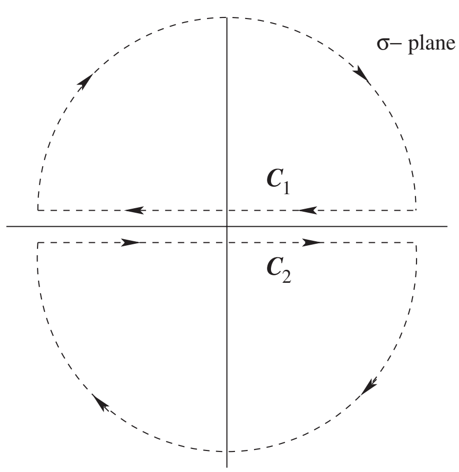

FIG. 8. The path of integration of integral (C15) in the complex σ-plane: $\sigma > 0$ semiaxis is the cut of $\mathcal{A}_1(-\sigma e^{\mathcal{C}})$ factor, and $\sigma < 0$ is the cut of the $[\mathcal{F}_{\mathcal{D}}(\sigma/Q^2) - \mathcal{F}_{\mathcal{D}}(0)]$ factor in the integral.

APPENDIX D: EXPLICIT EXPRESSIONS FOR VARIOUS COEFFICIENTS

Expansion (2) is solution of the perturbative RGE Eq. (1). If the conventional ("$\overline{\mathrm{MS}}$") reference scale $\bar{\Lambda}$ [14,15] is adopted, and RGE (1) is iteratively solved for large $Q^2/\bar{\Lambda}^2$ $[\ln(Q^2/\bar{\Lambda}^2) \gg 1]$ in an arbitrary RSch $(\beta_2, \beta_3, \ldots)$, this results in expansion (2) with coefficients $K_{k\ell}$ given in Eqs. (3) for $k \le 3$, and for $k = 4, 5, 6$ given below (notations: $c_j \equiv \beta_j/\beta_0$):

For $k = 4$:

$$\begin{aligned} K_{40} &= -\frac{1}{2}\left(\frac{c_1^3}{\beta_0^4}\right)\left(1 - \frac{c_3}{c_1^3}\right),\\ K_{41} &= -\left(\frac{c_1^3}{\beta_0^4}\right)\left(-2 + 3\frac{c_2}{c_1^2}\right),\\ K_{42} &= \frac{5}{2}\left(\frac{c_1^3}{\beta_0^4}\right),\\ K_{43} &= -\left(\frac{c_1^3}{\beta_0^4}\right). \end{aligned} \tag{D1}$$

For $k = 5$:

$$\begin{aligned} K_{50} &= \frac{1}{6\beta_0^5}(7c_1^4 - 18c_1^2c_2 + 10c_2^2 - c_1c_3 + 2c_4),\\ K_{51} &= \frac{c_1}{\beta_0^5}(4c_1^3 - 3c_1c_2 - 2c_3),\\ K_{52} &= -\frac{3}{2\beta_0^5}(c_1^4 - 4c_1^2c_2),\\ K_{53} &= -\frac{13c_1^4}{3\beta_0^5},\\ K_{54} &= \frac{c_1^4}{\beta_0^5}. \end{aligned} \tag{D2}$$

For $k = 6$:

$$\begin{aligned} K_{60} &= \frac{1}{12\beta_0^6}(17c_1^5 - 18c_1^3c_2 - c_1c_2^2 - 23c_1^2c_3 + 24c_2c_3\\ &\quad - 2c_1c_4 + 3c_5),\\ K_{61} &= \frac{1}{6\beta_0^6}(-11c_1^5 + 72c_1^3c_2 - 50c_1c_2^2 - 7c_1^2c_3 - 10c_1c_4),\\ K_{62} &= \frac{1}{2\beta_0^6}(-23c_1^5 + 27c_1^3c_2 + 10c_1^2c_3),\\ K_{63} &= \frac{1}{6\beta_0^6}(-11c_1^5 - 60c_1^3c_2),\\ K_{64} &= \frac{77c_1^5}{12\beta_0^6},\\ K_{65} &= -\frac{c_1^5}{\beta_0^6}. \end{aligned} \tag{D3}$$

In practical calculations, we use: (a) at loop level = 3: $c_3 = c_4 = c_5 = 0$ and we include in expansion (2) terms $K_{k\ell}$ up to $k_{\max} = 5$; (b) at loop level = 4: $c_4 = c_5 = 0$ and we include terms up to $k_{\max} = 6$.

The perturbation coefficients d_j ($j = 1, 2$) of the perturbation expansion for the massless Adler function $d_v(Q^2)$, cf. Eq. (40), in $\overline{\rm MS}$ RSch and at RScl $\mu^2 = Q^2$, are known exactly, Refs. [36,37], respectively,

$$\begin{aligned} d_1^{(\mathrm{Adl.})} &= \tfrac{1}{12} + 0.691\,772\beta_0,\\ d_2^{(\mathrm{Adl.})} &= -27.849 + 8.226\,12\beta_0 + 3.103\,45\beta_0^2. \end{aligned} \tag{D4}$$

The N^{3}LO coefficient d_3, in the aforementioned RSch and RScl, was obtained in an approximate form in Ref. [38] [Eqs. (20) and (12) in [38]]:

$$\begin{aligned} d_3^{(\mathrm{Adl.})} &= 46.1992 - 131.04\beta_0 + 49.5237\beta_0^2\\ &\quad + 2.180\,04\beta_0^3, \end{aligned} \tag{D5}$$

where the coefficients at β_0^3 and at β_0^2 are known exactly ([15,56,57]), and the other two coefficients were estimated in Ref. [38] by using the methods of the principle of minimal sensitivity [29], and of the effective charge [59,60].

The light-by-light contributions are not included in the coefficients (D4) and (D5). They have a different topology of diagrams and should probably be resummed separately (cf. Ref. [11]), and they appear for the first time at $\sim a^3$ [37]. They are proportional to the sum of the charges $\sum Q_f$. This sum is zero in the case $n_f = 3$ considered here.

Coefficients d_1 and d_2 for BjPSR $d_b(Q^2)$, in the aforementioned RSch and RScl, were obtained in Ref. [41] and are

$$\begin{aligned} d_1^{(\mathrm{Bj.})} &= -\tfrac{11}{12} + 2\beta_0,\\ d_2^{(\mathrm{Bj.})} &= -35.7644 + 10.5048\beta_0 + 6.388\,89\beta_0^2. \end{aligned} \tag{D6}$$

In the coefficient $d_3^{(\mathrm{Bj.})}$, only the leading-n_f part ($\propto n_f^3$) is known exactly [40] ($\Leftrightarrow$ the leading-β_0 part, $\propto \beta_0^3$). On this basis, the authors of Ref. [42] obtained estimates of $d_3^{(\mathrm{Bj.})}$ as a polynomial in β_0 by using NNA [43]: $n_f \mapsto -6\beta_0$. Several relations between BjPSR, Bjorken unpolarized sum rule, and Gross-Llewellyn Smith sum rule were found out and investigated in Ref. [61].

APPENDIX E: MASSLESS PART OF THE STRANGELESS TAU DECAY RATIO

In this Appendix we extract the measured value of the massless part of the QCD-canonical strangeless ratio $r_\tau(\triangle S = 0, m_q = 0)$ for the semihadronic decay, on the basis of the results of the final ALEPH data analysis [8,9].[11] This quantity is related to the ALEPH-measured [8,9] $(V + A)$-decay ratio

$$R_\tau(\triangle S = 0) \equiv \frac{\Gamma(\tau^- \to \nu_\tau \text{ hadrons } (\gamma))}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma))} - R_\tau(\triangle S \neq 0) \tag{E1}$$

[11] For an extraction of $r_\tau(\triangle S = 0, m_q = 0)$ based on the older set of measured results [7], see, for example, Ref. [62].

$$= \frac{(1 - B_e - B_\mu)}{B_e} - R_\tau(\triangle S \neq 0) = 3.482 \pm 0.014. \quad \text{(E2)}$$

These values were obtained in Ref. [9] from the measured leptonic branching ratio $B_e \equiv B(\tau^- \to \nu_\tau e^- \bar{\nu}_e) = (17.810 \pm 0.039)\%$ (ALEPH [8]), from $B_\mu \equiv B(\tau^- \to \nu_\tau \mu^- \bar{\nu}_\mu) = (17.332 \pm 0.049)\%$ (world average [9]), and from the strangeness-changing branching ratio $B_S = (2.85 \pm 0.11)\%$ (ALEPH [8]). The relation between the canonic massless quantity $r_\tau(\triangle S = 0, m_q = 0)$ and quantity (E1) and (E2) is

$$r_\tau(\triangle S = 0, m_q = 0) \equiv r_\tau(\triangle S = 0, m_q) - \delta r_\tau(\triangle S = 0, m_{u,d} \neq 0) \quad \text{(E3)}$$

$$= \frac{R_\tau(\triangle S = 0)}{3|V_{ud}|^2(1 + \delta_{\rm EW})} - (1 + \delta'_{\rm EW}) - \delta r_\tau(\triangle S = 0, m_{u,d} \neq 0). \quad \text{(E4)}$$

Here, $r_\tau(\triangle S = 0, m_q = 0)$ is QCD canonical, i.e., its pQCD expansion is $r_\tau(\triangle S = 0, m_q = 0)_{\rm pt} = a + \mathcal{O}(a^2)$; the Cabibbo-Kobayashi-Maskawa matrix element $|V_{ud}|$ has the value largely dominated by $0^+ \to 0^+$ nuclear beta decays [63]

$$|V_{ud}| = 0.9738 \pm 0.0003, \quad \text{(E5)}$$

the electroweak (EW) correction parameter is $1 + \delta_{\rm EW} = 1.0198 \pm 0.0006$ [8,9]; the residual EW correction parameter is $\delta'_{\rm EW} = 0.0010$ [64]; the $(V + A)$-channel corrections $\delta r_\tau(\triangle S = 0, m_{u,d} \neq 0)$ due to the nonzero quark masses are [9,65] the sum of the $D = 2$-, 4-, 6-, and 8-dimensional corrections $(\delta^{(D)}_{ud,V} + \delta^{(D)}_{ud,A})/2$ and their value is [9] either $\delta r_\tau(\triangle S = 0, m_{u,d} \neq 0) = (-5.2 \pm 1.7) \times 10^{-3}$ if the gluon condensate contribution is included, and $(-5.0 \pm 1.7) \times 10^{-3}$ if the gluon condensate contribution is not included (using for the gluon condensate the ALEPH value $\langle aGG \rangle = (-0.5 \pm 0.3) \times 10^{-2}$ [8,9]).

Inserting all the aforementioned values in relation (E4) and taking into account the value (E2), we extract the experimental prediction for $r_\tau(\triangle S = 0, m_q = 0)$ based on the most recent ALEPH data

$$r_\tau(\triangle S = 0, m_q = 0)_{\rm exp} = 0.204 \pm 0.005, \quad \text{(E6)}$$

where the uncertainties have been added in quadrature. The uncertainty in Eq. (E6) is dominated by the experimental uncertainty $\delta R_\tau = \pm 0.014$ (E2). The value (E6) remains unaffected up to the displayed digits when we either include or exclude from the above quantity the gluon condensate contribution.

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