Method of characteristics and solution of DGLAP evolution equation in leading and next to leading order at small x

R. Baishya*

Physics Department, J. N. College, Boko-781123, Assam, India

J. K. Sarma

Physics Department, Tezpur University, Napaam-784028, Assam, India

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In this paper the singlet and nonsinglet structure functions have been obtained by solving Dokshitzer, Gribov, Lipatov, Altarelli, Parisi (DGLAP) evolution equations in leading order and next to leading order at the small x limit. Here we have used Taylor series expansion and then the method of characteristics to solve the evolution equations. We have also calculated t and x evolutions of the deuteron structure function and the results are compared with the New Muon Collaboration data.

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I. INTRODUCTION

The solutions of DGLAP evolution equations give quark and gluon structure functions that ultimately produce proton, neutron, and deuteron structure functions [1–9]. Though various methods like brute force approaches, the Melline moments method, etc. are available in order to obtain a numerical solution of DGLAP evolution equations, exact analytical solutions of singlet equations are not known [10,11]. Because the evolution equations are partial differential equations (PDE), their ordinary solutions are not unique solutions, but rather a range of solutions. On the other hand, this limitation can be overcome by using the method of characteristics. The method of characteristics is an important technique for solving initial value problems of first order PDE. It turns out that if we change coordinates from (x, t) to suitable new coordinates (S, τ) then the PDE becomes an ordinary differential equation (ODE). Then we can solve the ODE by the standard method.

II. THEORY

The DGLAP evolution equations for singlet and nonsinglet structure functions in leading order (LO) and next to leading order (NLO) in standard form [5–7] are

$$\frac{\partial F_2^{\mathrm{S}}}{\partial t} - \frac{\alpha_{\mathrm{S}}(t)}{2\pi}\left[\frac{2}{3}\{3 + 4\ln(1-x)\}F_2^{\mathrm{S}}(x,t) + I_1^{\mathrm{S}}(x,t) + I_2^{\mathrm{S}}(x,t)\right] = 0, \quad (1)$$

$$\frac{\partial F_2^{\mathrm{NS}}}{\partial t} - \frac{\alpha_{\mathrm{S}}(t)}{2\pi}\left[\frac{2}{3}\{3 + 4\ln(1-x)\}F_2^{\mathrm{NS}}(x,t) + I_1^{\mathrm{NS}}(x,t)\right] = 0, \quad (2)$$

$$\frac{\partial F_2^{\mathrm{S}}}{\partial t} - \frac{\alpha_{\mathrm{S}}(t)}{2\pi}\left[\frac{2}{3}\{3 + 4\ln(1-x)\}F_2^{\mathrm{S}}(x,t) + I_1^{\mathrm{S}}(x,t) + I_2^{\mathrm{S}}(x,t)\right] - \left(\frac{\alpha_{\mathrm{S}}(t)}{2\pi}\right)^2 I_3^{\mathrm{S}} = 0, \quad (3)$$

$$\frac{\partial F_2^{\mathrm{NS}}}{\partial t} - \frac{A_f}{t}[\{3 + 4\ln(1-x)\}F_2^{\mathrm{NS}}(x,t) + I_1^{\mathrm{NS}}(x,t)] - \left(\frac{\alpha_{\mathrm{S}}(t)}{2\pi}\right)^2 I_2^{\mathrm{NS}} = 0, \quad (4)$$

where I_1^{S}, I_2^{S}, I_3^{S}, I_1^{NS}, I_2^{NS} are integral functions.

Using Taylor's expansion series we can rewrite $F_2^S(\frac{x}{\omega}, t)$ and $G(\frac{x}{\omega}, t)$ as [7,8]

$$F_2^{\mathrm{S}}\left(\frac{x}{\omega}, t\right) = F_2^{\mathrm{S}}(x,t) + \frac{xu}{1-u}\frac{\partial F_2^{\mathrm{S}}(x,t)}{\partial x}$$

$$\text{and} \quad G\left(\frac{x}{\omega}, t\right) = G(x,t) + \frac{xu}{1-u}\frac{\partial G(x,t)}{\partial x}.$$

Since x is small in our region of discussion, the terms containing x^2 and higher powers of x are neglected.

In order to solve Eq. (1), we need to relate the singlet distribution $F_2^{\mathrm{S}}(x,t)$ with the gluon distribution $G(x,t)$. For small x and high Q^2, the gluon is expected to be more dominant than the sea quark. But for lower Q^2, there is no such clear cut distinction between the two. Hence, for simplicity, let us assume $G(x,t) = k(x)F_2^{\mathrm{S}}(x,t)$, where $k(x)$ is a suitable function of x or may be a constant. We may assume $k(x) = k$, ax^b, ce^{dx} where k, a, b, c, d are suitable parameters which can be determined by phenomenological analysis. But the possibility of the breakdown of the relation cannot be ruled out either [8,9].

Performing u integrations, Eq. (1) becomes

$$-t\frac{\partial F_2^{\mathrm{S}}(x,t)}{\partial t} + A_f L(x)F_2^{\mathrm{S}}(x,t) + A_f M(x)\frac{\partial F_2^{\mathrm{S}}}{\partial x} = 0. \quad (5)$$

To introduce the method of characteristics, let us consider two new variables (S, τ) instead of (x, t), such that $\frac{dt}{dS} = -t$

*Email address: rjitboko@yahoo.co.in

and $\frac{dx}{dS} = A_f M(x)$. Putting these equations in (5), we get $\frac{dF_2^S(S,\tau)}{dS} + L(S,\tau)F_2^S(S,\tau) = 0$, which can be solved as $F_2^S(S,\tau) = F_2^S(\tau)(\frac{t}{t_0})^{L(S,\tau)}$.

Now we have to replace the coordinate system (S, τ) to (x, t) with the input function $F_2^S(\tau) = F_2^S(x, t_0)$ and we will get the t evolution of the singlet structure function in the LO as $F_2^S(x,t) = F_2^S(x,t_0)(\frac{t}{t_0})^{A_f L(x)}$. Similarly the x evolution of the singlet structure function will be

$$F_2^S(x,t) = F_2^S(x_0,t)\exp\left[-\int_{x_x}^{x}\frac{L(x)}{M(x)}dx\right].$$

Hence the t and x evolution of deuteron structure functions in LO can be obtained as

$$F_2^d(x,t) = F_2^d(x,t_0)\left(\frac{t}{t_0}\right)^{A_f L(x)}$$

$$\text{and}\quad F_2^d(x,t) = F_2^d(x_0,t)\exp\left[-\int_{x_x}^{x}\frac{L(x)}{M(x)}dx\right],$$

where $F_2^d(x,t_0) = \frac{5}{2}F_2^S(x,t_0)$ and $F_2^d(x_0,t) = \frac{5}{2}F_2^S(x_0,t)$ are input functions.

In the NLO, the t and x evolution of deuteron structure functions will be obtain as

$$F_2^d(x,t) = F_2^d(x,t_0)\left(\frac{t}{t_0}\right)^{(3/2)A_f[L(x)+T_0L_1(x)]},$$

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\left[-\int_{x_x}^{x}\frac{L(x)+T_0L_1(x)}{M(x)+T_0M_1(x)}dx\right],$$

where

$$L_1(x) = B_1(x) + k(x)B_2(x) + B_4(x)\partial k(x)/\partial x;$$

$$M_1(x) = B_3(x) + k(x)B_4(x);$$

$$B_1(x) = x\int_0^1 f(\omega)d\omega - \int_0^x f(\omega)d\omega + \frac{4}{3}N_f\int_x^1 F_{qq}(\omega)d\omega;$$

$$B_2(x) = \int_x^1 F_{qg}^S(\omega)d\omega;$$

$$B_3(x) = x\int_x^1\left[f(\omega) + \frac{4}{3}N_f F_{qg}^S(\omega)\right]\frac{1-\omega}{\omega}d\omega;$$

$$B_4(x) = x\int_x^1\frac{1-\omega}{\omega}F_{qg}^S(\omega)d\omega.$$

Here we consider an extra assumption $(\frac{\alpha_S(t)}{2\pi})^2 = T_0(\frac{\alpha_S(t)}{2\pi})$, where T_0 is a numerical parameter. By a suitable choice of T_0 we can reduce the error to a minimum.

III. RESULTS AND DISCUSSION

In this paper, we compare our result of t and x evolution of the deuteron structure function F_2^d measured by the New Muon Collaboration in muon deuteron deep inelastic scattering with incident momentum 90, 120, 200, 280 GeV [12]. For quantitative analysis, we consider the QCD cutoff parameter $\Lambda_{\overline{MS}} = 0.323$ GeV for $\alpha_S(M_Z^2) = 0.119 \pm 0.002$ and $N_f = 4$. It is observed that our result is very sensitive to arbitrary parameters k, a, b, c, and d. In Fig. 1, for t evolution, we have plotted computed values of F_2^d

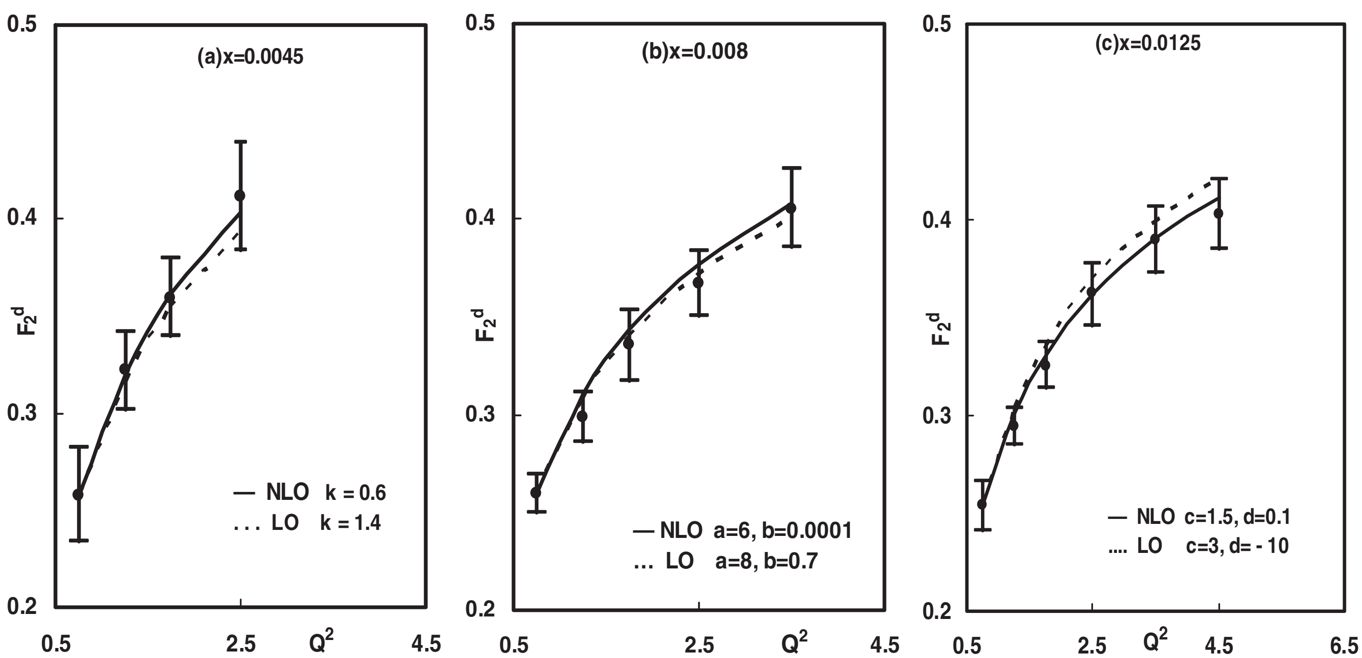

FIG. 1. t evolution of the deuteron structure function in LO and NLO.

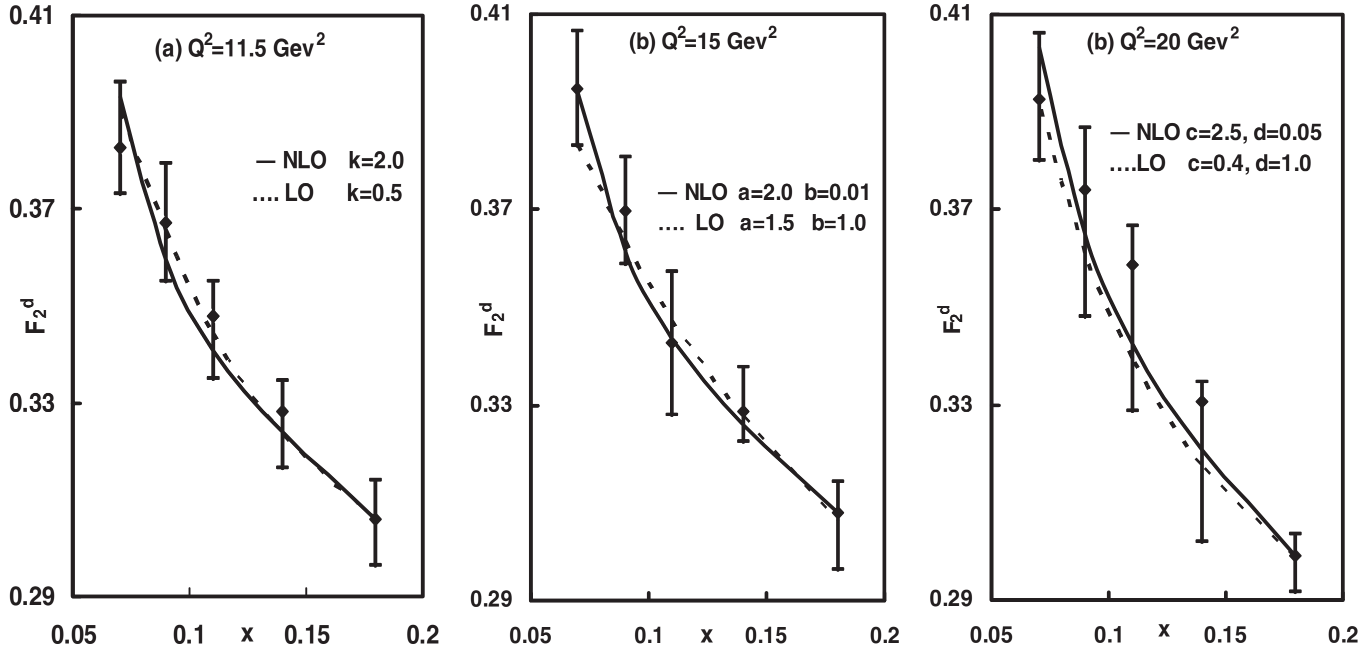

FIG. 2. x evolution of the deuteron structure function in LO and NLO.

against Q^2 values for a fixed x in LO and NLO. Here the solid lines represent the best fitting curves in NLO and the dotted lines represent those for LO evolutions. In Fig. 2, for x evolution, we have plotted computed values of F_2^d against the x values for a fixed Q^2. Here also the solid lines represent the best-fit curves for NLO and the dotted lines represent for LO evolutions.

Though there are various numerical methods to solve DGLAP evolution equations to obtain quark and gluon structure functions, the method of characteristics to solve these equations analytically is also a viable alternative. Though mathematically vigorous, it changes the integro-differential equations into ODE and then makes it possible to obtain unique solutions.

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