PHYSICAL REVIEW D **74,** 105018 (2006)

Constructing the cubic interaction vertex of higher spin gauge fields

I. L. Buchbinder,[1,*] Angelos Fotopoulos,[2,3,†] Anastasios C. Petkou,[2,‡] and Mirian Tsulaia[2,§]

[1]*Department of Theoretical Physics, Tomsk State Pedagogical University, 634041 Tomsk, Russia*

[2]*Department of Physics, University of Crete, 710 03 Heraklion, Crete, Greece*

[3]*Dipartimento di Fisica Teorica dell'Università di Torino and Istituto Nazionale di Fisica Nucleare, Sezione di Torino via P.Giuria 1, I-10125 Torino, Italy*

(Received 6 October 2006; published 21 November 2006)

We propose a method of construction of a cubic interaction in massless higher spin gauge theory both in flat and in AdS space-times of arbitrary dimensions. We consider a triplet formulation of the higher spin gauge theory and generalize the higher spin symmetry algebra of the free model to the corresponding algebra for the case of cubic interaction. The generators of this new algebra carry indexes which label the three higher spin fields involved into the cubic interaction. The method is based on the use of oscillator formalism and on the Becchi-Rouet-Stora-Tyutin (BRST) technique. We derive general conditions on the form of cubic interaction vertex and discuss the ambiguities of the vertex which result from field redefinitions. This method can in principle be applied for constructing the higher spin interaction vertex at any order. Our results are a first step towards the construction of a Lagrangian for interacting higher spin gauge fields that can be holographically studied.

DOI: 10.1103/PhysRevD.74.105018 PACS numbers: 11.10.Kk, 11.15.−q, 11.25.Hf, 11.25.Tq

I. INTRODUCTION

Classical higher spin gauge theories should describe consistent dynamics of free and interacting massive and massless particles with arbitrary values of spin (e.g. [1] for recent reviews of various aspects of higher spin field theory). One of the leading directions in this area is devoted to constructing a Lagrangian formulation for higher spin fields in flat and AdS space-times of arbitrary dimensions. Other than being a fascinating topic by itself, higher spin field theory has attracted a significant amount of attentio n due to its close relations with string—and M-theories. Also we point out the interesting links of higher spin field theory with holography ideas.

The study of higher spin (HS) gauge theories is notoriously difficult and demanding. Already for free HS gauge fields it is highly nontrivial to construct Lagrangians that yield HS field equations with enough gauge invariance to remove nonphysical polarizations—ghosts—from the spectrum. Moreover, the requirement of gauge invariance restricts severely the possible gravitational backgrounds where free fields with spin greater than two can consistently propagate. Up to date, only constant curvature backgrounds—Minkowski, de Sitter (dS), and anti-de Sitter (AdS) spaces—are known to support the consistent propagation of HS gauge fields.

Interacting HS gauge fields are much harder to deal with. An important landmark was reached with the understanding of [2,3] (see also [4]) that the AdS background can accommodate consistent self-interactions of massless HS fields. An important property of this construction is that the coupling constants of massless HS interactions are proportional to positive powers of the AdS radius and therefore this picture admits no flat space-time limit. This picture has two crucial features; the presence of an infinite tower of massless HS fields and nonlocality.

Studies of HS gauge fields can be grouped into two broad classes according to the particular formulation of HS gauge theory they use. In the Vasiliev formulation ("Frame-like formulation") a massless HS field with spin s is encoded into generalized spin- connections $\omega_\mu^{A_1,A_2,\ldots,A_{s-1},B_1,B_2,\ldots,B_{s-1}}$ and the free part of the theory is a generalization of the MacDowell—Mansouri formulation of gravity [5]. An alternative formulation ("metriclike" formulation), due to Fronsdal, uses conventional tensor fields $\phi_{\mu_1,\mu_2,\ldots\mu_3}(x)$ to construct the free gauge invariant Lagrangian both for flat space-time [6] and for 4-dimensional AdS space [7] (see [8] for field equations and [9,10] for Lagrangians in an arbitrary number of dimensions). It was recently shown that this formulation results from a partial gauge fixing of Maxwell—like geometric equations [11,12].

In this work we undertake the first step towards constructing explicitly the interaction vertex for HS gauge fields in AdS in the "metriclike" formulation. Some features of higher spin interaction have previously been studied in flat space both in covariant [13] and in the light-cone [14,15] formalisms. These studies have shown that for spin higher than two a group structure for nonabelian gauge transformations fails to exist unless one considers the full infinite tower of massless HS fields[1] [17,18].

*Electronic address: afotopou@tspu.edu.ru
†Electronic address: afotopou@physics.uoc.gr
‡Electronic address: petkou@physics.uoc.gr
§Electronic address: tsulaia@physics.uoc.gr

[1]Two examples of a consistent self-interactions of three fields of spin 3 was recently found in [16].

The construction of consistent higher spin field interactions is an old open problem of classical field theory. However, there exist some new motivations for the study of this problem. The first one is the holography of HS gauge theories. It is believed to be the appropriate framework for the holographic description of weakly coupled gauge theories [19]. In fact, we believe that one can holographically translate the wealth of knowledge on weakly coupled quantum field theories to information about HS gauge theories [20,21]. A second motivation is stipulated by the study of the tensionless limit of string theory. It is widely believed that in the $\alpha' \rightarrow \infty$ limit the true symmetries of string theory will emerge [22,23] and HS gauge theories should play a prominent role. It is natural to assume that a consistent tensionless limit can only be taken in the presence of a dimensionful parameter such as space-time curvature. For example, curvature provides an effective tension among string-bits which can compensate the absence of tension providing a stringy tensionless limit. Similar ideas appear in a number of recent works [24–27].

To construct the interaction vertex in AdS we use the covariant BRST approach [28–30] of the "metriclike formulation" to impose gauge invariance. The theory will be formulated in terms of the HS functional—an analogue of the string-field functional which contains an infinite tower of massless fields with arbitrary integer spins. At the free field level such a system describing totally symmetric reducible representations has been considered in [10,31]. We extend those studies to the interacting level.

The techniques used in the approach under consideration are analogous in some aspects to techniques of string-field theory [32,33]. However there are some crucial differences. Unlike string-field theory, a world sheet description of HS fields is not known and therefore there is no analogue of the string overlap conditions, which restrict the argument of the cubic interaction vertex to be quadratic in the oscillators. In our case, the interaction vertex is a general polynomial of the oscillator and ghost variables.

We emphasize that our approach is in a sense perturbative, the perturbation parameter being the dimensionful coupling g, whose physical meaning we explain below. That is the reason why our results in flat space-time and in AdS do not contradict the known no-go theorems for interacting HS gauge fields. To construct the fully gauge invariant action, i.e. gauge invariant to all orders in g, one probably has to add quartic and higher order interactions. We expect that the fully gauge invariant action would contain all the known features of interacting HS theories, such as an infinite tower of fields of all spin and possibly nonlocality. Also we point out that the symmetry algebra in HS theory is not the Virasoro one as in string-field theory. Throughout the paper we restrict to symmetric tensor fields. This suffices if we do not include fermions and fields with mixed symmetry [34,35]. We hope to address such issues in a future work.

The paper is organized as follows: In Sec. II we review the equations of motion, the Lagrangians and their gauge transformations describing reducible representations of the Poincaré and of the AdS group, using the triplet method for the description of HS fields. In Sec. III we formulate the general approach to constructing cubic interaction vertices for massless higher spin fields in flat and AdS backgrounds. We present the main equations of the BRST analysis of the vertex, which are used to constrain the form of the interaction vertex. We define the coefficients of the vertex using an appropriate expansion in ghost and "matter" oscillators. We explain in addition that not all possible interactions terms contain nontrivial information about the cubic vertex. Some expansion coefficients lead to total derivative terms, while some others lead to "fake" interactions which can be factored out using appropriate field redefinitions. We deal separately with the flat and AdS cases in Secs. IV and V respectively. In Sec. IV we demonstrate how one can use our formalism to solve the equations for gauge invariance of the vertex in the flat case, after fake interactions have been taken into account. These result can be used in the sequel to bring the vertex into a form directly applicable on the one hand to holography, and on the other hand to the high energy limit of string theory. The AdS case involves some extra complications which we discuss in Sec. V. In the appendix we present the detailed field redefinitions formulas used in order to factor out fake interactions from the cubic vertex in Sec. III.

II. FREE HIGHER SPIN GAUGE FIELDS IN FLAT AND ADS SPACE-TIMES

There are many ways used in the literature to present the theory of free HS gauge fields [1]. We believe that one of the most elegant and clear descriptions of HS gauge fields is the one based on the triplet construction which we review below. This construction was developed in flat space in [12] and in AdS in [10,31]. This system is named bosonic triplet and describes the propagation of reducible massless HS fields in flat and AdS backgrounds. The name "triplet" comes about because a gauge invariant description of massless fields with spins $s, s-2, s-4, \ldots$ requires in addition to a tensor field ϕ of rank s, the presence of two auxiliary tensor fields. We denote them as C (of rank $s-1$) and D (of rank $s-2$). After elimination of these auxiliary fields via the gauge transformations and/or via their own equations of motion one is left only with the degrees of freedom describing the physical polarization of higher spin fields with spins s, $s-2$, $s-4$, etc.

We restrict ourselves here to the case of totally symmetric fields on $\mathcal{D}$-dimensional AdS space-time. Such a tensor of rank-s is the coefficient of the following state in a Fock space

$$|\Phi^{(s)}\rangle = \frac{1}{(s)!}\varphi_{\mu_1\mu_2\ldots\mu_s}(x)\alpha^{\mu_1+}\ldots\alpha^{\mu_s+}|0\rangle. \qquad (2.1)$$

We will call the vector (2.1) a higher spin functional. To describe the triplet we introduce the tangent-space valued oscillators (α^a, α^{a+}), which satisfy

$$[\alpha^a, \alpha^{b+}] = \eta^{ab}, \qquad a, b = 0, \ldots, \mathcal{D} - 1. \qquad (2.2)$$

The oscillators $(\alpha^{\mu+}, \alpha^{\mu})$ are obtained using the AdS vielbein e^a_μ and inverse vielbein E^μ_a as

$$\alpha^a = e^a_\mu \alpha^\mu, \qquad \alpha^\mu = E^\mu_a \alpha^a, \qquad [\alpha^\mu, \alpha^{\nu+}] = g^{\mu\nu}, \qquad (2.3)$$

with $g_{\mu\nu}$ being the AdS metric. The ordinary partial derivative is now replaced by the operator [36],

$$p_\mu = -i(\nabla_\mu + \omega^{ab}_\mu \alpha^+_a \alpha_b), \qquad (2.4)$$

where ω^{ab}_μ is the spin connection of AdS and ∇_μ is the AdS covariant derivative. This operator satisfies the commutation relations

$$D_{\mu\nu} \equiv [p_\mu, p_\nu] = -[\nabla_\mu, \nabla_\nu] + \frac{1}{L^2}(\alpha^+_\mu \alpha_\nu - \alpha^+_\nu \alpha_\mu),$$
$$[p_\mu, \alpha^{\nu+}] = 0. \qquad (2.5)$$

The action of p_μ on the state (2.1) gives the AdS covariant derivative ∇_μ as

$$p_\mu|\Phi^{(s)}\rangle = -\frac{i}{(s)!}\alpha^{\mu_1+}\ldots\alpha^{\mu_s+}\nabla_\mu\varphi_{\mu_1\mu_2\ldots\mu_s}(x)|0\rangle. \qquad (2.6)$$

We also write down for later use the left action of p_μ on states

$$\langle\Phi^{(s)}|p_\mu = \frac{i}{(s)!}\langle 0|\alpha^{\mu_1}\ldots\alpha^{\mu_s}\nabla_\mu\varphi_{\mu_1\mu_2\ldots\mu_s}(x). \qquad (2.7)$$

Let us note also that the first term in the right hand side in (2.5) gives zero when acting on states (2.1) since the later has no free indexes. The reason behind the use of a covariant derivative in (2.4) will be clear when considering the case of interacting fields as we shall see below.

Next we introduce the following operators:

(i) The d'Alembertian operator

$$l_0 = g^{\mu\nu}p_\mu p_\nu, \qquad (2.8)$$

which acts on Fock-space states as

$$l_0|\Phi^{(s)}\rangle = -\frac{1}{(s)!}\alpha^{\mu_1+}\ldots\alpha^{\mu_s+}\Box\varphi_{\mu_1\mu_2\ldots\mu_s}(x)|0\rangle. \qquad (2.9)$$

(ii) The divergence operator

$$l = \alpha^\mu p_\mu, \qquad (2.10)$$

which acts on a state in the Fock space as

$$l|\Phi^{(s)}\rangle = -\frac{i}{(s-1)!}\alpha^{\mu_2+}\ldots\alpha^{\mu_s+}\nabla_{\mu_1}\varphi^{\mu_1}{}_{\mu_2\mu_3\ldots\mu_s}(x)|0\rangle. \qquad (2.11)$$

(iii) The symmetrized exterior derivative operator,

$$l^+ = \alpha^{\mu+}p_\mu, \qquad (2.12)$$

which acts on states in the Fock space as

$$l^+|\Phi^{(s)}\rangle = -\frac{i}{(s+1)!}\alpha^{\mu+}\alpha^{\mu_1+}\ldots\alpha^{\mu_s+}\nabla_\mu\varphi_{\mu_1\mu_2\mu_3\ldots\mu_s}(x)|0\rangle. \qquad (2.13)$$

The latter is Hermitian conjugate to the operator l with respect to the scalar product

$$\int d^{\mathcal{D}}x\sqrt{-g}\langle\Phi^{(s)}_1||\Phi^{(s)}_2\rangle. \qquad (2.14)$$

It is straightforward to obtain the following commutation relations among the operators just introduced

$$[l, l^+] = \tilde{l}_0, \qquad (2.15)$$

where the modified d'Alembertian $\tilde{l}_0$ is defined as

$$\tilde{l}_0 = l_0 - \frac{1}{L^2}\Big(-\mathcal{D} + \frac{\mathcal{D}^2}{4} + 4M^\dagger M - N^2 + 2N\Big), \qquad (2.16)$$

and

$$[M^\dagger, l] = -l^+, \qquad [\tilde{l}_0, l] = \frac{2}{L^2} l - \frac{4}{L^2} Nl + \frac{8}{L^2} l^+ M, \\ [N, l] = -l. \tag{2.17}$$

Relations (2.17) form a closed algebra which is the base for Lagrangian construction of the massless higher spin theory in AdS space-time. We will call it higher spin symmetry algebra in AdS space.

The operators

$$N = \alpha^{\mu+}\alpha_\mu + \frac{\mathcal{D}}{2}, \qquad M = \frac{1}{2}\alpha^\mu \alpha_\mu, \tag{2.18}$$

form an $SO(1, 2)$ subalgebra of the total nonlinear algebra.

$$[N, M] = -2M, \qquad [M^\dagger, N] = -2M^\dagger, \\ [M^\dagger, M] = -N. \tag{2.19}$$

Having this algebra at hand one can construct the corresponding nilpotent BRST charge. There are two distinct options however, leading to different physical results. The first option is to treat all operators, except N, as constraints. In other words we introduce ghost and antighost variables for each one of the operators, except of N. The operator N can not be treated as a constraint since it is strictly positive and it can not annihilate any physical state. Then, if one builds the nilpotent BRST charge for this nonlinear algebra one arrives to a Lagrangian description of a single higher spin field in AdS [9].

In order to describe a triplet on AdS one has to follow another line [10,31]—namely to introduce ghost and antighost variables *only* for the operators $\tilde{l}_0$, l and l^+. Then one constructs a nilpotent BRST charge in the following way. First one rewrites the second of the commutation relations (2.17) in an equivalent way:

$$[\tilde{l}_0, l] = -\frac{1}{L^2}(6 + 4N)l + \frac{8}{L^2} M l^+, \tag{2.20}$$

i.e., pushing the operators l and l^+ to the right. Then one uses the standard formula for the BRST charge,

$$Q = c^A G_A - \tfrac{1}{2} U_{AB}{}^C c^A c^B b_C, \qquad A, B = 1, 2, 3, \tag{2.21}$$

where $c^A = (c_0, c, c^+)$ and $b_A = (b_0, b^+, b)$ are Grassman odd ghost and antighost variables with ghost number $+1$ and -1 respectively. The ghost and antighost variables satisfy the anticommutation relations $\{c^A, b_B\} = \delta^A_B$ while U^{AB}_C are structure constants $[G_A, G_B] = U_{AB}{}^C G_C$. However since now we have structure functions rather than structure constants, the naive BRST charge (2.21) will not be nilpotent. Therefore one computes Q^2 and adds compensating terms to restore nilpotence. This procedure leads to the BRST charge [10]

$$\begin{aligned} Q = {} & c_0\Big(\tilde{l}_0 - \frac{4}{L^2}N + \frac{6}{L^2}\Big) + c l^+ + c^+ l - c^+ c b_0 \\ & - \frac{6}{L^2} c_0 c^+ b - \frac{6}{L^2} c_0 b^+ c + \frac{4}{L^2} c_0 c^+ b N \\ & + \frac{4}{L^2} c_0 b^+ c N - \frac{8}{L^2} c_0 c^+ b^+ M + \frac{8}{L^2} c_0 c b M^\dagger \\ & + \frac{12}{L^2} c_0 c^+ b^+ c b. \end{aligned} \tag{2.22}$$

Furthermore, we define the ghost vacuum as

$$c|0\rangle_{\rm gh} = 0, \qquad b|0\rangle_{\rm gh} = 0, \qquad b_0|0\rangle_{\rm gh} = 0. \tag{2.23}$$

Therefore the total vacuum is given by the product

$$|0\rangle = |0\rangle_\alpha \otimes |0\rangle_{\rm gh}, \qquad \alpha^a|0\rangle_\alpha = 0. \tag{2.24}$$

The triplet of spin-s, (which involves symmetric tensors of ranks s, $s-1$, and $s-2$), is now expressed through the following states in this enlarged Fock space[2]

$$|\Phi\rangle = |\phi_1\rangle + c_0|\phi_2\rangle, \tag{2.25}$$

where

$$\begin{aligned} |\phi_1\rangle = {} & \frac{1}{s!}\phi_{\mu_1\ldots\mu_s}(x)\alpha^{\mu_1+}\ldots\alpha^{\mu_s+}|0\rangle \\ & + \frac{1}{(s-2)!} D_{\mu_1\ldots\mu_{s-2}}(x)\alpha^{\mu_1+}\ldots\alpha^{\mu_{s-2}+}c^+b^+|0\rangle, \\ |\phi_2\rangle = {} & \frac{-i}{(s-1)!} C_{\mu_1\ldots\mu_{s-1}}(x)\alpha^{\mu_1+}\ldots\alpha^{\mu_{s-1}+}b^+|0\rangle. \end{aligned} \tag{2.26}$$

We will call the vector (2.25) a higher spin functional as well as the (2.1). The vacuum $|0\rangle$ and the state $|\Phi\rangle$ have ghost number zero. The corresponding gauge transformation parameter has ghost number -1

$$|\Lambda\rangle = \frac{i}{(s-1)!}\Lambda_{\mu_1\mu_2\ldots\mu_{s-1}}(x)\alpha^{\mu_1+}\alpha^{\mu_2+}\ldots\alpha^{\mu_{s-1}+}b^+|0\rangle. \tag{2.27}$$

Then the Lagrangian, that has ghost number zero, is

$$L = \int dc_0 \langle\Phi|Q|\Phi\rangle, \tag{2.28}$$

and it is invariant under

$$\delta|\Phi\rangle = Q|\Lambda\rangle. \tag{2.29}$$

Now it is straightforward to obtain the space-time Lagrangian

[2] To avoid overloading the notation, the state $|\Phi\rangle$ will denote henceforth the triplet of spin-s, unless explicitly stated otherwise.

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\phi\Box\phi + s\nabla\cdot\phi C + s(s-1)\nabla\cdot CD \\ &- \frac{s(s-1)}{2}D\Box D - \frac{s}{2}C^2 + \frac{s(s-1)}{2L^2}(\phi')^2 \\ &- \frac{s(s-1)(s-2)(s-3)}{2L^2}(D')^2 - \frac{4s(s-1)}{L^2}D\phi' \\ &- \frac{1}{2L^2}[(s-2)(\mathcal{D}+s-3)-s]\phi^2 \\ &+ \frac{s(s-1)}{2L^2}[s(\mathcal{D}+s-2)+6]D^2. \end{aligned} \tag{2.30}$$

The equations of motion resulting from (2.30) are

$$\begin{aligned}\Box\phi &= \nabla C + \frac{1}{L^2}\{8gD - 2g\phi' \\ &\quad + [(2-s)(3-\mathcal{D}-s)-s]\phi\}, \\ C &= \nabla\cdot\varphi - \nabla D, \\ D &= \nabla\cdot C + \frac{1}{L^2}\{[s(\mathcal{D}+s-2)+6]D - 4\phi' - 2gD'\}.\end{aligned} \tag{2.31}$$

In the equations above $\nabla\cdot$ gives the divergence and ∇ acts as the symmetrized covariant derivative. Further, the prime $0000'$ denotes the trace with respect to the AdS metric. Taking the infinite radius limit $L \to \infty$, the Lagrangian (2.30) and the equations of motion (2.31) reduce to their flat space-time counterparts.

The Lagrangian is invariant under the gauge transformations

$$\begin{aligned}\delta\phi = \nabla\Lambda, \qquad \delta D = \nabla\cdot\Lambda, \\ \delta C = \Box\Lambda + \frac{(s-1)(3-s-\mathcal{D})}{L^2}\Lambda + \frac{2}{L^2}g\Lambda',\end{aligned} \tag{2.32}$$

by virtue of the standard formula for the AdS covariant derivatives acting on a vector ξ_ρ

$$[\nabla_\mu, \nabla_\nu]\xi_\rho = \frac{1}{L^2}(g_{\nu\rho}\xi_\mu - g_{\mu\rho}\xi_\nu). \tag{2.33}$$

Finally, we should make some comments regarding the spectrum. Massless fields of spin-s in AdS saturate the unitary bound for representations of the AdS isometry group $O(2, \mathcal{D}-1)$ (see e.g. [34] for detailed discussions). Their wave equation has the form

$$\left(\Box - \frac{1}{L^2}[(2-s)(3-\mathcal{D}-s)-s]\right)\Phi_{\mu_1\ldots\mu_s}(x) = 0, \tag{2.34}$$

where s is the spin. Then, in complete analogy with the case of flat space-time one can show that the triplet Eqs. (2.31) correctly reproduce the unitary bound for all physical modes i.e., after the diagonalization of the equations and gauge transformations we obtain the propagation of massless fields with spins $s, s-2\ldots$ and proper unitary bound for each of them separately. Next, imposing "by hand" the extra condition

$$\phi' = 2D, \tag{2.35}$$

one can completely eliminate the lower spin fields from the triplet equations and one arrives to the so-called Fronsdal equations in AdS [7]. Note that after imposing (2.35) the parameter of gauge transformations is no more unrestricted, but rather satisfies the condition $\lambda' = 0$. This extra condition can be obtained, after partial gauge fixing [9,28] as an equation of motion from a larger Lagrangian which contains some additional auxiliary fields. To get a formulation for free HS theory in flat space it is sufficient to tend the parameter L to infinity in all relations corresponding to AdS space.

III. METHOD OF CONSTRUCTING THE CUBIC VERTEX FOR HIGHER SPIN GAUGE FIELDS

In this section we discuss a general construction of the cubic HS vertex which is based on generalization of the BRST method mainly used earlier only in free HS theory. This approach is analogous in some aspects to vertex construction in string-field theory, however, as we have pointed out earlier, in our case there exists no analog of the overlap conditions on the three-string interaction vertex that would strongly restrict its form. In the case of interacting massless HS fields the only guiding principle is gauge invariance which manifests itself in the requirement of BRST invariance of the vertex.

There is one crucial point regarding interacting HS fields. It appears that a length parameter is necessary for the construction of the interaction vertex, such that the latter has the right dimensions. For HS field in flat space there is no obvious candidate for this length parameter. One possibility would be to consider HS gauge fields emerging at the tensionless limit of string theory, in which case the role of the above mentioned parameter is played by the inverse of the string tension α'. On the other hand, for HS fields in curved space-times such a dimensionful parameter is naturally given by the inverse curvature. In particular, in the case of HS gauge fields on AdS space-times this parameter is naturally associated with the AdS radius L. Notice that the zero radius limit of such a construction is the large-curvature limit.

After these remarks we will proceed along the lines of [17]. We wish to construct the most general cubic vertex that includes both the case when all interacting fields have the same spin (self interaction) as well as the case when the interacting fields are different. For that we use three copies of the higher spin functional defined in (2.25) as $|\Phi_i\rangle$, $i = 1, 2, 3$. If we studied the quartic vertex we would use four copies of the higher spin functional $|\Phi\rangle$ and etc. The tensors fields in $|\Phi_i\rangle$ are all at the same space-time point. Then, the $|\Phi_i\rangle$ interacting among each other are expanded in terms of the set of oscillators α_μ^{i+}, c^{i+}, and b^{i+}

$$[\alpha^i_\mu, \alpha^{j,+}_\nu] = \delta^{ij} g_{\mu\nu}, \quad \{c^{i,+}, b^j\} = \{c^i, b^{j,+}\} = \{c^i_0, b^j_0\} = \delta^{ij}, \tag{3.1}$$

in complete analogy to the free field case. The BRST charge of our construction consists of three copies of the free BRST change. The full interacting Lagrangian can be written as [32,33]

$$L = \sum_i \int dc^i_0 \langle\Phi_i|Q_i|\Phi_i\rangle + g\left(\int dc^1_0 dc^2_0 dc^3_0 \langle\Phi_1|\langle\Phi_2|\langle\Phi_3||V\rangle + \text{H.c.}\right), \tag{3.2}$$

where $|V\rangle$ is the cubic vertex and g is a dimensionless coupling constant.[3]

It is straightforward to show that the Lagrangian (3.2) is invariant up to terms of order g^2 under the nonabelian gauge transformations

$$\delta|\Phi_1\rangle = Q_1|\Lambda_1\rangle - g\int dc^2_0 dc^3_0 [(\langle\Phi_2|\langle\Lambda_3| + \langle\Phi_3|\langle\Lambda_2|)|V\rangle] + O(g^2), \tag{3.3}$$

$$\delta|\Phi_2\rangle = Q_2|\Lambda_2\rangle - g\int dc^3_0 dc^1_0 [(\langle\Phi_3|\langle\Lambda_1| + \langle\Phi_1|\langle\Lambda_3|)|V\rangle] + O(g^2), \tag{3.4}$$

$$\delta|\Phi_3\rangle = Q_3|\Lambda_3\rangle - g\int dc^1_0 dc^2_0 [(\langle\Phi_1|\langle\Lambda_2| + \langle\Phi_2|\langle\Lambda_1|)|V\rangle] + O(g^2), \tag{3.5}$$

provided that the vertex V satisfies the BRST invariance condition

$$\sum_i Q_i|V\rangle = 0. \tag{3.6}$$

The gauge transformations (3.3), (3.4), and (3.5) are nonlinear deformations of previously considered abelian gauge transformations. We assume here that the tensor fields obtained after the expansion of the $|\Phi_i\rangle$ functionals in terms of the oscillators α^{i+}_μ are different from each other. One can consider cases when two or all three HS functionals contain the same tensor fields. We expect that in such cases the general interaction vertex will exhibit additional symmetry properties.

In order to ensure zero ghost number for the Lagrangian, the cubic vertex must have ghost number 3. We make the following ansatz for the cubic vertex

$$|V\rangle = V|-\rangle_{123} \tag{3.7}$$

where the vacuum $|-\rangle$, with ghost number 3 is defined as the product of the individual Hilbert space ghost vaccua

$$|-\rangle_{123} = c^1_0 c^2_0 c^3_0 |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3. \tag{3.8}$$

The function V has ghost number 0 and it is a function of the rest of the creation operators as well as of the operators p^i_μ. In string-field theory the right-hand side of (3.8) is multiplied by $\delta^{\mathcal{D}}(\sum_i p_i)$ which imposes momentum conservation on the three-string vertex. In our case the analogous constrain is to discard total derivative terms of the lagrangian which is certainly true for flat and AdS spacetimes. So in what follows we will impose "momentum" conservation in the sense described above.

The condition of BRST invariance (3.6) does not completely fix the cubic vertex. There is an enormous freedom due to Field Redefinitions (FR) just like in any field theory Langrangian. It is clear in the free theory case that any FR of the form

$$\delta\Phi_i = F(\Phi_i), \tag{3.9}$$

gives a gauge equivalent set of equations of motion for the fields Φ_i. Lagrangians obtained from the free one after the field redefinition (3.9) yield additional "fake interactions" and should be discarded. For the interacting case at hand we see, from (3.6), that the modified gauge variation (3.3), (3.4), and (3.5) can only determine the cubic vertex up to $\tilde{Q}$-exact cohomology terms:

$$\delta|V\rangle = \tilde{Q}|W\rangle, \tag{3.10}$$

where $\tilde{Q} = \sum_i Q_i$ and $|W\rangle$ is a state with total ghost charge 2. We will see in what follows that this FR freedom can lead into major simplifications for the functional form of the vertex.

Next, we expand the vertex operator $|V\rangle$ and the function $|W\rangle$ in terms of ghost variables or equivalently in terms of the following two ghost quantities with ghost number zero:

$$\gamma^{ij,+} = c^{i,+} b^{j,+}, \qquad \beta^{ij,+} = c^{i,+} b^j_0. \tag{3.11}$$

These are 3×3 matrices with no symmetry properties. For the cubic vertex we have the expansion

[3]Each term in the Lagrangian (3.2) has length dimension $-\mathcal{D}$. This requirement holds true for each space-time vertex contained in (3.2) after multiplication by an appropriate power of the length scale of the theory, as discussed before.

$$|V\rangle = \{X^1 + X^2_{ij}\gamma^{ij,+} + X^3_{ij}\beta^{ij,+} + X^4_{(ij);(kl)}\gamma^{ij,+}\gamma^{kl,+} + X^5_{ij;kl}\gamma^{ij,+}\beta^{kl,+} + X^6_{(ij);(kl)}\beta^{ij,+}\beta^{kl,+} + X^7_{(ij);(kl);(mn)}\gamma^{ij,+}\gamma^{kl,+}\gamma^{mn,+} + X^8_{(ij);(kl);mn}\gamma^{ij,+}\gamma^{kl,+}\beta^{mn,+} + X^9_{ij;(kl);(mn)}\gamma^{ij,+}\beta^{kl,+}\beta^{mn,+} + X^{10}_{(ij);(kl);(mn)}\beta^{ij,+}\beta^{kl,+}\beta^{mn,+}\}|-\rangle_{123}, \tag{3.12}$$

since the function V in (3.7) has ghost number zero. In our notation we put in parentheses pairs of indices which are symmetric under mutual exchange. For example, $X^4_{(ij);(kl)}$ is symmetric under $(ij) \leftrightarrow (kl)$. The coefficient $X^4_{(ij);(kl)}$ is also antisymmetric under $i \rightarrow k$ since $\{c^{i,+}, c^{k,+}\} = 0$ but we have not indicated these symmetries in order to avoid clustering notation.

In a similar manner we have the following expansion:

$$|W\rangle_{123} = \{W^1_i b^{i,+} + W^2_i b^i_0 + W^3_{i;jk} b^{i,+}\gamma^{jk,+} + W^4_{i;jk} b^{i,+}\beta^{jk,+} + W^5_{i;jk} b^i_0\beta^{jk,+} + W^6_{i;(jk);(lm)} b^{i,+}\gamma^{jk,+}\gamma^{lm,+} + W^7_{i;jk;lm} b^{i,+}\gamma^{jk,+}\beta^{lm,+} + W^8_{i;(jk);(lm)} b^{i,+}\beta^{jk,+}\beta^{lm,+} + W^9_{i;(jk);(lm)} b^i_0\beta^{jk,+}\beta^{lm,+} + W^{10}_{i;(jk);(lm);pn} b^{i,+}\gamma^{jk,+}\gamma^{lm,+}\beta^{pn,+} + W^{11}_{i;jk;(lm);(pn)} b^{i,+}\gamma^{jk,+}\beta^{lm,+}\beta^{pn,+} + W^{12}_{i;(jk);(lm);(pn)} b^{i,+}\beta^{jk,+}\beta^{lm,+}\beta^{pn,+}\}|-\rangle_{123}. \tag{3.13}$$

for the FR functional W.

IV. THE CUBIC VERTEX IN FLAT SPACE-TIME

We consider HS fields in flat space-time first. Each component of the vertex in (3.12) has an oscillator expansion in terms of matter oscillators $\alpha^{i,+}_\mu$ and derivatives p^i_μ, where the latter act to the left. As we have done throughout the paper we will restrict our study to the case of totally symmetric massless higher spin fields and therefore we have only to consider three different sets of oscillators and momenta.

A. Flat space-time generators and their algebra for the interacting case

The interaction vertex glues together three Hilbert spaces and for this reason it is convenient to define, in complete analogy to the free case, the following generators

$$l^{ij} = \alpha^{\mu i} p^j_\mu, \qquad l^{ij,+} = \alpha^{\mu,i+} p^j_\mu,$$
$$l^{ij}_0 = p^{\mu i} p^j_\mu, \qquad M^{ij} = \frac{1}{2}\alpha^{\mu i}\alpha^j_\mu, \tag{4.1}$$
$$M^{ij,+} = \frac{1}{2}\alpha^{\mu,i+}\alpha^{j+}_\mu, \qquad N^{ij} = \alpha^{\mu,i+}\alpha^j_\mu + \delta^{ij}\frac{\mathcal{D}}{2}.$$

We see that generators (4.1) are indexed by integers $i, j = 1, 2, 3$. The three values for i and j originate from the fact that we consider three field interaction. In general case of n-field interaction, we should take the same generators with $i, j = 1, 2 \ldots, n$. Using the generators above one can build all possible interaction terms between symmetric higher spin fields. Therefore our ansatz for the vertex is that of the most general polynomial made out from the operators l^{ij}_0, $l^{ij,+}$, and $M^{ij,+}$. This corresponds to the usual derivative expansion for the vertex, since the operators l^{ij}_0 have dimensions $[\text{Length}]^{-2}$ and the operators $l^{ij,+}$, $\beta^{ij,+}$ have dimension $[\text{Length}]^{-1}$. To make sense of such an expansion one needs to introduce a physical length parameter. In flat space-times it is not clear where does such a length scale may come from, nevertheless the hope is that it would be connected to the length scale of a fundamental theory such as string or M-theory.

The commutator algebra of the operators in (4.1) is

$$\begin{aligned}[][l^{ij}, l^{kl,+}] &= \delta^{ik} l^{jl}_0,\\ [N^{ij}, l^{kl}] &= -\delta^{ik} l^{jl},\\ [M^{ij,+}, l^{kl}] &= -\tfrac{1}{2}(\delta^{jk} l^{il,+} + \delta^{ik} l^{jl,+}),\\ [N^{ij}, M^{kl}] &= -(\delta^{ik} M^{jl} + \delta^{il} M^{kj}),\\ [M^{ij}, M^{kl,+}] &= -\tfrac{1}{4}(\delta^{jk} N^{il} - \delta^{jl} N^{ik} - \delta^{ik} N^{jl} - \delta^{il} N^{jk}).\end{aligned} \tag{4.2}$$

Algebra (4.2) generalizes the algebra of generators of the free HS theory and can be called the symmetry algebra of interacting HS theory. It is obvious that the diagonal subalgebra of (4.2)[4] consists of three copies of the algebra presented in (2.17) and (2.19).

Let us consider the constrains imposed by momentum conservation on the vertex. Clearly, not all generators in (4.1) are linearly independent once we consider the operatorial equation $\sum_i p^\mu_i = 0$, which means that we omit total derivatives, as discussed in section III. A convenient set of linearly independent generators is the following:

$$\begin{aligned} l^{ij}_0 &= (l^{11}_0, l^{22}_0, l^{33}_0) = (l^1_0, l^2_0, l^3_0)\\ l^{ij,+} &= (l^{1,+}, I^{1,+}, l^{2,+}, I^{2,+}, l^{3,+}, I^{3,+}),\\ l^{i,+} &= l^{ii,+},\\ I^{1,+} &= \alpha^{\mu,1+}(p^2_\mu - p^3_\mu),\\ I^{2,+} &= \alpha^{\mu,2+}(p^3_\mu - p^1_\mu)\\ I^{3,+} &= \alpha^{\mu,3+}(p^1_\mu - p^2_\mu)\\ M^{ij,+} &= (M^{11,+}, M^{22,+}, M^{33,+}, M^{12,+}, M^{13,+}, M^{23,+})\end{aligned} \tag{4.3}$$

[4] This algebra consists of generators ($l^{ii}_0, l^{ii}, l^{ii,+}, M^{ii,+}, M^{ii}, N^{ii,+}$) for $i = 1, 2, 3$.

Based on the above analysis we can write the most general form of the expansion coefficients $X^l_{(\dots)}$:

$$X^l_{(\dots)} = X^l_{n_1,n_2,n_3;m_1,k_1,m_2,k_2,m_3,k_3;p_1,p_2,p_3,r_{12},r_{13},r_{23}(\dots)}(l_0^1)^{n_1}\dots(l^{+,1})^{m_1}(I^{+,1})^{k_1}\dots(M^{+,11})^{p_1}\dots(M^{+,12})^{r_{12}}\dots \tag{4.4}$$

An analogous expansion can be written for $W^l_{(\dots)}$ as well. It is interesting to point out that the expansion in l^{ij}, $l^{ij,+}$, l_0^{ij} is an expansion in powers of space-time derivatives. Therefore it is naturally to expect that the cubic vertex as well as any interaction vertex will contain higher space-time derivatives. In principle, this circumstance allows us to develop a perturbation scheme for finding the vertex keeping in the expansion derivatives up to some fixed order.

An easy way to recognize interactions obtained from the free Lagrangian (''Fake interactions'') due to field redefinitions is the following [13]. Fake interactions vanish for the fields obeying free equations of motion—which is another way to state that the only nontrivial interaction are in the cohomology of the BRST charge $\tilde{Q}$. Fake interactions can in principle be completely eliminated using the field redefinitions in (3.13). In Appendix A we demonstrate how FR can bring the matrix element $X^n_{\dots}$ of the vertex in a convenient form, both for analyzing the Eqs. (4.6) and for writing the Lagrangian in a simpler form.

B. BRST invariance constraints for the cubic vertex

Using the explicit form of the BRST charges:

$$Q^i = c_0^i l_0^i + c^i l^{i,+} + c^{i,+} l^i - c^{i,+} c^i b_0^i, \qquad \text{(no sum)} \tag{4.5}$$

and Eqs. (3.6) and (4.4) we arrive to the following set of equations:

$$\begin{aligned}
c^{i,+}[l^i X^1 - l^{s,+}X^2_{is} - l_0^s X^3_{is}] &= 0,\\
c^{i,+}\gamma^{jk,+}[l^i X^2_{jk} - 2l^{s,+}X^4_{(is);(jk)} - l_0^s X^5_{jk;is}] &= 0,\\
c^{i,+}\beta^{jk,+}[-\delta_{jk}X^2_{ij} + l^i X^3_{jk} - l^{s,+}X^5_{is;jk} - 2l_0^s X^6_{(is);(jk)}] &= 0,\\
c^{i,+}\gamma^{jk,+}\gamma^{lm,+}[l^i X^4_{(jk);(lm)} - 3l^{s,+}X^7_{(is);(jk);(lm)} - l_0^s X^8_{(jk);(lm);is}] &= 0,\\
c^{i,+}\gamma^{jk,+}\beta^{lm,+}[-2\delta_{lm}X^4_{(il);(jk)} + l^i X^5_{jk;lm} - 2l^{+s}X^8_{(is);(jk);lm} - 2l_0^s X^9_{jk;(is);(lm)}] &= 0,\\
c^{i,+}\beta^{jk,+}\beta^{lm,+}[-\delta_{jk}X^5_{ji;lm} + l^i X^6_{(jk);(lm)} - l^{s,+}X^9_{is;(jk);(lm)} - 3l_0^s X^{10}_{(is);(jk);(lm)}] &= 0.
\end{aligned} \tag{4.6}$$

To simplify the analysis of these equations we define the operator:

$$\tilde{N} = \alpha^{\mu,i+}\alpha^i_\mu + b^{i,+}c^i + c^{i,+}b^i. \tag{4.7}$$

This operator commutes with the BRST charges Q_i and its eigenvalues count the degree of the $X^l_{(\dots)}$s in the $\alpha^{i,+}_\mu$ oscillator expansion. Namely, as it can be seen from the Eq. (3.12), if the degree of the coefficient X^1 in oscillators $\alpha^{i,+}_\mu$ is K, then the rest of the coefficients have the following degrees in the oscillators $\alpha^{i,+}_\mu$

$$X^1(K), \quad X^2(K-2), \quad X^3(K-1), \quad X^4(K-4),$$
$$X^5(K-3), \quad X^6(K-2), \quad X^7(K-6),$$
$$X^8(K-5), \quad X^9(K-4), \quad X^{10}(K-3).$$

For example the first equation has degree $K-1$, since l^{ij} reduces the value of K by one, $l^{ij,+}$ increases it by one and l_0^{ij} leaves it unchanged.

There is yet another number which can be used in a way similar to K. Namely if a term in the expansion of V has powers of operators l_0^{ij}, $l^{ij,+}$, $M^{ij,+}$, $\gamma^{ij,+}$, and $\beta^{ij,+}$ equal to s_1, s_2, s_3, s_4, and s_5 respectively, then the total number $s = s_1 + s_2 + s_3 + s_4 + s_5$ is unchanged under the action of the BRST charge.

The above observations can be used to classify Eqs. (4.5) by their degree K and by the number s. This means that the vertex can be expanded in a sum of contribution with fixed degrees K and s as

$$|V\rangle = \sum_{K,s}|V(K,s)\rangle. \tag{4.8}$$

Therefore the Eq. (3.6) can be split into the infinite sets of equations

$$\sum_i Q_i V(K,s) = 0. \tag{4.9}$$

for each value of K and s.

C. Determining the cubic vertex: An example

We will now show how the first of Eqs. (4.6) can be solved resulting in recursive relations which determine the expansion coefficients $X^2_{(\dots)}$ and $X^3_{(\dots)}$ in terms of $X^1_{(\dots)}$. Using the notation in (A3) we can write the first equation of (4.6) in the form:

$$m_i X^1(n_i{:}n_i+1, m_i{:}m_i-1) - \frac{k_i}{2}\epsilon_{ikl}(X^1(n_k{:}n_k+1, k_i{:}k_i-1) - X^1(n_l{:}n_l+1, k_i{:}k_i-1)) + p_i X^1(m_i{:}m_i+1, p_i{:}p_i-1)$$
$$-\frac{1}{4}\sum_l r_{il}X^1(r_{il}{:}r_{il}-1, m_l{:}m_l+1) + \frac{1}{4}\sum_l r_{il}\epsilon_{lim}X^1(r_{il}{:}r_{il}-1, k_l{:}k_l+1) - \sum_s X^2_{is}(m_s{:}m_s+1) - \sum_s X^3_{is}(n_s{:}n_s+1) = 0. \tag{4.10}$$

We have omitted all indices of the $X^n_{\dots}$ matrices which are not relevant above. Let us look at the equation for $i = 1$. Using the specific scheme described in (A4) we observe that first term in (4.10) vanishes since it would require nonzero matrix elements of the form $X^1_{m_i\geq 1\dots}$. Collecting all term with same exponents with respect to the generators, we find a set of equations between the various matrix elements $X^l_{n_1,n_2\dots;(\dots)}$. Namely terms which have $n_i = m_i = 0$ give:

$$r_{12}X^1_{\dots k_2,k_3+1;\dots r_{12},r_{13}-1,r_{23}} = r_{13}X^1_{\dots k_2+1,k_3;\dots r_{12}-1,r_{13},r_{23}}, \tag{4.11}$$

with the obvious restrictions that all indices of the matrix elements must be positive integer numbers.

We can similarly analyze all other relevant terms. In the table bellow we summarize our results for $i = 1$. The left column shows the exponents of the generators which we factor out from (4.10) and the right one the corresponding equation between matrix elements resulting from this process. Notice that we give the values of the n_i, m_i powers of the generators (l_0^i), (l^{+i}), respectively, only. All other exponents in (4.10) that are not shown here are taken to be arbitrary positive integer numbers. We have not included some equations which are directly deduced by symmetry from those bellow.

Powers of Generators	Equations for $i = 1$
$n_i = 0,\ m_i = 0$	$X^1_{k_2,k_3+1;\, r_{12},r_{13}-1,r_{23}} = \frac{r_{13}}{r_{12}} X^1_{k_2+1,\, k_3;r_{12}-1,r_{13},r_{23}}$
$n_2 = 1,\ n_1 = n_3 = m_i = 0$	$-k_1 X^1_{n_i=0,\, k_1} - X^3_{(12);\, n_i=0,\, k_1-1} = 0$
$n_3 = 1,\ n_1 = n_2 = m_i = 0$	$-k_1 X^1_{n_i=0,\, k_1} + X^3_{(13);\, n_i=0,\, k_1-1} = 0$
$m_1 = 1,\ m_2 = m_3 = n_i = 0$	$p_1 X^1_{m_i=0,\, p_1} - X^2_{(11);\, m_i=0,\, p_1-1} = 0$
$m_2 = 1,\ m_1 = m_3 = n_i = 0$	$r_{12} X^1_{m_i=0,\, r_{12}} + 4X^2_{(12);\, m_i=0,\, r_{12}-1} = 0$
$n_1 = 1,\ n_{2,3} = 0, m_i \geq 0$	$X^3_{(11);\, n_i=0,\, m_i\geq 0} = 0$
$n_2 = 1,\ m_1 \geq 1, n_{1,3} = m_{2,3} = 0$	$X^3_{(12);\, n_i=0,\, m_1\geq 1, m_{2,3}=0} = 0$
any two $m_i \geq 1$	$X^3_{(ij);n_i=0,\ \text{any}\ m_i\geq 1} = 0$

(4.12)

Performing the same analysis for $i = 2, 3$ in (4.10) one can see that all $X^3_{(ij);m_i\geq 1}$ elements can be set equal to zero. In addition the diagonal X^3_{ii} elements vanish altogether. All other ones $X^3_{(ij);m_i=0}$ and X^2_{ij} are determined in terms of one single unknown matrix X^1. There is only one constrain on X^1 at this level from (4.11). The remaining equations in (4.6) are very similar to the one we just analyzed. It might be possible to determine all $X^{n\geq 2}$ matrix elements in terms of a single infinite dimensional matrix X^1. However we postpone this rather lengthy analysis for a future publication.

One can simply check that the solution given in [18] emerges as a special case from the scheme described above. In [18] the function V was taken to be of the form

$$V = \exp(Y_{ij}l^{ij,+} + Z_{ij}\beta^{ij,+}). \tag{4.13}$$

Expanding the exponential in terms of $\beta^{rs,+}$ one can see that the only nonzero functions are X^1, X^3, X^6, and X^{10}. The requirement of the BRST invariance of the vertex gives us the following conditions on the coefficients Y^{rs} and Z^{rs}

$$Z_{i,i+1} + Z_{i,i+2} = 0, \tag{4.14}$$

$$Y_{i,i+1} = Y_{ii} - Z_{ii} - 1/2(Z_{i,i+1} - Z_{i,i+2}), \tag{4.15}$$

$$Y_{i,i+2} = Y_{ii} - Z_{ii} + 1/2(Z_{i,i+1} - Z_{i,i+2}), \tag{4.16}$$

Nevertheless, it is instructive to divide the exponent into two parts and use the basis (4.3):

$$\Delta_1 = \tilde{Y}_r I^{r,+} + \hat{Z}_{rs}\beta^{rs,+}, \tag{4.17}$$

$$\Delta_2 = \tilde{Y}_{rr}l^{r,+} + Z_{rr}\beta^{rr,+}, \tag{4.18}$$

where compared to the basis in (4.13), $\hat{Z}_{rs}$ is the offdiagonal part of Z_{rs},

$$\tilde{Y}_r = \frac{1}{2}\epsilon_{rkl}Y^{kl} \qquad \tilde{Y}_{rr} = Y_{rr} - \frac{1}{2}(Y_{rs} + Y_{sk}) \qquad (r, s, k) \text{ permutations of } (1, 2, 3) \tag{4.19}$$

as one can check using (4.3). In Δ_2 we have grouped all the diagonal terms of the exponent. The important point is that the two exponentials will lead into linearly independent conditions under BRST invariance. To prove the BRST invariance of the full vertex. it suffices to show that $\exp(\Delta_1)$ and $\exp(\Delta_2)$ are separately invariant. Since $\tilde{Q}\Delta_1$ commutes with Δ_1 we can easily show that

$$\tilde{Q}\exp(\Delta_1)|-\rangle_{123} = \exp(\Delta_1)(\tilde{Q}\Delta_1|-\rangle_{123}). \tag{4.20}$$

This implies that BRST invariance in (4.20) is equivalent to

$$\tilde{Q}\Delta_1|-\rangle_{123} = \sum_i c^{i,+}\left(-\frac{1}{2}\epsilon_{ikl}\tilde{Y}_i(l_0^k - l_0^l) - \hat{Z}_{is}l_0^s\right)|-\rangle_{123} = 0, \tag{4.21}$$

where we have used the commutation relations (A2). From the condition above we find the solution

$$\tilde{Y}_i = -\frac{1}{2}\epsilon_{ikl}\hat{Z}^{kl}, \qquad \hat{Z}_{ik} = -\hat{Z}_{il} \qquad (i, k, l) \text{ permutations of } (1, 2, 3). \tag{4.22}$$

In a similar manner we can show that BRST invariance of $\exp(\Delta_2)$ is equivalent to

$$\tilde{Q}\Delta_2|-\rangle_{123} = \sum_i c^{i,+}(Y_{ii}l_0^i - Z_{ii}l_0^i)|-\rangle_{123} = 0 \tag{4.23}$$

with solution

$$Z_{rr} = \tilde{Y}_{rr}. \tag{4.24}$$

We can easily verify, using the transformations in (4.19), that the conditions (4.22) and (4.24) are equivalent to those of (4.14). The two conditions (4.21) and (4.23) are obviously linearly independent in terms of $c^{i,+}$ and l_0^i and this is the reason they can be satisfied independently. This shows that the full vertex

$$|V\rangle = \exp(\Delta_1 + \Delta_2)|-\rangle_{123} \tag{4.25}$$

is BRST invariant with the coefficients satisfying (4.22) and (4.24). Note that Δ_2 is $\tilde{Q}$-exact, modulo terms which vanish acting on the vacuum $|-\rangle_{123}$. This means that the dependence of the vertex on Δ_2 can be eliminated via a FR. We can easily show that

$$|W\rangle = -Y_{jj}b^{j,+}\exp(\Delta_1)\sum_{l=0}^{\infty}\frac{\Delta_2^l}{(l+1)!}|-\rangle_{123} \tag{4.26}$$

leads to the FR

$$\delta|V\rangle = \tilde{Q}|W\rangle = -(\exp(\Delta_2) - 1)\exp(\Delta_1)|-\rangle_{123} \tag{4.27}$$

and gives $V' = \exp(\Delta_1)$. This is the scheme (A5) we have developed in the appendix to remove "fake interactions." We should point out that the specific scheme does not actually remove all diagonal ghost terms like $\beta^{ii,+}$. Such terms appear in the β-expansion of the exponent to quadratic order and beyond. This is because terms like $\beta^{ik,+}\beta^{ji,+}$ can be equivalently written as $\beta^{ii,+}\beta^{jk,+}$. It is only the $l^{i,+}$ terms which are removed.

Having done all of the above, it is straightforward to show that the first equation in (4.6) leads to the same constrains as in (4.24). The ansatz $V' = \exp(\Delta_1)$ is a particular case of the general solution in with

$$X^1 = \exp(\tilde{Y}_r I^{+r}), \qquad X_{ij}^2 = 0, \qquad X_{ij}^3\beta^{ij,+} = \tilde{Z}_{ij}\beta^{ij,+}X^1, \tag{4.28}$$

and the matrix elements are

$$X^1_{n_i=m_i=0,k_1,k_2,k_3} = \frac{1}{k_1!k_2!k_3!}. \tag{4.29}$$

The remaining equations in (4.6) are rather straight forward to solve in this particular case and determine X^6 and X^{10} in terms of X^1. These agree with the expansion of V' in ghost variables $\beta^{ij,+}$.

V. THE CUBIC VERTEX ON ADS

To construct the vertex on AdS we use the same procedure as in the flat case, in particular, we solve the same Eq. (3.6). In this case, however, care is needed when trying to extend the the algebra (4.2) to a nontrivial background.

A. AdS generators and their algebra in the interacting case

In order to compute the algebra it is useful to recall how various operators defined previously act on physical states. For example operator $l_0^{12} = p_\mu^1 p_\mu^2$, where p_μ is the operator (2.4), acts as follows:

$$l_0^{12}|\Phi_1\rangle\otimes|\Phi_2\rangle = \frac{i}{(s_1)!}\alpha^{\mu_1,1+}\ldots\alpha^{\mu_s,1+}\nabla^\mu\varphi^1_{\mu_1\mu_2\ldots\mu_{s_1}}(x)|0\rangle_1 \otimes\frac{i}{(s_2)!}\alpha^{\nu_1,2+}\ldots\alpha^{\nu_s,2+}\nabla_\mu\varphi^2_{\nu_1\nu_2\ldots\nu_{s_2}}(x)|0\rangle_2.$$

The operators p_μ^i act only on i -th Hilbert space and therefore

$$[p_\mu^i, p_\nu^j] = \delta^{ij}\left(-[\nabla_\mu^i, \nabla_\nu^i] + \frac{1}{L^2}(\alpha_\mu^{i,+}\alpha_\nu^i - \alpha_\nu^{i,+}\alpha_\mu^i)\right) = \delta^{ij}D_{\mu\nu}^i. \tag{5.1}$$

The other operators are defined in an analogous way. For example the operator $l^{12} = \alpha^{\mu,1}p_\mu^2$ acts as

$$l^{12}|\Phi_1\rangle \otimes |\Phi_2\rangle = \frac{1}{(s_1-1)!}\alpha^{\mu_2,1+}\ldots\alpha^{\mu_s,1+}\varphi^{1\mu}_{\mu_2\ldots\mu_{s_1}}(x)|0\rangle_1 \otimes -\frac{i}{(s_2)!}\alpha^{\nu_1,2+}\ldots\alpha^{\nu_s,2+}\nabla_\mu\varphi^2_{\nu_1\nu_2\ldots\nu_{s_2}}(x)|0\rangle_2,$$

the operator $l^{12+} = \alpha^{\mu,1+}p^2_\mu$ acts as

$$l^{12+}|\Phi_1\rangle \otimes |\Phi_2\rangle = \frac{1}{(s_1)!}\alpha^{\mu,1+}\alpha^{\mu_1,1+}\ldots\alpha^{\mu_s,1+}\varphi^1_{\mu_1\ldots\mu_{s_1}}(x)|0\rangle_1 \otimes -\frac{i}{(s_2)!}\alpha^{\nu_1,2+}\ldots\alpha^{\nu_s,2+}\nabla_\mu\varphi^2_{\nu_1\nu_2\ldots\nu_{s_2}}(x)|0\rangle_2,$$

and the operator $M^{12} = \frac{1}{2}\alpha^{\mu,1}\alpha^2_\mu$ acts as

$$M^{12}|\Phi_1\rangle \otimes |\Phi_2\rangle = \frac{1}{2}\frac{1}{(s_1-1)!}\alpha^{\mu_2,1+}\ldots\alpha^{\mu_s,1+}\varphi^{1\mu}{}_{\mu_2\ldots\mu_{s_1}}(x)|0\rangle_1 \otimes \frac{1}{(s_2-2)!}\alpha^{\nu_2,2+}\ldots\alpha^{\nu_s+}\varphi^2_{\mu\nu_2\ldots\nu_{s_2}}(x)|0\rangle_2.$$

The definition of the diagonal operators is the same as (2.8), (2.10), and (2.12).

At this point we think it is instructive to present an explicit example of a computation. Let us compute the commutator between l^{11} and l^{12+} acting on $|\Phi_1\rangle \otimes |\Phi_2\rangle$, where, for clarity, we take $|\Phi_1\rangle$ to be a vector and $|\Phi_2\rangle$ to be a scalar.

$$\begin{aligned}[][\alpha^{\mu,1}p^1_\mu, \alpha^{\nu,1+}p^2_\nu]\varphi^1_\rho\alpha^{\rho,1+}|0\rangle_1 \otimes \varphi^2|0\rangle_2 &= -i(\alpha^{\mu,1}[p^1_\mu, \alpha^{\nu,1+}]\phi^1_\rho\alpha^{\rho,1+} + [\alpha^{\mu,1}, \alpha^{\nu,1+}]p^1_\mu\phi^1_\rho\alpha^{\rho,1+})(\nabla_\nu\phi^2)|0\rangle_1 \otimes |0\rangle_2 \\ &= -\alpha^{\rho,1+}(\nabla_\nu\varphi^1_\rho)(\nabla_\nu\varphi^2)|0\rangle_1 \otimes |0\rangle_2 \end{aligned} \quad (5.2)$$

In obtaining the above result it was crucial that p^i_μ commutes with $\alpha^{\nu,j+}$.

Proceeding this way one obtains the algebra of operators

$$l^{ij}_0 = p^{\mu,i}p^j_\mu, \qquad l^{ij} = \alpha^{\mu,i}p^j_\mu, \qquad l^{ij,+} = \alpha^{\mu i,+}p^j_\mu, \quad (5.3)$$

on AdS for the interacting case

$$[l^{ij}, l^{mn,+}] = \delta^{im}l^{jn}_0 - \delta^{jn}\alpha^{\mu m,+}D^j_{\mu\nu}\alpha^{\nu i}, \quad (5.4)$$

$$[l^{mn}, l^{kl}] = \delta^{nl}\alpha^{\mu m}D^j_{\mu\nu}\alpha^{\nu k}, \quad (5.5)$$

$$[l^{ij}_0, l^{mn}] = \delta^{jn}\alpha^{\nu m}D^j_{\mu\nu}p^{\mu,i} + \delta^{in}\alpha^{\nu m}D^i_{\mu\nu}p^{\mu,j}, \quad (5.6)$$

$$\begin{aligned}[][l^{ij}_0, l^{kl}_0] = {}& \delta^{jk}p^{\mu,i}D^j_{\mu\nu}p^{\nu,l} + \delta^{ik}p^j_\mu D^i_{\mu\nu}p^l_\nu \\ & - \delta^{jl}p^{\mu,k}D^j_{\mu\nu}p^{\nu,i} - \delta^{il}p^{\mu,k}D^i_{\mu\nu}p^{\nu j} \\ & + \frac{(1-\mathcal{D})}{L^2}\delta^{ik}\delta^{ij}l^{il}_0 - \frac{(1-\mathcal{D})}{L^2}\delta^{jl}\delta^{ij}l^{ki}_0 \end{aligned} \quad (5.7)$$

supplemented by the part of the algebra (4.2) which involves commutators of M^{ij}, $M^{ij,+}$, and N^{ij}. We will call the algebra (5.4), (5.5), (5.6), and (5.7) the symmetry algebra of interacting HS theory in AdS space-time.

The commutation relations above differ from the corresponding flat space ones (4.2), in that they involve extra terms which when acting on states give $O(1/L^2)$ contributions. These terms are subleading in the $L \to \infty$ limit, hence the algebra (5.4) contracts to the flat space-time algebra (4.2) in the small curvature limit. This implies that free HS gauge fields in flat space-time can be viewed as the zero curvature limit of free HS gauge fields on AdS. However, the interacting HS gauge fields on AdS do not have a smooth $L \to \infty$ limit since the interaction vertices contain positive powers of L. Nevertheless, as we shall see below, the functional form of the cubic vertex of HS gauge fields on AdS differs from the cubic vertex in flat space-time by terms which are subleading as $L \to \infty$.

From the explicit form (5.1) the first term $[\nabla_\mu, \nabla_\nu]$ leaves the "scalar" Fock state invariant and only the oscillator piece contributes, which can be written in terms of the standard (4.1) generators. The algebra (5.4), (5.5), (5.6), and (5.7) acting on states becomes:

$$\begin{aligned}[][l^{ij}, l^{mn,+}] = {}& \delta^{im}l^{jn}_0 + \frac{1}{L^2}\delta^{jn}\Big[N^{mj}N^{mi} \\ & + ((\mathcal{D}-1)\delta^{ij} - 1)N^{mi} \\ & - \frac{\mathcal{D}}{2}(\delta^{mi} + \delta^{mj})N^{ij} - \frac{\mathcal{D}^2}{4}\delta^{mj}\delta^{mi} \\ & - 4M^{mj,+}M^{ij}\Big], \end{aligned} \quad (5.8)$$

$$\begin{aligned}[][l^{mn}, l^{kl}] = {}& \frac{1}{L^2}\delta^{nl}\Big[\Big(\frac{\mathcal{D}}{2} - 1\Big)(\delta^{kl}M^{ml} - \delta^{ml}M^{kl}) \\ & + N^{lm}M^{kl} - N^{kl}M^{ml}\Big], \end{aligned} \quad (5.9)$$

$$\begin{aligned}[][l^{ij}_0, l^{mn}] = {}& \frac{1}{L^2}\delta^{jn}(2l^{ji,+}M^{mn} - N^{jm}l^{ni}) \\ & + \frac{1}{L^2}\delta^{in}(2l^{ij,+}M^{mn} - N^{im}l^{nj}) \\ & + \frac{1}{L^2}(\delta^{jn}\delta^{jm}l^{ni})\Big(1 - \frac{\mathcal{D}}{2}\Big) \\ & + \frac{1}{L^2}(\delta^{in}\delta^{im}l^{ni})\Big(1 - \frac{\mathcal{D}}{2}\Big) \\ & - \frac{1}{L^2}(\delta^{in}\delta^{ij}l^{ni})(1 - \mathcal{D}), \end{aligned} \quad (5.10)$$

$$[l_0^{ij}, l_0^{kl}] = \frac{1}{L^2}\delta^{jk}(l^{ji,+}l^{jl} - l^{jl,+}l^{ji}) + \frac{1}{L^2}\delta^{ik}(l^{ij,+}l^{il} - l^{il,+}l^{ij}) + \frac{1}{L^2}\delta^{jl}(l^{ji,+}l^{jk} - l^{jk,+}l^{ji}) + \frac{1}{L^2}\delta^{il}(l^{ij,+}l^{ik} - l^{ik,+}l^{ij}) - \frac{1}{L^2}(\mathcal{D}-1)(\delta^{jk}\delta^{jl} + \frac{1}{L^2}\delta^{ik}\delta^{il})l_0^{ij} + \frac{1}{L^2}(\mathcal{D}-1)(\delta^{ik}\delta^{jk}l_0^{il} + \delta^{il}\delta^{jl}l_0^{ik}). \quad (5.11)$$

We find it again useful to demonstrate how the calculations are done in the interacting case on AdS with an example.

$$\begin{aligned}[l^{12}, l^{22,+}]l^{12,+}\varphi^1_\rho\alpha^{\rho,1+}|0\rangle_1 \otimes \varphi^2|0\rangle_2 &= \alpha^{2,+}_\mu D^2_{\mu\nu}\alpha^1_\nu l^{12,+}\varphi^1_\rho\alpha^{\rho 1,+}|0\rangle_1 \otimes \varphi^2|0\rangle_2 \\ &= -\frac{1}{L^2}\left(l^{22,+}\left(N^{11} - 1 + \frac{\mathcal{D}}{2}\right) - 2M^{12,+}l^{12}\right)\varphi^1_\rho\alpha^{\rho,1+}|0\rangle_1 \otimes \varphi^2|0\rangle_2 \\ &= \frac{i}{L^2}\alpha^{2,+}_\mu\alpha^{1,+}_\nu(\mathcal{D}\varphi^\nu_1(\nabla^\mu\varphi_2) - g^{\mu\nu}\varphi^\rho_1(\nabla_\rho\varphi_2))|0\rangle_1 \otimes |0\rangle_2. \end{aligned} \quad (5.12)$$

There is only a p^2_λ from the second Hilbert space involved in the example above. In the first equality we used (5.4). In the second equality we acted with $D^2_{\mu\nu}$ on the p^2_λ of the $l^{12,+}$ operator using (2.33) and (5.1). This was the only "tensor" operator in the 2nd Hilbert space, since φ^2 is a scalar. Consequently, we commuted operators α^i_σ and p^2_σ past each other to bring the result to the second line of (5.12). Finally we used the algebra of (5.8), (5.9), (5.10), and (5.11) to complete the calculations since no other "vector" operator, in the the 2nd Hilbert space, was left for l^{22+} or l^{12} to act upon.

From the manipulations above we conclude the following: The algebra of constraints being obviously more complicated than in the case of flat space-time shares its main property—namely it preserves the polynomial form of (3.12), (3.13), and (4.4). Therefore we can proceed in an analogous manner as in the flat case.

B. BRST invariance constrains for the cubic vertex on AdS

The next step is to choose an expansion of the cubic vertex in terms of the AdS generators (4.1) and (5.3). In the AdS case the creation generators of (4.3) do not commute among each other, unlike the flat case, as one can see from, i.e., (5.6). Nevertheless, we can choose a *standard* ordering as in (4.4). All other possible orderings can be brought in the *standard* form (i.e., use an analogue of the Weyl ordering in quantum mechanics), using the algebra (5.4), (5.5), (5.6), and (5.7) and the manipulations described in the previous subsection, modulo $\frac{1}{L^2}$ terms[5] that affect lower dimension terms in the L^2 expansion of X^n. These latter terms can again be brought in the *standard* form following the same procedure and finally be absorbed in the definition of the matrix elements with lower dimension than the one we started from.

In addition although naively we do not have momentum conservation in AdS space-time, we can still make use of the equation $\sum_i p^\mu_i = 0$, since it leads into total derivative terms in the Lagrangian.

To conclude, one can construct the same linearly independent set of generators as in (4.3). The expansion of the coefficients is exactly the same as in (4.4) with all generators the AdS equivalent of the flat ones. Using the explicit form of (A6) it is straightforward to write down the equations resulting from (3.6). They are the same as in flat case with the substitution $l_0 \to \hat{l}_0$ as in (A7) and some modifications analogous to those in (A8). The final result is

$$\begin{aligned} c^{i,+}\left[l^iX^1 - l^{s,+}X^2_{is} - \hat{l}^s_0X^3_{is} + \frac{16}{L^2}M^{s,+}X^5_{is;ss}\right] &= 0, \\ c^{i,+}\gamma^{jk,+}\left[l^iX^2_{jk} - 2l^{s,+}X^4_{(is);(jk)} - \hat{l}^s_0X^5_{jk;is} + \frac{8}{L^2}(\delta_{jk}M_jX^3_{jk} - 6M^{s,+}X^8_{(ss);(jk);is})\right] &= 0, \\ c^{i,+}\beta^{jk,+}\left[-\delta_{jk}X^2_{ij} + l^iX^3_{jk} - l^{s,+}X^5_{is;jk} - 2\hat{l}^s_0X^6_{(is);(jk)} - \frac{32}{L^2}M^{s,+}X^9_{ss;(jk);(is)}\right] &= 0, \\ c^{i,+}\gamma^{jk,+}\gamma^{lm,+}\left[l^iX^4_{(jk);(lm)} - 3l^{s,+}X^7_{(is);(jk);(lm)} - \hat{l}^s_0X^8_{(jk);(lm);is} - \frac{8}{L^2}\delta_{jk}M^jX^5_{lm;ij}\right] &= 0, \\ c^{i,+}\gamma^{jk,+}\beta^{lm,+}\left[-2\delta_{lm}X^4_{(il);(jk)} + l^iX^5_{jk;lm} - 2l^{s,+}X^8_{(is);(jk);lm} - 2\hat{l}^s_0X^9_{jk;(is);(lm)} + \frac{16}{L^2}\delta_{jk}M^jX^6_{(lm);(ij)}\right] &= 0, \\ c^{i,+}\beta^{jk,+}\beta^{lm,+}[-\delta_{jk}X^5_{ji;lm} + l^iX^6_{(jk);(lm)} - l^{s,+}X^9_{is;(jk);(lm)} - 3\hat{l}^s_0X^{10}_{(is);(jk);(lm)}] &= 0. \end{aligned} \quad (5.13)$$

[5] The action of $D^i_{\mu\nu}$ on "tensors" produces terms proportional to $\frac{1}{L^2}$ as on can easily verify from (2.33) and (5.1).

Combinations involving the operator $\hat{l}_0^s$ should be understood as follows. For example the term in the first equation $c^{i+}\hat{l}_0^s X_{is}^3$ is a result of an action of the operator $c_0^{i+}\hat{l}_0^i$ at $X_{mn}^3\beta^{+mn}$ and using the expression (A7)

$$\begin{aligned} c_0^{i,+}\hat{l}_0^i X_{mn}^3\beta^{mn,+} = & -c^{i,+}\Big(p^{\mu,s}p_\mu^s + \frac{1}{L^2}((\alpha^{\mu,s+}\alpha_\mu^s)^2 \\ & + \mathcal{D}\alpha^{\mu,s+}\alpha_\mu^s - 6\alpha^{\mu,s+}\alpha_\mu^s \\ & - (2\mathcal{D}-6)^s - 4M^{s,+}M^s)X_{is}^3 \\ & - \frac{1}{L^2}c^{i,+}(4\alpha^{\mu,i+}\alpha_\mu^i + (2\mathcal{D}-6)^i)X_{ii}^3. \end{aligned} \tag{5.14}$$

The equations in (5.13) are more difficult to analyze compared to flat case despite their apparent similarity. The main reason is obvious from the algebra (5.4), (5.5), (5.6), and (5.7) which has nontrivial commutators containing $D_{\mu\nu}^i$. This causes more a technical difficulty rather than a conceptual one. It would be interesting to find a solution in a closed compact form (if such a solution exists of course) but at the present moment we are content to have a well defined iteration procedure and a system of equations which can be straightforwardly solved via this procedure.

VI. SUMMARY AND OUTLOOK

In the present paper we have addressed the problem of constructing the cubic interaction vertex of higher spin Theory in the "metriclike formalism" on $\mathcal{D}$-dimensional flat and AdS spaces. We have discussed the free equations of motion for higher spin fields in flat and AdS space-times in triplet formalism. The only principle we have followed in this construction is the requirement of gauge invariance of the Lagrangian. These equations describe reducible representation of the Poincare and AdS groups. To obtain an irreducible representation on has to add certain off-shell constraints to the field equations.

Assuming the cubic interaction vertex to be a series in ghost and oscillator variables we have obtained the equations which determine the vertex. We outlined the way how these equations can be solved level by level in oscillator expansions. The vertex obtained in this way contains a part which produces "fake interactions" i.e. the ones which can be obtained form the free field Lagrangian via field redefinitions. We have shown how in practice this trivial part of the vertex can be factorized out by solving the cohomologies of the corresponding BRST operator which determines a free part of the Lagrangian. As a result, the gauge invariant formulation the Lagrangian contains alongside physical modes some auxiliary fields as well. Finding the form of the vertex is essentially based on the symmetry algebras of interacting HS fields in flat (4.2) and AdS (5.4), (5.5), (5.6), and (5.7) space-times.

There are several open problems to address namely

(i) Our results can used to focus on particular sets of fields, having in mind holography. In particular, it would be quite interesting to reproduce holographically the known results for the conformal 3-point functions of fields with spin 1 and 2 [37].

(ii) A crucial test would be is to compare the cubic interaction vertices obtained in the present approach to the ones obtained in a "framelike" formulation by Vasiliev [3].

(iii) Our approach can be used to discuss the interactions of HS fermionic fields, as well as of HS fields represented by tensors with mixed symmetry.

(iv) Finally, it would be very desirable if our calculations shed some light into the conjectured link of HS gauge theory with the high energy behavior of string theory in flat [22,23] and in AdS space-times.

ACKNOWLEDGMENTS

It is a pleasure to thank X. Bekaert, C. Coriano, N. Irges, A. Koshelev, P. Lavrov, M. Matone, P. Pasti, D. Sorokin, P. Sundell, M. Tonin, T. Tomaras, M. Vasiliev, P. West, and R. Woodard for useful discussions. This work was partially supported by the INTAS grant, project No. INTAS-03-51-6346. The work of I. L. B was partially supported by RFBR grant, Project No 06-02-16346, joint RFBR/DFG grant, Project No 06-02-04012, DFG grant, Project No 436 RUS 113/669/0-3 and grant LRSS, Project No 4489.2006.2. The work of A. C. P. was partially supported by the PYTHAGORAS II Research Program K.A.2101, of the Greek Ministry of Higher Education. The work of A. F. is supported by a PYTHAGORAS Research Program K.A.1955 of the Greek Ministry of Higher Education and partially supported by the European Commission, under RTN program MRTN-CT-2004-0051004 and by the Italian MIUR under the Contract No. PRIN 2005023102. The work of M. T. was supported by the European Contract No. MRTN-CT-2004-512194.

APPENDIX: FIELD REDEFINITIONS IN BRST FORMALISM

In this appendix we will demonstrate how one can use the FR freedom $|W\rangle$ to eliminate "fake interactions" and to bring $|V\rangle$ in a convenient form.

1. Equations for field redefinitions on flat space-time

A direct computation of (3.10) using (3.12), (3.13), and (4.5), leads to the following transformations for the expansion coefficients of the vertex:

$$\begin{aligned}
\delta X^1 &= l^{i,+}W_i^1 + l_0^i W_i^2,\\
[\delta X_{ij}^2]\gamma^{ij,+} &= [l_i W_j^1 + 2l^{s,+}W_{s;ij}^3 - l_0^s W_{i;js}^4]\gamma^{ij,+},\\
[\delta X_{ij}^3]\beta^{ij,+} &= [\delta_{ij}W_i^1 + l_i W_j^2 + l^{s,+}W_{s;ij}^4 + 2l_0^s W_{s;ij}^5]\beta^{ij,+},\\
[\delta X_{(ij);(kl)}^4]\gamma^{ij,+}\gamma^{kl,+} &= [l_i W_{j;(kl)}^3 + 3l^{s,+}W_{s;(ij);(kl)}^6 + l_0^s W_{l;(ij);ks}^7]\gamma^{ij,+}\gamma^{kl,+},\\
[\delta X_{ij;kl}^5]\gamma^{ij,+}\beta^{kl,+} &= [2\delta_{kl}W_{l;ij}^3 + l_i W_{j;kl}^4 + 2l^{s,+}W_{s;ij;kl}^7 - 2l_0^s W_{j;(kl);(is)}^8]\gamma^{ij,+}\beta^{kl,+},\\
[\delta X_{(ij);(kl)}^6]\beta^{ij,+}\beta^{kl,+} &= [\delta_{(ij)}W_{j;(kl)}^4 + l_{(i}W_{j);kl}^5 + l^{s,+}W_{s;(ij);(kl)}^8 + 3l_0^s W_{s;(ij);(kl)}^9]\gamma^{ij,+}\beta^{kl,+},\\
[\delta X_{(ij);(kl);(mn)}^7]\gamma^{ij,+}\gamma^{kl,+}\gamma^{mn,+} &= [l_i W_{j;(kl);(mn)}^6 - l_0^s W_{n;(ij);(kl);ms}^{10}]\gamma^{ij,+}\gamma^{kl,+}\gamma^{mn,+},\\
[\delta X_{(ij);(kl);mn}^8]\gamma^{ij,+}\gamma^{kl,+}\beta^{mn,+} &= [3\delta_{mn}W_{n;(ij);(kl)}^5 + l_i W_{j;(kl);mn}^7 + 3l^{s,+}W_{s;(ij);(kl);mn}^{10} - 2l_0^s W_{l;(ij);ks;mn}^{11}]\gamma^{ij,+}\gamma^{kl,+}\beta^{mn,+},\\
[\delta X_{ij;(kl);(mn)}^9]\gamma^{ij,+}\beta^{kl,+}\beta^{mn,+} &= [2\delta_{kl}W_{l;ij;(mn)}^7 + l_i W_{j;(kl);(mn)}^8 + 2l^{s,+}W_{s;ij;(kl);(mn)}^{11} - 3l_0^s W_{j;is;(kl);(mn)}^{12}]\gamma^{ij,+}\beta^{kl,+}\beta^{mn,+},\\
[\delta X_{(ij);(kl);(mn)}^{10}]\beta^{ij,+}\beta^{kl,+}\beta^{mn,+} &= [-\delta_{ij}W_{j;(kl);(mn)}^8 + l_i W_{j;(kl);(mn)}^9 + l^{s,+}W_{s;(ij);(kl);(mn)}^{12}]\beta^{ij,+}\beta^{kl,+}\beta^{mn,+}.
\end{aligned} \tag{A1}$$

In order to analyze these equations we need to determine the action of operators l^i, $l^{i,+}$, l_0^i on the matrix elements W^l. First let us write down the commutator relations for the basis in (4.3):

$$\begin{gathered}
[l^i, l^{j,+}] = \delta^{ij}l_0^j, \qquad [l^i, I^{j,+}] = -\tfrac{1}{2}\delta^{ij}\epsilon_{ikl}(l_0^k - l_0^l),\\
[l_i, M^{kl,+}] = \tfrac{3}{2}\delta^{ik}\delta^{kl}l^{k,+} - \tfrac{1}{4}(\delta^{li}l^{k,+} + \delta^{ki}l^{l,+}) + \tfrac{1}{4}(\delta^{li}\epsilon_{klm}I^{k,+} + \delta^{ik}\epsilon_{lkm}I^{l,+}),
\end{gathered} \tag{A2}$$

with all other commutators vanishing. The indexes i, j, k run over the three Hilbert spaces. Based on the algebra above we can deduce the following set of simple transformation rules:

$$l^0 W(n_i{:}n_i) \to W(n_i{:}n_i + 1), \qquad l^{+i}W(m_i{:}m_i) \to W(m_i{:}m_i + 1), \tag{A3}$$

$$\begin{aligned}
l^i W(n_i{:}n_i, m_i{:}m_i, k_i{:}k_i, p_i{:}p_i, r_{ij}{:}r_{ij}) \to{}& m_i W(n_i{:}n_i + 1, m_i{:}m_i - 1) + p_i W(m_i{:}m_i + 1, p_i{:}p_i - 1)\\
&- \frac{k_i}{2}\epsilon_{ikl}(W(n_k{:}n_k + 1, k_i{:}k_i - 1) - W(n_l{:}n_l + 1, k_i{:}k_i - 1))\\
&- \frac{1}{4}\sum_l r_{il}W(r_{il}{:}r_{il} - 1, m_l{:}m_l + 1) + \frac{1}{4}\sum_l r_{il}\epsilon_{lim}W(r_{il}{:}r_{il} - 1, k_l{:}k_l + 1).
\end{aligned}$$

Let us explain our notation in the above equation. We use $W(n_i{:}n_i)$ for an element which has an expansion as in (4.4): $W_{n_i,\ldots}(l_0^i)^{n_i}\ldots$. Then the transformation law in the second equation above means that we have on the right-hand side the expansion: $W(n_i{:}n_i + 1) = W_{n_i,\ldots}(l_0^i)^{n_i+1}\ldots$. Analogously are defined the other quantities in (A3).

As we have already mentioned the l_0^i generators act on states of the ith-Hilbert space as flat space-time Laplacian. In addition the l^{+i} operators act on the left, on bra states, as divergences resulting in $\nabla^\mu\Phi_{\mu\ldots}$ terms. We will show that an appropriate choice of the FR functions W^n, can be used to eliminate most of the dependence, of the X^n coefficients, on these operators.

From the first of (A1) and (A3) we can easily see that using all of the $W^1_{1;n_i\geq 0,m_i\geq 0}$ freedom we can eliminate the $m_1 > 0$ terms in X^1 and we are left with the element $X^1_{n_i\geq 0,m_1=0,m_{2,3}\geq 0}$. We can go on and use $W^1_{2;n_i\geq 0,m_1=0,m_{2,3}\geq 0}$ to eliminate $m_2 > 0$ terms. Next we use some of the W_3^1 freedom to restrict to $X^1_{n_i\geq 0,m_i=0}$. Notice that in the second and third steps we have not used all the $W^1_{2,3}$ freedom available. Proceeding in a similar fashion we can show that some of the W_i^2 freedom can be used to eliminate all $n_i > 0$ matrix elements of X^1. Working carefully with the remaining of (A1) we can eliminate most of the $n_i > 0$ and $m_i > 0$ dependence of the $X^n_{\ldots}$. The nonvanishing matrix elements for the specific scheme chosen are:

$$\begin{gathered}
X^1_{n_i=0,m_i=0\ldots}, \quad X^2_{n_i=0,m_i=0\ldots}, \quad X^3_{n_i=0,m_i\geq 0},\\
X^4_{n_i=0,m_i=0\ldots}, \quad X^5_{n_i=0,m_i\geq 0,}, \quad X^6_{n_i=0,m_i\geq 0},\\
X^7_{n_i=0,m_i\geq 0\ldots}, \quad X^8_{n_i=0,m_i\geq 0\ldots}, \quad X^9_{n_i=0,m_i\geq 0\ldots},\\
X^{10}_{n_i\geq 0,m_i\geq 0\ldots}.
\end{gathered} \tag{A4}$$

Another useful FR scheme is the following:

$$\begin{gathered}
X^1_{n_i=0,m_i=0\ldots}, \quad X^2_{n_i\geq 0,m_i=0\ldots}, \quad X^3_{n_i=0,m_i=0},\\
X^4_{n_i\geq 0,m_i=0\ldots}, \quad X^5_{n_i\geq 0,m_i=0,}, \quad X^6_{n_i=0,m_i=0},\\
X^7_{n_i\geq 0,m_i\geq 0\ldots}, \quad X^8_{n_i\geq 0,m_i=0\ldots}, \quad X^9_{n_i\geq 0,m_i=0\ldots},\\
X^{10}_{n_i\geq 0,m_i=0\ldots}.
\end{gathered} \tag{A5}$$

We should emphasize that there are various FR schemes like (A4) and (A5). We have chosen the specific scheme because we believe it is a very economical off-shell Lagrangian action where most redundant terms are absent. Obviously two interaction vertices which differ by a FR are equivalent on-shell.

2. Equations for field redefinitions on AdS

For AdS FR equations can be computed in an analogous way as for flat case. Equation (2.22) can be written in the following more compact form:

$$Q = c_0\hat{l}_0 + cl^+ + c^+l - \frac{8}{L^2}c_0(\gamma^+M + \gamma M^+) - c^+cb_0. \tag{A6}$$

With the BRST charge written in this form it is straightforward to compute the analogous of (A1) for the AdS case. We only need to substitute

$$\begin{aligned} l_0 \to \hat{l}_0 = p^\mu p_\mu + \frac{1}{L^2}((\alpha^{\mu+}\alpha_\mu)^2 + \mathcal{D}\alpha^{\mu+}\alpha_\mu - 6\alpha^{\mu+}\alpha_\mu - 2\mathcal{D} + 6 - 4M^+M + c^+b(4\alpha^{\mu+}\alpha_\mu + 2\mathcal{D} - 6) \\ + b^+c(4\alpha^{\mu+}\alpha_\mu + 2\mathcal{D} - 6) + 12c^+bb^+c) \end{aligned} \tag{A7}$$

in (A1) and compute the only modification coming from the third term of the last line in (A6). The final result is

$$\begin{aligned}
\delta X^{1;\text{AdS}} &= \delta X^1 - \frac{8}{L^2}M^{s,+}W^4_{s;ss}, \\
[\delta X^{2;\text{AdS}}_{ij}]\gamma^{ij,+} &= \left[\delta X^2_{ij} - \frac{8}{L^2}(\delta_{ij}M_iW^2_i + 4M^{s,+}W^7_{s;ij;ss})\right]\gamma^{ij,+}, \\
[\delta X^{3;\text{AdS}}_{ij}]\beta^{ij,+} &= \left[\delta X^3_{ij} - \frac{32}{L^2}M^{s,+}W^8_{s;(ss);(ij)}\right]\beta^{ij,+}, \\
[\delta X^{4;\text{AdS}}_{(ij);(kl)}]\gamma^{ij,+}\gamma^{kl,+} &= \left[\delta X^4_{(ij);(kl)} + \frac{8}{L^2}(\delta_{ij}M_iW^4_{l;ki} - 9M^{s,+}W^{10}_{s;(ij);(kl);ss})\right]\gamma^{ij,+}\gamma^{kl,+}, \\
[\delta X^{5;\text{AdS}}_{ij;kl}]\gamma^{ij,+}\beta^{kl,+} &= \left[\delta X^5_{ij;kl} - \frac{16}{L^2}(\delta_{ij}M_iW^5_{i;kl} + 6M^{s,+}W^{11}_{s;ij;(kl);(ss)})\right]\gamma^{ij,+}\beta^{kl,+}, \\
[\delta X^{6;\text{AdS}}_{(ij);(kl)}]\beta^{ij,+}\beta^{kl,+} &= \left[\delta X^6_{(ij);(kl)} - \frac{72}{L^2}M^{s,+}W^{12}_{s;(ss);(ij);(kl)}\right]\beta^{ij,+}\beta^{kl,+}, \\
[\delta X^{7;\text{AdS}}_{(ij);(kl);(mn)}]\gamma^{ij,+}\gamma^{kl,+}\gamma^{mn,+} &= \left[\delta X^7_{(ij);(kl);(mn)} + \frac{8}{L^2}\delta_{ij}M_iW^7_{l;mn;ki}\right]\gamma^{ij,+}\gamma^{kl,+}\gamma^{mn,+}, \\
[\delta X^{8;\text{AdS}}_{(ij);(kl);mn}]\gamma^{ij,+}\gamma^{kl,+}\beta^{mn,+} &= \left[\delta X^8_{(ij);(kl);mn} + \frac{16}{L^2}\delta_{ij}M_iW^8_{l;(ki);(mn)}\right]\gamma^{ij,+}\gamma^{kl,+}\beta^{mn,+}, \\
[\delta X^{9;\text{AdS}}_{ij;(kl);(mn)}]\gamma^{ij,+}\beta^{kl,+}\beta^{mn,+} &= \left[\delta X^9_{ij;(kl);(mn)} - \frac{24}{L^2}\delta_{ij}M_iW^9_{i;(kl);(mn)}\right]\gamma^{ij,+}\beta^{kl,+}\beta^{mn,+}, \\
[\delta X^{10;\text{AdS}}_{(ij);(kl);(mn)}]\beta^{ij,+}\beta^{kl,+}\beta^{mn,+} &= [\delta X^{10}_{(ij);(kl);(mn)}]\beta^{ij,+}\beta^{kl,+}\beta^{mn,+}.
\end{aligned} \tag{A8}$$

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