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Chiral Lagrangian at finite temperature from the Polyakov-chiral quark model

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We analyze the consequences of the inclusion of the gluonic Polyakov loop in chiral quark models at low temperature in the light of chiral perturbation theory. Specifically, the low-energy effective chiral Lagrangian from two such quark models is computed. The tree level vacuum energy density, quark condensate, pion decay constant, and Gasser-Leutwyler coefficients are found to acquire a temperature dependence. This dependence is, however, exponentially small for temperatures below the mass gap in the full unquenched calculation. The introduction of the Polyakov loop and its quantum fluctuations is essential to achieve this result and also the correct large N_c counting for the thermal corrections. We find that new coefficients are introduced at $\mathcal{O}(p^4)$ to account for the Lorentz breaking at finite temperature. As a byproduct, we obtain the effective Lagrangian which describes the coupling of the Polyakov loop to the Goldstone bosons.

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I. INTRODUCTION

At zero temperature, confinement and spontaneous breaking of chiral symmetry emerge as distinct features of QCD. This explains the absence of free quarks and gluons as well as the observed mass gap in the hadron spectrum between the pseudoscalar mesons and the rest of particles and resonances. At a given critical temperature, lattice simulations predict a deconfinement phase transition where chiral symmetry is simultaneously restored (for reviews and recent results see e.g. Ref. [1–4] and references therein). This remarkable coincidence between the deconfinement and chiral phase transitions remains so far unexplained from the theoretical side. Nevertheless, there are two extreme limits where each of the phase transitions can be characterized by an order parameter. For extremely light quarks, the quark condensate is used as an order parameter for the chiral phase transition where one goes from a nonvanishing to a vanishing value across the phase transition. In the opposite limit of infinitely heavy quarks, the deconfinement phase transition can be characterized by a breaking of the center symmetry of the gauge group and the order parameter is the Polyakov loop which evolves from a vanishing value to unity above the critical temperature. The real situation for light quarks is in between but one still observes sudden changes both in the chiral condensate as well as in the Polyakov loop. In the present paper we want to address in a quantitative manner a rather remarkable feature that arises at low temperatures in quark models when the chiral flavor symmetry and color gauge center symmetry are jointly considered.

Besides the existence of a mass gap, a further outstanding consequence of spontaneous chiral symmetry breaking is that the would-be Goldstone bosons interact weakly at low energies and effective field theory methods such as chiral perturbation theory (ChPT) [5,6] (for a review see e.g. Ref. [7] and references therein), can successfully be applied in terms of unknown low-energy constants (LEC's) which cannot be explained on the basis of the symmetry alone. This implies neglecting the explicit effects of states about the mass gap which in the case of two flavors might be identified with the scalar or vector meson masses, and limits the maximum energy at which standard ChPT may confidently be applied. Actually, the bulk of the values of the LEC's can be saturated by the low-energy contribution stemming from the exchange of resonances located in the mass gap region. At finite, but low, temperatures the physics of QCD is believed to consist of a gas of heated hadrons and one still expects the dominant role to be played by the pseudoscalar mesonic thermal excitations [8–10]; effects of resonances are exponentially suppressed by a Boltzmann factor in the mass gap $e^{-m_\rho/T}$. Obviously, the applicability of such an approach requires that there still be a mass gap and that confinement still holds. This entitles, in particular, to consider the LEC's of the chiral Lagrangian as temperature independent couplings and finite (low) temperature model independent predictions are deduced [11–17]. From this point of view finite temperature ChPT provides a strong theoretical constraint on the QCD physics well below the phase transition. Extrapolations based on ChPT suggest a melting of the condensate. However, it is unclear whether this vanishing meets the very requirement of a phase transition regarding the quark condensate. Moreover, such a purely hadronic based description can never account, by construction, for the deconfinement phase transition expected from lattice simulations [1].

As already mentioned, the deconfinement phase transition can be characterized for infinitely heavy quarks by a breaking of the center symmetry of the color gauge group. The interplay between this center symmetry and chiral symmetry requires explicit consideration of quark degrees of freedom and can quantitatively be assessed in chiral

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quark models where the spontaneous chiral symmetry breaking is implemented (see e.g. Ref. [18] for a review and references therein) and the mass gap can be identified as twice the constituent quark mass M. Chiral quark model Lagrangians are invariant under the flavor chiral group $SU_R(N_f) \otimes SU_L(N_f)$ and have often been used to provide some semiquantitative understanding of hadronic features in the low-energy domain. In addition, at zero and finite temperature the standard chiral quark models are invariant under *global* $SU(N_c)$ transformations. At some critical temperature chiral quark models predict already at the one loop level a chiral phase transition [19,20] at realistic temperatures. Pion corrections where first considered in Ref. [21] (for a review see e.g. Ref. [22]). This has been traditionally considered a big phenomenological success for these models, but they suffer from the unphysical contribution of states which are not color singlets, so that even at lowest temperatures the hot environment corresponds to a plasma of multiquarks, characterized by Boltzmann factors $\sim e^{-nM/T}$ and not color neutral hadronic states [23]. This unphysical feature can be avoided by noting the relevance of large and local $SU(N_c)$ gauge invariance at finite temperature. Basically, this corresponds to include nonperturbative finite temperature gluons. Based on previous works [24–26] (see also Refs. [27–30]) we have discussed at length [23] how large gauge invariance can be efficiently implemented by the coupling of the Polyakov loop to the standard chiral quark models hence providing a cooling mechanism to the chiral-deconfinement phase transition in such a way that the melting of the chiral condensate is shifted to higher temperature values although with large uncertainties. An increase in the critical temperature is in fact required by recent lattice results [4] where the quoted value is $T_c = 192(7)(4)$ MeV for $2 + 1$ flavors. Without Polyakov loop coupling the classical chiral quark model value was about 150 MeV, and in [23] this value becomes 250 ± 50 MeV (for two light flavors). The overshooting of the central value is presumably due to a lack of feedback of the quarks on the gluonic action, which would eventually make less efficient the Polyakov cooling mechanism. Although Polyakov-chiral quark models might be improved along these lines by suitable refinement of the gluonic action, given the estimated uncertainties, it is not obvious whether the discrepancy is significant, and in any case the region of low temperatures would hardly be affected. This is due to the dominance of the group integration measure, as already discussed in great detail in Ref. [23] (see also below).

The coupling of QCD distinctive order parameters at finite temperature to hadronic properties has been the subject of much attention over the recent past [24,26,31–34]. Effective actions for the Polyakov loop as a confinement-deconfinement order parameter have been proposed because of their relevance in describing the phase transition from above the critical temperature [35–38]. These works focus naturally on the phase transition, but do not investigate the relation to well established low temperature or large N_c constraints within ChPT at finite temperature. Actually there is a challenge of simultaneously accounting for a phase transition and complying to ChPT at low temperatures.

The Polyakov-chiral quark models, unlike ChPT, are known to predict a rapid change of the mass gap as well as a sudden rise of the Polyakov loop expectation value at about the same temperature [33], complying on a semi-quantitative level to lattice QCD simulations [39,40]. Given this successful link between (flavor) chiral and (color) center symmetries, it is intriguing how the Polyakov loop couples to the lightest Goldstone bosons at low energies, and what is the net effect at low temperatures in the effective chiral Lagrangian. In the present paper we analyze these issues. Amazingly, the inclusion of Polyakov loops, a color source, at a *quantum level* [23][1] reproduces the expectations of low temperature ChPT based on the existence of a mass gap, namely, the tree level coefficients of the chiral Lagrangian are exponentially suppressed in the mass gap. This supports the ChPT-expected dominance of pion fluctuations at low temperatures, and as we will show, the large N_c behavior is the correct one. Thus Polyakov-chiral quark models not only provide a physical picture, but can also be regarded as an interpolating description between ChPT on the low temperature-low-energy side and lattice data around the critical temperature, when the quantum and local nature of the Polyakov loop is taken into account.

The paper is organized as follows. In Sec. II we describe shortly the Polyakov-chiral quark models, and more specifically the simplest version of them. In Sec. III we set up the calculational framework of the low-energy chiral Lagrangian at finite temperature and display already the general structure of the main result. In Sec. IV we use the technique of heat kernel expansion at finite temperature with Polyakov loops and the associated large gauge invariance is preserved. This enables the calculation of the effective Lagrangian in terms of the Polyakov loop and pseudoscalar mesons as basic and independent degrees of freedom. In Sec. V we go further and integrate over the Polyakov loop variable in a gauge invariant manner, providing the form of the effective (tree level) Lagrangian of ChPT at finite temperature. Finally, in Sec. VI we summarize our conclusions.

II. POLYAKOV-CHIRAL QUARK MODELS

In this section we review the coupling of the Polyakov loop to chiral quark models as a quantum and independent

[1] All other existing implementations [24–30] of the Polyakov-chiral quark models are carried out at the mean field level. See discussion in Ref. [23] on the advantages and shortcomings of such an approximation.

variable as suggested in previous work [23–26]. We follow Ref. [23] for the basic ingredients involved in the construction of such a model. Since the coupling to quarks is rather universal we will restrict ourselves to the simplest chiral model, namely, a constituent quark model (CQ). The corresponding Lagrangian reads[2]

$$\mathcal{L}_{\rm CQ} = \bar{q}(i\not{\partial} + \not{v}^f + \not{a}^f\gamma_5 - MU^{\gamma_5} - \hat{M}_0)q =: \bar{q}i\mathbf{D}q, \tag{2.1}$$

where $q = (u, d, s, \ldots)$ represents a quark spinor with N_c colors and N_f flavors. $\hat{M}_0 = \mathrm{diag}(m_u, m_d, m_s, \ldots)$ stands for the current quark mass matrix. The symbols (v_μ^f, a_μ^f) denote external vector and axial-vector fields, in flavor space. M is the constituent quark mass and $U = e^{i\sqrt{2}\Phi/f_\pi}$ (f_π being the pion weak decay constant in the chiral limit) is the flavor matrix representing the pseudoscalar octet of mesons in the nonlinear representation:

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \tag{2.2}$$

For vanishing current quark masses $\mathcal{L}_{\rm CQ}$ is invariant under local $\mathrm{U}(N_f)_R \otimes \mathrm{U}(N_f)_L$ transformations. In addition there is a global $\mathrm{SU}(N_c)$ symmetry.

After formally integrating out the quarks one gets the effective action

$$\Gamma_{\rm CQ} = -i\,\mathrm{Tr}\log(i\mathbf{D}) = \int d^4x \mathcal{L}_\Omega(x), \tag{2.3}$$

where the corresponding Lagrangian, $\mathcal{L}_\Omega(x)$, has also been introduced. We use Tr for the full functional trace, tr_f stands for the trace in flavor space, and tr_c for the trace in color space. The ultraviolate divergences introduced by the functional determinant can be conveniently handled by the Pauli-Villars method. Let us note that the issue of regularization is not so crucial in the present case, $T \ll \Lambda$ [20], since the divergences affect only the zero temperature contributions [41,42].

The chiral quark model coupled to the Polyakov loop corresponds to introducing a non trivial color component of the vector field in the temporal direction

$$v_\mu^f \to v_\mu^f + v_\mu^c, \qquad v_\mu^c = \delta_{\mu 0} v_0^c \tag{2.4}$$

in the Dirac operator, Eq. (2.1). The field $v_0^c(x)$ acts as a chemical potential in color space [26]. Upon Wick rotation to pass to imaginary time (which we use to introduce finite temperature) this field gives rise to the Polyakov loop

$$\Omega(\vec{x}, x_4) = \mathcal{T} \exp\left(i \int_{x_4}^{x_4+\beta} dx_4' v_4^c(\vec{x}, x_4') \right), \tag{2.5}$$

where $\mathcal{T}$ indicates the Euclidean temporal ordering and $\beta = 1/T$. The Polyakov loop plays an important role as an order parameter for the deconfinement transition. In the present context it appears quite naturally, since as shown in Refs. [43,44] a generic effective action with gauge fields at finite temperature will depend not only on the standard zero temperature local gauge covariant operators $\mathcal{O}_n(x)$ constructed with the covariant derivative, but also on the Polyakov loop $\Omega(x)$ as a new finite temperature gauge covariant operator. Specifically, at the one loop level the effective Lagrangian takes the generic form [44]

$$\mathcal{L}(x) = \sum_n \mathrm{tr}[f_n(\Omega)\mathcal{O}_n]. \tag{2.6}$$

In the Polyakov-chiral quark model the partition function takes the form

$$Z = \int DUD\Omega e^{i\Gamma_G[\Omega]} e^{i\Gamma_{\rm CQ}[U,\Omega]}, \tag{2.7}$$

where DU is the Haar measure over the flavor group $\mathrm{SU}(N_f)$ (actually, the product of local Haar measures) and $D\Omega$ is essentially the Haar measure of the color group $\mathrm{SU}(N_c)$ (see however Sec. V), Γ_G is the effective gluon action whereas $\Gamma_{\rm CQ}$ stands for the quark effective action in Eq. (2.3). The model is motivated by the underlying idea that all zero temperature gluon degrees of freedom have been integrated out to yield the constituent quark mass, leaving unintegrated the Polyakov loop as the only specifically finite temperature gluonic degree of freedom. Ideally $\Gamma_G[\Omega]$ would be the result of such a partial integration in pure gluodynamics. The Polyakov loop integration implements the gauge invariant integration over the v_0^c field [45]. Since gluons are vector fields, no color axial-vector component is introduced in Eq. (2.4).

As discussed in [23] the integration over the Polyakov loop coupled to quarks suppresses unwanted colored quark states and mimics the confinement mechanism within the quark model. Note that the Polyakov loop measure $D\Omega e^{i\Gamma_G[\Omega]}$ preserves center symmetry, i.e., invariance under 't Hooft transformations [46] under which $\Omega \to z\Omega$, with $z^{N_c} = 1$. This symmetry is preserved by hadrons but not by quarks. In the Polyakov-chiral quark model picture, as a valence quark propagates at finite temperature, one can at any place insert a path starting and ending at the same point x which winds a number n of times around the thermal cylinder, picking up a factor $\langle(-\Omega)^n\rangle$ that goes with the propagator factor $e^{-|n|M/T}$ (modulo subleading polynomial corrections). In the presence of sea quarks there are further thermal windings, giving roughly (being L the number of sea quark loops)

$$\langle(-\Omega)^{n+n_1+\cdots+n_L}\rangle e^{-(|n|+|n_1|+\cdots+|n_L|)M/T}. \tag{2.8}$$

The average over the gauge group is that for a pure Yang-Mills theory and so preserves triality in the confined phase that we are considering, i.e., $n + n_1 + \cdots + n_L$ must be a

[2]In Minkowski space we will use Bjorken-Drell conventions throughout the paper.

multiple of N_c. For $N_c \geq 3$, the leading thermal corrections comes from contributions of the form $n = -n_1 = 1$, $n_2 = \ldots = n_L = 0$. This describes a thermal $q\bar{q}$ pair, with a propagator factor $e^{-2M/T}$. Taking into account quark interactions the leading thermal corrections will be of the type $e^{-m_\pi/T}$, being the pion the lightest meson.

A remaining technical point deserves comment. As noted above the presence of the Polyakov loop in the effective action is an inescapable consequence of having a gauge theory at finite temperature. So in addition to the standard color Polyakov loop, there will be also Polyakov loops associated to the vector and axial-vector flavor external fields. Such *chiral* Polyakov loops take the form

$$\Omega_{R,L}(x) = \mathcal{T} \exp\Big(i \int_{x_4}^{x_4+\beta} dx_4'(v_4^f(\vec{x}, x_4') \pm a_4^f(\vec{x}, x_4'))\Big), \tag{2.9}$$

and they appear automatically in any gauge invariant one loop computation at finite temperature. Such operators (as well as the *vector* flavor Polyakov loop) would introduce interesting new effects such as *center symmetry in flavor space*, preserved by mesons but not necessarily by baryons, depending on the number of flavors. Since spin 1 mesons are relatively heavy we do not expect the flavor Polyakov loops to be essential in the low temperature regime to be studied in this work, so it will be disregarded in what follows. Nevertheless they could play a more active role near the transition temperature.

III. THE STRUCTURE OF THE LOW-ENERGY CHIRAL LAGRANGIAN AT FINITE TEMPERATURE

As already shown in previous works at zero temperature [47–50] (see e.g. [51] for an updated list of references) chiral quark models may provide a quantitative and microscopic understanding of the structure of the low-energy effective Lagrangian of ChPT for the pseudoscalar mesons at *tree level*, namely, providing numerical values for the leading N_c contributions to the low-energy constants (LEC's). In such a framework, external currents are minimally coupled at the level of the more microscopic quark degrees of freedom. On top of this, meson loops would provide subleading $1/N_c$ contributions to the LEC's in addition to the standard unitarity corrections of ChPT [52].

In this section we extend the zero temperature results to the finite temperature case and consider also the influence of the Polyakov loop in the scheme described in the previous section. More explicitly, the partition function in Eq. (2.7) can be written as

$$Z = \int DUD\Omega e^{i\Gamma_G[\Omega]} e^{i\int d^4x \mathcal{L}_\Omega(x)} \tag{3.1}$$

and then

$$Z = \int DU e^{i\int d^4x \mathcal{L}^*(x)}. \tag{3.2}$$

The Lagrangian $\mathcal{L}_\Omega(x)$ is obtained after integration of the quarks (see Eq. (2.3)) and depends on the fields $U(x)$ and $\Omega(x)$. The traditional CQ model can be recovered from the full Lagrangian $\mathcal{L}_\Omega(x)$ by setting $\Omega = 1$ and leaving out the $\Omega(x)$ integration. In such a case one would obtain the ChPT Lagrangian at finite temperature corresponding to the CQ model without coupling to the Polyakov loop.[3] If the Polyakov loop is retained, as done in this work, $\mathcal{L}_\Omega(x)$ provides also the coupling of Goldstone mesons to the Polyakov loop field, and interesting subject by itself.[4]

The computation of $\mathcal{L}_\Omega(x)$ can be carried out by following the methods developed in Ref. [43,44] and already applied to QCD in Ref. [53]. The procedure is detailed in the next section. It is worth noticing that $\mathcal{L}_\Omega(x)$, either directly, after setting $\Omega = 1$, or averaged over the Polyakov loop, represents a one quark-loop result and hence a *quenched* approximation to the chiral Lagrangian.

Integration over the Polyakov loop field in Eq. (3.1) gives the (unquenched) finite temperature ChPT Lagrangian of the model, denoted $\mathcal{L}^*(x)$, which is the main purpose of this work. In ChPT at finite temperature it is usually assumed that the corresponding low-energy constants are *temperature independent* [8] (see also Refs. [16,54]). This is a quite natural assumption because the applicability of ChPT is based on the existence of a mass gap between the Goldstone bosons and the rest of the hadronic spectrum. For nonstrange mesons the gap is provided by the ρ meson mass M_V, so one expects the temperature dependence of the LEC's to be of the order of $e^{-M_V/T}$. In a chiral quark model, however, the pseudoscalar mesons are composite particles of constituent quarks of a mass M, and the finite temperature also influences their microscopic quark substructure. As a consequence the LEC's become temperature dependent of the order $e^{-2M/T}$. Our calculation below makes these remarks quantitative and also provides an understanding on how the Polyakov loop cooling mechanism works in favor of the ChPT expectations at finite temperature.

Although the calculation is straightforward, reducing effectively to evaluation of Dirac and flavor traces, it is technically involved so we discuss here the structure of the final result for the general case. More details will be elaborated in the next sections.

The low-energy effective Lagrangian written in the Gasser-Leutwyler [5] notation and in Minkowski space reads as follows (we often use an asterisk as upperscript

[3] This is obtained as a byproduct of our calculation. For such a calculation at zero temperature see Refs. [47–51].

[4] It is not clear, however, how such an effective Lagrangian could be formulated in a model independent way since in QCD the Polyakov loop is renormalized and in general it will lie outside the SU(N_c) manifold.

for finite temperature quantities),

$$\mathcal{L}^{*(0)}(x) = B^*, \tag{3.3}$$

$$\mathcal{L}^{*(2)}(x) = \frac{f_\pi^{*2}}{4} \operatorname{tr}_f(\mathbf{D}_\mu U^\dagger \mathbf{D}^\mu U + \chi^\dagger U + \chi U^\dagger), \tag{3.4}$$

$$\begin{aligned}\mathcal{L}^{*(4)}(x) = {} & L_1^*(\operatorname{tr}_f(\mathbf{D}_\mu U^\dagger \mathbf{D}^\mu U))^2 + L_2^*(\operatorname{tr}_f(\mathbf{D}_\mu U^\dagger \mathbf{D}_\nu U))^2 + L_3^* \operatorname{tr}_f(\mathbf{D}_\mu U^\dagger \mathbf{D}^\mu U \mathbf{D}_\nu U^\dagger \mathbf{D}^\nu U) \\ & + \bar{L}_3^* \operatorname{tr}_f(\mathbf{D}_0 U^\dagger \mathbf{D}_0 U \mathbf{D}_\mu U^\dagger \mathbf{D}^\mu U) + L_4^* \operatorname{tr}_f(\mathbf{D}_\mu U^\dagger \mathbf{D}^\mu U) \operatorname{tr}_f(\chi^\dagger U + \chi U^\dagger) + L_5^* \operatorname{tr}_f(\mathbf{D}_\mu U^\dagger \mathbf{D}^\mu U(\chi^\dagger U + U^\dagger \chi)) \\ & + \bar{L}_5^* \operatorname{tr}_f(\mathbf{D}_0 U^\dagger \mathbf{D}_0 U(\chi^\dagger U + U^\dagger \chi)) + \bar{L}_5^{\prime *} \operatorname{tr}_f(\chi \mathbf{D}_0 \mathbf{D}_0 U^\dagger + \chi^\dagger \mathbf{D}_0 \mathbf{D}_0 U) + L_6^*(\operatorname{tr}_f(\chi^\dagger U + \chi U^\dagger))^2 \\ & + L_7^*(\operatorname{tr}_f(\chi^\dagger U - \chi U^\dagger))^2 + \bar{L}^{\prime *} \operatorname{tr}_f(U^\dagger \mathbf{D}_0 \mathbf{D}_0 U - U \mathbf{D}_0 \mathbf{D}_0 U^\dagger) \operatorname{tr}_f(\chi^\dagger U - \chi U^\dagger) \\ & + L_8^* \operatorname{tr}_f(\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger) - i L_9^* \operatorname{tr}_f(F_{\mu\nu}^R \mathbf{D}^\mu U^\dagger \mathbf{D}^\nu U + F_{\mu\nu}^L \mathbf{D}^\mu U \mathbf{D}^\nu U^\dagger) \\ & - i \bar{L}_9^* \operatorname{tr}_f(E_i^R(\mathbf{D}_0 U^\dagger \mathbf{D}^i U - \mathbf{D}^i U^\dagger \mathbf{D}_0 U) + E_i^L(\mathbf{D}_0 U \mathbf{D}^i U^\dagger - \mathbf{D}^i U \mathbf{D}_0 U^\dagger)) \\ & - i \bar{L}_9^{\prime *} \operatorname{tr}_f(\mathbf{D}_0 E_i^R U^\dagger \mathbf{D}^i U + \mathbf{D}_0 E_i^L U \mathbf{D}^i U^\dagger) + L_{10}^* \operatorname{tr}_f(U^\dagger F_{\mu\nu}^L U F^{\mu\nu R}) + H_1^* \operatorname{tr}_f((F_{\mu\nu}^R)^2 + (F_{\mu\nu}^L)^2) \\ & + \bar{H}_1^* \operatorname{tr}_f((E_i^R)^2 + (E_i^L)^2) + H_2^* \operatorname{tr}_f(\chi^\dagger \chi). \end{aligned} \tag{3.5}$$

Here, tr_f means flavor trace and we have introduced the standard chiral covariant derivatives and gauge field strength tensors,

$$\begin{aligned} \mathbf{D}_\mu U &= D_\mu^L U - U D_\mu^R = \partial_\mu U - i V_\mu^L U + i U V_\mu^R, \\ F_{\mu\nu}^r &= i[D_\mu^r, D_\nu^r] = \partial_\mu V_\nu^r - \partial_\nu V_\mu^r - i[V_\mu^r, V_\nu^r], \end{aligned} \tag{3.6}$$

with $r = L, R$, and $V_\mu^{R,L} = v_\mu^f \pm a_\mu^f$. Finally, as at zero temperature, we have normalized the explicit chiral breaking field $\chi = 2B_0^* \hat{M}_0$ so that $\mathcal{L}^{*(2)}$ takes a standard form. This introduces a LEC $B_0^* = |\langle \bar{q} q \rangle^*| / f_\pi^{*2}$, where $\langle \bar{q} q \rangle^*$ is the quark condensate for one flavor at finite temperature.

The general form of the chiral low-energy effective Lagrangian, Eq. (3.5), requires some remarks. As we see there are terms which can be written as the zero temperature Lagrangian but with temperature dependent effective couplings. In addition, we also have new terms, which due to the finite temperature generated by a heat bath at rest, necessarily break Lorentz invariance. This feature was already pointed out in [44]. The remarkable, not yet understood, feature is that there appear less terms breaking the Lorentz symmetry than one might naively suggest (for instance a term of the type $\operatorname{tr}_f(\mathbf{D}_0 U^\dagger \mathbf{D}_0 U)$ is missing). Thus, some accidental symmetry may be at work and it would be interesting to find it explicitly.

Although the form of the Lagrangian is quite general, the particular values of the low-energy coefficients obviously depend on the specific model. We will consider two particular chiral quark models: the constituent quark model and the spectral quark model [55]. The results for the latter model will be presented in Appendix B, and we concentrate on the CQ model in the following sections.

IV. HEAT KERNEL EXPANSION AT FINITE TEMPERATURE IN THE PRESENCE OF POLYAKOV LOOPS

The calculation of the effective chiral Lagrangian of Eqs. (3.3)–(3.5) can be divided into several steps. First, we construct a Klein-Gordon operator out of the Dirac operator and its adjoint for the nonanomalous part of the effective action. After a generalized proper-time representation we are naturally lead to a heat kernel of the Klein-Gordon operator, for which a heat kernel expansion is particularly suited. This identification allows to directly apply the results of Refs. [44,53] and to work out the traces after applying some relevant matrix identities which reduce the number of independent operators. Finally, the equations of motion are also considered to take into account that the pion fields are on the mass shell. In the present section we assume a fixed value of the Polyakov loop. This provides $\mathcal{L}_\Omega(x)$ of (3.1). In Sec. V the Polyakov loop integration will be addressed, to yield the unquenched chiral Lagrangian $\mathcal{L}^*(x)$ of Eq. (3.2).

In this section and the next one we revert to Euclidean space, $x_4 = i x_0$, since it is much more convenient for calculations at finite temperature in the imaginary time formalism. The partition function is numerically unchanged, but takes the form

$$Z = \int D U e^{-\int d^4 x \mathcal{L}^*(x)} \quad \text{(Euclidean)}. \tag{4.1}$$

Likewise, the low-energy coefficients are also unchanged. See e.g.,

$$\mathcal{L}^{*(0)}(x) = -B^* \quad \text{(Euclidean)},$$

$$\mathcal{L}^{*(2)}(x) = \frac{f_\pi^{*2}}{4} \operatorname{tr}_f(\mathbf{D}_\mu U^\dagger \mathbf{D}_\mu U - \chi^\dagger U - \chi U^\dagger),$$

with the same values of B^*, f_π^*, etc.

A. The effective Klein-Gordon operator

The Dirac operator appearing in the fermion determinant behaves covariantly under chiral transformations. This implies that, in principle, one has to consider both vector and axial-vector couplings. A great deal of simplification is achieved if the conventions of Refs. [56,57] are considered, where it is shown that it suffices to carry out the calculation in the simpler case of a vectorlike coupling and then reconstruct the total chiral invariant result in a suitable way.

Let us hence consider the following Dirac operator with a vectorlike coupling,[5]

$$\mathbf{D} = \not{D} + h, \qquad h = m + z, \tag{4.3}$$

where h includes the pion field m, which we take $\mathcal{O}(p^0)$ in the chiral counting, and the chiral symmetry breaking mass term z which we take $\mathcal{O}(p^2)$. We also introduce the following useful notation

$$m_{LR} = MU, \qquad m_{RL} = MU^\dagger, \tag{4.4}$$

$$z_{LR} = \frac{1}{2B_0^*}\chi, \qquad z_{RL} = \frac{1}{2B_0^*}\chi^\dagger, \tag{4.5}$$

where χ has been introduced after Eq. (3.6). (In the notation of Refs. [56,57], the symbol m is to be interpreted as m_{LR} or m_{RL} depending on its position in the formula, and similarly for the other vectorlike symbols.) It is convenient to introduce the adjoint Dirac operator

$$\mathbf{D}^\dagger = -\not{D} + h. \tag{4.6}$$

This definition allows us to separate the effective action into γ_5-even and γ_5-odd components, corresponding to normal and abnormal parity processes, since they correspond to the real and imaginary parts of the Euclidean effective action, respectively, (see e.g. Ref. [58]).

We will focus on the normal parity component of the effective action, which from Eq. (2.3) is formally given by

$$\Gamma^+_{\rm CQ} = -\frac{1}{2}\,\mathrm{Tr}\log(\mathbf{D}^\dagger\mathbf{D}) =: \int_0^\beta dx_4 \int d^3\mathbf{x}\,\mathcal{L}_\Omega(x), \tag{4.7}$$

with the relevant Klein-Gordon operator

$$\mathbf{D}^\dagger\mathbf{D} = -D_\mu^2 + \frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu} - \gamma_\mu\mathbf{D}_\mu h + z^2 + \{m,z\} + M^2. \tag{4.8}$$

We use the notation $\mathbf{D}_\mu h = [D_\mu, h]$. The field strength tensor is defined as $F_{\mu\nu} = i[D_\mu, D_\nu]$. The operator $\mathbf{D}^\dagger\mathbf{D}$ is of the Klein-Gordon type, with mass term

$$\frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu} - \gamma_\mu\mathbf{D}_\mu h + z^2 + \{m,z\} + M^2. \tag{4.9}$$

[5]Dirac gammas in Euclidean space are taken Hermitian, and $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu}$.

Therefore a heat kernel expansion becomes appropriate. After a (generalized) proper-time regularization, the effective Lagrangian in Euclidean space becomes

$$\begin{aligned}\mathcal{L}_\Omega(x) &= \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\phi(\tau)\,\mathrm{tr}\langle x|e^{-\tau\mathbf{D}^\dagger\mathbf{D}}|x\rangle \\ &= \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\phi(\tau)\frac{e^{-\tau M^2}}{(4\pi\tau)^2}\sum_n \tau^n\,\mathrm{tr}(b_n^*(x)).\end{aligned} \tag{4.10}$$

This proper-time representation has been chosen to accommodate the Pauli-Villars regularization as used in e.g. Ref. [48],

$$\phi(\tau) = \sum_i c_i e^{-\tau\Lambda_i^2}, \qquad c_0 = 1, \qquad \Lambda_0 = 0, \tag{4.11}$$

but still using full advantage of the heat kernel. The conditions $\sum_i c_i\Lambda_i^n = 0$, for $n = 0, 2, 4$, allow to render finite the logarithmic, quadratic and quartic divergencies, respectively. To $\mathcal{O}(p^4)$ terms we obtain the following contributions for the thermal Seeley-DeWitt coefficients [44,53] after the Dirac trace has been implemented,

$$\begin{aligned}
b_0^* &= 4\varphi_0(\Omega), \\
b_{1/2}^* &= 0, \\
b_1^* &= -4\varphi_0(\Omega)(\{m,z\} + z^2), \\
b_{3/2}^* &= 0, \\
b_2^* &= 2\varphi_0(\Omega)((m_\mu)^2 + \{m_\mu, z_\mu\} + \{m,z\}^2 + \tfrac{1}{3}F_{\mu\nu}^2) \\
&\quad + \tfrac{2}{3}\bar\varphi_2(\Omega)E_i^2 + \mathcal{O}(p^6), \\
b_{5/2}^* &= \mathcal{O}(p^5), \\
b_3^* &= -\tfrac{2}{3}\varphi_0(\Omega)(m_\mu\{m_\mu,\{m,z\}\} + \{m,z\}(m_\mu)^2 \\
&\quad - i\{F_{\mu\nu}, m_\mu m_\nu\} + i m_\mu F_{\mu\nu} m_\nu + \tfrac{1}{2}(m_{\mu\nu})^2) \\
&\quad + \tfrac{1}{3}\bar\varphi_2(\Omega)(m_{4\mu})^2 + \mathcal{O}(p^5), \\
b_{7/2}^* &= \mathcal{O}(p^5), \\
b_4^* &= \tfrac{1}{6}\varphi_0(\Omega)((m_\mu)^4 + m_\mu(m_\nu)^2 m_\mu - (m_\mu m_\nu)^2) \\
&\quad + \mathcal{O}(p^5).
\end{aligned} \tag{4.12}$$

In these formulas $E_i = F_{4i}$ is the "electric field" and a notation of the type $X_{\mu\nu\alpha}$ has been used to mean $\mathbf{D}_\mu\mathbf{D}_\nu\mathbf{D}_\alpha X = [D_\mu,[D_\nu,[D_\alpha, X]]]$, e.g. $m_{4\mu} = \mathbf{D}_4\mathbf{D}_\mu m$, $F_{\alpha\mu\nu} = \mathbf{D}_\alpha F_{\mu\nu}$. The functions φ_0 and $\bar\varphi_2$ are defined in Appendix A.

Note that, in principle, since D_μ contains both flavor and color gauge fields, the Polyakov loop and the field strength tensor $F_{\mu\nu}$ (and derivatives) in the heat kernel coefficients will also contain flavor and color contributions. As explained before in the Polyakov loop we retain only its color part. In addition, in the strength tensor and derivatives we keep only the flavor part. This neglects some gluonic corrections. This kind of QCD corrections have been considered for instance in Ref. [49] for the Nambu-Jona-

Lasinio (NJL) model. We disregard them in the present treatment since they are not specific of finite temperature.

B. Effective Lagrangian

The effective Lagrangian can be written as

$$\mathcal{L}_\Omega(x) = \mathcal{L}_\Omega^{(0)}(x) + \mathcal{L}_\Omega^{(2)}(x) + \mathcal{L}_\Omega^{(4)}(x) + \cdots, \qquad (4.13)$$

where $\mathcal{L}_\Omega^{(n)}$ is of $\mathcal{O}(p^n)$. Making use of the expression (4.10), the flavor trace of the Seeley-DeWitt coefficients obtained in Eq. (A10) and after evaluating the proper-time integrals using the Pauli-Villars regularization (see Appendix A), we get for the Lagrangian $\mathcal{L}_\Omega(x)$ an expression formally identical to (the Euclidean version of) that in (3.3)–(3.5), except that the coefficients depend on the Polyakov loop, that is, $B^*(\Omega)$ instead of B^*, etc.

At zeroth order one finds

$$B^*(\Omega) = -\frac{2N_f M^4}{(4\pi)^2}\,\mathrm{tr}_c\,\mathcal{I}_{-4}(\Omega). \qquad (4.14)$$

The second order gives

$$f_\pi^{*2}(\Omega) = \frac{M^2}{4\pi^2}\,\mathrm{tr}_c\,\mathcal{I}_0(\Omega). \qquad (4.15)$$

Note that we have not yet averaged over the Polyakov loop and the (Polyakov loop independent) normalization constant B_0^* (needed to fix the definition of χ) is determined after this average, as we will explain in the next section. However the combination $\chi/2B_0^* = \hat{M}_0$ is unambiguous. It contributes to $\mathcal{L}_\Omega^{(2)}$ (Euclidean) with

$$-\frac{M^3}{(4\pi)^2 B_0^*}\,\mathrm{tr}_c\,\mathcal{I}_{-2}(\Omega)\,\mathrm{tr}_f(\chi^\dagger U + U^\dagger \chi). \qquad (4.16)$$

For the fourth order one obtains

$$L_1^*(\Omega) = \frac{1}{24(4\pi)^2}\,\mathrm{tr}_c\,\mathcal{I}_4(\Omega), \qquad (4.17)$$

$$L_2^*(\Omega) = 2L_1^*(\Omega), \qquad (4.18)$$

$$L_3^*(\Omega) = -8L_1^*(\Omega) + \tfrac{1}{2}L_9^*(\Omega), \qquad (4.19)$$

$$\bar{L}_3^*(\Omega) = -\frac{1}{6(4\pi)^2}\,\mathrm{tr}_c\,\bar{\mathcal{I}}_2(\Omega), \qquad (4.20)$$

$$L_4^*(\Omega) = 0, \qquad (4.21)$$

$$L_5^*(\Omega) = \frac{M}{2B_0^*}\left(\frac{f_\pi^{*2}(\Omega)}{4M^2} - 3L_9^*(\Omega)\right), \qquad (4.22)$$

$$\bar{L}_5^*(\Omega) = \bar{L}_5'^*(\Omega) = \tfrac{1}{2}\bar{L}_3^*(\Omega), \qquad (4.23)$$

$$L_6^*(\Omega) = 0, \qquad (4.24)$$

$$L_7^*(\Omega) = \frac{1}{8N_f}\left(-\frac{f_\pi^{*2}(\Omega)}{2B_0^* M} + L_9^*(\Omega)\right), \qquad (4.25)$$

$$\bar{L}'^*(\Omega) = -\frac{1}{4N_f}\bar{L}_3^*(\Omega), \qquad (4.26)$$

$$L_8^*(\Omega) = \frac{1}{16B_0^*}\left(\frac{1}{M} - \frac{1}{B_0^*}\right) f_\pi^{*2}(\Omega) - \frac{1}{8}L_9^*(\Omega), \qquad (4.27)$$

$$L_9^*(\Omega) = \frac{1}{3(4\pi)^2}\,\mathrm{tr}_c\,\mathcal{I}_2(\Omega), \qquad (4.28)$$

$$\bar{L}_9^*(\Omega) = \bar{L}_9'^*(\Omega) = -\bar{L}_3^*(\Omega), \qquad (4.29)$$

$$L_{10}^*(\Omega) = -\tfrac{1}{2}L_9^*(\Omega), \qquad (4.30)$$

$$H_1^*(\Omega) = -\frac{f_\pi^{*2}(\Omega)}{24M^2} + \frac{1}{4}L_9^*(\Omega), \qquad (4.31)$$

$$\bar{H}_1^*(\Omega) = -\frac{1}{6(4\pi)^2}\,\mathrm{tr}_c\,\bar{\mathcal{I}}_0(\Omega), \qquad (4.32)$$

$$H_2^*(\Omega) = -\frac{f_\pi^{*2}(\Omega)}{8B_0^{*2}} + \frac{1}{4}L_9^*(\Omega). \qquad (4.33)$$

Note that all new Lorentz breaking terms, except $\bar{H}_1^*$, are proportional among them. On the other hand, the standard Gasser-Leutwyler coefficients can be expressed in terms of f_π^{*2}, B_0^*, L_1^* and L_9^* or, equivalently, in terms of the integrals $\mathrm{tr}_c\,\mathcal{I}_n$ for $n = -2$, 0, 2, 4, which are computed in Appendix A.

V. POLYAKOV LOOP INTEGRATION

A. Color group averaging

In order to proceed to the computation of the chiral (Euclidean) Lagrangian we need to carry out the Polyakov loop integration

$$e^{-\int d^4x \mathcal{L}^*(x)} = \int D\Omega\, e^{-\Gamma_G[\Omega]} e^{-\int d^4x \mathcal{L}_\Omega(x)}. \qquad (5.1)$$

To this end we can take advantage of the chiral counting to expand the exponential with $\mathcal{L}_\Omega(x)$ to fourth order. There is an obstruction, however, since $\mathcal{L}_\Omega^{(0)}$ will appear to all orders, making it difficult to take the Polyakov loop average. To sort this problem in the low temperature regime, we introduce a further *thermal* counting, in addition to the chiral one. The thermal counting suppresses terms which become irrelevant at low temperatures and so it is compatible with the chiral expansion.

The thermal counting is defined as follows. $\mathcal{L}_\Omega(x)$ comes from one quark-loop. As noted at the end of Sec. II, any such loop will wind ℓ times around the thermal cylinder (in imaginary time). Paths with $\ell = 0$ give the zero temperature Lagrangian $\mathcal{L}_0$ which depends on U but not on Ω. The other paths pick up a factor $\mathrm{tr}_c[(-\Omega)^\ell]$ which is accompanied by a suppression factor (at low temperature) of the order of $e^{-|\ell|M/T}$ from the quark propagator. They give the thermal component of the Lagrangian.

$$\mathcal{L}_\Omega(x) = \mathcal{L}_0(x) + \sum_{\ell\geq 1}\mathcal{L}_\ell(x). \tag{5.2}$$

In this thermal counting the terms are ordered by the number ℓ of Polyakov loops they carry with $\mathrm{tr}_c[(-\Omega)^\ell]$ as order $|\ell|$. The discussion of the incidence of the gluonic part of the action $\Gamma_G[\Omega]$ on this thermal expansion will be postponed till Sec. V D.

Combining the chiral and thermal expansions, one has

$$\mathcal{L}_\Omega(x) = \mathcal{L}_0(x) + \sum_{n,\ell}\mathcal{L}_\ell^{(n)}(x). \tag{5.3}$$

(Note that the chiral expansion of the zero temperature component $\mathcal{L}_0(x)$ is not required since it does not depend on the Polyakov loop.) This is to be introduced in the Boltzmann factor exponential of Eq. (5.1) and expanded. We will retain terms of $\mathcal{O}(p^4)$ in the chiral expansion and of leading order in the thermal expansion. Because the Haar measure $D\Omega$ (and actually also the gluonic correction) preserves center symmetry, the first thermal correction with a single quark loop (i.e., keeping just one Lagrangian in the expansion of the exponential) will be of order $\ell = N_c$ (a baryon-loop like contribution). For $N_c \geq 3$ this is not the dominant contribution. The leading thermal correction comes instead from correlation of a quark loop with an anti quark loop, a meson-loop like configuration. Schematically, these corrections have thus the following structure

$$\mathcal{L}_1^{(0)}\mathcal{L}_1^{(0)} + \mathcal{L}_1^{(0)}\mathcal{L}_1^{(2)} + (\mathcal{L}_1^{(0)}\mathcal{L}_1^{(4)} + \mathcal{L}_1^{(2)}\mathcal{L}_1^{(2)}) \tag{5.4}$$

contributing to $\mathcal{L}^{*(0)}$, $\mathcal{L}^{*(2)}$, and $\mathcal{L}^{*(4)}$, respectively. Note that the two quark loops occur at different spatial points, so the correlation between two Polyakov loops is needed. This we model as [23]

$$\int D\Omega\, \mathrm{tr}_c\Omega(\vec{x})\, \mathrm{tr}_c\Omega^{-1}(\vec{y}) = e^{-\sigma|\vec{x}-\vec{y}|/T}, \tag{5.5}$$

where $\sigma = (425\ \mathrm{MeV})^2$ is the string tension. (This, of course, implies that $D\Omega$ is not exactly the product of local Haar measures.) This prescription is consistent with the zero temperature quark-anti quark potential at medium and large distances,[6] while at coincident points it reproduces the $\mathrm{SU}(N_c)$ group integration formula [59],

$$\int d\Omega\, \mathrm{tr}_c\Omega\, \mathrm{tr}_c\Omega^{-1} = 1. \tag{5.6}$$

[6]The short distance Coulombian part of the $q\bar{q}$ potential cannot be reproduced with unrenormalized Polyakov loops lying on the group manifold.

Effectively, the consequence of using Eq. (5.5) in a two-point computation of type (5.4) is to introduce a correlation domain of spatial size V

$$\int d^4x d^4y \int D\Omega\, \mathrm{tr}_c\Omega(\vec{x})\,\mathrm{tr}_c\Omega^{-1}(\vec{y}) = \int d^4x \frac{V}{T}, \qquad V = \frac{8\pi T^3}{\sigma^3}. \tag{5.7}$$

Application of the procedure just described produces, for the zeroth order of the effective Lagrangian in the chiral expansion

$$B^* = B + \frac{N_f^2 M^4 T^3 V}{\pi^4} K_2^2. \tag{5.8}$$

In this formula $K_2 = K_2(M/T)$ and $K_n(z)$ refers to the n-th order Bessel function [60]. We use the same convention in all forthcoming expressions for the LEC. Not surprisingly, the thermal correction to the vacuum energy density is proportional to the correlation volume V and contains exactly two Bessel functions from the two correlated quark loops. This holds also for all LEC to follow.

To $\mathcal{O}(p^2)$ we find

$$f_\pi^{*2} = f_\pi^2 - \frac{N_f M^4 T V}{\pi^4} K_0 K_2, \tag{5.9}$$

$$B_0^* = B_0 + \frac{N_f M^4 T V}{\pi^4 f_\pi^2} K_2(B_0 K_0 - 2T K_1). \tag{5.10}$$

And therefore, for the single flavor quark condensate at finite temperature

$$\langle\bar{q}q\rangle^* = \langle\bar{q}q\rangle + \frac{2N_f M^4 T^2 V}{\pi^4} K_1 K_2. \tag{5.11}$$

At very low temperatures, the asymptotic form of the Bessel function [60]

$$K_n(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \tag{5.12}$$

applies, and so the first thermal correction in the low-energy coefficients has the exponential suppression $e^{-2M/T}$. This indicates that meson thermal loops dominate at low temperatures (in our treatment we ignore completely the quark binding effects so the nature of the mesons involved cannot be resolved within the present approach). Note that the group integration produces an N_c suppression with respect to the zero temperature contribution [23].[7] We warn, however, that the results obtained with the full

[7]Naively, a two quark-loop term would be $\mathcal{O}(N_c^2)$, and in fact, ignoring the Polyakov loop dependence, i.e., setting $\Omega = 1$ as in the standard NJL model, would give N_c^2, instead of unity on the right-hand side of Eq. (5.6). Thus the introduction of the Polyakov loop proves essential to reproduce the $\mathcal{O}(N_c^0)$ result that one should expect for the thermal corrections coming from colorless hadronic excitations.

Bessel functions and with their asymptotic form differ beyond very low temperatures, especially as n increases.

Because of center symmetry of the Polyakov loop measure, the higher thermal contributions are of the general type $\int D\Omega\, \mathrm{tr}_c(\Omega^{n_1})\cdots \mathrm{tr}_c(\Omega^{n_\ell})$, with $\sum_i n_i$ a multiple of N_c. Among other, there are

(i) Two or more mesonlike loops, of the type $\int D\Omega (\mathrm{tr}_c\Omega\, \mathrm{tr}_c\Omega^{-1})^n$, $n = 2, 3, \ldots$, with a thermal suppression factors $e^{-2nM/T}$.

(ii) Baryonlike loop excitations, of the type $\int D\Omega\, \mathrm{tr}_c\Omega^{\pm N_c}$, with thermal suppression $e^{-N_c M/T}$. They come from a single quark looping N_c times the thermal cylinder. These are the leading thermal corrections of hadronic type in the *quenched* version of the theory [61,62].

(iii) Gluonic corrections, with thermal suppression $e^{-m_G/T}$, $m_G \sim 650$ MeV, from $\Gamma_G[\Omega]$.

In passing, we note that the leading thermal contribution to the Polyakov loop expectation value can also be easily computed. It comes from the coupling of the Polyakov loop observable with a quark loop. Schematically, from $\langle \Omega \mathcal{L}_1^{(0)} \rangle$. An easy computation gives, at leading order

$$\left\langle \frac{1}{N_c} \mathrm{tr}_c\Omega \right\rangle = \frac{N_f}{N_c} \frac{M^2 TV}{\pi^2} K_2. \tag{5.13}$$

This is a measure of the explicit breaking of center symmetry by quarks, and is controlled by the ratio M/T in the exponential factor.

B. Low-energy coefficients

Let us come now to $\mathcal{O}(p^4)$ terms. The result we find for the Gasser-Leutwyler coefficients for the constituent quark model reads

$$L_1^* = L_1 + \frac{M^4 V}{64\pi^4 T}\left(K_0^2 - \frac{N_f}{6} K_2^2\right), \tag{5.14}$$

$$L_2^* = L_2 - \frac{N_f M^4 V}{192\pi^4 T} K_2^2, \tag{5.15}$$

$$L_3^* = L_3 + \frac{N_f M^3 V}{48\pi^4 T} K_2 (M K_2 - T K_1), \tag{5.16}$$

$$\bar{L}_3^* = \frac{N_f M^4 V}{48\pi^4 T} K_0 K_2, \tag{5.17}$$

$$L_4^* = \frac{M^4 V}{16\pi^4 B_0} K_0 K_1, \tag{5.18}$$

$$L_5^* = L_5 - \epsilon L_5 + \frac{N_f M^3 V}{16\pi^4 B_0} K_2 (M K_1 - 2T K_0), \tag{5.19}$$

$$\bar{L}_5^* = \bar{L}_5^{\prime *} = \tfrac{1}{2}\bar{L}_3^*, \tag{5.20}$$

$$L_6^* = \frac{M^4 TV}{64\pi^4 B_0^2} K_1^2, \tag{5.21}$$

$$L_7^* = L_7 + \frac{f_\pi^2}{16 N_f M B_0}\epsilon + \frac{M^3 V}{192\pi^4} K_2\left(\frac{12T}{B_0} K_0 - K_1\right), \tag{5.22}$$

$$\bar{L}^{\prime *} = -\frac{1}{4N_f} \bar{L}_3^*, \tag{5.23}$$

$$L_8^* = L_8 + \frac{f_\pi^2}{16 B_0}\left(\frac{2}{B_0} - \frac{1}{M}\right)\epsilon + \frac{N_f M^3 V}{192\pi^4} K_2 \times \left(K_1 + \frac{12MT}{B_0}\left(\frac{1}{B_0} - \frac{1}{M}\right) K_0\right), \tag{5.24}$$

$$L_9^* = L_9 - \frac{N_f M^3 V}{24\pi^4} K_1 K_2, \tag{5.25}$$

$$\bar{L}_9^* = \bar{L}_9^{\prime *} = -\bar{L}_3^*, \tag{5.26}$$

$$L_{10}^* = -\tfrac{1}{2} L_9^*, \tag{5.27}$$

$$H_1^* = H_1 + \frac{N_f M^2 V}{96\pi^4} K_2 (4T K_0 - M K_1), \tag{5.28}$$

$$\bar{H}_1^* = \frac{N_f M^3 V}{24\pi^4} K_1 K_2, \tag{5.29}$$

$$H_2^* = H_2 + \frac{f_\pi^2}{4B_0^2}\epsilon + \frac{N_f M^3 V}{96\pi^4} K_2\left(\frac{12MT}{B_0^2} K_0 - K_1\right), \tag{5.30}$$

where we have defined the quantity

$$\epsilon = (B_0^* - B_0)/B_0. \tag{5.31}$$

As usual, to bring the fourth order Lagrangian to the standard form (3.5) we have used the equation of motion for the nonlinear U field. This is obtained by minimizing $\mathcal{L}^{*(2)}(x)$ (and hence after the average over the Polyakov loop) and reads

$$u_{\mu\mu} u + u_\mu u_\mu - B_0^*[u, z] + \frac{B_0^*}{N_f} \mathrm{tr}_f([u, z]) = 0 \tag{5.32}$$

(where the vectorlike notation has been used, with $u_{LR} = U$ and $u_{RL} = U^\dagger$). The last term arises from the constraint $\det(U) = 1$, since we are considering a $SU(N_f)$ flavor group.

Explicit expressions for the zero temperature coefficients appear in Ref. [51]. Once again, we observe that the finite temperature corrections are N_c-suppressed as compared to the zero temperature values, as expected from hadronic excitations, and this requires a proper

Polyakov loop average. The terms with N_f, and also L_7^* and $\bar{L}'^*$, are those coming from $\mathcal{L}_1^{(0)}\mathcal{L}_1^{(4)}$ while those without come from $\mathcal{L}_1^{(2)}\mathcal{L}_1^{(2)}$. It is worth noticing that some structures which were absent in the quenched approximation $\mathcal{L}_\Omega^{(4)}(x)$, e.g. L_4^* and L_6^*, are allowed in the unquenched result, from $\mathcal{L}_1^{(2)}\mathcal{L}_1^{(2)}$.

C. Volume rule

Clearly, to carry out the functional integration over the Polyakov loop indicated in (5.1), besides requiring a careful specification of the Polyakov loop action, would be an exceedingly demanding task. However, we can make systematic relations of the type (5.7) using a simple model. This will allow us to go beyond the leading thermal corrections and to include gluonic corrections from $\Gamma_G[\Omega]$. Specifically, we assume that the space-time is decomposed into correlation domains of size V/T, in such a way that two Polyakov loops are completely correlated if they lie within the same domain and are completely decorrelated otherwise. So for a partition function of the form

$$Z = \int D\Omega e^{-\int d^4x \mathcal{L}(x)} := e^{-\int d^4x \mathcal{L}'(x)}, \qquad (5.33)$$

we take

$$Z = \int \prod_n d\Omega_n e^{-\sum_n \Gamma_n}, \qquad (5.34)$$

where n labels the correlation domain, $d\Omega$ is the Haar measure, and $\Gamma_n = (V/T)\mathcal{L}(x_n)$, x_n lying in the domain n. This gives

$$\begin{aligned} Z &= \int \prod_n d\Omega_n\Big(1 - \sum_n \Gamma_n + \frac{1}{2}\sum_{n,n'}\Gamma_n\Gamma_{n'} + \cdots\Big) \\ &= 1 - \sum_n \langle\Gamma_n\rangle + \frac{1}{2}\sum_n\langle\Gamma_n^2\rangle + \frac{1}{2}\sum_{n\neq n'}\langle\Gamma_n\rangle\langle\Gamma_{n'}\rangle + \cdots, \end{aligned} \qquad (5.35)$$

where $\langle\ \rangle$ indicates average of Ω over SU(N_c). Obvious rearrangement gives, finally

$$\mathcal{L}'(x) = \langle\mathcal{L}(x)\rangle - \frac{1}{2}\frac{V}{T}(\langle\mathcal{L}^2(x)\rangle - \langle\mathcal{L}(x)\rangle^2) + \cdots. \qquad (5.36)$$

This is the standard cumulant expansion. Equivalently, $\mathcal{L}'(x)$ can be read off from

$$e^{-(V/T)\mathcal{L}'(x)} = \langle e^{-(V/T)\mathcal{L}(x)}\rangle_{\mathrm{SU}(N_c)}. \qquad (5.37)$$

In concrete configurations the right-hand side can be computed numerically (being the integration manifold very well behaved.) Points to be noted are: (i) The effective action is an extensive quantity. It is given by $\int d^4x\mathcal{L}'(x)$ and not $(V/T)\mathcal{L}'(x)$. (ii) If $\mathcal{L}(x)$ contains a piece $\mathcal{L}_0(x)$ which is Polyakov loop independent, this term simply adds in $\mathcal{L}'(x)$. It does not appear in the V-dependent contributions. And, iii) previous results of this section, e.g. Eq. (5.7), are correctly reproduced.

D. Gluonic corrections

Up to now in Eq. (5.1) we have neglected the factor $e^{-\Gamma_G}$ in the Polyakov loop measure. Following Ref. [26] we adopt a simple model inspired on the lattice strong coupling expansion [63,64]. This model has one parameter which is fitted in Ref. [26] to reproduce the transition temperature in gluodynamics in the mean field approximation. (A more refined fit involving more terms is discussed in Ref. [27].) Because our calculation does not rely on the mean field approach, our implementation of the model is not identical to that in those works and so refitting would probably be required. Nevertheless, this will not be needed here to make low temperature estimates since the gluonic corrections are much suppressed. Specifically, since we want to have extensivity, we introduce a local Lagrangian

$$\Gamma_G[\Omega] = \int d^4x\mathcal{L}_g(x), \qquad (5.38)$$

which is modeled as

$$\mathcal{L}_g(x) = -\frac{6}{a^3}Te^{-\sigma a/T}|\mathrm{tr}_c(\Omega)|^2 \qquad (5.39)$$

with an adjustable parameter $a^{-1} = 272$ MeV. Triality is preserved, but, unlike the pure Haar measure, at higher temperatures the action favors center-of-the-group values for the Polyakov loop. (The spontaneous breaking of center symmetry in gluodynamics indicates that a Lagrangian with a different form would be needed in the high temperature regime.) At low temperature we can see an exponential thermal suppression controlled by a gluonic mass $m_G = \sigma a = 664$ MeV.

The gluonic corrections can be accounted for by including $\mathcal{L}_g(x)$ in the Lagrangian of the volume rule (5.37). The leading gluonic correction is $\mathcal{O}(e^{-m_G/T})$ and goes to the vacuum energy density. Using Eq. (5.6) one easily finds $\delta_g B^* = -\langle\mathcal{L}_g(x)\rangle = 6a^{-3}Te^{-m_G/T}$. At low temperature this would be comparable with the meson-loop like terms, however, recall that the constituent quark mass M (actually M^*) gets reduced as the temperature approaches the chiral transition, thereby enhancing the role played by mesonic loops.

The gluonic corrections to f_π^{*2}, B_0^* and the other low-energy coefficients L_i^* are of $\mathcal{O}(e^{-(2M+m_G)/T})$, as is easily verified, thus they can be neglected as compared with the hadronic-loop corrections.

Instead of modeling gluonic corrections through a local Lagrangian, another approach suggested by the volume rule (5.37) is to remove the Polyakov loop action from the Lagrangian and instead, to replace the Haar measure average over the color group by a weighted average, $d\Omega \to d\Omega e^{-\Gamma_G[\Omega]}$. In principle such a weight could be obtained through a proper sampling of the Polyakov loop distribution on the group manifold within lattice computations of gluodynamics. At present this is not feasible due to severe numerical and conceptual problems related to renormalization issues. An advantage of this approach would be that expectation values of the type $\langle f(\Omega(x))\rangle$ (a single point) would be exact when the quarks are switched off. However, it misses the normalization of the measure, which is needed for its contribution to the vacuum energy density. From Eq. (5.37) it is clear that such weight is equivalent to a local Lagrangian $\mathcal{L}_g(x) = (T/V)\Gamma_G$.

VI. CONCLUSIONS

The previous results clarify the question that chiral quark model practitioners might legitimately ask, namely, whether the tree level low-energy coefficients in the chiral Lagrangian do genuinely depend on temperature or not. According to our present calculation this temperature dependence at very small temperatures is tiny. It is exponentially suppressed by a scale which is *exactly the mass gap*, and becomes inessential on temperature scales below the deconfining transition (the region of temperatures where the Polyakov cooling mechanism acts) if the Polyakov loop is first introduced and color singlet projection is subsequently carried out in a gauge invariant manner. In a quark model without Polyakov loop the temperature dependence of the low-energy constants behaves as $L_i^* - L_i \sim N_c e^{-M/T}$, due to spurious contributions of color non singlet multiquark states while the Polyakov cooling, i.e., the explicit suppression of colorful excitations provides instead the behavior $L_i^* - L_i \sim e^{-2M/T}$ as expected from ChPT arguments. Remarkably, this cooling mechanism produces also the correct N_c counting. As shown in previous work [23] this effect at low temperatures manifests quantitatively in the chiral symmetry restoration-center symmetry breaking phase transition.

To see how the agreement of Polyakov-chiral quark models to the theoretical assumptions of ChPT at finite temperature well below the phase transition materializes in practice we have calculated the chiral effective Lagrangian at finite temperature. As a byproduct the interaction between Polyakov loop and Goldstone bosons may be analyzed in some detail. The resulting chiral Lagrangian can be decomposed into a zero temperature like looking piece with temperature dependent low-energy constants and a new Lorentz breaking contribution with novel structures generated by the heat bath at rest. We remind that these calculations aim at describing the tree level effective action of ChPT at finite temperature. In any case, the corresponding temperature induced effects on the low-energy constants at this level of approximation exhibit the Polyakov cooling. In other words, below the phase transition any temperature dependence on the tree level low-energy constants may be neglected. This is precisely the starting assumption of ChPT theory, a purely hadronic theory, and as we have shown it arises naturally when the local and quantum nature of the Polyakov loop is taken into account.

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APPENDIX A: FLAVOR TRACES AND PROPER-TIME INTEGRALS

In this appendix we compute the flavor traces of the Seeley-DeWitt coefficients of Eq. (4.12), and the proper time integrals which appear in Eq. (4.10), in order to obtain the effective Lagrangian of Sec. IV B. Euclidean signature is used throughout.

1. Flavor traces and useful identities

For $N_f = 3$ flavors we have the SU(3) identity

$$\mathrm{tr}\,(ABAB) = -2\,\mathrm{tr}(A^2B^2) + \tfrac{1}{2}\mathrm{tr}(A^2)\,\mathrm{tr}(B^2) + (\mathrm{tr}(AB))^2, \tag{A1}$$

where A and B are 3×3 Hermitian traceless matrices. From here one gets

$$\mathrm{tr}_f((m_\mu m_\nu)^2) = -2\,\mathrm{tr}_f((m_\mu)^4) + \tfrac{1}{2}(\mathrm{tr}_f((m_\mu)^2))^2 + (\mathrm{tr}_f(m_\mu m_\nu))^2, \tag{A2}$$

$$\mathrm{tr}_f((m_4 m_\mu)^2) = -2\,\mathrm{tr}_f((m_4)^2(m_\mu)^2) + \tfrac{1}{2}\mathrm{tr}_f((m_4)^2)\,\mathrm{tr}_f((m_\mu)^2) + (\mathrm{tr}_f(m_4 m_\mu))^2. \tag{A3}$$

Further useful identities are

$$\mathrm{tr}_f((m_{\mu\nu})^2) = \mathrm{tr}_f((m_{\mu\mu})^2) + 2i\,\mathrm{tr}_f(F_{\mu\nu}m_\mu m_\nu) - \mathrm{tr}_f((mF_{\mu\nu})^2) + M^2\,\mathrm{tr}_f(F_{\mu\nu}^2), \tag{A4}$$

$$\mathrm{tr}_f((m_{4\mu})^2) = \mathrm{tr}_f(m_{44}m_{\mu\mu}) + 2i\,\mathrm{tr}_f(E_i[m_4, m_i]) + 2i\,\mathrm{tr}_f(E_{4i}mm_i), \tag{A5}$$

where we have applied the identity $X_{\mu\nu} = X_{\nu\mu} - i[F_{\mu\nu}, X]$, being X any operator. Using the equation of motion, Eq. (5.32), we obtain the following identities

$$\mathrm{tr}_f(m_\mu z_\mu) = \frac{1}{2B_0^* M^2}\,\mathrm{tr}_f((m_\mu)^2 mx) - \frac{1}{4B_0^* M}\,\mathrm{tr}_f((mx)^2) + \frac{M}{4B_0^*}\,\mathrm{tr}_f(x^2) + \frac{1}{8MN_f B_0^*}(\mathrm{tr}_f([m,x]))^2, \tag{A6}$$

$$\mathrm{tr}_f((m_{\mu\mu})^2) = \frac{1}{M^2}\,\mathrm{tr}_f((m_\mu)^4) - \frac{1}{2}\,\mathrm{tr}_f((mx)^2) + \frac{M^2}{2}\,\mathrm{tr}_f(x^2) + \frac{1}{4N_f}(\mathrm{tr}_f([m,x]))^2, \tag{A7}$$

$$\mathrm{tr}_f(m_{44}m_{\mu\mu}) = \frac{1}{M^2}\,\mathrm{tr}_f((m_4)^2(m_\mu)^2) - M\,\mathrm{tr}_f(m_{44}x) - \frac{1}{M}\,\mathrm{tr}_f((m_4)^2 mx) + \frac{1}{2MN_f}\,\mathrm{tr}_f(m_{44}m)\,\mathrm{tr}_f([m,x]), \tag{A8}$$

where the normalized field $x = 2B_0^* z$ has been introduced and so

$$x_{LR} = \chi, \qquad x_{RL} = \chi^\dagger. \tag{A9}$$

Applying Eqs. (A2)–(A8) the flavor trace of the Seeley-DeWitt coefficients can be worked out, yielding

$$\begin{aligned}
\mathrm{tr}_f b_0^* &= 4N_f\varphi_0(\Omega),\\
\mathrm{tr}_f b_1^* &= -\varphi_0(\Omega)\Big(\frac{4}{B_0^*}\,\mathrm{tr}_f(mx) + \frac{1}{B_0^{*2}}\,\mathrm{tr}_f(x^2)\Big),\\
\mathrm{tr}_f b_2^* &= 2\varphi_0(\Omega)\,\mathrm{tr}_f((m_\mu)^2) + \frac{2}{B_0^* M^2}\,\varphi_0(\Omega)\,\mathrm{tr}_f((m_\mu)^2 mx) + \frac{1}{B_0^*}\Big(\frac{1}{B_0^*} - \frac{1}{M}\Big)\varphi_0(\Omega)\,\mathrm{tr}_f((mx)^2) + \frac{M}{B_0^*}\Big(\frac{M}{B_0^*} + 1\Big)\varphi_0(\Omega)\,\mathrm{tr}_f(x^2)\\
&\quad + \frac{2}{3}\,\varphi_0(\Omega)\,\mathrm{tr}_f(F_{\mu\nu}^2) + \frac{2}{3}\,\bar\varphi_2(\Omega)\,\mathrm{tr}_f(E_i^2) + \frac{1}{2MN_f B_0^*}\,\varphi_0(\Omega)(\mathrm{tr}_f([m,x]))^2,\\
\mathrm{tr}_f b_3^* &= \frac{4}{3}\,i\varphi_0(\Omega)\,\mathrm{tr}_f(F_{\mu\nu}m_\mu m_\nu) + \frac{1}{3}\,\varphi_0(\Omega)\,\mathrm{tr}_f((mF_{\mu\nu})^2) - \frac{1}{3}M^2\varphi_0(\Omega)\,\mathrm{tr}_f(F_{\mu\nu}^2) - \frac{1}{6}M^2\varphi_0(\Omega)\,\mathrm{tr}_f(x^2)\\
&\quad + \frac{1}{6}\,\varphi_0(\Omega)\,\mathrm{tr}_f((mx)^2) - \frac{2}{B_0^*}\,\varphi_0(\Omega)\,\mathrm{tr}_f((m_\mu)^2 mx) - \frac{1}{3M}\,\bar\varphi_2(\Omega)\,\mathrm{tr}_f((m_4)^2 mx) - \frac{M}{3}\,\bar\varphi_2(\Omega)\,\mathrm{tr}_f(m_{44}x)\\
&\quad + \frac{2}{3}\,i\bar\varphi_2(\Omega)\,\mathrm{tr}_f(E_i[m_4,m_i]) + \frac{2}{3}\,i\bar\varphi_2(\Omega)\,\mathrm{tr}_f(E_{4i}mm_i) - \frac{1}{3M^2}\,\varphi_0(\Omega)\,\mathrm{tr}_f((m_\mu)^4) + \frac{1}{3M^2}\,\bar\varphi_2(\Omega)\,\mathrm{tr}_f((m_4)^2(m_\mu)^2)\\
&\quad - \frac{1}{12N_f}\,\varphi_0(\Omega)(\mathrm{tr}_f([m,x]))^2 + \frac{1}{6MN_f}\,\bar\varphi_2(\Omega)\mathrm{tr}_f(m_{44}m)\,\mathrm{tr}_f([m,x])),\\
\mathrm{tr}_f b_4^* &= -\frac{1}{12}\,\varphi_0(\Omega)(\mathrm{tr}_f((m_\mu)^2))^2 - \frac{1}{6}\,\varphi_0(\Omega)(\mathrm{tr}_f(m_\mu m_\nu))^2 + \frac{2}{3}\,\varphi_0(\Omega)\,\mathrm{tr}_f((m_\mu)^4).
\end{aligned} \tag{A10}$$

2. Proper-time integrals with Polyakov loop

The basic proper-time integrals are defined by

$$\mathcal{I}_{2l}(\Lambda, M, \Omega) := M^{2l}\int_0^\infty \frac{d\tau}{\tau}\,\phi(\tau)\tau^l e^{-\tau M^2}\varphi_0(\Omega), \tag{A11}$$

$$\bar{\mathcal{I}}_{2l}(\Lambda, M, \Omega) := M^{2l}\int_0^\infty \frac{d\tau}{\tau}\,\phi(\tau)\tau^l e^{-\tau M^2}\bar\varphi_2(\Omega), \tag{A12}$$

where Ω is a $SU(N_c)$ matrix in color space, and we define φ_0 and $\bar\varphi_2$ as follows:

$$\varphi_0(\Omega) = \sum_{n\in\mathbb{Z}} e^{-(n^2\beta^2/4\tau)}(-\Omega)^n, \tag{A13}$$

$$\bar\varphi_2(\Omega) = \frac{\beta^2}{2\tau}\sum_{n\in\mathbb{Z}} n^2 e^{-(n^2\beta^2/4\tau)}(-\Omega)^n. \tag{A14}$$

The $n = 0$ term in the sum corresponds to the zero temperature contribution, and for such a term the Pauli-Villars regularization can be applied. In the remaining $n \neq 0$ terms the regularization will be removed, since these thermal terms are ultraviolet finite. This approximation is justified for temperatures well below the cutoff. The calculation of the integral yields

$$
\begin{aligned}
M^4 \mathcal{I}_{-4}(\Lambda, M, \Omega) &= -\frac{1}{2}\sum_i c_i(\Lambda_i^2 + M^2)^2 \log(\Lambda_i^2 + M^2) + 8(MT)^2 \sum_{n=1}^{\infty} \frac{K_2(nM/T)}{n^2}((-\Omega)^n + (-\Omega)^{-n}),\\
M^2 \mathcal{I}_{-2}(\Lambda, M, \Omega) &= \sum_i c_i(\Lambda_i^2 + M^2)\log(\Lambda_i^2 + M^2) + 4MT \sum_{n=1}^{\infty} \frac{K_1(nM/T)}{n}((-\Omega)^n + (-\Omega)^{-n}),\\
\mathcal{I}_0(\Lambda, M, \Omega) &= -\sum_i c_i \log(\Lambda_i^2 + M^2) + 2\sum_{n=1}^{\infty} K_0(nM/T)((-\Omega)^n + (-\Omega)^{-n}),\\
\mathcal{I}_{2l}(\Lambda, M, \Omega) &= \Gamma(l)\sum_i c_i\left(\frac{M^2}{\Lambda_i^2 + M^2}\right)^l + 2\left(\frac{M}{2T}\right)^l \sum_{n=1}^{\infty} n^l K_l(nM/T)((-\Omega)^n + (-\Omega)^{-n}), \qquad \mathrm{Re}(l) > 0,\\
\bar{\mathcal{I}}_{2l}(\Lambda, M, \Omega) &= 4\left(\frac{M}{2T}\right)^{l+1} \sum_{n=1}^{\infty} n^{l+1} K_{l-1}(nM/T)((-\Omega)^n + (-\Omega)^{-n}), \qquad l \in \mathbb{R}.
\end{aligned}
\tag{A15}
$$

APPENDIX B: RESULTS FOR THE SPECTRAL QUARK MODEL

In this appendix we present the results of the low-energy coefficients for the spectral quark model (SQM). In the SQM the effective action reads

$$\Gamma_{\rm SQM} = -i\int d\omega \rho(\omega)\,\mathrm{Tr}\log(i\mathbf{D}), \tag{B1}$$

where the Dirac operator is given by

$$i\mathbf{D} = i\partial\!\!\!/ + v\!\!\!/ + a\!\!\!/\gamma_5 - \omega U^{\gamma_5} - \hat{M}_0 \tag{B2}$$

and $\rho(\omega)$ is the spectral function of a generalized Lehmann representation of the quark propagator with ω the spectral mass defined on a suitable contour of the complex plane [50,55,65,66]. The use of certain spectral conditions guarantees finiteness of the action.

A judicious choice of the spectral function based on vector meson dominance generates a quark propagator with no-poles (analytic confinement). More details of the SQM at zero and finite temperature relevant for the present paper are further developed at Ref. [23]. The partition function for the SQM can be written as

$$Z = \int DUD\Omega e^{i\Gamma_G[\Omega]} e^{i\Gamma_{\rm SQM}[U,\Omega]}. \tag{B3}$$

In this model one has to compute an average over the constituent quark mass with an spectral weight function $\rho(\omega)$ satisfying a set of spectral conditions Note that M appears as argument in the integrals $\mathcal{I}_{2l}$ and $\bar{\mathcal{I}}_{2l}$, but also as additional multiplicative factors in the Gasser-Leutwyler coefficients. This generates a larger number of independent functions as compared to the NJL model.

We write first the results for the effective Lagrangian before the group average. We get for the zeroth order Lagrangian

$$\mathcal{L}_\Omega^{(0)} = -\frac{2N_f}{(4\pi)^2}\langle\omega^4\,\mathrm{tr}_c\,\mathcal{I}_{-4}(\Omega)\rangle. \tag{B4}$$

To simplify the notation we indicate with $\langle\quad\rangle$ the spectral average $\int d\omega\rho(\omega)$. The second order Lagrangian becomes

$$\mathcal{L}_\Omega^{(2)} = \frac{f_\pi^{*2}(\Omega)}{4}\mathrm{tr}_f((u_\mu)^2) + \frac{2}{(4\pi)^2 B_0^*} \times\langle\omega^3\,\mathrm{tr}_c\,\mathcal{I}_{-2}(\Omega)\rangle\,\mathrm{tr}_f(ux), \tag{B5}$$

with

$$f_\pi^{*2}(\Omega) = \frac{1}{4\pi^2}\langle\omega^2\,\mathrm{tr}_c\,\mathcal{I}_0(\Omega)\rangle. \tag{B6}$$

As in the CQ model, B_0^* will be obtained after the average in Ω. The fourth order Lagrangian has the low-energy coefficients

$$
\begin{aligned}
L_1^*(\Omega) &= \frac{1}{24(4\pi)^2}\langle\mathrm{tr}_c\,\mathcal{I}_4(\Omega)\rangle,\\
\bar{L}_3^*(\Omega) &= -\frac{1}{6(4\pi)^2}\langle\mathrm{tr}_c\,\bar{\mathcal{I}}_2(\Omega)\rangle,\\
L_5^*(\Omega) &= \frac{1}{2(4\pi)^2 B_0^*}(\langle\omega\,\mathrm{tr}_c\,\mathcal{I}_0(\Omega)\rangle - \langle\omega\mathrm{tr}_c\,\mathcal{I}_2(\Omega)\rangle),\\
L_7^*(\Omega) &= \frac{1}{2(4\pi)^2 N_f}\left(-\frac{1}{2B_0^*}\langle\omega\mathrm{tr}_c\,\mathcal{I}_0(\Omega)\rangle + 4\pi^2 L_9^*(\Omega)\right),\\
L_8^*(\Omega) &= \frac{1}{4(4\pi)^2 B_0^*}\langle\omega\mathrm{tr}_c\,\mathcal{I}_0(\Omega)\rangle - \frac{f_\pi^{*2}(\Omega)}{16 B_0^{*2}} - \frac{1}{8}L_9^*(\Omega),\\
L_9^*(\Omega) &= \frac{1}{3(4\pi)^2}\langle\mathrm{tr}_c\,\mathcal{I}_2(\Omega)\rangle,\\
H_1^*(\Omega) &= -\frac{1}{6(4\pi)^2}\langle\mathrm{tr}_c\,\mathcal{I}_0(\Omega)\rangle + \frac{1}{4}L_9^*(\Omega),\\
\bar{H}_1^*(\Omega) &= -\frac{1}{6(4\pi)^2}\langle\mathrm{tr}_c\,\bar{\mathcal{I}}_0(\Omega)\rangle,\\
H_2^*(\Omega) &= \frac{1}{2(4\pi)^2 B_0^*}\left(\frac{1}{B_0^*}\langle\omega^2\,\mathrm{tr}_c\,\mathcal{I}_{-2}(\Omega)\rangle - \langle\omega\,\mathrm{tr}_c\,\mathcal{I}_0(\Omega)\rangle\right)\\
&\quad - \frac{f_\pi^{*2}(\Omega)}{8B_0^{*2}} + \frac{1}{4}L_9^*(\Omega).
\end{aligned}
\tag{B7}
$$

The remaining coefficients satisfy the same geometric relations obtained in the NJL, Eqs. (4.17)–(4.33). We also get the relation

$$L_7^*(\Omega) = -\frac{1}{N_f}\left(\frac{f_\pi^{*2}(\Omega)}{16B_0^{*2}} + L_8^*(\Omega)\right), \tag{B8}$$

both in the SQM and in NJL.

The next step is to carry out the integration over Polyakov loops, as in Sec. V. For the temperature independent part the result can be expressed in terms of the spectral log-moments as was already done in Ref. [50]. For the temperature dependent contributions we will explicitly use the meson dominant form of the spectral function $\rho(\omega)$. After computing all averages (spectral and color group average), we get for the zeroth order effective Lagrangian at leading order in the thermal expansion

$$B^* = B + \frac{N_f^2 T^3 V}{\pi^4}\langle\omega^2 K_2\rangle^2. \tag{B9}$$

In this expression $K_2 = K_2(\omega/T)$. We use the same convention for the symbol K_n in subsequent expressions. The spectral average can be computed and gives

$$\langle\omega^2 K_2\rangle = \frac{T^2}{24}(48 + 24x_V + 6x_V^2 + x_V^3)e^{-x_V/2}. \tag{B10}$$

We use the notation $x_V = M_V/T$.

The pion weak decay constant and the normalization constant read respectively

$$\frac{f_\pi^{*2}}{f_\pi^2} = 1 - \frac{N_f TV}{\pi^4 f_\pi^2}\langle\omega^2 K_0\rangle\langle\omega^2 K_2\rangle, \tag{B11}$$

$$\frac{B_0^*}{B_0} = 1 + \frac{N_f TV}{\pi^4 f_\pi^2}\langle\omega^2 K_2\rangle \times \left(\langle\omega^2 K_0\rangle - \frac{2T}{B_0}\langle\omega^2 K_1\rangle\right), \tag{B12}$$

with the spectral averages

$$\langle\omega^2 K_0\rangle = \frac{T^2}{24}x_V^2(2 + x_V)e^{-x_V/2}, \tag{B13}$$

$$\langle\omega^2 K_1\rangle = \frac{\rho_3'}{4T}e^{-x_S/2}, \tag{B14}$$

and $x_S = M_S/T$. The equations of motion are those in Eq. (5.32). (Note that B_0^*, and hence the normalization of the field χ depends on the model.) The low-energy coefficients, including the first thermal correction, read

$$L_1^* = L_1 + \frac{2V}{3(4\pi)^4 T}(6\langle\omega^2 K_0\rangle^2 - N_f\langle\omega^2 K_2\rangle^2), \tag{B15}$$

$$L_2^* = L_2 - \frac{N_f V}{3(8\pi^2)^2 T}\langle\omega^2 K_2\rangle^2, \tag{B16}$$

$$L_3^* = L_3 + \frac{N_f V}{3(2\pi)^4 T}\langle\omega^2 K_2\rangle(\langle\omega^2 K_2\rangle - T^2\langle\omega K_1\rangle), \tag{B17}$$

$$\bar{L}_3^* = \frac{N_f V}{3(2\pi)^4 T}\langle\omega^2 K_0\rangle\langle\omega^2 K_2\rangle, \tag{B18}$$

$$L_4^* = \frac{V}{(2\pi)^4 B_0}\langle\omega^2 K_0\rangle\langle\omega^2 K_1\rangle, \tag{B19}$$

$$L_5^* = (1-\epsilon)L_5 - \frac{N_f V}{(2\pi)^4 B_0}\langle\omega^2 K_2\rangle(2T\langle\omega K_0\rangle - \langle\omega K_1\rangle), \tag{B20}$$

$$\bar{L}_5^* = \bar{L}_5'^* = \tfrac{1}{2}\bar{L}_3^*, \tag{B21}$$

$$L_6^* = \frac{TV}{64\pi^4 B_0^2}\langle\omega^2 K_1\rangle^2, \tag{B22}$$

$$L_7^* = L_7 + \frac{L_5}{2N_f}\epsilon + \frac{V}{192\pi^4}\langle\omega^2 K_2\rangle\left(\frac{12T}{B_0}\langle\omega K_0\rangle - \langle\omega^2 K_1\rangle\right), \tag{B23}$$

$$\bar{L}'^* = -\frac{1}{4N_f}\bar{L}_3^*, \tag{B24}$$

$$L_8^* = L_8 + \left(\frac{f_\pi^2}{8B_0^2} - \frac{L_5}{2}\right)\epsilon + \frac{N_f V}{192\pi^4}\langle\omega^2 K_2\rangle \times \left(\langle\omega^2 K_1\rangle + \frac{12T}{B_0}\left(\frac{1}{B_0}\langle\omega^2 K_0\rangle - \langle\omega K_0\rangle\right)\right), \tag{B25}$$

$$L_9^* = L_9 - \frac{N_f TV}{24\pi^4}\langle\omega K_1\rangle\langle\omega^2 K_2\rangle, \tag{B26}$$

$$\bar{L}_9^* = \bar{L}_9'^* = -\bar{L}_3^*, \tag{B27}$$

$$L_{10}^* = -\tfrac{1}{2}L_9^*, \tag{B28}$$

$$H_1^* = H_1 + \frac{N_f V}{6(2\pi)^4}\langle\omega^2 K_2\rangle(4T\langle K_0\rangle - \langle\omega^2 K_1\rangle), \tag{B29}$$

$$\bar{H}_1^* = \frac{N_f V}{24\pi^4}\langle\omega K_1\rangle\langle\omega^2 K_2\rangle, \tag{B30}$$

$$H_2^* = H_2 + \left(L_5 + \frac{f_\pi^2}{2B_0^2}\right)\epsilon + \frac{N_f V}{96\pi^4}\langle\omega^2 K_2\rangle \times \left[\frac{12T}{B_0}\left(\langle\omega K_0\rangle + \frac{1}{B_0}\langle\omega^2 K_0\rangle\right) - \langle\omega^2 K_1\rangle - \frac{24T^2}{B_0^2}\langle\omega K_1\rangle\right], \tag{B31}$$

where $\epsilon = (B_0^* - B_0)/B_0$. The spectral averages are

$$\langle K_0 \rangle = -\frac{1}{2}\gamma_E - \log(x_V/4) + \frac{1}{2}\psi(5/2) + \frac{x_V^5}{7200}\,{}_1F_2[5/2; 7/2, 7/2; (x_V/4)^2] + \frac{x_V^2}{48}\,{}_2F_3[1, 1; -1/2, 2, 2; (x_V/4)^2], \qquad \text{(B32)}$$

$$\langle \omega K_0 \rangle = \frac{\rho_3'}{2T^2 x_S^2}(2 + x_S)e^{-x_S/2}, \qquad \text{(B33)}$$

$$\langle \omega K_1 \rangle = \frac{T}{12}(12 + 6x_V + x_V^2)e^{-x_V/2} \qquad \text{(B34)}$$

${}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z]$ is the generalized hypergeometric function. The expressions for the zero temperature coefficients appear in Ref. [50].

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