Genus dependence of superstring amplitudes

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The problem of the consistency of the finiteness of the supermoduli space integral in the limit of vanishing super-fixed point distance and the genus-dependence of the integral over the super-Schottky coordinates in the fundamental region containing a neighborhood of $|K_n| = 0$ is resolved. Given a choice of the categories of isometric circles representing the integration region, the exponential form of bounds for superstring amplitudes is derived.

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I. INTRODUCTION

Superstring amplitudes are finite in the degeneration limits [1–3] of super-Riemann surfaces and do not require regularization. While a minimum-length condition is satisfied by the categories of isometric circles [4]

$$
\begin{aligned}
&\text{(i)}\ \frac{\epsilon_0}{g^{1-2q'}} \leq |K_n| \leq \frac{\epsilon_0'}{g^{1-2q'}} \qquad \delta_0 \leq |\xi_{1n} - \xi_{2n}| \leq \delta_0' \\
&\quad 0 \leq q' \leq \frac{1}{2} \\
&\text{(ii)}\ \frac{\epsilon_0}{g^{1-2q'}} \leq |K_n| \leq \frac{\epsilon_0'}{g^{1-2q'}} \qquad \frac{\delta_0}{g^q} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{g^q} \\
&\quad 0 \leq q \leq q' < \frac{1}{2} \\
&\text{(iii)}\ \epsilon_0 \leq |K_n| \leq \epsilon_0' \qquad \frac{\delta_0}{\sqrt{g}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{\sqrt{g}}
\end{aligned}
\tag{1.1}
$$

the sizes of the handles decrease to zero even in the intrinsic metric in the following ranges for the parameters [5]

$$
\begin{aligned}
&\frac{\epsilon_0}{n^{q''}} \leq |K_n| \leq \frac{\epsilon_0'}{n^{q''}}, \qquad q'' > 1, \qquad n = 1, 2, 3, \ldots \\
&\frac{\delta_0}{n^{q'''}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{n^{q'''}} \qquad n = 1, 2, 3, \ldots.
\end{aligned}
\tag{1.2}
$$

Estimates of the amplitudes have been derived for the configurations of isometric circles in Eq. (1.1) [4]. The conditions for the fundamental region require combinatorial factors which reduce the integrals by a factorial function of the genus. For each spin structure, an exponential genus-dependence is found only if the leading-order dependence of the fixed-point integral is also factorial. Nevertheless, it has been established that the superstring integral must be finite in each of the degeneration limits $|K_n| \to 0$, and $|\xi_{1n} - \xi_{2n}| \to 0$ [3]. This property holds since any potential divergences in these limits are cancelled in a sum over the spin structures. However, given that it is necessary to maintain an exponential genus-dependence for the entire superstring amplitude after the sum over spin structures, it must be demonstrated that the genus-dependence of the fixed-point integrals does not affect the finiteness in these limits. While the result does not immediately follow from the form of the superstring integrand, it can be proven using the finiteness of the supermoduli space integral in the coincidence limit of the handles.

From the formula for the handle operator, the limit of the product of two operators for approaching handles at large genus shall be evaluated in Sec. II. The effect of restricting the values of the parameter fixed-point integral will be considered in Sec. III. Finally, in Sec. IV, the exponential bound on superstring amplitudes in the super-Schottky parametrization shall be verified.

II. THE HANDLE OPERATOR FOR SUPERSTRING THEORY

From the representation of Riemann surfaces as states in the Hilbert space of a free fermion theory [6], the handle operator is defined such to be a map from the g-vacuum to the $g + 1$-vacuum.

In the Reggeon formalism, the loop is constructed by sewing coordinate fields $\partial X^{(B)}$ and $\partial X^{(C)}$. In the coherent state representation, the unit operator [7] is

$$
\begin{aligned}
\mathbb{I}_{BC} = \int d\tilde{\alpha}_0 \prod_{n=1}^{g} d\hat{a}_n^* d\hat{a}_n \\
\times \exp\bigg\{ - \oint_{C_0} \frac{dx}{2\pi i} \oint_{C_0} \frac{dy}{2\pi i} \partial_y \bigg[\hat{X}\bigg(\frac{1}{y}\bigg) \bigg] \\
\times \ln(1 - yx) \partial \hat{X}(x) |\hat{\alpha}_n, \hat{\alpha}_0\rangle_{BC} \langle \hat{\alpha}_0, \hat{\alpha}_n| \bigg\}
\end{aligned}
\tag{2.1}
$$

where the coherent states have the properties, $\alpha_n |\hat{\alpha}_n, \hat{\alpha}_0\rangle = \hat{\alpha}_n |\hat{\alpha}_n, \hat{\alpha}_0\rangle$ for $n \geq 0$ and $\hat{\alpha}_n = \sqrt{|n|}\hat{a}_n$ and $\hat{a}_{-n} = \hat{a}_n^*$.

The coordinate part of the handle operator is

$$\Omega_{X,AD} = (\det C)^{-d}\,{}_D\langle q=0|\delta(\alpha_0^{(A)} - \alpha_0^{(D)}) \int dk\colon \exp\Big\{ i\pi k^2\tau + 2\pi i k \oint_{C_0} \frac{dz}{2\pi i}\Big[\partial_z\Big[X^{(A)}\Big(\frac{1}{z}\Big)\Big]\varphi(\Gamma(z), z_0) + \partial X^{(D)}(z)\varphi(z, z_0)\Big]$$
$$+ \oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\Big[\frac{1}{2}\partial X^{(D)}(x)\partial X^{(D)}(y)[\ln E(x,y) - \ln(x-y)] + \frac{1}{2}\partial_x\Big[X^{(A)}\Big(\frac{1}{x}\Big)\Big]\partial_y\Big[X^{(A)}\Big(\frac{1}{y}\Big)\Big]$$
$$\times [\ln xyE(\Gamma(x),\Gamma(y) - \ln(x-y)] + \partial X^{(D)}(x)\partial_y\Big[X^A\Big(\frac{1}{y}\Big)\Big]\ln yE(x,\Gamma(y))\Big]\Big\}\colon |q=0\rangle_A. \tag{2.2}$$

where Γ is a projective transformation. Since

$$\prod_{n=1}^{\infty} d\hat{a}_n^* d\hat{a}_n \exp[-\hat{a}_n^* C_{nm}\hat{a}_m + \hat{a}_n^* A_n + B_n\hat{a}_n] = (\det C)^{-1}\det(BC^{-1}A) \tag{2.3}$$

where

$$A_n = -\oint_{C_0}\frac{dw}{2\pi i}\oint\frac{dz}{2\pi i}\sqrt{n}w^{-n-1}\Big\{\ln(1-zw)\partial X^D(z) + \ln\Big(\frac{T\Gamma(z)-\Gamma(w)}{\sqrt{\partial T\Gamma(z)\partial\Gamma(w)}}\Big)\partial_z\Big[X^{(A)}\Big(\frac{1}{z}\Big)\Big] - \frac{k}{z}\ln\Big(\frac{\Gamma T(z) - w}{\sqrt{\partial\Gamma T(z)}}\Big)\Big\}$$
$$B_m = \oint_{C_0}\frac{dw}{2\pi i}\oint_{C_0}\frac{dz}{2\pi i}\sqrt{m}z^{-m-1}\Big\{\ln(1-zw)\partial_w\Big[\partial X^{(A)}\Big(\frac{1}{w}\Big)\Big] + \ln\Big(\frac{T(z)-w}{\sqrt{\partial T(z)}}\Big)\partial X^{(D)}(w) - \frac{k}{z}\ln\Big(\frac{T(w)-\Gamma(z)}{\sqrt{\partial T(w)}\sqrt{\Gamma(z)}}\Big)\Big\} \tag{2.4}$$

and, given that the elements of the matrix C are

$$C_{nm} = \oint_{C_0}\frac{dx}{2\pi i}\oint\frac{dy}{2\pi i}\Big\{\ln\Big(\frac{T(x)-T(y)}{\sqrt{\partial T(x)\partial\Gamma(y)}}\Big) - \ln(1-xy)\Big\}, \tag{2.5}$$

it follows that

$$\Omega_{X,AD} = {}_D\langle q=0|\delta(\alpha_0^{(A)} - \alpha_0^{(D)})(\det C)^{-d}\int dk\colon \exp\Big\{-k^2[\ln(T^{k+1}(0), T^{-1}\Gamma(0), T^k(0), \Gamma(0)) - \ln\sqrt{\partial T(0)}]$$
$$+ k\oint_{C_0}\frac{dz}{2\pi i}\Big\{\partial X^{(D)}(z)\Big[\ln\frac{\prod_{k=0}^{\infty}(T^{k+1}(0), T^{-1}(z), T^k(0), \Gamma(0))}{\prod_{k=0}^{\infty}(T^k(x), T^{-1}\Gamma(0), T^k(0), \Gamma(0))}\Big]$$
$$+ \partial_z\Big[X^{(A)}\Big(\frac{1}{z}\Big)\Big]\Big[\ln\frac{\prod_{k=0}^{\infty}(T^{k+1}(0), T^{-1}(z), T^k(0), \Gamma(0))}{\prod_{k=0}^{\infty}(T^k(x), T^{-1}\Gamma(0), T^k(0), \Gamma(0))} + \ln\sqrt{\partial T\Gamma(z)}\Big]\Big\}$$
$$+ \oint_{C_0}\frac{dz}{2\pi i}\oint_{C_0}\frac{dw}{2\pi i}\Big\{\partial X^{(D)}(z)\partial X^{(D)}(w)\ln\frac{\prod_{k=0}^{\infty}(T^{k+1}(0), T^{-1}(z), T^k(0), \Gamma(0))}{\prod_{k=0}^{\infty}(T^k(x), T^{-1}\Gamma(0), T^k(0), \Gamma(0))} + \partial_z\Big[X^{(A)}\Big(\frac{1}{z}\Big)\Big]\partial_w\Big[X^{(A)}\Big(\frac{1}{w}\Big)\Big]$$
$$\times \ln\prod_{k=0}^{\infty}(T^{K+1}\Gamma(z), \Gamma(w), T^k(0), \Gamma(0)) + \ln\Big(\frac{T\Gamma(w) - z}{\sqrt{\partial T}\Gamma(w)}\Big)\Big\}\Big\}\colon |q=0\rangle_A \tag{2.6}$$

where (a, b, c, d) is the cross-ratio $\frac{a-b}{a-d}\frac{c-d}{c-b}$.

The Abelian integral of the first kind is

$$\varphi(z, z_0) = \frac{1}{2\pi i}\Big\{\ln\frac{\prod_{k=1}^{\infty}(T^k(z), u, T^k(z_0), T(u))}{\prod_{k=1}^{\infty}(T^{-k}(z), u, T^{-k}(z_0), T(u))} - \ln\Big(\frac{z-u}{z_0-u}\frac{z_0-T(u)}{z-T(u)}\Big)\Big\}$$
$$= \frac{1}{2\pi i}\Big\{\ln\prod_{k=1}^{\infty}\frac{T^k(z)-u}{T^k(z)-T(u)}\frac{T^k(z_0)-T(u)}{T^k(z_0)-u} - \ln\Big(\frac{z-u}{z-T(u)}\frac{z_0-T(u)}{z_0-u}\Big) - \ln\prod_{k=1}^{\infty}\frac{T^{-k}(z)-u}{T^{-k}(z_0)-T(u)}\frac{T^{-k}(z_0)-T(u)}{T^{-k}(z_0)-u}\Big\}$$
$$= \frac{1}{2\pi i}\Big\{\ln\prod_{k=1}^{\infty}\frac{T^k(z)-u}{T^k(z)-T(u)}\frac{T^k(z_0)-T(u)}{T^k(z_0)-u} - \ln\prod_{k=0}^{\infty}\frac{T^{-k}(z)-u}{T^{-k}(z_0)-T(u)}\frac{T^{-k}(z_0)-T(u)}{T^{-k}(z_0)-u}\Big\} \tag{2.7}$$

while the period matrix, multiplier and Abelian differential of the third kind are

$$\tau = -\frac{1}{\pi i}\left\{\ln\prod_{k=1}^{\infty}\left(T^{k+1}(0), T^{-1}\Gamma(0), T^{k}(0), \Gamma(0)\right) + \ln\sqrt{\partial T(0)}\right\} \qquad \partial T(0) = K_T$$
$$\omega_{y,y_0}(z,0) = \ln\prod_{k=-\infty}^{\infty}(T^k(z), y_0, T^k(z_0), y) \qquad \omega_{ab}(z,z_0) = \ln\left(\frac{E(z,b)}{E(z,a)}\frac{E(z_0,a)}{E(z_0,b)}\right) \tag{2.8}$$

where $E(x, y)$ is the prime form in the Schottky parametrization. After the bosonization of the fermions in Neveu-Schwarz-Ramond action, that part of the handle operator can be constructed for an arbitrary spin structure [7]

$$\begin{aligned}\Omega_{\phi,AD} &= (\det C)^{-d/2}{}_D\langle q=0|\delta(\alpha_0^{(A)}-\alpha_0^{(D)})\sum_{k\ \text{integer}} :\ \exp\left\{i\pi(k+a)\tau(k+a) + 2\pi i(k+a)\left[b + \oint_{C_0}\frac{dz}{2\pi i}\left\{\partial_z\left[\phi^{(A)}\left(\frac{1}{z}\right)\right]\right.\right.\right.\\ &\quad\left.\left.\times\,\varphi(\Gamma(z),z_0) + \partial\phi^{(D)}(z)\varphi(z,z_0)\right\}\right] + \frac{1}{2}\partial_x\left[\phi^{(A)}\left(\frac{1}{x}\right)\right]\partial_y\left[\phi^{(A)}\left(\frac{1}{y}\right)\right][\ln xyE(\Gamma(x),\Gamma(y)) - \ln(x-y)]\\ &\quad\left. + \partial\phi^{(D)}(x)\partial_y\left[\phi^{(A)}\left(\frac{1}{y}\right)\right]\ln\{yE(x;\Gamma(y))\}:\ |q=0\rangle_A\right\}\\ &= (\det C)^{-5}{}_D\langle q=0|:\ \prod_{i=1}^{5}\delta(\alpha_{i;0}^{(A)}-\alpha_{i;0}^{(D)})\left[\vartheta\begin{bmatrix}a\\b\end{bmatrix}\left(\tau\ \middle|\ \oint_{C_0}\frac{dz}{2\pi i}\left\{\partial_z\left[\phi_i^{(A)}\left(\frac{1}{z}\right)\right]\varphi(\Gamma(z),z_0) + \partial\phi_i^{(D)}(z)\varphi(z,z_0)\right)\right.\right.\\ &\quad\times\exp\left\{\sum_{i=1}^{5}\oint\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\left[\frac{1}{2}\partial\phi_i^{(D)}(x)\partial\phi_i^{(D)}(y)\{\ln E(x,y)-\ln(x-y)\}\right] + \frac{1}{2}\partial_x\left[\phi_i^{(A)}\left(\frac{1}{x}\right)\right]\partial_y\left[\phi_i^{(A)}\left(\frac{1}{y}\right)\right]\right.\\ &\quad\left.\times\{\ln xyE(\Gamma(x),\Gamma(y)) - \ln(x-y)\} + \partial\phi_i^{(D)}(x)\partial\left[\phi_i^{(A)}\left(\frac{1}{y}\right)\right]\ln yE(x,\Gamma(y)):\right\} \qquad |q=0\rangle_A\end{aligned} \tag{2.9}$$

where the fields ϕ_i, $i = 1, \dots, 5$ are the bosonized fermions, and (D) and (A) label incoming and outgoing states, and

$$\begin{aligned}{}_A\langle 0|\Omega_{\phi,AD} &= (\det C)^{-(d/2)}{}_D\langle 0|:\ \prod_{i=1}^{d/2}\left[\vartheta\begin{bmatrix}a\\b\end{bmatrix}\left(\tau\ \middle|\ \oint_C\frac{dz}{2\pi i}\partial\phi_i^{(D)}(z)\varphi(z,z_0)\right)\right]\\ &\quad\times\exp\left\{\sum_{i=1}^{d/2}\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\frac{1}{2}\partial\phi_i^{(D)}(x)\partial\phi_i^{(D)}(y)[\ln E(x,y)-\ln(x-y)]\right\}.\end{aligned} \tag{2.10}$$

The product of two operators [7] yields

$$\begin{aligned}{}_A\langle 0|\Omega_{AD}^{2\text{-loop}} &= {}_A\langle 0|\Omega_{AB}\mathcal{I}_{BC}\Omega_{CD}\\ &= (\det C^{2\text{-loop}})^{-d}{}_D\langle 0|\int :\ \exp\left\{i\pi k_\mu k_\nu\tau_{\mu\nu}^{2\text{-loop}} + 2\pi i k_\nu\oint_{C_0}\frac{dz}{2\pi i}\partial X^{(D)}(z)\varphi^{2\text{-loop}}(z,z_0)\right.\\ &\quad\left. + \frac{1}{2}\oint_{C_0}\frac{dx}{2\pi i}\oint\frac{dy}{2\pi i}\partial X^{(D)}(x)\partial X^{(D)}(y)[E^{2\text{-loop}}(x,y)-\ln(x-y)]\right\}:\end{aligned} \tag{2.11}$$

where

$$\det C^{2\text{-loop}} = \det[C_1(1-AB)^{1/2}C_2] \qquad A_{nm} = \oint_{C_0}\frac{dx}{2\pi i}\oint\frac{dy}{2\pi i}\sqrt{n}x^{-n-1}\ln\left(\frac{xyE^{(2)}(\Gamma(x),\Gamma(y))}{x-y}\right)\sqrt{m}y^{-m-1}$$
$$B_{nm} = \oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\sqrt{n}x^{-n-1}\ln\left(\frac{E^{(1)}(x,y)}{x-y}\right)\sqrt{m}y^{-m-1} \qquad \det C_1 = \prod_{n=1}^{\infty}(1-K_1^n) \qquad \det C_2 = \prod_{n=1}^{\infty}(1-K_2^n) \tag{2.12}$$

whereas the g-loop states are

$$ {}_A\langle 0|\Omega^{g\text{-loop}}_{X,AD} = (\det C^{g\text{-loop}})^{-d}{}_B\langle 0|\prod_{\nu=1}^{g} dk_\nu :\ \exp\left\{ i\pi \sum_{\mu,\nu=1}^{g} k_\mu k_\nu \tau^{g\text{-loop}}_{\mu\nu} + 2\pi i \sum_{\nu=1}^{g} k_\nu \oint_{C_0}\frac{dz}{2\pi i}\partial X^{(D)}(z)\varphi^{g\text{-loop}}_\nu(z,z_0) + \frac{1}{2}\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\partial X^{(D)}(x)\partial X^{(D)}(y)[E^{g\text{-loop}}(x,y) - \ln(x-y)]\right\}: $$

$$ {}_A\langle 0|\Omega^{g\text{-loop}}_{\phi,AD} = (\det C^{g\text{-loop}})^{-(d/2)}_D\langle 0|\prod_{i=1}^{d/2} :\ \vartheta\begin{bmatrix} a_\nu \\ b_\nu \end{bmatrix}\left(\tau_{\mu\nu} \,\middle|\, \oint_{C_0}\frac{dz}{2\pi i}\partial\phi^{(D)}_i(z)\varphi_\nu(z,z_0)\right) \times \exp\left(\frac{1}{2}\sum_{i=1}^{d/2}\oint_{C_0}\frac{dx}{2\pi i}\oint_{C_0}\frac{dy}{2\pi i}\partial\phi^{(D)}_i(x)\partial\phi^{(D)}_i(y)[\ln E^{g\text{-loop}}(x,y) - \ln(x-y)]\right): \tag{2.13} $$

with

$$ \begin{bmatrix} a_\nu \\ b_\nu \end{bmatrix} $$

denoting the spin structure and

$$ \varphi^{g\text{-loop}}_\nu(z,z_0) = \frac{1}{2\pi i}\ln\prod_\alpha (T_\alpha(z), T_\nu(u), T_\alpha(z_0), u) \qquad \tau^{g\text{-loop}}_{\mu\nu} = \frac{1}{2\pi i}\ln\prod_\alpha (T_\alpha T_\mu(u), T_\nu(v), T_\alpha(u), v) $$

$$ E^{g\text{-loop}}(x,y) = \frac{\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau^{g\text{-loop}}_{\mu\nu}|\int_x^y d\varphi^{g\text{-loop}}_\nu)}{[\varphi^{g\text{-loop}}_\nu(x)\partial_\nu\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau^{g\text{-loop}}_{\mu\nu}|0)\varphi^{g\text{-loop}}_\mu(y)\partial_\mu\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau^{g\text{-loop}}_{\mu\nu}|0)]^{1/2}}. \tag{2.14} $$

The period matrix is

$$ \tau_{\mu\nu} = \frac{1}{2\pi i}\left[\ln K_\mu\delta_{\mu\nu} + \sum_\alpha^{(\mu,\nu)}\ln\frac{\xi_{1\mu} - V_\alpha\xi_{1\nu}}{\xi_{1\mu} - V_\alpha\xi_{2\nu}}\frac{\xi_{1\mu} - V_\alpha\xi_{2\nu}}{\xi_{2\mu} - V_\alpha\xi_{1\nu}}\right] \tag{2.15} $$

and

$$ \ln\left[\frac{\xi_{1\mu} - V_\alpha\xi_{1\nu}}{\xi_{1\mu} - V_\alpha\xi_{2\nu}}\frac{\xi_{2\mu} - V_\alpha\xi_{2\nu}}{\xi_{2\mu} - V_\alpha\xi_{1\nu}}\right] = \ln\left[1 + \frac{(\xi_{2\mu} - \xi_{1\mu})(\xi_{2\nu} - \xi_{1\nu})\gamma_\alpha^{-2}}{(\xi_{1\nu} + \frac{\delta_\alpha}{\gamma_\alpha})(\xi_{2\nu} + \frac{\delta_\alpha}{\gamma_\alpha})(\xi_{1\mu} - V_\alpha\xi_{2\nu})(\xi_{2\mu} - V_\alpha\xi_{1\nu})}\right]. \tag{2.16} $$

Since $V_\alpha\xi_{1\mu}\notin D_{T_\mu}$, $D_{T_\mu^{-1}}$, D_{T_ν}, $D_{T_\nu^{-1}}$ and $V_\alpha\xi_{2\mu}\notin D_{T_\mu}$, $D_{T_\mu^{-1}}$, D_{T_ν}, $D_{T_\nu^{-1}}$, when the distance between pairs of circles decrease as $\frac{1}{n^s}$,

$$ |\xi_{1\mu} - \xi_{2\mu}| \sim \frac{1}{n^q} \qquad |\xi_{1\nu} - \xi_{2\nu}| \sim \frac{1}{n^q} $$
$$ \left|\xi_{1\nu} + \frac{\delta_\alpha}{\gamma_\alpha}\right| \geq \frac{1}{n^s} \qquad \left|\xi_{2\nu} + \frac{\delta_\alpha}{\gamma_\alpha}\right| \geq \frac{1}{n^s} $$
$$ |\xi_{1\mu} - V_\alpha\xi_{1\nu}| \geq \frac{1}{n^s} \qquad |\xi_{2\mu} - V_\alpha\xi_{2\nu}| \geq \frac{1}{n^s} \tag{2.17} $$
$$ |\gamma_\alpha|^{-2} \sim \frac{1}{n} $$

and

$$ \left|\frac{(\xi_{2\mu} - \xi_{1\mu})(\xi_{1\nu} - \xi_{2\nu})\gamma_\alpha^{-2}}{(\xi_{1\nu} + \frac{\delta_\alpha}{\gamma_\alpha})(\xi_{2\nu} + \frac{\delta_\alpha}{\gamma_\alpha})(\xi_{1\mu} - V_\alpha\xi_{1\nu})(\xi_{2\mu} - V_\alpha\xi_{2\nu})}\right| \geq \frac{1}{n^{2q+1-4s}} \tag{2.18} $$

which decreases with n if $s < \frac{2q+1}{4}$. The sum of the logarithms converges only if $\sum_n \frac{1}{n^{2q+1-4s}}$ is finite, implying that $s < \frac{q}{2}$. Therefore, the supermoduli space integral by a surface in the category defined for each value of q is well defined.

The handle operator will be well defined even for configurations of isometric circles with $\frac{\delta_0}{n^{q'''}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{n^{q'''}}$ provided that $s < \frac{q'''}{2}$. The product of handle operators then would be bounded by an exponential function of the number of handles.

As the handles become smaller, the integral around C_0 decreases in magnitude and $(\det C)^{2\text{-loop}} \to \det C_1 \det C_2 = \prod_n (1 - K_\mu^n)(1 - K_\nu^n)$. When $K_\mu, K_\nu \ll 1$. Consequently, the removal or creation of handles gives rise to an integral which differs by an exponential function of the corresponding set of handles.

The finiteness of the handle operator has been verified also in the degeneration limits [8].

III. THE FIXED-POINT INTEGRAL

To determine the dependence of the superstring amplitude on the fixed-point distance, consider first the holomorphic of part Neveu-Schwarz measure multiplied by the inverse of the fifth power of the determinant of the superperiod matrix [9,10]

$$\prod_{n=1}^{g}\frac{dK_n}{K_n^{3/2}}\prod_{m=2}^{g}\frac{dB_m}{B_m^{3/2}}\prod_{m=2}^{g-1}\frac{dH_m}{H_m^{3/2}}\prod_{i=2}d\vartheta_{1i}\prod_{i=1}^{g}d\vartheta_{2i}\left(\frac{1-K_n}{1-(-1)^{B_n}K_n^{1/2}}\right)^2[\det(\mathrm{Im}\mathcal{T})]^{-5}\prod_{\alpha}{}'\prod_{p=1}^{\infty}\left(\frac{1-(-1)^{N_\alpha^B}K_\alpha^{p-(1/2)}}{1-K_\alpha^p}\right)^{-10}$$
$$\times\prod_{\alpha}{}'\prod_{p=2}^{\infty}\left(\frac{1-K_\alpha^p}{1-(-1)^{N_\alpha^B}K_\alpha^{p-(1/2)}}\right)^2 \qquad (3.1)$$
$$\xi_{1n}=\frac{\xi_{2n}}{1-H_n-\sqrt{H_n}\vartheta_{1n}\vartheta_{2n}}\qquad \xi_{2n}=\prod_{j=2}^{n}B_j\qquad \mathcal{T}_{mn}=\frac{1}{2\pi i}\left[\ln K_m\delta_{mn}+\sum_{\alpha}{}^{(m,n)}\ln\left(\frac{Z_{1m}-V_\alpha Z_{1n}}{Z_{1m}-V_\alpha Z_{1n}}\frac{Z_{2m}-V_\alpha Z_{2n}}{Z_{2m}-V_\alpha Z_{1n}}\right)\right]$$

where $(-1)^{B_n}$ determines the periodicity of fermions about the B_n-cycles and $\sum_{n=1}^{g}B_nN_\alpha^n$, with N_α^n being defined by the number of times T_n or T_n^{-1} occurs in V_α. Inverting the relation for ξ_{1n},

$$H_n=\frac{\xi_{1n}-\xi_{2n}}{\xi_{1n}}-\frac{(\xi_{1n}-\xi_{2n})^{1/2}}{\xi_{1n}^{1/2}}\vartheta_{1n}\vartheta_{2n}$$
$$|H_n|^{-1}=\left|\frac{\xi_{1n}}{\xi_{1n}-\xi_{2n}}\right|\left[1+\frac{1}{2}\frac{\xi_{1n}^{1/2}}{(\xi_{1n}-\xi_{2n})^{1/2}}\vartheta_{1n}\vartheta_{2n}+\frac{1}{2}\frac{\overline{\xi_{1n}}^{1/2}}{\overline{(\xi_{1n}-\xi_{2n})}^{1/2}}\bar{\vartheta}_{1n}\bar{\vartheta}_{2n}+\frac{1}{4}\frac{|\xi_{1n}|}{|\xi_{1n}-\xi_{2n}|}\vartheta_{1n}\vartheta_{2n}\bar{\vartheta}_{1n}\bar{\vartheta}_{2n}\right]. \qquad (3.2)$$

Since

$$\int\frac{dH_n\wedge d\bar{H}_n}{|H_n|^3}=-2\pi|H_n|^{-1} \qquad (3.3)$$

and

$$-\int|H_n|^{-1}d\vartheta_{1n}d\vartheta_{2n}d\bar{\vartheta}_{1n}d\bar{\vartheta}_{2n}=-\int\left[1+\frac{1}{2}\frac{\xi_{1n}^{1/2}}{(\xi_{1n}-\xi_{2n})^{1/2}}\vartheta_{1n}\vartheta_{2n}+\frac{1}{2}\frac{\overline{\xi_{1n}}^{1/2}}{\overline{(\xi_{1n}-\xi_{2n})}^{1/2}}\bar{\vartheta}_{1n}\bar{\vartheta}_{2n}\right.$$
$$\left.+\frac{1}{4}\frac{|\xi_{1n}|}{|\xi_{1n}-\xi_{2n}|}\vartheta_{1n}\vartheta_{2n}\bar{\vartheta}_{1n}\bar{\vartheta}_{2n}\right]d\vartheta_{1n}d\vartheta_{2n}d\bar{\vartheta}_{1n}d\bar{\vartheta}_{2n}$$
$$=-\frac{1}{4}\frac{|\xi_{1n}|^2}{|\xi_{1n}-\xi_{2n}|^2} \qquad (3.4)$$

by Berezin integration. Because of the factor $\frac{1}{|\xi_{1n}-\xi_{2n}|^2}$, the fixed point integral over the range $\frac{\delta_0}{g^{q_i}}\le|\xi_{1n}-\xi_{2n}|\le\frac{\delta_0'}{g^{q_i}}$ yields $(\frac{g^{2q_i}}{\delta_0^2}-\frac{g^{2q_i}}{\delta_0'^2})$.

While finiteness of the supermoduli space integral implies that the leading-order divergences cancel in a sum over spin structures in the limit $|\xi_{1n}-\xi_{2n}|\to 0$, the previously derived genus-dependence would be valid for each spin structure and also perhaps in the integration range bounded away from the degeneration limit. Furthermore, given the restriction to the fundamental region of the modular group, the genus-dependence $g^{2\sum_i q_i n_i}(\ln g)^{-5N_0}$ of the entire superstring amplitude would follow, even though the divergences, which may have resulted from the factor $\frac{1}{|\xi_{1n}-\xi_{2n}|^2}$ in the integrand, were canceled in the limit.

At one loop, a copy of the fundamental region including $\tau=0$ or $K=1$ can be transformed to the domain in the upper-half plane with boundaries $\mathrm{Re}\tau=-\frac{1}{2}$, $\mathrm{Re}\tau=\frac{1}{2}$ and $|\tau|=1$, which contains the limit $\tau\to i\infty$ or $K\to 0$. Both limits $K\to 0$ and $K\to 1$ introduce divergences in the bosonic string partition function. At higher genus, the standard fundamental region will be bounded away from $|K_n|\to 1$ and therefore $|\xi_{1n}-\xi_{2n}|\to 0$. Since it is not possible to have simultaneously $|K_n|\to 0$ and $|\xi_{1n}-\xi_{2n}|\to 0$ simultaneously in one fundamental region of the modular group with finite-size handles, the problem of divergences in the fixed-point integral should not arise.

The regularization of the bosonic string partition function depends on the genus-independent lower bound on the length of closed geodesics. With a metric of curvature -1,

$$A = \int d^2z\sqrt{g} = -\int d^2z\sqrt{g}R = 2\pi(2g-2) \quad (3.5)$$

and the areas increase linearly with the genus. Multiplication of the areas by a factor of g in the intrinsic metric is valid for surfaces constructed by attaching handles to the sphere but not for ladder diagrams, where the increasing area results from the linear extent of the surface. As the S-matrix theory is based on a finite-size interaction region, the class of surfaces, defined by attaching handles to the sphere, should be used.

In the Schottky parametrization, with a fundamental domain defined in the extended complex plane or the sphere, a cutoff of $|\gamma_n|^{-2} \geq \frac{1}{g}$ is required of the squares of the radii of the isometric circles. For the first three categories of isometric circles (1.1), if $|\gamma_n|^{-2} \sim \frac{1}{g}$, a restriction on the range of the magnitudes of the multipliers, $\frac{\epsilon_0}{g^{1-2q'}} \leq |K_n| \leq \frac{\epsilon_0'}{g^{1-2q'}}$ implies that $\frac{\delta_0}{g^{q'}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{g^{q'}}$. Extending the configurations of isometric circles to $\frac{\epsilon_0}{n^{q''}} \leq |K_n| \leq \frac{\epsilon_0'}{n^{q''}}$, $n = 1, 2, 3, \ldots$ for a genus-g surface in the limit $g \to \infty$ with the handles labeled by n, the corresponding dependence of the distance between the fixed points is $|\xi_{1n} - \xi_{2n}| \sim \frac{1}{\sqrt{g}} n^{q''/2}$. When $n = \mathcal{O}(g)$, $|\xi_{1n} - \xi_{2n}| = \mathcal{O}(\frac{1}{g^{1/2-q''/2}})$. When $q'' > 1$, $|\xi_{1n} - \xi_{2n}|$ would increase with the genus for $n = \mathcal{O}(g)$. It follows tat the category of circles with $\frac{\delta_0}{n^{q'''}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{n^{q'''}}$, $n = 1, 2, 3, \ldots$, $q''' > \frac{1}{2}$ is excluded in a single copy of the fundamental region when $|\gamma_n|^{-2} \sim \frac{1}{g}$.

Since areas are multiplied by a factor of g in the intrinsic metric, the sizes of bases the handles would be $\frac{1}{n^{s'}} \cdot g$ when $|\gamma_n|^{-2} \sim \frac{1}{n^{s'}}$ in the Schottky covering of the surface. When $n = \mathcal{O}(g)$, the thickness then is $\mathcal{O}(\frac{1}{g^{s'-1}})$.

The handles in the Reggeon formalism are constructed by sewing vertices that are defined to have a leg of equal size connected to the external legs, in accordance with sewing based on a path integral and with a light-cone diagram. Although virtual in the interaction region, a genus-dependent reduction in the size of the string would not be possible in Reggeon diagrams because conservation of momentum is required, based on the form of the vertex operators.

In the light-cone diagram, the total momentum, proportional to the sum of the lengths of the strings, is conserved even in the interaction region, but the strings can combine and split at intermediate times and have individual propagators may be represented by arbitrarily thin strips. The light-cone diagrams are known to provide a complete covering of a single copy of moduli space.

It would appear then that infinite-genus surfaces with $s' > 1$ and accumulating handles could be included in path integral, which is not regularized in superstring theory. Estimates of the superstring amplitudes for configurations of isometric circles in the Schottky covering sheets of these surfaces reveal that the integrals can be bounded. Together with the exponential bounds at large genus for the categories in Eq. (1.1), it follows that a path integral given by a sum of smooth and noded surfaces may be bounded for the appropriate value of the coupling. Therefore, the conditions $|\gamma_n|^{-2} \geq \frac{1}{g}$ or $|\gamma_n|^{-2} \leq \frac{1}{n}$ are valid for isometric circles in the zero-odd Grassmann component of the covering sheet of the super-Schottky uniformization of the super-Riemann surfaces.

The inequalities defining the fundamental region of the modular group lead to the elimination of several of the categories of the isometric circles. By the formula

$$\tau_{nn} = \frac{1}{2\pi i}\left[\ln K_n + \sum_{\alpha}{}^{(n,n)} \ln \frac{\xi_{1n} - V_\alpha \xi_{1n}}{\xi_n - V_\alpha \xi_{1n}} \frac{\xi_{2n} - V_\alpha \xi_{1n}}{\xi_{2n} - V_\alpha \xi_{2n}}\right] \quad (3.6)$$

large values of $|\tau_{nn}|$ would be mapped to small values of $|\tau_{nn}|$. under the modular transformation $\tau \to -\frac{1}{\tau}$. When $|\tau_{nn}|$ is large, $|K_n^{-1}|$ is significantly greater than one and $|K_n| \ll 1$. If $|\tau_{nn}| \ll 1$, $|K_n| = \mathcal{O}(1)$ and cancellations with the sum of the logarithms of the cross-ratios must occur. It is not necessary then to sum over categories with $\frac{\epsilon_0}{g^{q_i}} \leq |K_n| \leq \frac{\epsilon_0'}{g^{q_i}}$ and $\frac{\delta_0}{n^{q'''}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{n^{q'''}}$ for sufficiently large values of q'''.

To avoid overcounting of the configurations of isometric circles, the parameter q''' will be restricted by the inequalities

$$\frac{\ln(\ln n)}{\ln(\frac{\delta_0'}{\delta_0})} \leq \tilde{N}'_{\max} \leq x \frac{\ln n}{\ln(\frac{\delta_0'}{\delta_0})}. \quad (3.7)$$

The genus-dependence of the fixed-point integrals is determined by the regions in Eqs. (1.1) and (1.2). Its consistency with the finiteness of the supermoduli space integrals in the limit $|\xi_{1n} - \xi_{2n}| \to 0$ can be established as follows. While the coefficient of $\frac{1}{|\xi_{1n} - \xi_{2n}|^2}$ must vanish after a sum over the spin structures in this limit, the integral also should be evaluated at an upper limit for $|\xi_{1n} - \xi_{2n}|$. As the fundamental regions do not intersect, the upper limit must have the form $\frac{\delta_0}{g^{q'''}}$ with $q''' \geq \frac{1}{2}$. With an integrand which contains $\frac{1}{|\xi_{1n} - \xi_{2n}|^2}$, the evaluation of the integral at the upper limit again would yield a factorial dependence on the genus, which, after division by a combinatorial factor, would give an exponential function of the genus for the supermoduli space integral. Therefore, earlier estimates of the integral over the super-Schottky coordinates, based on fundamental region of the modular group, are sufficient.

IV. ESTIMATES OF THE SUPERMODULI SPACE INTEGRAL

For the Neveu-Schwarz sector, the fermion fields are periodic around all of the A_n-cycles, while the traversal of the B_n cycles can give either sign. The weighted sum of the measures (4.1) over the 2^g spin structures contains the expression

$$\sum_{NS}(-1)^{B_n}\left(\frac{1-K_n}{1-(-1)^{B_n}K_n^{1/2}}\right)^2\prod_\alpha{}'\prod_{p=1}^\infty\left(\frac{1-(-1)^{N_\alpha^B}K_\alpha^{p-(1/2)}}{1-K_\alpha^p}\right)^{10}\prod_\alpha{}'\prod_{p=2}^\infty\left(\frac{1-K_\alpha^p}{1-(-1)^{N_\alpha^B}K_\alpha^{p-(1/2)}}\right)^2. \tag{4.1}$$

Since

$$\begin{aligned}\exp\left(-\frac{32}{1-\epsilon_0'}\sum_\alpha{}'|K_\alpha|\right)\prod_\alpha{}'\frac{|1-K_\alpha^{1/2}|^{16}}{|1+K_\alpha|^4} &< \left|\prod_\alpha{}'\prod_{p=1}^\infty\left(\frac{1-K_\alpha^{p-(1/2)}}{1-K_\alpha^p}\right)^{16}\prod_\alpha{}'(1+K_\alpha^{1/2})^{-4}\right| \\ &< \exp\left(\frac{32}{1-\epsilon_0'}\sum_\alpha{}'|K_\alpha|\right)\prod_\alpha{}'\frac{|1-K_\alpha^{(1/2)}|^{16}}{|1+K_\alpha|^4} \\ \exp\left(\frac{32}{1-\epsilon_0'}\sum_\alpha{}'|K_\alpha|\right)\prod_\alpha{}'|1-K_\alpha^{1/2}|^{12} &< \left|\prod_\alpha{}'\prod_{p=1}^\infty\left(1+\frac{K_\alpha^{p-(1/2)}}{1-K_\alpha^p}\right)^{16}\prod_\alpha{}'(1-K_\alpha^{1/2})^{-4}\right| \\ &< \exp\left(\frac{32}{1-\epsilon_0'}\sum_\alpha{}'|K_\alpha|\right)\prod_\alpha{}'|1-K_\alpha^{1/2}|^{12}\end{aligned} \tag{4.2}$$

and it has been demonstrated that the products $\prod_\alpha{}'|1+K_\alpha^{1/2}|$ and $\prod_\alpha{}'|1-K_\alpha^{1/2}|$ are exponential functional of the genus [5], the sum (4.1) is bounded by the product of an exponential and

$$2^{g-1}\left[\frac{1}{[1-K_n^{1/2}]^2}-\frac{1}{[1+K_n^{1/2}]^2}\right]=2^{g+1}\frac{K_n^{1/2}}{[1-K_n]^2}. \tag{4.3}$$

For the Ramond sector, the sum over the spin structures will yield a holomorphic measure of the form $2^g\prod_{n=1}^g\frac{dK_n}{K_n}\cdot$ (primitive element products). The measure for the other sectors will be a linear combination of these two differential elements. Combining the holomorphic and antiholomorphic parts of the superstring measure, the multiplier integral for each handle, excluding exponential factors, equals

$$\int\frac{d^2K_n}{|K_n|^2|1-K_n|^4(\ln(\frac{1}{|K_n|}))^5}=\frac{\pi}{2}\left[\ln\left(\frac{1}{|K_n|^4}\right)\right]^{-4} \tag{4.4}$$

which is bounded [11] by

$$\int\frac{d^2K_n}{|K_n|^2[1-|K_n|]^4(\ln(\frac{1}{|K_n|}))^5}\approx\frac{\pi}{2}\left[\ln\left(\frac{1}{|K_n|^4}\right)\right]^{-4}+2\pi\left\{\frac{|K_n|}{6\ln|K_n|}+\frac{|K_n|}{6(\ln|K_n|)^2}+\frac{|K_n|}{3(\ln|K_n|)^3}+\frac{|K_n|}{(\ln|K_n|)^4}+\frac{li(x)}{3}\right\} \tag{4.5}$$

for the Neveu-Schwarz sector and

$$\int\frac{d^2K_n}{|K_n|^2(\ln(\frac{1}{|K_n|}))^5}=\frac{\pi}{2}\left[\ln\left(\frac{1}{|K_n|^4}\right)\right]^{-4} \tag{4.6}$$

for the Ramond sector. The integral over configurations of isometric circles in the first and second categories equals

$$\begin{aligned}&2\pi((1-2q')\ln g)^{-5}\ln\left(\frac{\epsilon_0'}{\epsilon_0}\right)+2\pi\left\{\left[\frac{\epsilon_0}{6g^{1-2q_i'}\ln(\frac{\epsilon_0}{g^{1-2q_i'}})}-\frac{\epsilon_0'}{6g^{1-2q_i'}\ln(\frac{\epsilon_0'}{g^{1-2q_i'}})}\right]+\left[\frac{\epsilon_0}{6g^{1-2q_i'}(\ln(\frac{\epsilon_0}{g^{1-2q_i}}))^2}-\frac{\epsilon_0'}{6g^{1-2q_i'}(\ln(\frac{\epsilon_0'}{g^{1-2q_i'}}))^2}\right]\right.\\&+\left[\frac{\epsilon_0}{3g^{1-2q_i'}(\ln(\frac{\epsilon_0}{g^{1-2q_i'}}))^3}-\frac{\epsilon_0'}{3g^{1-2q_i'}(\ln(\frac{\epsilon_0'}{g^{1-2q_i'}}))^3}\right]+\left[\frac{\epsilon_0}{g^{1-2q_i'}(\ln(\frac{\epsilon_0}{g^{1-2q_i'}}))^4}-\frac{\epsilon_0'}{g^{1-2q_i'}(\ln(\frac{\epsilon_0'}{g^{1-2q_i'}}))^4}\right]\\&\left.+\frac{1}{3}\left[li\left(\frac{\epsilon_0'}{g^{1-2q_i'}}\right)-li\left(\frac{\epsilon_0}{g^{1-2q_i'}}\right)\right]+\mathcal{O}(((1-2q_i')\ln g)^{-6})\right\}\end{aligned} \tag{4.7}$$

and it is

$$\frac{\pi}{2}\bigg[\bigg(\ln\frac{1}{\epsilon_0}\bigg)^{-4}-\bigg(\ln\frac{1}{\epsilon_0}\bigg)^{-4}\bigg]+2\pi\bigg\{\bigg[\frac{\epsilon_0}{6\ln(\epsilon_0)}-\frac{\epsilon_0'}{6\ln(\epsilon_0')}\bigg]+\bigg[\frac{\epsilon_0}{6(\ln(\epsilon_0))^2}-\frac{\epsilon_0'}{6(\ln(\epsilon_0'))^2}+\bigg[\frac{\epsilon_0}{3(\ln(\epsilon_0))^3}-\frac{\epsilon_0'}{3(\ln(\epsilon_0')^3}\bigg]$$
$$+\bigg[\frac{\epsilon_0}{(\ln(\epsilon_0))^4}-\frac{\epsilon_0'}{(\ln(\epsilon_0'))^4}\bigg]+\frac{1}{3}[li(\epsilon_0')-li(\epsilon_0)]\bigg]\bigg\} \tag{4.8}$$

in the third category.

In categories (i) and (ii), the integral (4.6) equals

$$2\pi((1-2q')\ln g)^{-5}\ln\bigg(\frac{\epsilon_0'}{\epsilon_0}\bigg)+\mathcal{O}(((1-2q')\ln g)^{-6}) \tag{4.9}$$

while it is

$$\frac{\pi}{2}\bigg[\bigg(\ln\frac{1}{\epsilon_0'}\bigg)^{-4}-\bigg(\ln\frac{1}{\epsilon_0}\bigg)^{-4}\bigg] \tag{4.10}$$

in the third category. Similarly, the integrals for the Ramond sector can be evaluated in these categories.

By the formula (3.2) for H_n,

$$\frac{dH_n\wedge d\bar{H}_n}{|H_n|^3}\sim\frac{[|\xi_{2n}|^2d^2\xi_{1n}-\xi_{1n}\bar{\xi}_{2n}d\xi_{2n}\wedge d\bar{\xi}_{1n}-\xi_{2n}\bar{\xi}_{1n}d\xi_{1n}\wedge d\bar{\xi}_{2n}+|\xi_{1n}|^2d^2\xi_{2n}]}{|\xi_{1n}||\xi_{1n}-\xi_{2n}|^3}$$
$$+\frac{3}{4}\bigg|\frac{\xi_{1n}}{\xi_{1n}-\xi_{2n}}\bigg|^3\vartheta_{1n}\vartheta_{2n}\bar{\vartheta}_{1n}\bar{\vartheta}_{2n}d|H_n|d\theta_n^H. \tag{4.11}$$

Since

$$d\bigg|\frac{\xi_{1n}-\xi_{2n}}{\xi_{1n}}\bigg|=\frac{d|\xi_{1n}-\xi_{2n}|}{|\xi_{1n}|}-\frac{1}{|\xi_{1n}|^2}|\xi_{1n}-\xi_{2n}|d|\xi_{1n}| \tag{4.12}$$

$$\frac{dB_2\wedge d\bar{B}_2\wedge\ldots\wedge dB_g\wedge d\bar{B}_g}{|B_2|^3\ldots|B_g|^3}=\frac{1}{|\xi_{2g}|}\prod_{n=2}^{g}\frac{d\xi_{2n}\wedge d\bar{\xi}_{2n}}{|\xi_{2n}|^2}. \tag{4.13}$$

Specifying the fixed points with specific values to be ξ_{11}, ξ_{1g} and ξ_{21}, the product of the integrals over H_n and B_m are

$$\bigg(\frac{3}{2}\pi\bigg)^{g-3}4^g\int d^2\xi_{11}\prod_{n=2}^{g-1}\frac{|\xi_{1n}|^2}{|\xi_{1n}-\xi_{2n}|^3}d|\xi_{1n}-\xi_{2n}|d^2\xi_{1g}\cdot d^2\xi_{21}\frac{1}{|\xi_{2g}|}\prod_{n=2}^{g}\frac{d^2\xi_{2n}}{|\xi_{2n}|^2}|\xi_{11}-\xi_{1g}|^2|\xi_{11}-\xi_{21}|^2|\xi_{1g}-\xi_{21}|^2$$
$$\times\delta^2(\xi_{11}-\xi_{11}^0)\delta^2(\xi_{1g}-\xi_{1g}^0)\delta^2(\xi_{21}-\xi_{21}^0)-\bigg(\frac{3}{2}\pi\bigg)^{g-3}4^g\int d^2\xi_{11}\prod_{n=2}^{g-1}\frac{|\xi_{1n}|}{|\xi_{1n}-\xi_{2n}|^2}d^2\xi_{1g}\cdot d^2\xi_{21}\frac{1}{|\xi_{2g}|^2}$$
$$\times\prod_{n=2}^{g}\frac{d^2\xi_{2n}}{|\xi_{2n}|^2}|\xi_{11}-\xi_{1g}|^2|\xi_{11}-\xi_{21}|^2|\xi_{1g}-\xi_{21}|^2\delta^2(\xi_{11}-\xi_{11}^0)\delta^2(\xi_{1g}-\xi_{1g}^0)\delta^2(\xi_{21}-\xi_{21}^0)$$
$$<4^3(6\pi)^{g-3}|\xi_{11}^0-\xi_{1g}^0|^2|\xi_{11}^0-\xi_{21}^0|^2|\xi_{1g}^0-\xi_{21}^0|^2\bigg\{\pi^{N_0-1}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{N_0-1}\prod_i\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+2\frac{\delta_0'}{g^{q_i}}(\delta_2'-\delta_2)$$
$$+\frac{\delta_0'^2}{g^{2q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}\frac{2\pi}{\delta_0'^2}(\delta_2'-\delta_2)g^{2\sum_i q_in_i}\cdot\pi^{g-N_0}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{g-N_0}\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+\frac{\delta_0'^2}{g}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{g-N_0}$$
$$-\pi^{N_0-1}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)^{N_0-1}\prod_i\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{g^{q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}\frac{\pi}{\delta_0'^2}(\delta_2'-\delta_2)\pi^{g-N_0}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)$$
$$\times\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{\sqrt{g}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{\sum_i q_in_i}g^{(g-N_0)/2}\bigg\}. \tag{4.14}$$

Integration of the Neveu-Schwarz measure for all of the configurations consisting of n_i isometric circles with parameter q_i, $\sum_i n_i=N_0$, $n_i=1,2,\ldots,N_0$, $N=0,1,2,\ldots,g$ and $g-N_0$ circles from the third category yields the upper bound of

$$\sum_{N_0=0}^{g}\sum_{\substack{\{n_i\}\\ \sum_i n_i=N_0}}\frac{1}{n_1!\ldots n_r!(g-N_0)!}(2\pi)^g\bigg[\prod_i\bigg\{((1-2q_i')\ln g)^{-5}\ln\bigg(\frac{\epsilon_0'}{\epsilon_0}\bigg)+\bigg\{\bigg[\frac{\epsilon_0}{6g^{1-2q_i'}\ln(\frac{\epsilon_0}{g^{1-2q_i'}})}-\frac{\epsilon_0'}{6g^{1-2q_i'}\ln(\frac{\epsilon_0'}{g^{1-2q_i'}})}\bigg]$$
$$+\bigg[\frac{\epsilon_0}{6g^{1-2q_i'}(\ln(\frac{\epsilon_0}{g^{1-2q_i}}))^2}-\frac{\epsilon_0'}{6g^{1-2q_i'}(\ln(\frac{\epsilon_0'}{g^{1-2q_i'}}))^2}\bigg]+\bigg[\frac{\epsilon_0}{3g^{1-2q_i'}(\ln(\frac{\epsilon_0}{g^{1-2q_i'}}))^3}-\frac{\epsilon_0'}{3g^{1-2q_i'}(\ln(\frac{\epsilon_0'}{g^{1-2q_i'}}))^3}\bigg]$$
$$+\bigg[\frac{\epsilon_0}{g^{1-2q_i'}(\ln(\frac{\epsilon_0}{g^{1-2q_i'}}))^4}-\frac{\epsilon_0'}{g^{1-2q_i'}(\ln(\frac{\epsilon_0'}{g^{1-2q_i'}}))^4}\bigg]+\frac{1}{3}\bigg[li\bigg(\frac{\epsilon_0'}{g^{1-2q_i'}}\bigg)-li\bigg(\frac{\epsilon_0}{g^{1-2q_i'}}\bigg)\bigg]+\mathcal{O}(((1-2q_i')\ln g)^{-6})\bigg\}\bigg\}^{n_i}\bigg]g^{2\sum_i q_in_i}$$
$$\times\bigg\{\frac{1}{4}\bigg[\bigg(\ln\frac{1}{\epsilon_0'}\bigg)^{-4}-\bigg(\ln\frac{1}{\epsilon_0}\bigg)^{-4}\bigg]+\bigg\{\bigg[\frac{\epsilon_0}{6\ln(\epsilon_0)}-\frac{\epsilon_0'}{6\ln(\epsilon_0')}\bigg]+\bigg[\frac{\epsilon_0}{6(\ln(\epsilon_0))^2}-\frac{\epsilon_0'}{6(\ln(\epsilon_0'))^2}\bigg]+\bigg[\frac{\epsilon_0}{3(\ln(\epsilon_0))^3}-\frac{\epsilon_0'}{3(\ln(\epsilon_0')^3}\bigg]$$
$$+\bigg[\frac{\epsilon_0}{(\ln(\epsilon_0))^4}-\frac{\epsilon_0'}{(\ln(\epsilon_0'))^4}\bigg]+\frac{1}{3}[li(\epsilon_0')-li(\epsilon_0)]\bigg\}\bigg\}^{g-N_0}\cdot\bigg\{4^3(6\pi)^{g-3}|\xi_{11}^0-\xi_{1g}^0|^2|\xi_{11}^0-\xi_{21}^0|^2|\xi_{1g}^0-\xi_{21}^0|^2$$
$$\times\bigg\{\pi^{N_0-1}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{N_0-1}\prod_i\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+2\frac{\delta_0'}{g^{q_i}}(\delta_2'-\delta_2)+\frac{\delta_0'^2}{g^{2q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}\frac{2\pi}{\delta_0'^2}(\delta_2'-\delta_2)g^{2\sum_i q_in_i}$$
$$\cdot\pi^{g-N_0}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{g-N_0}\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+\frac{\delta_0'^2}{g}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{g-N_0}-\pi^{N_0-1}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)^{N_0-1}\prod_i\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{g^{q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}$$
$$\times\frac{\pi}{\delta_0'^2}(\delta_2'-\delta_2)\pi^{g-N_0}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{\sqrt{g}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{\sum_i q_in_i}g^{(g-N_0)/2}\bigg\}$$
$$\cdot 2^{g+1}(\text{exponential primitive}-\text{element products}) \tag{4.15}$$

and the bound is

$$\sum_{N_0=0}^{g}\sum_{\substack{\{n_i\}\\ \sum_i n_i=N_0}}\frac{1}{n_1!\ldots n_r!(g-N_0)!}(1-2\bar{q}')^{N_0}(2\pi)^g\bigg[\ln\bigg(\frac{\epsilon_0'}{\epsilon_0}\bigg)^{N_0}(\ln g)^{-5N_0}\bigg]\bigg[\frac{1}{4}\bigg(\ln\frac{1}{\epsilon_0'}\bigg)^{-4}-\frac{1}{4}\bigg(\ln\frac{1}{\epsilon_0}\bigg)^{-4}\bigg]^{g-N_0}$$
$$\cdot\bigg\{4^3(6\pi)^{g-3}|\xi_{11}^0-\xi_{1g}^0|^2|\xi_{11}^0-\xi_{21}^0|^2|\xi_{1g}^0-\xi_{21}^0|^2\bigg\{\pi^{N_0-1}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{N_0-1}\prod_i\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+2\frac{\delta_0'}{g^{q_i}}(\delta_2'-\delta_2)$$
$$+\frac{\delta_0'^2}{g^{2q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}\frac{2\pi}{\delta_0'^2}(\delta_2'-\delta_2)g^{2\sum_i q_in_i}\cdot\pi^{g-N_0}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{g-N_0}\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+\frac{\delta_0'^2}{g}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{g-N_0}$$
$$-\pi^{N_0-1}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)^{N_0-1}\prod_i\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{g^{q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}\frac{\pi}{\delta_0'^2}(\delta_2'-\delta_2)\pi^{g-N_0}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)$$
$$\times\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{\sqrt{g}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{\sum_i q_in_i}g^{(g-N_0)/2}\bigg\}\bigg\}\cdot 2^g(\text{exponential primitive}-\text{element products})$$
$$(1-2\bar{q}')^{N_0}=\prod_{m=1}^{N_0}(1-2q_m') \tag{4.16}$$

for the Ramond sector, where division by factorial factors results from the conditions for the fundamental region $(\mathrm{Im}\tau)_{ss}\geq(\mathrm{Im}\tau)_{rr}$, $s\geq r$. The other conditions only reduce the integral by an exponential function of the genus and imply a restriction on ϵ_0'.

For the surfaces with handles of decreasing size, the multiplier integrals are

$$\int_{|K_n|\sim(1/n^{q''_{\tilde N}})} \frac{d^2K_n}{|K_n|(\ln\frac{1}{|K_n|})^5} = 2\pi\left(\frac{\ln n}{\ln(\frac{\epsilon'_0}{\epsilon})} + \tilde N\right)^{-5}\left[\ln\left(\frac{\epsilon'_0}{\epsilon_0}\right)\right]^{-4} + 2\pi\Bigg\{\left[\frac{\epsilon_0}{6n^{q''_{\tilde N}}\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}})} - \frac{\epsilon'_0}{6n^{q''_{\tilde N}}\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}})}\right]$$
$$+ \left[\frac{\epsilon_0}{6n^{q''_{\tilde N}}(\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}}))^2} - \frac{\epsilon'_0}{6n^{q''_{\tilde N}}(\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}))^2}\right] + \left[\frac{\epsilon_0}{3n^{q''_{\tilde N}}(\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}}))^3} - \frac{\epsilon'_0}{3n^{q''_{\tilde N}}(\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}))^3}\right]$$
$$+ \left[\frac{\epsilon_0}{n^{q''_{\tilde N}}(\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}}))^4} - \frac{\epsilon'_0}{n^{q''_{\tilde N}}(\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}))^4}\right] + \frac{1}{3}\left[li\left(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}\right) - li\left(\frac{\epsilon_0}{n_{q''_{\tilde N}}}\right)\right] + \mathcal{O}(q''_{\tilde N}\ln n)^{-6})\Bigg\} \qquad (4.17)$$

for the Neveu-Schwarz sector and

$$\int_{|K_n|\sim(1/n^{q''_{\tilde N}})} \frac{d|K_n|}{|K_n|(\ln\frac{1}{|K_n|})^5} = \left(\frac{\ln n}{\ln(\frac{\epsilon'_0}{\epsilon})} + \tilde N\right)^{-5}\left[\ln\left(\frac{\epsilon'_0}{\epsilon_0}\right)\right]^{-4} \qquad (4.18)$$

for the Ramond sector.

An upper bound for the fixed-point integral would be

$$4^3|\xi^0_{11} - \xi^0_{1g}|^2|\xi^0_{11} - \xi^0_{21}|^2|\xi^0_{1g} - \xi^0_{21}|^2\Bigg\{\prod_n (6\pi)^{n-3}\pi\left(\frac{1}{\delta_0^2} - \frac{1}{\delta_0'^2}\right)n^{2q'''_{\tilde N'}}\left[\left(\frac{\delta_2'^2}{2} - \frac{\delta_2^2}{2}\right) + \frac{\delta_0'^2}{n^{2q'''_{\tilde N'}}}\ln\left(\frac{\delta'_2}{\delta_2}\right)\right]$$
$$- \pi\left(\frac{1}{\delta_0} - \frac{1}{\delta'_0}\right)\left[(\delta'_2 - \delta_2) + \frac{\delta'_0}{q'''_{\tilde N}}\ln\left(\frac{\delta'_2}{\delta_2}\right)\right]n^{q'''_{\tilde N'}}\Bigg\}. \qquad (4.19)$$

The constraints $(\mathrm{Im}\tau)_{ss} \geq (\mathrm{Im}\tau)_{rr}$, $s \geq r$, are imposed on overlapping intervals $[\frac{\epsilon_0}{n_r^{q''_{\tilde N}}}, \frac{\epsilon'_0}{n_r^{q''_{\tilde N}}}]$ and $[\frac{\epsilon_0}{n_s^{q''_{\tilde N}}}, \frac{\epsilon'_0}{n_s^{q''_{\tilde N}}}]$. Since an overlap would imply that $\frac{n_s}{n_r} \leq (\frac{\epsilon'_0}{\epsilon_0})^{1/q''_{\tilde N}}$, for $n_r > [(\frac{\epsilon'_0}{\epsilon_0})^{1/q''_{\tilde N}} - 1]^{-1}$, the intervals $[\frac{\epsilon_0}{n_r^{q''_{\tilde N}}}, \frac{\epsilon'_0}{n_r^{q''_{\tilde N}}}]$ and $[\frac{\epsilon'_0}{(n_r+1)^{q''_{\tilde N}}}, \frac{\epsilon'_0}{(n_r+1)^{q''_{\tilde N}}}]$ overlap. Consider the following integral

$$\int_{\epsilon_0/(n_0+2)^{q''_{\tilde N}}}^{|K_{n_0+3}|}\int_{\epsilon_0/(n_0+1)^{q''_{\tilde N}}}^{|K_{n_0+2}|}\int_{\epsilon_0/n_0^{q''_{\tilde N}}}^{|K_{n_0+1}|} \frac{d|K_{n_0}|}{|K_{n_0}|(\ln(\frac{1}{|K_{n_0}|}))^5}\frac{d|K_{n_0+1}|}{|K_{n_0+1}|(\ln(\frac{1}{|K_{n_0+1}|}))^5}\frac{d|K_{n_0+2}|}{|K_{n_0}|(\ln(\frac{1}{|K_{n_0+2}|}))^5}$$
$$= \frac{1}{4\cdot8\cdot12}\left(\ln\frac{1}{|K_{n_0}|}\right)^{-12} - \frac{1}{4\cdot8\cdot12}\left(\ln\left(\frac{(n_0+2)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-12} - \frac{1}{4\cdot8\cdot4}\left(\ln\left(\frac{(n_0+1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-8}\left(\ln\left(\frac{1}{|K_{n_0+3}|}\right)\right)^{-4}$$
$$+ \frac{1}{4\cdot8\cdot4}\left(\ln\left(\frac{(n_0+1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-8}\left(\ln\left(\frac{(n_0+2)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4} - \frac{1}{4\cdot4\cdot8}\left(\ln\left(\frac{n_0^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4}\left(\ln\left(\frac{1}{|K_{n_0+3}|}\right)\right)^{-8}$$
$$+ \frac{1}{4\cdot4\cdot8}\left(\ln\left(\frac{n_0^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4}\left(\ln\left(\frac{(n_0+2)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-8} + \frac{1}{4\cdot4\cdot4}\left(\ln\left(\frac{n_0^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4}\left(\ln\left(\frac{(n_0+1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4}\left(\ln\left(\frac{1}{|K_{n_0+3}|}\right)\right)^{-4}$$
$$- \frac{1}{4\cdot4\cdot4}\left(\ln\left(\frac{n_0^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4}\left(\ln\left(\frac{(n_0+1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4}\left(\ln\left(\frac{(n_0+2)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4} \qquad (4.20)$$

which approximately equals

$$-\frac{5}{4}\frac{(q''_{\tilde N}\epsilon_0(\frac{1}{\epsilon_0} - \frac{1}{\epsilon'_0}))^3}{n_0(n_0+1)(n_0+2)}\left(\ln\left(\frac{n_0^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}\left(\ln\left(\frac{(n_0+1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}\left(\ln\left(\frac{(n_0+2)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}. \qquad (4.21)$$

Suppose that after n_1 integrals, the approximate value, evaluated with the upper limit of the integral over $|K_{n_0+n_1}|$ equal to $\frac{\epsilon'_0}{(n_0+n_1)^{q''_{\tilde N}}}$ is

$$\alpha_{n_0+n_1}\frac{(q''_{\tilde N}\epsilon_0(\frac{1}{\epsilon_0} - \frac{1}{\epsilon'_0}))^{n_1}}{n_0(n_0+1)\ldots(n_0+n_1+1)}\left(\ln\left(\frac{n_0^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}\ldots\left(\ln\left(\frac{(n_0+n_1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}. \qquad (4.22)$$

Since the next multiplier integral with the upper limit $\frac{\epsilon'_0}{(n_0+n_1+1)^{q''_{\tilde N}}}$ is

$$\int_{\epsilon_0/(n_0+n_1+1)^{q''_{\tilde N}}}^{\epsilon'_0/(n_0+n_1)^{q''_{\tilde N}}} \frac{d|K_{n_0+n_1+1}|}{|K_{n_0+n_1+1}|(\ln(\frac{1}{|K_{n_0+n_1+1}|}))^5} = \left[\frac{1}{4}\left(\ln\left(\frac{(n_0+n_1+1)^{q''_{\tilde N}}}{\epsilon'_0}\right)\right)^{-4} - \frac{1}{4}\left(\ln\left(\frac{(n_0+n_1+1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-4}\right]$$
$$= 4\epsilon_0\left(\frac{1}{\epsilon_0}-\frac{1}{\epsilon'_0}\right)\frac{1}{n_0+n_1+1}\left(\ln\left(\frac{(n_0+n_1+1)^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5} \tag{4.23}$$

it follows by induction that the value after n' integrals

$$\alpha_{n_0+n'}\frac{(q''_{\tilde N}\epsilon_0(\frac{1}{\epsilon_0}-\frac{1}{\epsilon'_0}))^{n'}}{n_0(n_0+1)\ldots(n_0+n')}\left(\ln\left(\frac{n_0^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}\ldots\left(\ln\left(\frac{(n_0+n')^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}. \tag{4.24}$$

Then the entire integral over the super-Schottky coordinates including all of the categories of isometric circles bounded by

$$\begin{aligned}
&\Bigg[\sum_{N_0=0}^{g}\sum_{\substack{\{n_i\}\\ \sum_i n_i=N_0}}\frac{1}{n_1!\ldots n_r!(g-N_0)!}(2\pi)^g\Bigg[\prod_i\Bigg\{((1-2q'_i)\ln g)^{-5}\ln\left(\frac{\epsilon'_0}{\epsilon_0}\right)+\Bigg\{\Bigg[\frac{\epsilon_0}{6g^{1-2q'_i}\ln(\frac{\epsilon_0}{g^{1-2q'_i}})}-\frac{\epsilon'_0}{6g^{1-2q'_i}\ln(\frac{\epsilon'_0}{g^{1-2q'_i}})}\Bigg]\\
&+\left[\frac{\epsilon_0}{6g^{1-2q'_i}(\ln(\frac{\epsilon_0}{g^{1-2q_i}}))^2}-\frac{\epsilon'_0}{6g^{1-2q'_i}(\ln(\frac{\epsilon'_0}{g^{1-2q'_i}}))^2}\right]+\left[\frac{\epsilon_0}{3g^{1-2q'_i}(\ln(\frac{\epsilon_0}{g^{1-2q'_i}}))^3}-\frac{\epsilon'_0}{3g^{1-2q'_i}(\ln(\frac{\epsilon'_0}{g^{1-2q'_i}}))^3}\right]\\
&+\left[\frac{\epsilon_0}{g^{1-2q'_i}(\ln(\frac{\epsilon_0}{g^{1-2q'_i}}))^4}-\frac{\epsilon'_0}{g^{1-2q'_i}(\ln(\frac{\epsilon'_0}{g^{1-2q'_i}}))^4}\right]+\frac{1}{3}\left[li\left(\frac{\epsilon'_0}{g^{1-2q'_i}}\right)-li\left(\frac{\epsilon_0}{g^{1-2q'_i}}\right)\right]+\mathcal{O}(((1-2q'_i)\ln g)^{-6})\Bigg\}\Bigg\}^{n_i}\Bigg]\\
&\times\Bigg\{\frac{1}{4}\left[\left(\ln\frac{1}{\epsilon'_0}\right)^{-4}-\left(\ln\frac{1}{\epsilon_0}\right)^{-4}\right]+\Bigg\{\left[\frac{\epsilon_0}{6\ln(\epsilon_0)}-\frac{\epsilon'_0}{6\ln(\epsilon'_0)}\right]+\left[\frac{\epsilon_0}{6(\ln(\epsilon_0))^2}-\frac{\epsilon'_0}{6(\ln(\epsilon'_0))^2}\right]+\left[\frac{\epsilon_0}{3(\ln(\epsilon_0))^3}-\frac{\epsilon'_0}{3(\ln(\epsilon'_0)^3}\right]\\
&+\left[\frac{\epsilon_0}{(\ln(\epsilon_0))^4}-\frac{\epsilon'_0}{(\ln(\epsilon'_0))^4}\right]+\frac{1}{3}[li(\epsilon'_0)-li(\epsilon_0)]\Bigg\}\Bigg\}^{g-N_0}\\
&+\lim{}_{g\to\infty}\prod_{n=1}^{g}\sum_{\tilde N=1}^{\infty}\Bigg\{\alpha_n\frac{q''_{\tilde N}\epsilon_0(\frac{1}{\epsilon_0}-\frac{1}{\epsilon'_0})^{n-[(\epsilon'_0/\epsilon_0)^{1/(q''_{\tilde N})}-1]^{-1}}}{[[(\frac{\epsilon'_0}{\epsilon_0})^{1/q''_{\tilde N}}-1]^{-1}]\ldots(n-1)n}\left(\ln\left(\left[\left[\left(\frac{\epsilon'_0}{\epsilon_0}\right)^{1/q''_{\tilde N}}-1\right]^{-1}\right]\right)\right)^{-5}\ldots\left(\ln\left(\frac{n^{q''_{\tilde N}}}{\epsilon_0}\right)\right)^{-5}\Bigg\}\\
&\times\Bigg\{2\pi\left(\frac{\ln n}{\ln(\frac{\epsilon'_0}{\epsilon_0})}+\tilde N\right)^{-5}\left[\ln\left(\frac{\epsilon'_0}{\epsilon_0}\right)\right]^{-4}+2\pi\Bigg\{\left[\frac{\epsilon_0}{6n^{q''_{\tilde N}}\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}})}-\frac{\epsilon'_0}{6n^{q''_{\tilde N}}\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}})}\right]+\left[\frac{\epsilon_0}{6q''_{\tilde N}(\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}}))^2}-\frac{\epsilon'_0}{6n^{q_{\tilde N}}(\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}))^2}\right]\\
&+\left[\frac{\epsilon_0}{3n^{q''_{\tilde N}}(\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}}))^3}-\frac{\epsilon'_0}{3n^{q''_{\tilde N}}(\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}))^3}\right]+\left[\frac{\epsilon_0}{n^{q''_{\tilde N}}(\ln(\frac{\epsilon_0}{n^{q''_{\tilde N}}}))^4}-\frac{\epsilon'_0}{n^{q''_{\tilde N}}(\ln(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}))^4}\right]+\frac{1}{3}\left[li\left(\frac{\epsilon'_0}{n^{q''_{\tilde N}}}\right)-li\left(\frac{\epsilon_0}{n_{q''_{\tilde N}}}\right)\right]+\mathcal{O}(q''_{\tilde N}\ln n)^{-6})\Bigg\}\Bigg\}\Bigg]\\
&\cdot\Bigg[\Bigg\{4^3(6\pi)^{g-3}|\xi^0_{11}-\xi^0_{1g}|^2|\xi^0_{11}-\xi^0_{21}|^2|\xi^0_{1g}-\xi^0_{21}|^2\Bigg\{\pi^{N_0-1}\left(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\right)^{N_0-1}\prod_i\Bigg[\left(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\right)+2\frac{\delta'_0}{g^{q_i}}(\delta'_2-\delta_2)\\
&+\frac{\delta_0'^2}{g^{2q_i}}\ln\left(\frac{\delta'_2}{\delta_2}\right)\Bigg]^{n_i}\frac{2\pi}{\delta_0'^2}(\delta'_2-\delta_2)g^{2\sum_i q_in_i}\cdot\pi^{g-N_0}\left(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\right)^{g-N_0}\left[\left(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\right)+\frac{\delta_0'^2}{g}\ln\left(\frac{\delta'_2}{\delta_2}\right)\right]^{g-N_0}g^{g-N_0}\\
&-\pi^{N_0-1}\left(\frac{1}{\delta_0}-\frac{1}{\delta'_0}\right)^{N_0-1}\prod_i\left[(\delta'_2-\delta_2)+\frac{\delta'_0}{g^{q_i}}\ln\left(\frac{\delta'_2}{\delta_2}\right)\right]^{n_i}\frac{\pi}{\delta_0'^2}(\delta'_2-\delta_2)\pi^{g-N_0}\left(\frac{1}{\delta_0}-\frac{1}{\delta'_0}\right)\\
&\times\left[(\delta'_2-\delta_2)+\frac{\delta'_0}{\sqrt{g}}\ln\left(\frac{\delta'_2}{\delta_2}\right)\right]^{g-N_0}g^{\sum_i q_in_i}g^{g-N_0/2}\Bigg\}\\
&+\lim{}_{g\to\infty}\prod_{n=1}^{g}\Bigg\{4^3|\xi^0_{11}-\xi^0_{1g}|^2|\xi^0_{11}-\xi^0_{21}|^2|\xi^0_{1g}-\xi^0_{21}|^2\Bigg\{(6\pi)^{n-3}\pi\left(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\right)n^{2q'''_{\tilde N'}}\left[\left(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\right)+\frac{\delta_0'^2}{n^{2q'''_{\tilde N'}}}\ln\left(\frac{\delta'_2}{\delta_2}\right)\right]\\
&-\pi\left(\frac{1}{\delta_0}-\frac{1}{\delta'_0}\right)\left[(\delta'_2-\delta_2)+\frac{\delta'_0}{q'''_{\tilde N}}\ln\left(\frac{\delta'_2}{\delta_2}\right)\right]n^{q'''_{\tilde N'}}\Bigg\}\Bigg\}\Bigg]\cdot 2^{g+1}(\text{exponential primitive}-\text{element products})
\end{aligned}\tag{4.25}$$

for the Neveu-Schwarz sector and

$$\Bigg[\sum_{N_0=0}^{g}\sum_{\substack{\{n_i\}\\ \sum_i n_i=N_0}}\frac{1}{n_1!\ldots n_r!(g-N_0)!}(1-2\bar{q}')^{N_0}(2\pi)^g\bigg[\ln\bigg(\frac{\epsilon_0'}{\epsilon_0}\bigg)^{N_0}(\ln g)^{-5N_0}\bigg]$$
$$\times\bigg[\frac{1}{4}\bigg(\ln\frac{1}{\epsilon_0'}\bigg)^{-4}-\frac{1}{4}\bigg(\ln\frac{1}{\epsilon_0}\bigg)^{-4}\bigg]^{g-N_0}+\lim_{g\to\infty}\prod_{n=1}^{g}\sum_{\tilde{N}=1}^{\infty}\bigg\{\alpha_n\frac{q_{\tilde{N}}''\epsilon_0(\frac{1}{\epsilon_0}-\frac{1}{\epsilon_0'})^{n-[(\epsilon_0'/\epsilon_0)^{1/(q_{\tilde{N}}'')}-1]^{-1}}}{[[(\frac{\epsilon_0'}{\epsilon_0})^{1/q_{\tilde{N}}''}-1]^{-1}]\ldots(n-1)n}$$
$$\times\bigg(\ln\bigg(\frac{[[(\frac{\epsilon_0'}{\epsilon_0})^{1/q_{\tilde{N}}''}-1]^{-1}]^{q_{\tilde{N}}''}}{\epsilon_0}\bigg)\bigg)^{-5}\ldots\bigg(\ln\bigg(\frac{n^{q_{\tilde{N}}''}}{\epsilon_0}\bigg)\bigg)^{-5}\bigg\}\bigg]\cdot\bigg[\bigg\{4^3(6\pi)^{g-3}|\xi_{11}^0-\xi_{1g}^0|^2|\xi_{11}^0-\xi_{21}^0|^2|\xi_{1g}^0-\xi_{21}^0|^2$$
$$\times\bigg\{\pi^{N_0-1}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{N_0-1}\prod_i\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+2\frac{\delta_0'}{g^{q_i}}(\delta_2'-\delta_2)+\frac{\delta_0'^2}{g^{2q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}\frac{2\pi}{\delta_0'^2}$$
$$\times(\delta_2'-\delta_2)g^{2\sum_i q_in_i}\cdot\pi^{g-N_0}\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)^{g-N_0}\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+\frac{\delta_0'^2}{g}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{g-N_0}$$
$$-\pi^{N_0-1}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)^{N_0-1}\prod_i\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{g^{q_i}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{n_i}\frac{\pi}{\delta_0'^2}(\delta_2'-\delta_2)\pi^{g-N_0}\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)$$
$$\times\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{\sqrt{g}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]^{g-N_0}g^{\sum_i q_in_i}g^{(g-N_0/2)}\bigg\}\bigg\}+\bigg\{4^3|\xi_{11}^0-\xi_{1g}^0|^2|\xi_{11}^0-\xi_{21}^0|^2|\xi_{1g}^0-\xi_{21}^0|^2$$
$$\times\bigg\{\prod_n(6\pi)^{n-3}\pi\bigg(\frac{1}{\delta_0^2}-\frac{1}{\delta_0'^2}\bigg)n^{2q_{\tilde{N}'}'''}\bigg[\bigg(\frac{\delta_2'^2}{2}-\frac{\delta_2^2}{2}\bigg)+\frac{\delta_0'^2}{n^{2q_{\tilde{N}'}'''}}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]-\pi\bigg(\frac{1}{\delta_0}-\frac{1}{\delta_0'}\bigg)\bigg[(\delta_2'-\delta_2)+\frac{\delta_0'}{q_{\tilde{N}}'''}\ln\bigg(\frac{\delta_2'}{\delta_2}\bigg)\bigg]n^{q_{\tilde{N}'}'''}\bigg\}\bigg\}\bigg]$$
$$\cdot 2^g(\text{exponential primitive} - \text{element products}) \tag{4.26}$$

for the Ramond sector.

The integrals for the other sectors would be bounded by a linear combination of the two expressions (4.25) and (4.26). Based on the study of the allowed configurations of isometric circles, the sum of the bounds for the 2^g sectors is sufficient for the supermoduli space integral.

It follows that the allowed class of surfaces in the string path integral should be larger than that defined by the three categories of isometric circles (1.1), since the bounds remain exponential for $q''' \leq \frac{1}{2}+5\frac{\ln\,\ln n}{\ln n}$.

The weighted sum over the 2^{2g} spin structures of the integrals of the superstring measure actually must vanish at each order in perturbation theory. However, the sum of absolute values of integrals in the bounds for each sector together with the sum over the sectors removes the cancellations and gives rise to an exponential function of the genus. While the N-point superstring amplitudes continue to vanish for $N < 4$ [12], these exponential bounds do provide the dominant genus-dependence for the nonvanishing amplitudes with $N \geq 4$.

V. CONCLUSION

The problem of the consistency of the genus-dependence of the fixed-point integral in one fundamental domain with its finiteness in the limit $|\xi_{1n}-\xi_{2n}|\to 0$ in another domain is resolved. Through handle operators, it is apparent that the amplitude would be finite even for isometric circles that are spaced apart in the Schottky covering sheet by an infinitesimal distance decreasing with either the genus or the order in an infinite sequence. It is shown that the genus-dependence is attained also in the copy of the fundamental domain that is a neighborhood of the limiting point of $|\xi_{1n}-\xi_{2n}|\to 0$. The physical dimensions of the string and the restriction to a single copy of the fundamental region also lead to the elimination of several categories of infinite-genus surfaces with accumulating handles decreasing in size too rapidly. Given this choice of the categories of isometric circles in the integral over super-Schottky coordinates, an exponential bound has been derived for the supermoduli space integral.

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