# Chiral symmetry breaking and vacuum polarization in a bag

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We study the effects of a finite quark mass in the hedgehog configuration in the two phase chiral bag model. We discuss the chiral properties, such as the fractional baryon number and the chiral Casimir energy, by using the Debye expansion for the analytical calculation and the Strutinsky's smearing method for the numerical computation. It is shown that the fractional baryon number carried by massive quarks in the vacuum is canceled by that in the meson sector. A finite term of the chiral Casimir energy is obtained with subtraction of the logarithmic divergence term.

## I. INTRODUCTION

Strangelets of finite volume quark matter with strangeness have been one of the most interesting subjects of exotic particles in hadron and quark physics [1–4]. In recent studies of strangelets, a model has been presented for a discussion of chiral symmetry breaking inside a quark droplet of finite volume quark matter [5–8]. There, in addition to quark confinement for a finite volume system, it was allowed that the quarks acquire a finite mass by dynamical chiral symmetry breaking. In this picture, it was assumed that the quark interaction was provided by the Nambu-Jona-Lasinio (NJL) type interaction inside a quark droplet. In earlier works, the quark wave function was given by the MIT bag in Refs. [5–7]. Later, in order to overcome the chiral symmetry breaking at the bag surface, the chiral bag model was introduced in Ref. [8]. The latter model was called as the NJL chiral bag model. Such an idea was first presented by T. Kunihiro in 1983 [9,10].

In the discussion of strangelets by the NJL chiral bag model, the hedgehog ansatz was used for the pion and *ud* quarks to include nonlinear interaction of pions and quarks [8]. In the history of the study of the chiral bag model since [11,12], effects of vacuum polarization, such as the anomalous baryon number and the chiral Casimir effects, have been discussed extensively for massless quarks [13–21]. It was shown that the correct behavior of the baryon number was provided by the contribution from the sea quarks [18–21]. However, their discussions were limited only in the massless quark. One motivation in the present paper is to discuss the baryon number conservation for the massive quark.

In the development of the chiral bag model, several techniques for calculation of the sea quark contribution were also developed. In Ref. [18], the fractional baryon number was derived by using the Multiple Reflection Expansion method. After that, the Debye expansion was used [20,21]. The Casimir energy is also affected by the pion cloud. It was shown that the chiral Casimir energy had

a term of logarithmic divergence [20–23]. The similar situation can occur in the case of massive quarks. However, these properties for massive quarks have not been investigated so far. In the framework of the NJL chiral bag model, the dynamical quark mass is an order parameter for the chiral symmetry breaking in the chiral bag [8]. In order to search the real stable state, we need to take a variation of the energy with respect to the dynamical quark mass. There we need to clarify the effect of vacuum polarization (the chiral Casimir energy) for the finite quark mass. The result of dynamical symmetry breaking in a chiral bag was already given in the proceeding paper [8]. The calculation of the chiral Casimir energy is a complicated problem which needs some techniques. It is our second motivation to discuss a detailed procedure to solve this problem with keeping the dynamical quark mass as some fixed value.

We organize this paper as follows. In Sec. II, we propose a model Lagrangian of the chiral bag with massive quarks. In this paper, we describe the quarks with a finite Dirac mass, which is generated through the NJL type interaction in the chiral bag. In Sec. III, we discuss a fractional baryon number for massive quarks by using both numerical computation and analytical formulation. In Sec. IV, we discuss the chiral Casimir energy. Deriving the terms of logarithmic divergence the finite term of the chiral Casimir energy is obtained numerically. We summarize our discussion in Sec. V.

## II. THE CHIRAL BAG WITH MASSIVE QUARKS

In the NJL chiral bag model [8–10], the quark mass in a finite volume system is induced by the NJL pointlike interaction [24]. We consider the *ud* quark sector with a finite quark mass induced by the mean field approximation in the scalar channel. The problem which we would like to discuss here is the chiral vacuum polarization induced by finite mass of the dynamical quarks. The generation mechanism of the dynamical quark mass was discussed in [8]. Therefore, we take the dynamical quark mass as a constant value in order to concentrate on the chiral vacuum polarization. The Lagrangian in the *ud* quark sector is

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written as

$$\mathcal{L} = \bar{\psi}(i\slashed{\partial} - \hat{m})\psi\theta(R - r) - \frac{1}{2}\bar{\psi}U^{\gamma_5}\psi\delta(r - R), \quad (1)$$

which includes a boundary condition explicitly. Here, $\psi = (u, d)^t$ is the $ud$ quark fields, and the $ud$ quark mass matrix is given by $\hat{m} = \text{diag}(m_u, m_d)$ with flavor symmetry $m = m_u = m_d$. The $ud$ quark mass is the sum of the current mass and the constituent mass which is generated in the nonzero expectation value of the quark scalar condensate $\bar{q}q$. In general, the constituent quark mass can be positioned dependent. Here, in order to simplify the essential discussion of the finite quark mass, we treat the quark mass as a constant. For a bag configuration, we assume a static spherical bag with radius $R$. The step function is multiplied in the first term in order to confine quarks inside the bag. Here, $r$ is a distance from the center of the bag. The second term with the delta function realizes the chiral invariant boundary condition at the bag surface, where we define

$$U^{\gamma_5} = e^{i\vec{\tau}\cdot\vec{\pi}\gamma_5}, \quad (2)$$

with the $\pi$ meson field $\vec{\pi}$ and the Pauli matrix $\vec{\tau}$. In this paper, we do not explicitly show the Lagrangian in the pion sector, since our current interest is to study the chiral vacuum polarization of the quarks in the bag.

In order to consider the nonlinear effect of the pion, we assume the hedgehog ansatz in the $\pi$ meson sector, where the pion field conserves the grand spin $\vec{K} = \vec{J} + \vec{I}$ with total angular momentum $\vec{J}$ and isospin $\vec{I}$. The hedgehog pion field for $r > R$ is written as

$$\vec{\pi} = F(r)\vec{n}, \quad (3)$$

with a chiral angle $F(r)$ and a unit radial vector $\vec{n}$ in the real space [25,26].

According to the $\pi$ meson sector, we introduce the hedgehog basis set in the $ud$ quark sector [19,27]. We construct the quark wave function $\psi^{(\kappa)}$ for natural ($\kappa = +$) and unnatural ($\kappa = -$) assignment for grand spin $K$, respectively;

$$\psi^{(+)} = \begin{pmatrix} a_0 j_K(pr)|0\rangle + a_1 j_K(pr)|1\rangle \\ a_2 j_{K+1}(pr)|2\rangle + a_3 j_{K-1}(pr)|3\rangle \end{pmatrix}, \quad (4)$$

and

$$\psi^{(-)} = \begin{pmatrix} b_2 j_{K+1}(pr)|2\rangle + b_3 j_{K-1}(pr)|3\rangle \\ b_0 j_K(pr)|0\rangle + b_1 j_K(pr)|1\rangle \end{pmatrix}. \quad (5)$$

The coefficients $a_i$ and $b_i$ ($i = 0, \cdots, 3$) are determined by satisfying the equation of motion of the $ud$ quark for $\kappa = \pm$, respectively. $j_K(pr)$ is the spherical Bessel function and $p$ is the $ud$ quark momentum. Here, the two-component spinors $|0\rangle, \cdots, |3\rangle$ are given by

$$|0\rangle = Y_{KM}(\theta, \phi)\chi_0^0,$$

$$|1\rangle = \sum_{\mu=-1,0,1} (KM - \mu 1\mu|KM)Y_{KM-\mu}(\theta, \phi)\chi_\mu^1,$$

$$|2\rangle = \sum_{\mu=-1,0,1} (K + 1M - \mu 1\mu|KM)Y_{K+1M-\mu}(\theta, \phi)\chi_\mu^1,$$

$$|3\rangle = \sum_{\mu=-1,0,1} (K - 1M - \mu 1\mu|KM)Y_{K-1M-\mu}(\theta, \phi)\chi_\mu^1,$$

$$(6)$$

where $Y_{LM}(\theta, \phi)$ is the spherical harmonics with spherical coordinate $(\theta, \phi)$. $\chi_\mu^G$ are eigenstates of $\vec{G} = \vec{S} + \vec{I} = \vec{\sigma}/2 + \vec{\tau}/2$,

$$\chi_0^0 = \frac{1}{\sqrt{2}}(|\uparrow\rangle|d\rangle - |\downarrow\rangle|u\rangle), \qquad \chi_1^1 = |\uparrow\rangle|u\rangle,$$

$$\chi_0^1 = \frac{1}{\sqrt{2}}(|\uparrow\rangle|d\rangle + |\downarrow\rangle|u\rangle), \qquad \chi_{-1}^1 = |\downarrow\rangle|d\rangle. \quad (7)$$

The sign of naturalness $\kappa$ corresponds to the parity $P = (-)^{K+\kappa}$.

In order to satisfy the equation of motion, the coefficients $a_i$ for naturalness assignment $\kappa = +$ are subjected to

$$\begin{pmatrix} E - m & 0 & -iPp & -iQp \\ 0 & E - m & -iQp & iPp \\ iPp & iQp & E + m & 0 \\ iQp & -iPp & 0 & E + m \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0, \quad (8)$$

where we define

$$P = \sqrt{\frac{K + 1}{2K + 1}}, \qquad Q = \sqrt{\frac{K}{2K + 1}}, \qquad E = \sqrt{p^2 + m^2}. \quad (9)$$

Then, we obtain two independent solutions

$$\vec{a}' = \begin{pmatrix} a_0' \\ a_1' \\ a_2' \\ a_3' \end{pmatrix} = N' \begin{pmatrix} \frac{p}{E-m}iP \\ \frac{p}{E-m}iQ \\ 1 \\ 0 \end{pmatrix},$$

$$\vec{a}'' = \begin{pmatrix} a_0'' \\ a_1'' \\ a_2'' \\ a_3'' \end{pmatrix} = N'' \begin{pmatrix} \frac{p}{E-m}iQ \\ -\frac{p}{E-m}iP \\ 0 \\ 1 \end{pmatrix}. \quad (10)$$

Here, the normalization constants $N'$ and $N''$ are determined by the condition $\int d^3x\psi^{(+)\dagger}\psi^{(+)} = 1$,

$$N'^{-2} = R^3 \frac{E}{E-m}\left[ j_K(pR)^2 + j_{K+1}(pR)^2 \right.$$
$$\left. - \frac{1}{pR}\left(2(K+1) - \frac{m}{E}\right)j_K(pR)j_{K+1}(pR) \right],$$

$$N''^{-2} = R^3 \frac{E}{E-m}\left[ j_{K-1}(pR)^2 + j_K(pR)^2 \right.$$
$$\left. - \frac{1}{pR}\left(2K + \frac{m}{E}\right)j_{K-1}(pR)j_K(pR) \right]. \quad (11)$$

The final solution is expressed as a linear combination of $\vec{a}'$ and $\vec{a}''$,

$$\vec{a} = c'\vec{a}' + c''\vec{a}''$$

$$= c'N'\begin{pmatrix} \frac{p}{E-m}iP \\ \frac{p}{E-m}iQ \\ 1 \\ 0 \end{pmatrix} + c''N''\begin{pmatrix} \frac{p}{E-m}iQ \\ -\frac{p}{E-m}iP \\ 0 \\ 1 \end{pmatrix}, \quad (12)$$

with constants $c'$ and $c''$. We obtain a solution for the unnaturalness assignment $\kappa = -$ in the same way.

The eigenenergy of the hedgehog $ud$ quark is given by the boundary condition at the bag surface $r = R$,

$$i\vec{n}\cdot\vec{\gamma}\psi^{(\kappa)} = -e^{i\vec{\tau}\cdot\vec{n}F(R)\gamma_5}\psi^{(\kappa)}. \quad (13)$$

By substituting the solution of Eq. (12), we obtain an equation for the eigenvalue, which is given by

$$\cos F(R)\left\{ j_K(pR)^2 - \left(\frac{E-\kappa m}{p}\right)^2 j_{K+1}(pR)j_{K-1}(pR) \right\}$$

$$- \kappa\frac{E-\kappa m}{p}j_K(pR)\{j_{K+1}(pR) - j_{K-1}(pR)\}$$

$$+ \frac{E-\kappa m}{p}\frac{\sin F(R)}{2K+1}j_K(pR)\{j_{K+1}(pR) + j_{K-1}(pR)\} = 0. \quad (14)$$

The equation for $K = 0$ is obtained by setting $j_{-1}(pR) = 0$. Equation (14) determines the energy spectrum of a quark in the bag. We mention that there is a symmetry of the energy level in the positive and negative energy

$$E_{K^P}(F) = -E_{K^{-P}}(-F). \quad (15)$$

This relation has its origin in the invariance of the Lagrangian (1) under the transformation $U \to U^*$ or $F \to -F$. Another symmetry under $F \to F + \pi$ and $\kappa \to -\kappa$ is conserved in the massless case. However, this symmetry is not conserved for the finite mass. This is due to an asymmetry of the quark energy levels, which is shown in Fig. 1 in [8].

### III. BARYON NUMBER CONSERVATION

In the limit of the zero bag radius, the meson field in the Skyrme Lagrangian gives a mapping from $R^3 \sim S^3$ to $SU(2) \sim S^3$ by imposing a boundary condition $U = 1$ at $r \to \infty$. The winding number in the mapping is identified

with the baryon number [25,26]. The fractional baryon number due to the pion cloud is given as [25,26]

$$B_\pi = -\frac{1}{\pi}\left[ F - \frac{1}{2}\sin 2F \right], \quad (16)$$

where $F$ is the chiral angle at the bag surface. In the pure Skyrmion, the chiral angle is given as $n\pi$ with integer $n$.

On the other hand, in the chiral bag with finite bag radius, the chiral angle is not generally equal to $n\pi$. Therefore, the meson carries only a fractional baryon number. There, it has been known that the total baryon number is composed of the fractional baryon numbers of the pion, of the vacuum and of the valence quarks in a bag [18–21,27]. The baryon number carried by the vacuum quarks is defined as

$$B_q(m, F) = \frac{1}{2}\langle\mathcal{O}|[\psi^\dagger, \psi]|\mathcal{O}\rangle$$

$$= -\frac{1}{2}\lim_{t\to 0_+}\sum_n \text{sgn}(E_n)e^{-t|E_n|}. \quad (17)$$

Here, $|\mathcal{O}\rangle$ is a vacuum state of an empty bag filled with negative energy quarks, $E_n = E_n(m, F)$ an $n$-th state energy of a quark with mass $m$, and at the chiral angle $F$ at the bag surface. The quantum number $n$ labels the grand spin and parity. The sum is taken over for all the quark states $n$ with positive and negative energies in the bag. In order to obtain the convergence in the sum, an exponential type regularization is multiplied.

In the studies of the chiral bag model with massless quarks, it was shown that the baryon number of the vacuum quarks was canceled by that of the meson cloud [13–16,18–21,23]. Accordingly, the total baryon number is a conserved quantity. This result should also be the case for the chiral bag with massive quarks. It is expected that the baryon number conservation is not affected by the finite quark mass, because $U(1)_B$ symmetry is conserved in our Lagrangian (1). In the following, we show explicitly the cancellation of the fractional baryon number between the vacuum massive quarks and the pion cloud. We use both numerical computation and analytical procedure.

First, we show the numerical computation. In the studies of chiral bag model with massless quarks, the regularization, such as the Gaussian type [22], heat kernel type [23], have been used with success. In this paper, we use the Strutinsky's smearing method [28,29]. This method has an advantage that the states necessary for computation are limited up to the grand spin $K_{max} \sim 40$ in order to obtain a good convergence, while we need $K_{max} \sim 100$ at least for the other regulators. We rewrite the baryon number (16) by introducing a delta function for a density of states to pick up the discrete levels

$$B_q(m, F) = -\frac{1}{2}\int_{-\infty}^{\infty} dx\sum_n \text{sgn}(E_n)\delta(x - E_nR) \quad (18)$$

Then, we replace the delta function by a gaussian function,

$$B_q(m, F) = \int_{-\infty}^{\infty} dx \rho_x(m, F), \qquad (19)$$

where we define a function for a density of states

$$\rho_x(m, F) = -\frac{1}{2} \sum_n \text{sign}(E_n) \frac{e^{-(x-E_nR)^2/\gamma^2}}{\gamma\sqrt{\pi}}, \qquad (20)$$

where $\gamma \simeq 2 - 3$ is a smearing parameter. We obtain $B_q = 0$ at $F = 0$, since the energy spectrum is symmetric for positive and negative energies. In order to obtain a rapid convergence in the sum, it is convenient to define the difference of $\rho_x(m, F)$ between finite $F$ and $F = 0$,

$$\tilde{\rho}_x(m, F) = \rho_x(m, F) - \rho_x(m, 0). \qquad (21)$$

Therefore, the baryon number defined by Eq. (16) is given by an alternative formulation

$$B_q(m, F) = \int_{-\infty}^{\infty} dx \tilde{\rho}_x(m, F). \qquad (22)$$

For a numerical computation, we restrict the range of the $x$-integral in a finite interval $x \in [-x_{\max}, x_{\max}]$.

$$B_q(m, F) = \int_{-x_{\max}}^{x_{\max}} dx \tilde{\rho}_x(m, F). \qquad (23)$$

We use $x_{\max} \simeq 20$ to obtain sufficient convergence. Carrying out numerical computation, we obtain the result which is given by

$$B_q(m, F) = \frac{1}{\pi} \left[ F - \frac{1}{2} \sin2F \right] \qquad (24)$$

for $F \in [F_0, F_0 + \pi]$. The value of $F_0$ depends on the quark mass; $F_0/\pi = -0.5, -0.695, -0.816$ and $-0.878$ for $mR = 0, 1, 2$ and 3, respectively. The critical chiral angle is given by a crossing point of the first $0^+$ state at $E = 0$ [8]. The value of the critical chiral angle is modified from $\pi/2$ due to asymmetry in the Dirac spectrum induced by finite quark mass. The fractional baryon number (24) is independent of quark mass $m$, and coincides with the opposite sign of Eq. (16) in the meson sector. Therefore, the fractional baryon number by meson and vacuum quark cancel each other for any quark mass.

The total baryon number is equal to $A$ for a quark droplet with $3A$ valence quarks for any chiral angle. Let us investigate the case of a baryon. The baryon number carried by valence and vacuum quarks for chiral angle $F_0 < F < 0$ is given as a sum of the valence quark and the fractional baryon number

$$B(m, F)_{\text{val+sea}} = 1 + \frac{1}{\pi} \left[ F - \frac{1}{2} \sin2F \right]. \qquad (25)$$

For $-\pi < F < F_0$, the valence quark is absent since the $0^+$ state is absorbed in the vacuum. Then, the baryon number is obtained by substituting $F + \pi$ instead of $F$ from a periodicity,

$$B(m, F)_{\text{val+sea}} = \frac{1}{\pi} \left[ (F + \pi) - \frac{1}{2} \sin2(F + \pi) \right]$$

$$= 1 + \frac{1}{\pi} \left[ F - \frac{1}{2} \sin2F \right]. \qquad (26)$$

Therefore, the total baryon number is one for any chiral angle. In the same way, the conservation of baryon number holds for any quark droplets.

Second, we present an analytical calculation of the baryon number (17) by using the Debye expansion. This method was originally developed in the chiral bag with massless quark [20,21]. We apply this formalism to the massive case. We rewrite Eq. (17) in terms of the quark propagator as in [20,21]. Then, we obtain

$$B_q(m, F) = \frac{1}{2} \lim_{\tau \to 0_+} \int_{-\infty}^{\infty} \frac{dx}{2i\pi} e^{i\tau x} \sum_{K=0}^{\infty} (2K + 1) \frac{d}{dx}$$

$$\times \ln \left[ \frac{S_K(m, F; ix)}{S_K^*(m, F; ix)} \right] \qquad (27)$$

Here, $S_K(m, F; p)$ is given as a product of the left hand side of the eigenvalue equation Eq. (14) for natural ($\kappa = +1$) and unnatural ($\kappa = -1$) states. For the sake of using the Debye expansion, we substitute an imaginary number $ix$ for $pR$ in the spherical Bessel function. We explicitly show

$$S_0(F; ix) = \cos F(j_1(ix)^2 - j_0(ix)^2)$$

$$- 2\frac{E \sin F + \bar{m}}{p} j_0(ix) j_1(ix) \qquad (28)$$

for $K = 0$, and

$$S_K(m, F; ix) = \left[ \frac{\bar{E}}{\bar{p}} \cos F(j_K(ix)^2 - j_{K+1}(ix) j_{K-1}(ix)) \right.$$

$$\left. + \frac{\sin F}{2K + 1} j_K(ix)(j_{K+1}(ix) + j_{K-1}(ix)) \right]^2$$

$$+ \left[ \frac{\bar{m}}{\bar{p}} (j_K(ix)^2 + j_{K+1}(ix) j_{K-1}(ix)) \right.$$

$$\left. - j_K(j_{K+1}(ix) - j_{K-1}(ix)) \right]^2. \qquad (29)$$

for $K \neq 0$. For a short notation, we write $\bar{m} = mR$, $\bar{p} = x$ and $\bar{E} = \sqrt{\bar{m}^2 + x^2}$. The $x$-integral picks up an eigenvalue as a residue instead of solving directly the boundary condition. We note that Eq. (27) is an exact formulation, but it is not practical to consider the $x$-integral and the sum over $K$ without approximation. In order to obtain the final result, it is sufficient to use an asymptotic behavior of the quark energy spectrum.

In the following manipulation, we discuss the $K = 0$ and $K \geq 1$ components separately for convenience. First, we consider the $K = 0$ component, which is defined by

$$B_q^{(K=0)}(m, F) = \frac{1}{2} \lim_{\tau \to 0_+} \int_{-\infty}^{\infty} \frac{dx}{2i\pi} e^{i\tau x} \frac{d}{dx} \ln \left[ \frac{S_0(m, F; ix)}{S_0^*(m, F; ix)} \right]. \qquad (30)$$

It is convenient to write the function $S_0(m, F; ix)$ in a polar coordinate

$$S_0(m, F; ix) = R_0(x)e^{i\Phi_0(x)}. \tag{31}$$

By using the asymptotic form

$$j_1(ix) \simeq \frac{i}{2x}(e^x + e^{-x}), \tag{32}$$

we obtain

$$\Phi_0(x) \simeq \arctan\left[\frac{e^{2x} - e^{-2x}}{e^{2x} + 2^{-2x}}\frac{\sqrt{x^2 + \bar{m}^2}\sin F + \bar{m}}{x\cos F}\right]. \tag{33}$$

Then, it is straightforward to perform the integral, giving

$$B_q^{(K=0)}(m, F) \simeq \frac{F}{\pi}, \tag{34}$$

for any quark mass $m$.

Second, for $K \geq 1$, we use the Debye expansion for the modified Bessel function in estimation of Eq. (29). In our notation, the modified Bessel function $I_\nu(x)$ is defined by

$$j_K(ix) = \sqrt{\frac{\pi}{2x}}i^K I_\nu(x), \tag{35}$$

with $\nu = K + 1/2$. The Debye expansion is a uniform asymptotic expansion with no constraint between $\nu$ and $x$. The Debye expansion gives

$$I_\nu(x) \simeq \frac{\Gamma(1/2)}{\pi\sqrt{2t}}e^{f_\nu(x)}, \tag{36}$$

in the lowest order, where we define $t = \sqrt{\nu^2 + t^2}$ and $f_\nu(x) = t - \nu\sinh^{-1}(\nu/x)$.

Now, let us define

$$B_q^{(K\geq 1)}(m, F) = \frac{1}{2}\lim_{\tau\to 0_+}\int_{-\infty}^{\infty}\frac{dx}{2i\pi}e^{i\tau x}\sum_{K\geq 1}\frac{d}{dx}$$
$$\times \ln\left[\frac{S_K(m, F; ix)}{S_K^*(m, F; ix)}\right]. \tag{37}$$

Here, we write $S_K(m, F; ix)$ in a polar coordinate

$$S_K(m, F; ix) = R_K(x)e^{i\Phi_K(x)}. \tag{38}$$

Then, we apply the Debye expansion (35). After a little tedious calculation, we pick up only the terms of the leading order of $\bar{m}$, and obtain

$$\Phi_K(x) \simeq \arctan\left[-\frac{1}{2K+1}\left(\frac{\nu x}{2t^3}\sin 2F - \frac{4\nu\bar{m}}{t}\right)\right]. \tag{39}$$

We consider that the quark mass $m$ is smaller as compared with the momentum $p$. Indeed, this is a good approximation, since we are interested in the asymptotic behavior of the quark energy spectrum. The $x$-integral for $\Phi_K(x)$ is integrated out and the integral is determined only by $\Phi_K(\infty) - \Phi_K(-\infty)$. Concerning the term which has no quark mass, we use an identity

$$\sum_{K\geq 1}\frac{\cos(\tau\nu z)}{\nu} = -\ln\tan(\tau z/4) - 2\cos(\tau z/2), \tag{40}$$

and

$$\lim_{\tau\to 0_+}\int_{-\infty}^{\infty}\frac{dx}{2\pi i}e^{-i\tau x}\sum_{K\geq 1}\frac{d}{dx}\left[-i\frac{\nu x}{2t^3}\right] = -\frac{1}{2\pi}. \tag{41}$$

Concerning the term proportional to $\bar{m}$, we use

$$\lim_{\tau\to 0_+}\int_{-\infty}^{\infty}\frac{dx}{2\pi i}e^{-i\tau x}\sum_{K\geq 1}\frac{d}{dx}\frac{4\nu\bar{m}}{t} = 0, \tag{42}$$

which indicates the mass term does not contribute to the integral. Consequently, we obtain the result

$$B_q^{(K\geq 1)}(m, F) = -\frac{\sin 2F}{2\pi}. \tag{43}$$

Finally, by Eqs. (34) and (43), the fractional baryon number obtained in the analytical procedure coincides with the numerical result Eq. (24).

## IV. CHIRAL CASIMIR ENERGY

The chiral Casimir energy arises as a result of the modification of quark energy levels for finite pion cloud. Following the regularization scheme in the baryon number, the chiral Casimir energy is defined by

$$E_C(m, F) = \frac{1}{2}\langle\mathcal{O}|[\psi^\dagger, H\psi]|\mathcal{O}\rangle$$
$$= -\frac{1}{2}\lim_{t\to 0_+}\sum_n \text{sgn}(E_n)E_n e^{-t|E_n|}, \tag{44}$$

as in Refs. [13–16,19–21,23,27,30].

We show the numerical procedure by using the Strutinsky's smearing method. By using the Gaussian function (20), we rewrite Eq. (44) as

$$E_C(m, F) = \int_{-\infty}^{\infty}dx x\rho_x(m, F). \tag{45}$$

By choosing the reference point at $F = 0$, we obtain an alternative formulation

$$\Delta E_C(m, F) = E_C(m, F) - E_C(m, 0) = \int_{-\infty}^{\infty}dx x\tilde{\rho}_x(m, F). \tag{46}$$

Restricting the range of integral in a finite range of $[-x_{\max}, x_{\max}]$ with sufficient convergence, we obtain,

$$\Delta E_C(m, F) = \int_{-x_{\max}}^{x_{\max}}dx x\tilde{\rho}_x(m, F). \tag{47}$$

In order to achieve a rapid convergence of the integral, it is much more practical to use the partial integration.[1] We

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[1]We use a relationship $\int_{-\infty}^{\infty}dx x\tilde{\rho}_x(m, F) = -\int_{-\infty}^{\infty}dx\frac{x^2}{2}\tilde{\rho}_x'(m, F)$ with $\tilde{\rho}_x(m, F) \to 0$ for $|x| \to \pm\infty$.

mention that we need $x_{max} \sim 40$ to obtain the convergence in the calculation. More states are necessary for the chiral Casimir energy as compared for the baryon number ($x_{max} \sim 20$).

For massless quarks, it has been known that there is a logarithmic divergence term in the Casimir energy (44). It was shown that the divergence term is proportional to $\sin^2 F$ [13–16,20–23,27,30]. Here, we also derive the logarithmic divergence term for massive quark by using the Debye expansion developed in [20,21]. Following the procedure in [20,21], we rewrite the Casimir energy as

$$\Delta E_C(m, F) = \frac{1}{2R} \lim_{\tau \to 0_+} \int_{-\infty}^{\infty} \frac{dx}{2\pi} x e^{i\tau x} \sum_{K=0}^{\infty} (2K + 1) \frac{d}{dx}$$
$$\times \ln \left| \frac{S_K(m, F; ix)}{S_K(m, 0; ix)} \right|. \tag{48}$$

Just as in the case of the baryon number, we consider the calculation for $K = 0$ and $K \geq 1$, separately for convenience.

First, we investigate the $K = 0$ sector,

$$\Delta E_C^{(K=0)}(m, F) = \frac{1}{2R} \lim_{\tau \to 0_+} \int_{-\infty}^{\infty} \frac{dx}{2\pi} x e^{i\tau x} \frac{d}{dx}$$
$$\times \ln \left| \frac{S_0(m, F; ix)}{S_0(m, 0; ix)} \right|. \tag{49}$$

We apply the asymptotic form of the spherical Bessel function (32). By expanding $|S_0(m, F; ix)/S_0(m, 0; ix)|$ in the lowest order of $m$, we obtain

$$\Delta E_C^{(K=0)}(m, F) \simeq \frac{1}{2R} \lim_{\tau \to 0_+} \int_{-\infty}^{\infty} \frac{dx}{2\pi} x e^{i\tau x} \frac{d}{dx} \left[ \cos^2 F \right.$$
$$\left. + \sin F \tanh^2(2x) + \sin F \frac{2\bar{m}}{x} \tanh^2(2x) \right]. \tag{50}$$

The first two terms in the integrand, which does not contain quark mass, are given in [20,21]. The integral of the third term proportional to $\bar{m}$ is easily shown to be equal to zero. Consequently, we obtain the result

$$\Delta E_C^{(K=0)}(m, F) \simeq \frac{F^2}{4\pi R}. \tag{51}$$

The divergence term does not appear in the $K = 0$ sector.

Second, we investigate the sum over $K \geq 1$,

$$\Delta E_C^{(K \geq 1)}(m, F) = \frac{1}{2R} \lim_{\tau \to 0_+} \int_{-\infty}^{\infty} \frac{dx}{2\pi} x e^{i\tau x} \sum_{K=1}^{\infty} (2K + 1) \frac{d}{dx}$$
$$\times \ln \left| \frac{S_K(m, F; ix)}{S_K(m, 0; ix)} \right|. \tag{52}$$

After a little troublesome calculation by using the Debye expansion and expanding in the lowest order of $\bar{m}$, we obtain

$$\ln \left| \frac{S_K(m, F; ix)}{S_K(m, 0; ix)} \right| \simeq -\sin^2 F \left[ \frac{(1 - L_+ L_-)^2}{(1 + L_+^2)(1 + L_-^2)} + \frac{1}{(2K+1)^2} \frac{(L_+ - L_-)^2}{(1 + L_+^2)(1 + L_-^2)} \right] + \frac{\bar{m}^2}{2x^2} \sin^2 F \frac{(1 - L_+ L_-)^2}{(1 + L_+)^2(1 + L_-)^2}$$
$$= -\frac{1}{2} \sin^2 F \frac{1}{t^2} \left[ \left( 2 - \frac{\nu^2}{t^2} \right) + \left( 1 - \frac{\nu^2}{t^2} \right) \left( 2 - \frac{3\nu^2}{t^2} \right) \frac{1}{t} + \mathcal{O}(1/t^2) \right]$$
$$+ \frac{\bar{m}^2}{2} \sin^2 F \frac{1}{4} \left( 1 - \frac{\nu^2}{t^2} \right) \frac{1}{t^2} \left[ 1 + \left( 1 - \frac{3\nu^2}{t^2} \frac{1}{t} \right) + \mathcal{O}(1/t^2) \right]. \tag{53}$$

Here, we define $L_{\pm} = I_{\nu \pm 1}/I_{\nu}$ for a short notation. The first term, which does not contain quark mass, coincides with the case of massless quarks [20,21]. The second term proportional to $\bar{m}$ is further analyzed as followings. By taking a derivative with respect to $x$, we obtain

$$x \frac{d}{dx} \ln \left| \frac{S_K(m, F; ix)}{S_K(m, 0; ix)} \right| \simeq -\frac{1}{2} \sin^2 F \left( -4 \frac{x^4}{t^6} - \frac{6}{t^3} + \frac{31\nu^2}{t^5} - \frac{46\nu^4}{t^7} + \frac{21\nu^6}{t^9} \right)$$
$$+ \frac{\bar{m}^2}{8} \sin^2 F \left( -\frac{2}{t^2} - \frac{3}{t^3} + \frac{6\nu^2}{t^4} + \frac{23\nu^2}{t^5} - \frac{4\nu^4}{t^6} - \frac{41\nu^4}{t^7} + \frac{21\nu^6}{t^9} \right). \tag{54}$$

Then, introducing new variables $h = t/\nu$ and $z = x/\nu$, we have

$$\Delta E_C^{(K \geq 1)}(m, F) \simeq \frac{\sin^2 F}{\pi R} \lim_{\tau \to 0_+} \int_0^{\infty} dz \frac{2z^4}{h^6} \sum_{K=1}^{\infty} \cos(\tau \nu z) + \frac{\sin^2 F}{\pi R} \lim_{\tau \to 0_+} \int_0^{\infty} dz \left( \frac{6}{h^3} - \frac{31}{h^5} + \frac{46}{h^7} - \frac{21}{h^9} \right) \sum_{K=1}^{\infty} \frac{\cos(\tau \nu z)}{\nu}$$
$$+ \frac{\bar{m}^2}{8\pi R} \sin^2 F \lim_{\tau \to 0_+} \int_0^{\infty} dz \left( -\frac{1}{h^2} + \frac{6}{h^4} - \frac{4}{h^6} \right) \sum_{K=1}^{\infty} \cos(\tau \nu z)$$
$$+ \frac{\bar{m}^2}{8\pi R} \sin^2 F \lim_{\tau \to 0_+} \int_0^{\infty} dz \left( -\frac{1}{h^3} + \frac{23}{h^5} - \frac{41}{h^7} + \frac{21}{h^9} \right) \sum_{K=1}^{\infty} \frac{\cos(\tau \nu z)}{\nu}. \tag{55}$$

The summation over $K \geq 1$ and the integral is achieved by the formula Eq. (40). Finally, we arrive at the result

$$\Delta E_C^{(K \geq 1)}(m, F) \simeq \left[ -\frac{3}{16R} + \left( -\frac{2}{15\pi R} \ln\tau + \frac{12\ln 2 - 9}{30\pi R} \right) \right.$$
$$+ \frac{\bar{m}^2}{64R} - \frac{1}{15} \frac{\bar{m}^2}{8\pi R} (\ln\tau + 1 - 4\ln 2) \right]$$
$$\times \sin^2 F \qquad (56)$$

Therefore, we obtain a logarithmic divergence term with coefficient of $\sin^2 F$ for a finite quark mass. Our result is consistent with that for the massless quark [20,21]. The logarithmic divergence appears as a consequence of the UV behavior of the quark modes. On the other hand, the mass effect influence the IR behavior. Therefore, it is a natural consequence that the chiral Casimir energy for finite quark mass has a logarithmic divergence as in the massless case.

Now, in order to remove the logarithmic divergence, we subtract the second derivative of the chiral Casimir energy with respect to the chiral angle at the reference point $F = 0$, and we obtain a finite contribution [20–22,28–30]

$$E_C^{\text{fin}}(m, F) = \Delta E_C(m, F) - \frac{1}{2} \sin^2 F \left. \frac{\partial^2 E_C(m, F)}{\partial F^2} \right|_{F=0}.$$
$$(57)$$

The regularization scheme in Eq. (57) was shown to consistent with the condition that axial flux through the bag surface remains finite at the zero bag radius limit [19,22,23]. In addition, the linear divergence in axial flux can be subtracted by the this regularization. The same regularization scheme was adopted to the massive quark case.

In Fig. 1, we show the chiral Casimir energy $E_C^{\text{fin}}$ as a function of the chiral angle $F$. The lines are distinguished by the quark masses $mR = 0$ (dashed line), $mR = 1$ (solid line), respectively. It is a remarkable point that the Casimir energy for the massive quark takes a nonzero value at $F = -\pi$, while that of the massless quark becomes zero at $F = -\pi$. This is because that the massive quark has an asymmetric energy spectrum between $F = 0$ and $-\pi$, while the massless quark has a symmetric spectrum, as shown in Fig. 1 in [8].

Here, we recall the Cheshire Cat picture in the chiral bag model with massless quarks. There, the continuous transformation from the chiral bag to the Skyrmion was induced in the limit of small bag radius [13–15,31–34]. This picture seems not to be applied to the case of the massive quarks, since the chiral Casimir energy takes a finite value at the zero bag radius, or $F = -\pi$. However, this observation does not make us abandon our discussion. Indeed, in our formulation, the quark mass is a dynamical variable which should be determined by the energy variation. Our previous result suggested that the chiral symmetry was
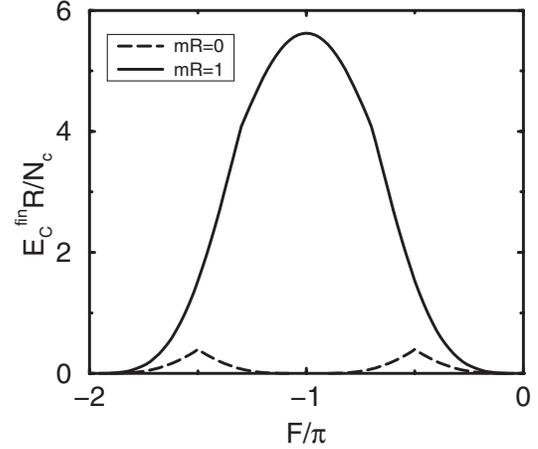


FIG. 1. The Casimir energy as a function of the chiral angle at the bag surface. The dashed and solid lines indicate the quark masses $mR = 0$ and $mR = 1$, respectively.

restored in the limit of small bag radius [8]. Therefore, the NJL chiral bag model holds the Cheshire Cat picture.

## V. SUMMARY

We discussed the chiral vacuum properties in the hedgehog configuration in the chiral bag model which contained massive quarks. This study gave a detailed description in order to complete the discussion in the chiral bag model with the NJL interaction, which was called as the NJL chiral bag model [8]. There, the chiral vacuum polarization for the finite quark mass should be understood in order to obtain the stable state for dynamical chiral symmetry breaking. In the present paper, we took the dynamical quark mass as a constant value, and investigated the chiral vacuum polarization effects, such as the fractional baryon number and the chiral Casimir energy, for the finite quark mass with an emphasis on the technical procedure.

We showed the numerical calculation by the Strutinsky's smearing method and the analytical technique by the Debye expansion. It was shown numerically and analytically that the fractional baryon number carried by vacuum massive quarks inside the bag was canceled that of the $\pi$ meson outside the bag. Therefore, the total baryon number is exactly conserved. By using the Debye expansion, it was shown that the chiral Casimir energy had a logarithmic divergence term. The chiral Casimir energy was obtained numerically by the Strutinsky's smearing method by removing the divergence term. It is a point different from massless quark that the chiral Casimir energy for massive quark had a nonzero value at the chiral angle $F = -\pi$.

As further development, we plan to discuss a fully chiral symmetric equation of motion for a finite quark mass. In our present analysis of the NJL chiral bag model, we performed a mean field approximation only in the scalar channel in the four point interaction [8]. However, in the hedgehog configuration, it may be allowed to have a finite

expectation value of $\bar{q}q$, not only in the sigma channel, but also in the pion channel. In the latter case, the expectation value of $\bar{q}q$ with pion quantum number can be given as a finite value in a basis set of the hedgehog quark wave function. This study is now in progress. The quantization of the hedgehog configuration for massive quark is an important subject in order to obtain the state with definite spin and isospin. It is also an interesting subject to include the strangeness sector in our framework. These subjects are left as future works.

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