

Anomalous fermion number nonconservation: Paradoxes in the level crossing picture

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In theories with anomalous fermion number nonconservation, the level-crossing picture is considered a faithful representation of the fermionic quantum number variation. It represents each created fermion by an energy level that crosses the zero-energy line from below. If several fermions of various masses are created, the level-crossing picture contains several levels that cross the zero-energy line and cross each other. However, we know from quantum mechanics that the corresponding levels cannot cross if the different fermions are mixed via some interaction potential. The simultaneous application of these two requirements on the level behavior leads to paradoxes. For instance, a naive interpretation of the resulting level-crossing picture gives rise to charge nonconservation. In this paper, we resolve this paradox by a precise calculation of the transition probability, and discuss what are the implications for the electroweak theory. In particular, the nonperturbative transition probability is higher if top quarks are present in the initial state.

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I. INTRODUCTION

When a classical conservation law is broken by quantum corrections, it is said that the associated symmetry is anomalous. An anomaly in a current associated with gauge symmetry ruins the consistency of the theory. The requirement that all gauge anomalies cancel strongly restricts the possible physical theories. On the other hand, anomalies arising in other type of currents can lead to interesting physics. For instance in strong interactions, the anomaly in the chiral current is important in the well-known pion decay to two photons. In weak interactions, there is an anomaly in the baryon number current. Although anomalous baryon number violating transitions are strongly suppressed at small energies, they could be at the origin of the baryon asymmetry of the universe.

Anomalous transitions leading to fermion number nonconservation arise in the electroweak theory or any other model with a similar vacuum structure. The crucial feature is the existence of many degenerate vacua, separated by energy barriers and the transition between them leads to the creation, or destruction, of fermions. The energy barrier can be passed by either by tunneling, which is represented by an instanton [1], or by thermal excitations [2,3]. In the second case, the relevant configuration is the sphaleron [4]. It is defined as the maximum energy configuration along the path of minimal energy connecting two neighboring vacua.

To visualize anomalous fermion number nonconservation, let us consider the path in the bosonic field space, parametrized by τ , that relates two neighboring vacua via the sphaleron configuration. If the bosonic fields evolve very slowly along this path, the fermionic states can be found by solving the static Dirac equation $H_D \Psi_n = E_n \Psi_n$. This equation has positive as well as negative energy states.

A way to represent the fermionic vacuum state is the Dirac sea. All states with negative energy are filled, whereas all positive energy states are empty. We are interested in the variation of the Dirac sea as a function of τ . On a graph containing all energy levels as function of τ , it may happen that an initially negative (therefore occupied) energy level crosses the zero-energy line and becomes a real particle. This is the level-crossing picture representation of the anomalous fermion number nonconservation [5]. In the case of the electroweak theory, one level for each existing fermionic doublet crosses the zero-energy line in the transition between two adjacent vacua [6].

The level-crossing picture can be thought of as a quantum mechanical description of fermion creation. This description is assumed to match the complete quantum field theory when the background evolves very slowly.

Consider now the case of two fermions Ψ_i , where $i = 1, 2$ is the flavor index. We first assume that the different flavors are not mixed by any interaction term, that is to say the Dirac equation, which generally reads $H_{ij} \Psi_j = E \Psi_i$, can be diagonalized in flavor space for any τ . We will call fermions for which H is diagonal, independent. On the level-crossing picture for two fermions with different masses,¹ we see that two energy levels cross the zero-energy line and cross each other. A simplified level-crossing picture containing these two levels is given in Fig. 1(a).²

On the other hand, if the two fermions are mixed, for instance by the interaction between them and the background sphaleron fields, and the Dirac Hamiltonian is not diagonalizable, we know from quantum mechanics that the energy levels cannot cross each other. Therefore, in this

¹We consider here fermions made massive through their Yukawa coupling to the Higgs field and not by a tree mass term.

²The full level-crossing picture of the theory we consider in the following is given in Fig. 2.

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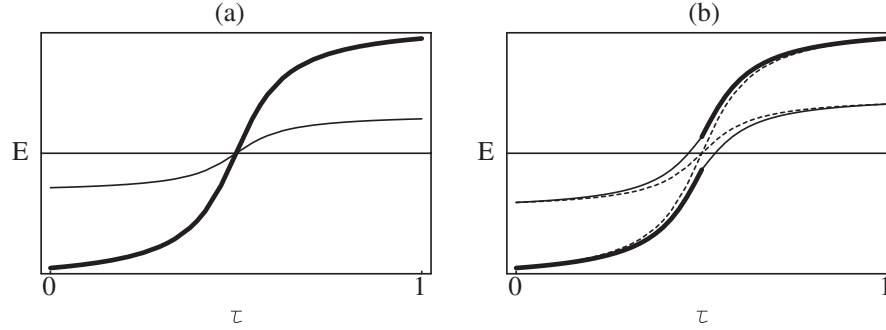


FIG. 1. Naive picture of level crossing containing only the two levels that cross the zero-energy line. The heavy particle is represented by a thick line. Two cases are pictured; without mixing (a) and with mixing between the two fermions (b). In the latter case the energy levels cannot cross, therefore the heavy particle becomes a light one and vice-versa.

case, the heavy fermion becomes the light one and the light one becomes the heavy one, see Fig. 1(b). This is so on the simplified level-crossing picture but, if we reintroduce the other energy level, we see that, generally there are excited states of the light particle and the light fermion evolves to one of them, see Fig. 3. Therefore, we conclude from the level-crossing picture that two light fermions are created in the case with mixing instead of a light and a heavy one.

Suppose now that we introduce another gauge field B_μ , which shall be Abelian, not spontaneously broken and free from any anomaly. We further assume that the different fermions have different charges with respect to this field. The cancellation of the anomaly for this gauge field B_μ requires that the sum of all the fermionic charges vanishes and therefore an anomalous transition leading to the creation of one of each existing fermions perfectly respects the B -gauge symmetry. However, in the case of mixing between the fermions, the creation of only light fermions leads to B -charge violation.

Note that the mixing between the fermions is only possible if the background has a nonvanishing charge with respect to the B -gauge symmetry. It means that the B -gauge is broken in the sphaleron or instanton core. This is possible, and is generally the case, even if the B -gauge symmetry is unbroken in vacuum.

Two points deserve further investigations. Firstly, if the level-crossing picture changes qualitatively, it is interesting to see if the transition probability undergoes such changes. Secondly, we have to understand how charge conservation is ensured.

These questions are mainly model independent, therefore we choose to resolve them in a simple $1 + 1$ dimensional anomalous Abelian Higgs model with two chiral fermions, which contains the above paradox. In this particular model, we will show that the probability for the creation of two light fermions is zero, unless it is accompanied by the emission of some other particles that compensate for the charge asymmetry. We will also see that the transition probability is larger if there are heavy fermions in the initial or final state, again in contradiction to what the level-crossing picture suggests.

These paradoxes also arise in the electroweak theory. Indeed, the quarks have various electric charges and in the background of the electroweak sphaleron or instanton the $SU(2) \times U(1)$ -gauge symmetry is completely broken by the presence of a charged weak field background. The resolution of these questions is of great interest for electroweak baryogenesis.³

This paper is organized as follows. In Sec. II, we design a $1 + 1$ dimensional model adapted to our purposes. The level-crossing picture of this model is derived in Sec. III. In Sec. IV, we compute the nonperturbative transition rate in the instanton picture and resolve the paradoxes mentioned before. The implications of our results for the electroweak baryogenesis are discussed in Sec. V.

II. THE MODEL

We construct here a simple model which contains the paradoxes mentioned in the introduction. A good candidate for studying the creation of massive fermions is the chiral Abelian Higgs model studied in [3,9–11],

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\bar{\Psi}^j \gamma^\mu \left(\partial_\mu - i\frac{e}{2} \gamma^5 A_\mu \right) \Psi^j - V(\phi) \\ & + \frac{1}{2} |D_\mu \phi|^2 + if^j \bar{\Psi}^j \frac{1 + \gamma^5}{2} \Psi^j \phi^* \\ & - if^j \bar{\Psi}^j \frac{1 - \gamma^5}{2} \Psi^j \phi, \end{aligned} \quad (1)$$

where $D_\mu \phi = \partial_\mu - ieA_\mu$, $V(\phi) = \frac{\lambda}{4} (|\phi|^2 - v^2)^2$ and

$$\Psi^j = \begin{pmatrix} \Psi_L^j \\ \Psi_R^j \end{pmatrix}$$

with $j = 1, 2$ labeling the different flavors. We use the following representation for the γ -matrices:

³Even though with the current constraints on the Higgs mass, producing the observed baryonic asymmetry within the minimal Standard Model is impossible, electroweak baryogenesis may still work in some of its extensions [7] and in supersymmetric theories [8].

$$\gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0 \gamma_1. \quad (2)$$

This model displays similar nonperturbative properties as the electroweak theory.

We also need a mixing term between the different flavors, which would, at the semiclassical level, prevent the energy levels of the fermions from crossing each other. Interaction terms between fermions are not possible in this simple model. Indeed, if we write interaction terms between different fermions like $f_{ij} \Psi_{i,L}^\dagger \Psi_{j,R} \phi^* + \text{h.c.}$ we could redefine the fields to diagonalize the matrix f_{ij} and the theory would inevitably lead to independent fermions.⁴ We therefore have to introduce another scalar field, allowing for other Yukawa couplings. This new scalar field should be nonzero in the instanton and sphaleron background to provide a semiclassical mixing term. To this aim, we couple it to the Higgs field with the interaction term $\frac{h}{2} |\chi|^2 (|\phi|^2 - v^2)$. We will also introduce a $U(1)^B$ gauge field B_μ to give different charges to the two different flavors. The bosonic sector reads:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{A\mu\nu} F_A^{\mu\nu} - \frac{1}{4} F_{B\mu\nu} F_B^{\mu\nu} + \frac{1}{2} |(\partial_\mu - ieA_\mu)\phi|^2 \\ & + \frac{1}{2} |(\partial_\mu - ieA_\mu - ie'B_\mu)\chi|^2 - \frac{\lambda}{4} (|\phi|^2 - v^2)^2 \\ & - \frac{\Lambda}{4} |\chi|^4 - \frac{M^2}{2} |\chi|^2 - \frac{h}{2} |\chi|^2 (|\phi|^2 - v^2). \end{aligned} \quad (3)$$

Note that the χ field should be charged with respect to B_μ to break the B -gauge symmetry in the center of the instanton and sphaleron. We have now to specify the charge of each of the four spinor components $\Psi_{L,R}^{1,2}$ with respect to $U(1)^A$ and $U(1)^B$. Let us note $\alpha_{L,R}^{1,2}$ and $\beta_{L,R}^{1,2}$ the charges with respect to A_μ and B_μ . The following choice turns out to serve our aim:

$$\begin{aligned} \alpha_R^1 = -\frac{e}{2}, \quad \alpha_L^1 = \frac{e}{2}, \quad \alpha_R^2 = -\frac{e}{2}, \\ \alpha_L^2 = \frac{e}{2}, \quad \text{and} \quad \beta_R^1 = \frac{e'}{2}, \quad \beta_L^1 = \frac{e'}{2}, \\ \beta_R^2 = -\frac{e'}{2}, \quad \beta_L^2 = -\frac{e'}{2}. \end{aligned} \quad (4)$$

The gauge symmetries imply that there are two classically conserved electric currents

$$\begin{aligned} j_A^\mu = & \alpha_L^i \bar{\Psi}_L^i \gamma^\mu \Psi_L^i + \alpha_R^i \bar{\Psi}_R^i \gamma^\mu \Psi_R^i, \\ j_B^\mu = & \beta_L^i \bar{\Psi}_L^i \gamma^\mu \Psi_L^i + \beta_R^i \bar{\Psi}_R^i \gamma^\mu \Psi_R^i. \end{aligned}$$

These currents are in general anomalous but are conserved with our particular choice of charges.

⁴Moreover the first scalar field in (2) vanishes in the center of the instanton, which would allow the levels to cross at the center.

$$\begin{aligned} \partial_\mu j_A^\mu = & \frac{1}{4\pi} \varepsilon_{\mu\nu} F_A^{\mu\nu} \sum_i [(\alpha_R^i)^2 - (\alpha_L^i)^2] \\ & + \frac{1}{4\pi} \varepsilon_{\mu\nu} F_B^{\mu\nu} \sum_i [\alpha_R^i \beta_R^i - \alpha_L^i \beta_L^i] = 0, \\ \partial_\mu j_B^\mu = & \frac{1}{4\pi} \varepsilon_{\mu\nu} F_B^{\mu\nu} \sum_i [(\beta_R^i)^2 - (\beta_L^i)^2] \\ & + \frac{1}{4\pi} \varepsilon_{\mu\nu} F_A^{\mu\nu} \sum_i [\alpha_R^i \beta_R^i - \alpha_L^i \beta_L^i] = 0. \end{aligned}$$

The fermionic current

$$j_F^\mu = \bar{\Psi}_L^i \gamma^\mu \Psi_L^i + \bar{\Psi}_R^i \gamma^\mu \Psi_R^i. \quad (5)$$

is conserved at the classical level, however, its anomaly does not vanish:

$$\begin{aligned} \partial_\mu j_F^\mu = & \frac{1}{4\pi} \varepsilon_{\mu\nu} F_A^{\mu\nu} \sum_i [(\alpha_R^i) - (\alpha_L^i)] \\ & + \frac{1}{4\pi} \varepsilon_{\mu\nu} F_B^{\mu\nu} \sum_i [(\beta_R^i) - (\beta_L^i)] \\ = & \frac{-e}{2\pi} \varepsilon_{\mu\nu} F_A^{\mu\nu}. \end{aligned} \quad (6)$$

This is indeed what we need; there is no gauge anomaly but the fermion number current is anomalous. We can now write down an interaction between fermions and scalar field, and the fermionic Lagrangian reads:

$$\begin{aligned} \mathcal{L}_{\text{ferm}}^{\text{Mink}} = & +i\bar{\Psi}^1 \gamma^\mu \left(\partial_\mu - e\frac{i}{2} \gamma_5 A_\mu - e'\frac{i}{2} B_\mu \right) \Psi^1 \\ & + i\bar{\Psi}^2 \gamma^\mu \left(\partial_\mu - e\frac{i}{2} \gamma_5 A_\mu + e'\frac{i}{2} B_\mu \right) \Psi^2 \\ & + if_j \bar{\Psi}^j \frac{1+\gamma_5}{2} \Psi^j \phi^* - if_j \bar{\Psi}^j \frac{1-\gamma_5}{2} \Psi^j \phi \\ & - if_3 \bar{\Psi}^1 \frac{1-\gamma_5}{2} \Psi^2 \chi + if_3 \bar{\Psi}^2 \frac{1+\gamma_5}{2} \Psi^1 \chi^*. \end{aligned} \quad (7)$$

The fermionic spectrum consists of two fermions of different mass $F^j = v f^j$, $j = 1, 2$ interacting with each other by Yukawa coupling to the scalar field χ . The vacuum structure of the model given by the Lagrangians (3) and (7) is nontrivial [12]. Taking the $A_0 = 0$ gauge and putting the theory in a spatial box of length L with periodic boundary conditions, one finds that there is an infinity of degenerate vacuum states $|n\rangle$, $n \in \mathbf{Z}$ with the gauge-Higgs configurations given by

$$A_1 = \frac{2\pi n}{eL}, \quad \phi = v e^{i(2\pi n x/L)}. \quad (8)$$

The transition between two neighboring vacua leads to the creation of two fermions as intended: If the vector field A_μ undergoes the variation

$$\delta A_1 = -\frac{2\pi}{Le}, \quad (9)$$

which corresponds to the difference between two adjacent vacua, the fermion number anomaly is

$$\begin{aligned} \delta N_F &= \int \frac{-e}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu}^A d^2x = \frac{-e}{2\pi} \int 2\partial_0 A_1 d^2x \\ &= \frac{-e}{2\pi} 2\delta A_1 L = 2. \end{aligned} \quad (10)$$

III. LEVEL-CROSSING PICTURE

We build a path in the bosonic field space that goes adiabatically from one vacuum to the neighboring one. To this aim, we find the sphaleron and construct a path that relates it with the initial and final vacua. Such configurations are relevant for high temperature dynamics [13].

Using the $A_0 = B_0 = 0$ gauge, the sphaleron in this model reads

$$\begin{aligned} \phi^{\text{cl}} &= -ive^{-(\pi ix/L)} \tanh(Mx), \\ \chi^{\text{cl}} &= i\alpha e^{-(\pi ix/L)} \cosh^{-1}(Mx), \\ A_1^{\text{cl}} &= -\frac{\pi}{eL}, \quad B_1^{\text{cl}} = 0, \end{aligned} \quad (11)$$

with $\alpha = \sqrt{\frac{1}{\hbar}(\lambda v^2 - 2M^2)}$. It can be found using results on solitons with two scalar fields of Ref. [14]. Note that this solution is only valid for a restricted parameter space

$$\lambda v^2 > 2M^2, \quad (12)$$

$$2M^2 + \Lambda\alpha^2 - \hbar v^2 = 0. \quad (13)$$

An example of a path going from vacuum $n = 0$ at $\tau = 0$ to vacuum $n = -1$ at $\tau = 1$ via the sphaleron at $\tau = 1/2$ is

$$\begin{aligned} \phi^{\text{cl}} &= ve^{-(2\pi i x \tau/L)} [\cos(\pi\tau) + i \sin(\pi\tau) \\ &\quad \times \tanh(Mx \sin(\pi\tau))], \\ \chi^{\text{cl}} &= -i\alpha e^{-(2\pi i x \tau/L)} \sin(\pi\tau) \cosh^{-1}(Mx \sin(\pi\tau)), \\ A_1^{\text{cl}} &= -\frac{2\pi\tau}{eL}, \quad B_1^{\text{cl}} = 0. \end{aligned} \quad (14)$$

These configurations represent a set of static background fields interpolating between vacua in which the fermions evolve. The equations of motion for the fermions are

$$i\partial_0 \Psi = H\Psi, \quad (15)$$

with

$$\Psi = \begin{pmatrix} \Psi_1 = \Psi_L^1 \\ \Psi_2 = \Psi_R^1 \\ \Psi_3 = \Psi_L^2 \\ \Psi_4 = \Psi_R^2 \end{pmatrix} \quad (16)$$

and

$$H = \begin{pmatrix} -i\partial_1 - \frac{\epsilon}{2}A_1 - \frac{\epsilon'}{2}B_1 & f_1\phi & 0 & f_3\chi \\ f_1\phi^* & i\partial_1 - \frac{\epsilon}{2}A_1 + \frac{\epsilon'}{2}B_1 & 0 & 0 \\ 0 & 0 & -i\partial_1 - \frac{\epsilon}{2}A_1 + \frac{\epsilon'}{2}B_1 & f_2\phi \\ f_3\chi^* & 0 & f_2\phi^* & i\partial_1 - \frac{\epsilon}{2}A_1 - \frac{\epsilon'}{2}B_1 \end{pmatrix}, \quad (17)$$

the Dirac Hamiltonian. In the limit of slow transition $\dot{\tau} \sim 0$, the Hamiltonian is time-independent and the spectrum of the static Dirac equation $H\Psi_n = E_n\Psi_n$ for each τ leads to the level-crossing picture. Of course, an analytic solution to this eigenvalue problem is not possible for each τ . We therefore give the analytic solutions at a few values of τ , check with perturbation theory that the interaction potential lifts the degeneracy of the levels where they cross each other, and then give the complete level-crossing picture resulting from numerical computation.

A. Fermionic spectrum in the vacuum $\tau = 0, 1$

For $\tau = 0, 1$ the fermionic spectrum is the one of non-interacting fermions Ψ^i , $i = 1, 2$ [11] and is labeled by an integer n .

$$E_n^i = \pm \sqrt{F_i^2 + k_n^2} \quad k_n = \begin{cases} \frac{2\pi n}{L}, & \tau = 0, \\ \frac{2\pi(n-\frac{1}{2})}{L}, & \tau = 1. \end{cases} \quad (18)$$

Note that the spectrum is different in the states $\tau = 0$ and $\tau = 1$. The configuration $\tau = 0$ is not a true vacuum for fermions, the fermionic contribution to vacuum energy being larger for $\tau = 0$ than for $\tau = 1$. This difference however vanishes in the limit of infinite system size [11]. All states are doubly degenerate in energy except for $\tau = 0$ in the case $n = 0$.

B. Sphaleron configuration, $\tau = 1/2$

The Dirac Hamiltonian in the background of the sphaleron reads

$$H = \begin{pmatrix} -i\partial_1 + \frac{\pi}{2L} & iF_1 e^{-i(\pi x/L)} \tanh(Mx) & 0 & -iF_3 \cosh(Mx)^{-1} e^{-i(\pi x/L)} \\ -iF_1 e^{i(\pi x/L)} \tanh(Mx) & i\partial_1 + \frac{\pi}{2L} & 0 & 0 \\ 0 & 0 & -i\partial_1 + \frac{\pi}{2L} & iF_2 e^{-i(\pi x/L)} \tanh(Mx) \\ iF_3 \cosh(Mx)^{-1} e^{i(\pi x/L)} & 0 & -iF_2 e^{i(\pi x/L)} \tanh(Mx) & i\partial_1 + \frac{\pi}{2L} \end{pmatrix}. \quad (19)$$

In the $F_3 = \alpha f_3 = 0$ case, the Dirac equations decouple and can be solved separately for each fermion. In the limit $L \rightarrow \infty$, one finds two zero-modes, one for each fermion:

$$\Psi_0^j = \begin{pmatrix} e^{-(i\pi/2L)x} [\cosh(Mx)]^{-(F_j/M)} \\ e^{(i\pi/2L)x} [\cosh(Mx)]^{-(F_j/M)} \end{pmatrix}, \quad j = 1, 2. \quad (20)$$

The interaction can be introduced perturbatively. To this aim, the Dirac Hamiltonian is separated in two parts $H = H_0 + W$ with $H_0 = H(f_3 = 0)$. In the Ψ_0^1, Ψ_0^2 subspace, the interaction matrix reads

$$M_{ij} = \frac{1}{n_i n_j} \langle \Psi^i | W | \Psi^j \rangle = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix},$$

with

$$I = \frac{1}{n_1 n_2} \int_{-L/2}^{L/2} (F_3) [\cosh(Mx)]^{-(F_1+F_2/M)-1} dx \\ = \sqrt{\frac{\Gamma[\frac{F_1+M}{2M}] \Gamma[\frac{F_2+M}{2M}]}{\Gamma[\frac{F_1}{2M}] \Gamma[\frac{F_2}{2M}]}} \frac{\Gamma[\frac{F_1+F_2+M}{2M}]}{\Gamma[1 + \frac{F_1+F_2}{2M}]}, \quad (21)$$

and $n_i = \langle \Psi^i | \Psi^i \rangle^{1/2}$. The eigenstates of the matrix M_{ij} are

$$\Psi_+ = -i\Psi_1 + \Psi_2 \quad \text{with energy } E_+ = I,$$

$$\Psi_- = \Psi_1 - i\Psi_2 \quad \text{with energy } E_- = -I.$$

We see here that the interaction between the fermions lifts the degeneracy between the states and avoids that the levels cross each other.

C. Numerical results

The energy levels may be found numerically for each value of τ solving the static Dirac equation with the Hamiltonian (17) and periodic boundary conditions in the interval of length L .

The results (Fig. 2 and 3) show, in the cases of independent and mixed fermions, the creation of two fermions (two levels cross the zero-energy line). In the independent case, one of each fermion is created (Fig. 2), whereas two light ones are created in the mixed case (Fig. 3). The latter process violates charge conservation.⁵ For charge conservation to be preserved, the transition probability of such a process must vanish. As a precise calculation of the transition probability is difficult in the sphaleron picture, we will use the instanton approach in the following, which leads to a well-defined semiclassical expansion. Note that the instanton picture will be similar to the adiabatic sphaleron transition if the fermionic masses are large and their associated time-scale small in comparison to the instanton size.

IV. INSTANTON PICTURE

We first derive the Euclidean properties of the model and then compute the transition probability for a few representative processes. In Euclidean space, the bosonic Lagrangian reads

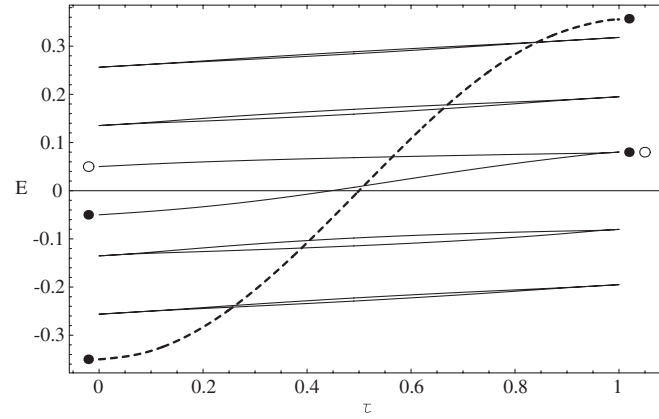


FIG. 2. Level crossing of two fermions without mixing, $F_1 = 0.05$, $F_2 = 0.35$, $f_3 = 0$, $L = 50$ and $h = m = e = e' = 1$, $M = 0.5$. One of each fermion is created when going from $\tau = 0$ to $\tau = 1$ and the heavy fermion level (dashed) crosses many energy levels of the light fermion.

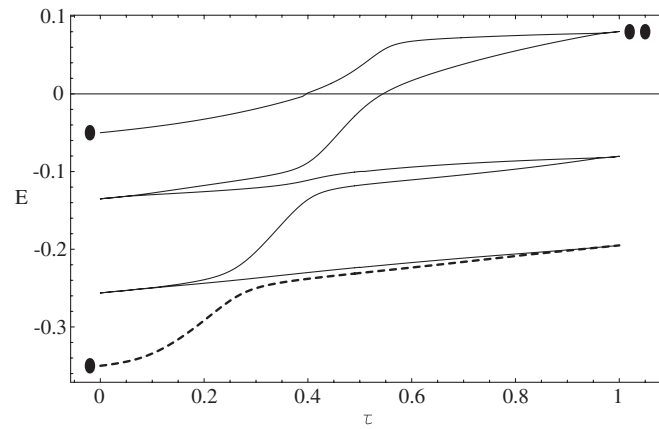


FIG. 3. Level crossing of two fermions with mixing, $F_1 = 0.05$, $F_2 = 0.35$, $f_3 = 0.24$. The heavy fermion (dashed) energy level cannot cross the light fermion levels and two light fermions are created.

⁵Two light fermions of charge $-1/2$ with respect to the B gauge field are created

$$\begin{aligned} \mathcal{L}_{\text{bos}}^{\text{Eucl}} = & \frac{1}{4} F_{A\mu\nu} F_{A\mu\nu} + \frac{1}{4} F_{B\mu\nu} F_{B\mu\nu} + \frac{1}{2} |(\partial_\mu - ieA_\mu)\phi|^2 \\ & + \frac{1}{2} |(\partial_\mu - ieA_\mu - ie'B_\mu)\chi|^2 + \frac{\lambda}{4} |\phi|^4 \\ & - \frac{m^2}{2} |\phi|^2 + \frac{\Lambda}{4} |\chi|^4 + \frac{M^2}{2} |\chi|^2 + \frac{h}{2} |\chi|^2 (|\phi|^2 - v^2), \end{aligned} \quad (22)$$

and the fermionic part

$$\begin{aligned} \mathcal{L}_{\text{ferm}}^{\text{Eucl}} = & +i\Psi^{\dagger 1} \gamma_\mu^E \left(\partial_\mu - e \frac{i}{2} \gamma_5 A_\mu - e' \frac{i}{2} B_\mu \right) \Psi^1 \\ & + i\Psi^{\dagger 2} \gamma_\mu^E \left(\partial_\mu - e \frac{i}{2} \gamma_5 A_\mu + e' \frac{i}{2} B_\mu \right) \Psi^2 \\ & - if_j \bar{\Psi}^j \frac{1 + \gamma_5}{2} \Psi^j \phi^* + if_j \bar{\Psi}^j \frac{1 - \gamma_5}{2} \Psi^j \phi \\ & + if_3 \bar{\Psi}^1 \frac{1 - \gamma_5}{2} \Psi^2 \chi - if_3 \bar{\Psi}^2 \frac{1 + \gamma_5}{2} \Psi^1 \chi^*, \end{aligned} \quad (23)$$

with $\gamma_0^E = i\gamma_0$ and $\gamma_1^E = \gamma_1$.

A. Bosonic sector

In order to find the instanton solution, let us point out the following: if $\chi = B = 0$, we know the solution of the remaining equations, it is the Nielsen-Olesen vortex [15]. We search here for a solution of the same type, adding some generic form for B and χ :

$$\begin{aligned} \phi_{\text{cl}}(r, \theta) = f(r) e^{-i\theta}, \quad A_{\text{cl}}^i(r, \theta) = \varepsilon^{ij} \hat{r}^j A(r), \\ \chi_{\text{cl}}(r, \theta) = g(r), \quad B_{\text{cl}}^i(r, \theta) = \varepsilon^{ij} \hat{r}^j B(r), \end{aligned} \quad (24)$$

with polar coordinates ($it = \tau = r \cos\theta$, $x = r \sin\theta$), \hat{r} the unit vector in the direction of r and ε^{ij} the completely antisymmetric tensor with $\varepsilon^{01} = 1$. Some details can be found in Appendix A, only the main results will be given

$$\begin{aligned} H = & \begin{pmatrix} H_1 & I_2 \\ I_1 & H_2 \end{pmatrix} \\ H_1 = & \begin{pmatrix} -if_1 \phi^* & i\partial_0 - \partial_1 + \frac{e}{2}(-A_0 - iA_1) + \frac{e'}{2}(B_0 + iB_1) \\ -i\partial_0 - \partial_1 + \frac{e}{2}(-A_0 + iA_1) + \frac{e'}{2}(-B_0 + iB_1) & if_1 \phi \end{pmatrix} \\ H_2 = & \begin{pmatrix} -if_2 \phi^* & i\partial_0 - \partial_1 + \frac{e}{2}(-A_0 - iA_1) + \frac{e'}{2}(-B_0 - iB_1) \\ -i\partial_0 - \partial_1 + \frac{e}{2}(-A_0 + iA_1) + \frac{e'}{2}(B_0 - iB_1) & if_2 \phi \end{pmatrix} \\ I_1 = & \begin{pmatrix} -if_3 \chi^* & 0 \\ 0 & 0 \end{pmatrix}, \\ I_2 = & \begin{pmatrix} 0 & 0 \\ 0 & if_3 \chi \end{pmatrix}. \end{aligned} \quad (27)$$

The zero-modes are found solving the equation $H\Psi = 0$, with H the Dirac operator in the background of the instanton. We use polar coordinates (r, θ) and expand fermionic fluctuations in partial waves $\Psi = \sum_{m=-\infty}^{\infty} e^{im\theta} \Psi_m$. This leads to the following system of equations:

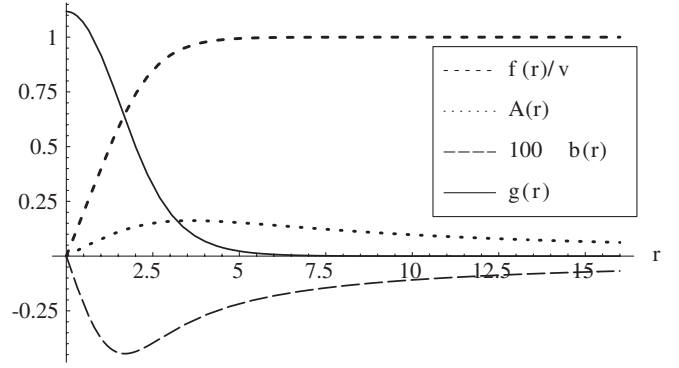


FIG. 4. Instanton shape (with a different scale for the field B) for the following values for dimensionless constants (see Appendix A): $m^2 = \frac{M^2}{\lambda v^2} = 1$, $\mu = \frac{\lambda}{e^2} = 4$, $\mu' = \frac{\lambda}{e'^2} = 4$, $\rho = \frac{\Lambda}{h} = 1$, $H = \frac{h}{\lambda} = 3$.

here. An example of profile is given in Fig. 4 and the asymptotic form of the different functions are

$$\begin{aligned} f(r) \xrightarrow{r \rightarrow 0} f_0 r + \mathcal{O}(r^3), \quad A(r) \xrightarrow{r \rightarrow 0} a_0 r + \mathcal{O}(r^3), \\ g(r) \xrightarrow{r \rightarrow 0} g_0 + \mathcal{O}(r^2), \quad B(r) \xrightarrow{r \rightarrow 0} b_0 r + \mathcal{O}(r^3), \end{aligned} \quad (25)$$

$$\begin{aligned} f(r) \xrightarrow{r \rightarrow \infty} 1 + f_\infty K_0(\sqrt{2\lambda}vr), \quad A(r) \xrightarrow{r \rightarrow \infty} \frac{1}{er} + a_\infty K_1(evr), \\ g(r) \xrightarrow{r \rightarrow \infty} g_\infty K_1(Mr), \quad B(r) \xrightarrow{r \rightarrow \infty} \frac{b_\infty}{r}, \end{aligned} \quad (26)$$

where $f_0, a_0, b_0, g_0, f_\infty, g_\infty, a_\infty, b_\infty$ are constants found by computing the exact instanton profile.

B. Fermions

The fermionic fluctuations in the background of the instanton (24) are given by $H\Psi = E\Psi$, with:

$$\begin{aligned}
 \left(\frac{\partial}{\partial r} + \frac{m}{r} + \frac{e}{2}A(r) + \frac{e'}{2}B(r)\right)\Psi_m^1 - f_1 f(r)\Psi_m^2 - f_3 g(r)\Psi_{m-1}^4 &= 0, \\
 \left(\frac{\partial}{\partial r} - \frac{m}{r} + \frac{e}{2}A(r) - \frac{e'}{2}B(r)\right)\Psi_m^2 - f_1 f(r)\Psi_m^1 &= 0, \\
 \left(\frac{\partial}{\partial r} + \frac{m-1}{r} + \frac{e}{2}A(r) - \frac{e'}{2}B(r)\right)\Psi_{m-1}^3 - f_2 f(r)\Psi_{m-1}^4 &= 0, \\
 \left(\frac{\partial}{\partial r} - \frac{m-1}{r} + \frac{e}{2}A(r) + \frac{e'}{2}B(r)\right)\Psi_{m-1}^4 - f_2 f(r)\Psi_{m-1}^3 - f_3 g(r)\Psi_m^1 &= 0.
 \end{aligned} \tag{28}$$

In the case $f_3 = 0$ and $B(r) = 0$, the two fermions decouple and their zero modes are [16]:

$$\begin{aligned}
 \psi^j(r) \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp\left[-\int_0^r dr' \left(v f_j f(r') + \frac{e}{2}A(r')\right)\right], \\
 j = 1, 2.
 \end{aligned} \tag{29}$$

If $f_3 \neq 0$ the fermions are coupled and the zero-modes cannot be found analytically. Their existence can be checked using the method of Ref. [16] and their asymptotic forms for $r \rightarrow \infty$ read⁶

$$\begin{aligned}
 \psi_{\text{cl}}^1 &= \frac{\alpha_1}{\sqrt{r}} \begin{pmatrix} e^{-F_1 r} \\ -e^{-F_1 r} \\ -\beta_1 e^{-F_2 r} e^{-i\theta} \\ \beta_1 e^{-F_2 r} \left(1 + \frac{1}{F_2 r}\right) e^{-i\theta} \end{pmatrix}, \\
 \psi_{\text{cl}}^2 &= \frac{\alpha_2}{\sqrt{r}} \begin{pmatrix} \beta_2 e^{-F_1 r} \left(1 + \frac{1}{F_1 r}\right) e^{i\theta} \\ -\beta_2 e^{-F_1 r} e^{i\theta} \\ -e^{-F_2 r} \\ e^{-F_2 r} \end{pmatrix},
 \end{aligned} \tag{30}$$

where $\alpha_{1,2}$ are normalization constants and $\beta_{1,2}$ parametrize the mixing of the two fermions. β_1 and β_2 vanish in the limit of decoupled fermions ($f_3 \rightarrow 0$) and have to be computed numerically solving the system of Eqs. (28) for arbitrary value of f_3 . The values of $\beta_{1,2}$ found numerically are given as a function of the fermion masses in Fig. 5 and as function of the coupling f_3 in Fig. 6. We will see that the constant β_1 arises as a multiplicative factor in the probability of creating two heavy fermions and β_2 in the probability of creating two light ones. The factors $\beta_{1,2}$ are therefore the most important parameters to compare the transition probabilities. It is then useful to get a good understanding of their dependence on the different parameters. We will therefore provide an analytical approximation for them.

For small coupling f_3 , and small instanton size a in units of fermion mass, we can get a rough approximation by perturbation theory. We checked numerically that it corresponds reasonably well to the exact case and will be

⁶We consider here the approximation $B = 0$ (or $e' = 0$), which does not lead to observable changes (see Fig. 4)

sufficient for the following discussion. We are interested in the case where the first fermion is very light in comparison to the second one and in comparison to the scalar field, $F_1 \ll m_\chi$. The calculations in Appendix B give

$$\beta_1 \cong f_3 v \int_0^\infty dx g(x) \sinh(f_2 x) e^{-f_1 x}. \tag{31}$$

If the inverse fermion mass $\frac{1}{f_1 v}$ is small in comparison to

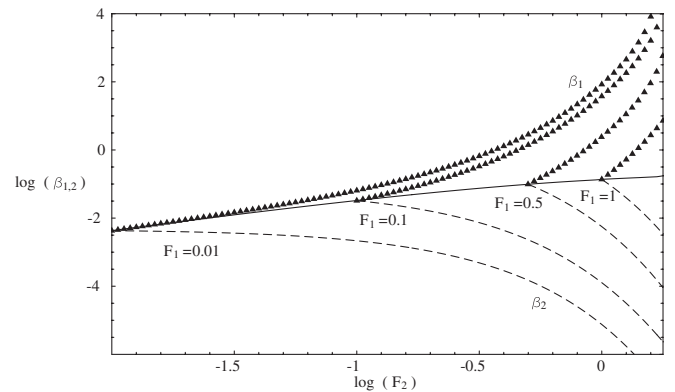


FIG. 5. Coefficients β_2 (dashed lines) and β_1 (triangles) as a function of the mass F_2 for some different light fermion masses $F_1 = 0.01, 0.1, 0.5, 1$ and for $f_3 = 0.2$. The line represent $\beta_1 = \beta_2$ in the degenerate case $F_1 = F_2$. The constants $F_{1,2}, f_3$ are in units of $\sqrt{\lambda}v$.

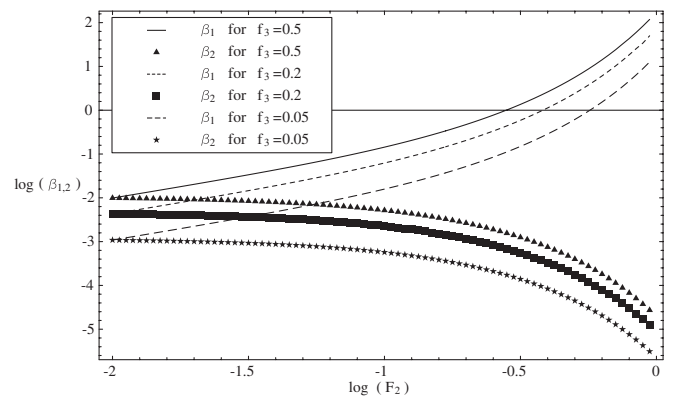


FIG. 6. Coefficients $\beta_{1,2}$ as a function of the mass F_2 for some different couplings f_3 and for $f_1 = 0.01$. The constants $F_{1,2}, f_3$ are in units of $\sqrt{\lambda}v$.

the typical extent r_{inst} of the function $g(r)$, we have:

$$\beta_1 \equiv f_3 v \int_0^\infty dx g(x) \sinh(f_2 x). \quad (32)$$

In the case of a large mass f_2 and large instanton size, the constant β_1 can be large from the presence of the sinh. In the case of small f_2 , the integral can be further simplified to $\beta_1 = f_3 F_2 \int x g(x) dx$. A similar computation can be performed for ψ_{cl}^2 ,

$$\beta_2 = f_3 v \int dx g(x) \sinh(f_1 x) e^{-f_2 x}. \quad (33)$$

If $f_1 r_{\text{inst}} \ll 1$ we have $\beta_2 = f_3 F_1 \int dx x g(x) e^{-f_2 x}$, which is generically small and can be large for a large instanton size only; it is further suppressed by a large fermion mass f_2 .

C. Transition probability

We start with two decoupled fermions ($f_3 = 0$) and introduce the interaction perturbatively. It is clear what happens here; the interaction term $-if_3 \bar{\Psi}^1 \frac{1-\gamma_5}{2} \Psi^2 \chi + \text{h.c.}$ allows for the decay of the heavy fermion into a light fermion and a χ boson, a process which conserves the charge and which can be taken into account in final state corrections (see [17], for inclusion of fermions see [18]). This is not what we are interested in here. In the non-perturbative regime, two light fermions are created in the χ background, where the $U(1)^B$ gauge is broken and a χ boson should be emitted from the instanton tail as the $U(1)^B$ gauge symmetry is restored far from the instanton center. We will show here that processes violating charge conservation have vanishing probability.

1. Green's function

Green's functions with creation of two fermions and an arbitrary number of other particles read

$$\begin{aligned} G^{ab}(x_1, x_2, y_1 \dots y_n, z_1 \dots z_m, w_1, \dots, w_l) \\ = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \mathcal{D}\chi \mathcal{D}\chi^* \mathcal{D}\eta e^{-S[\Psi, \bar{\Psi}, \chi, \chi^*, \eta]} \Psi^a(x_1) \Psi^b(x_2) \\ \times \prod_{i=1}^n \chi(y_i) \prod_{j=1}^m \chi^*(y_j) \prod_{k=1}^l \eta(y_k), \end{aligned} \quad (34)$$

where η stands for all neutral bosonic degrees of freedom, $\prod_{k''=1}^l \eta(y_k'')$ may contain the field A , ϕ and neutral pairs of fermions and the variable Ψ is a spinor containing the two fermions as in (16) and $a, b = 1, \dots, 4$.

The main contribution to the Green's function for the creation of two fermions comes from the sector with one instanton ($q = -1$).⁷ In this sector, fermions have two

zero-modes (30). The Gaussian path integral over fermionic degrees of freedom can be evaluated, leading to the fermionic determinant with zero-modes excluded and the product of the fermionic zero-mode wave functions,⁸

$$\begin{aligned} G(x_1, x_2, y_1 \dots y_n, z_1 \dots z_m, w_1, \dots, w_l) \\ = \int_{q=-1} \mathcal{D}\chi \mathcal{D}\chi^* \mathcal{D}\eta e^{-S[\Psi, \bar{\Psi}, \chi, \chi^*, \eta]} \det'(K[\chi, \eta]) \\ \times \psi_{\chi, \eta}^1(x_1) \psi_{\chi, \eta}^2(x_2) \prod_{i=1}^n \chi(y_i) \prod_{j=1}^m \chi^*(y_j) \prod_{k=1}^l \eta(y_k). \end{aligned} \quad (35)$$

2. Collective coordinates in the one instanton sector

The bosonic action is expanded around the instanton configuration. Gaussian integration over the quadratic fluctuations gives a determinant $\det(D_{\text{bos}}^2)^{-(1/2)}$. However, zero modes associated to symmetries require introduction of collective coordinates. There are two translation zero modes and one coming from $U(1)^B$ gauge. Performing an infinitesimal global gauge transformation, we get

$$\delta\chi = e^{i\beta} \chi - \chi \sim i\beta e^{i\theta} g(r), \quad \delta\phi = \delta A = \delta B = 0. \quad (36)$$

Note that the $U(1)^A$ gauge is broken, there is no normalizable zero mode associated to this symmetry. Rotation symmetry does not lead to a further zero mode.⁹

Collective coordinates are introduced as follows. The integral over the translation zero modes are replaced by an integral over the instanton position x_0 . The integral over the $U(1)^B$ gauge zero-mode is replaced by an integral over all possible global gauge transformations β . The Green's function reads:

$$\begin{aligned} G(x_1, x_2, y_1 \dots y_n, z_1 \dots z_m, w_1, \dots, w_l) \\ = \int d^2 x_0 d\beta e^{-S_{\text{cl}}} \det'(K_{\text{inst}}) N_B N_{\text{tr}} \det'(D_{\text{bos}}^2)^{-(1/2)} \\ \times \psi_{\beta}^1(x_1 - x_0) \psi_{\beta}^2(x_2 - x_0) \prod_{i=1}^n \chi_{\beta}(y_i - x_0) \\ \times \prod_{j=1}^m \chi_{\beta}^*(y_j - x_0) \prod_{k=1}^l \eta_{\beta}(y_k - x_0), \end{aligned} \quad (37)$$

with

$$\chi_{\beta} = e^{i\beta} \chi_{\text{cl}}, \quad \chi_{\beta}^* = e^{-i\beta} \chi_{\text{cl}}, \quad (38)$$

$$\eta_{\beta} = \eta_{\text{cl}}, \quad \psi_{\beta}^j = e^{i(\beta/2)\Gamma_5} \psi_{\text{cl}}^j, \quad j = 1, 2, \quad (39)$$

⁷More precisely, in the dilute instanton gas approximation the result from the one instanton sector can be exponentiated to give the complete Green's function [19].

⁸Note that the zero-mode functions still depend on the background $\psi^i = \psi^i[\chi, \chi^*, \eta]$.

⁹Rotations give the same zero-mode as $U(1)^B$ gauge transformations

and N_B, N_{tr} the normalization factor coming from variable change to collective coordinates. To simplify the notations, we also introduced the matrices $\Gamma_i, i = 1, 2, 5$ acting on the four dimensional spinor (16) as:

$$\begin{aligned}\Gamma_1 &= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}, & \Gamma_2 &= \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \\ \Gamma_5 &= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},\end{aligned}\quad (40)$$

where $\mathbb{1}_2$ is the identity on a two dimensional subspace.

3. Fourier transformation of the Green's function

The Fourier transformation of the Green's function after integration over the instanton location x_0 reads (writing spinor indices explicitly)

$$\begin{aligned}G^{ab}(k_1, k_2, p_1 \dots p_n, p'_1 \dots p'_m, q_1, \dots, q_l) \\ &= (2\pi)^2 \delta^{(2)}(P) \int_0^{2\pi} d\beta \kappa (e^{i(\beta/2)\Gamma_5} \tilde{\psi}_{\text{cl}}^1(k_1))^a \\ &\quad \times (e^{i(\beta/2)\Gamma_5} \tilde{\psi}_{\text{cl}}^2(k_2))^b \prod_{i=1}^n e^{i\beta} \tilde{\chi}_{\text{cl}}(p_i) \prod_{j=1}^m e^{-i\beta} \tilde{\chi}_{\text{cl}}^*(p'_j) \\ &\quad \times \prod_{k=1}^l \tilde{\eta}_{\text{cl}}(q_k),\end{aligned}\quad (41)$$

where $\kappa = e^{-S_{\text{cl}}} \det'(K_{\text{inst}}) N_B N_{\text{tr}} \det(D_{\text{bos}}^2)^{-(1/2)}$ and $P = k_1 + k_2 + \sum_{i=1}^n p_i + \sum_{i=1}^m p'_i + \sum_{i=1}^l q_i$. The integration over the instanton location leads to momentum conservation. In a similar way, integration over gauge rotation β enforces charge conservation. Indeed, the integral over β is nonzero only if the powers of $e^{i\beta}$ cancel, that is to say, if charge with respect to the gauge field B_μ is conserved.¹⁰

As the different components of the spinors have different powers of $e^{i\beta}$, different cases have to be considered. We will concentrate here on three interesting situations, from which we will be able to derive some general conclusions.

D. Examples of allowed matrix elements

First consider a process involving one ϕ scalar as initial state, which decays into two fermions. In this case the integration over the coordinate β leads to:

$$\begin{aligned}G^{ab}(k_1, k_2, q_1) &= (2\pi)^2 \delta^{(2)}(P) \kappa \tilde{\phi}_{\text{cl}}(q_1) ((\Gamma_1 \tilde{\psi}_{\text{cl}}^1(k_1))^a \\ &\quad \times (\Gamma_2 \tilde{\psi}_{\text{cl}}^2(k_2))^b \\ &\quad + (\Gamma_2 \tilde{\psi}_{\text{cl}}^1(k_1))^a (\Gamma_1 \tilde{\psi}_{\text{cl}}^2(k_2))^b).\end{aligned}\quad (42)$$

Applying the reduction formula, we get a nonvanishing

¹⁰Note that this do not depend on the existence of the B_μ field, but on the existence of the associated global symmetry. Therefore the requirement of charge conservation will persist in the limit $e' \rightarrow 0$.

matrix element for two different fermions only by multiplying the Green's function by two fermionic legs $\bar{u}^1(k_1), \bar{u}^2(k_1)$,

$$\begin{aligned}iM(k_1, k_2, q_1) &= (2\pi)^2 \delta^{(2)}(q - k_1 - k_2) \kappa i (q^2 + m_H^2) \\ &\quad \times \tilde{\phi}_{\text{cl}}(q) (i\bar{u}^1(k_1)(\hat{k}_1 + F)\Gamma_1 \psi_{\text{cl}}^1(k_1) \\ &\quad \times i\bar{u}^2(k_2)(\hat{k}_2 + F)\Gamma_1 \psi_{\text{cl}}^2(k_2)).\end{aligned}\quad (43)$$

A straightforward calculation gives (see Appendix C)

$$|M(k_1, k_2, q_1)|^2 = (2(2\pi)^3 \kappa f_\infty \alpha_1 \alpha_2 (1 + \beta_1 \beta_2))^2. \quad (44)$$

The decay rate is after integration of the phase space (supposing $m_1 \ll m_\chi$):

$$\begin{aligned}\Gamma_\phi &= \frac{1}{2m_\phi} \int d\text{Lips} |M(k_1, k_2, q_1)|^2 \\ &= \frac{1}{2m_\phi (m_\phi^2 - F^2)} (2(2\pi)^3 \kappa f_\infty \alpha_1 \alpha_2 (1 + \beta_1 \beta_2))^2.\end{aligned}\quad (45)$$

Secondly, we consider a process involving one χ scalar as initial state. The Fourier transformation of the Green's function reads

$$\begin{aligned}G^{ab}(k_1, k_2, q) &= (2\pi)^2 \delta^{(2)}(q - k_1 - k_2) \kappa \tilde{\chi}_{\text{cl}}^*(q) \\ &\quad \times (\Gamma_1 \tilde{\psi}_{\text{cl}}^1(k_1))^a (\Gamma_1 \tilde{\psi}_{\text{cl}}^2(k_2))^b.\end{aligned}$$

Applying the reduction formula, we get the matrix element for the creation of two light fermion by multiplying the Green's function by two light fermion legs $\bar{u}^1(k_1), \bar{u}^1(k_2)$,

$$\begin{aligned}iM(k_1, k_2, q_1) &= (2\pi)^2 \delta^{(2)}(q - k_1 - k_2) \kappa i (q^2 + m_\chi^2) \tilde{\chi}_{\text{cl}}^*(q) \\ &\quad \times i\bar{u}^1(k_1)(\hat{k}_1 + F)\Gamma_1 \psi_{\text{cl}}^1(k_1) \\ &\quad \times i\bar{u}^1(k_2)(\hat{k}_2 + F)\Gamma_1 \psi_{\text{cl}}^2(k_2).\end{aligned}\quad (46)$$

A straightforward calculation gives (see Appendix C)

$$|M(k_1, k_2, q_1)|^2 = (2(2\pi)^3 \kappa g_\infty \alpha_1 \alpha_2 \beta_2)^2. \quad (47)$$

The decay rate after integration of the phase space (supposing $m_1 \ll m_\chi$) reads

$$\Gamma_\chi = \frac{(2(2\pi)^3 \kappa g_\infty \alpha_1 \alpha_2 \beta_2)^2}{2m_\chi^2 \sqrt{m_\chi^2 - 4m_1^2}}. \quad (48)$$

A similar process involves the scalar χ^* , which decays into two heavy fermions:

$$\Gamma_{\chi^*} = \frac{(2(2\pi)^3 \kappa g_\infty \alpha_1 \alpha_2 \beta_1)^2}{2m_\chi^2 \sqrt{m_\chi^2 - 4m_2^2}}. \quad (49)$$

Generalizing these three examples, we see that any process leading to the creation of two light fermions contains a factor β_2 whereas a process leading the creation of heavy fermions has a factor β_1 . Processes leading to the creation of one of each fermion contain a factor $1 + \beta_1 \beta_2 \sim 1$. Apart from these factors $\beta_{1,2}$, we have a phase factor, which depends on the exact process but which is subdominant in two dimensions.

E. Discussion of the different transition probabilities

The integration over the collective coordinate associated with the gauge symmetry $U(1)^B$ leads necessarily to charge conservation. Therefore the process described by the level-crossing picture (Fig. 3) cannot take place without the emission of some other particle that compensates the additional $U(1)^B$ charge. The possible initial and final states are more restricted than suggested by the level-crossing picture.

We shall now discuss the transition probability of allowed processes. We leave aside for the moment the phase space factors, they are not large in $1 + 1$ dimensions. The main factors that distinguish the transition rates (45), (48), and (49) for the three possible fermionic final states are the constants $\beta_{1,2}$. This is also true if more complicated processes are considered. As expected, if the fermions are light and weakly coupled, the probability to create one of each fermion is much larger; it is proportional to $1 + \beta_1 \beta_2 \cong 1$, see Fig. 5 and 6. However, in the case where one fermion is very heavy, β_1 can exceed 1 (see Fig. 5 and 6) and in this case it is favored to create two heavy fermions rather than one of each. The creation of two light fermions suggested by the level-crossing picture is indeed suppressed, the factor β_2 having a chance to reach 1 only in the case of very slow transitions (heavy fermion masses $F_{1,2}$ or large instanton radius r_{inst}) and almost degenerate masses ($F_1 \sim F_2$).

V. CONCLUSION

In the model considered here, the level-crossing picture suggests a particular transition which must not and does not occur. A possible way out would be to reinterpret it as follows. The level-crossing picture only knows about fermions and the correct bosonic content of the initial and final states should be added by hand when dealing with a physical transition. More precisely, all symmetries that are broken by the fermionic initial and final states should be restored by supplementary bosonic operators. However, even with this extra requirement, the level-crossing picture suggests the creation of two light fermions, a transition that turns out to be suppressed. The most probable transition is

to create one of each fermion as long as the fermionic mixing and the time scale of the transition are not large. If the transition is slow, the mixing is large, and the mass hierarchy is large ($\frac{F_2}{F_1} \gg 1$) the factor β_1 can reach 1 and the probability of creating two heavy fermions is larger than to create one of each (see Figs. 5 and 6). On the level-crossing picture, creating two heavy fermions, or one of each, can occur only if the energy levels cross each other several times (see Fig. 2) in spite of the interaction potential. Note that this is perfectly possible in quantum field theory although forbidden in the adiabatic quantum mechanical description.

The results for the transition probability are rather surprising; for heavy fermions, such as the top quark, or adiabatic process $r_{\text{inst}} F_2 \gg 1$ (Sphaleron at high temperature) the probability of creating two heavy fermions is large. In the realistic electroweak theory, the phase space factor may be dominant and may change this conclusion. It is therefore very interesting to reproduce similar computations in the frame of the electroweak theory at high temperature, or at high energies.

A more interesting setup would be to include heavy quarks in the initial states. The phase space factor as well as the matrix element are then large. In this case, the nonperturbative transition rate can be enhanced by a huge factor (see Fig. 5). A high top quark density could therefore catalyze the nonperturbative transition rate. This phenomenon is relevant for baryogenesis at the electroweak phase transition. It could provide a mechanism to enhance the baryon number violating transition rate in the symmetric phase, while suppressing it in the broken phase. Indeed, while bubbles of true asymmetric vacuum expand in the symmetric universe, it may be that top quarks are more reflected by the bubble wall and are rare inside the bubble, and over-dense outside. This density asymmetry will render the nonperturbative rate faster outside the bubble, while slower inside.

It should be noted that the present calculation deals with the instanton rate, although at high temperature, the sphaleron rate is the relevant quantity. It would therefore be very interesting to find out if the sphaleron rate also displays these interesting features.

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APPENDIX A: THE INSTANTON

The bosonic Lagrangian (22) in the $\partial_\mu A_\mu = \partial_\mu B_\mu = 0$ gauge gives the following equations of motion,

$$\begin{aligned}
& -\partial_\mu \partial_\mu \phi + 2ieA_\mu \partial_\mu \phi + e^2 A_\mu^2 \phi - \lambda v^2 \phi + \lambda |\phi|^2 \phi + h |\chi|^2 \phi = 0, \\
& -\partial_\mu \partial_\mu \chi + 2i(eA_\mu + e'B_\mu) \partial_\mu \chi + (eA_\mu + e'B_\mu)^2 \chi + M^2 \chi + \Lambda |\chi|^2 \chi + h(|\phi|^2 - v^2) \chi = 0, \\
& -\partial_v \partial_v A_\mu + i \frac{e}{2} (\phi^* \overleftrightarrow{\partial}_\mu \phi + \chi^* \overleftrightarrow{\partial}_\mu \chi) + e^2 A_\mu (|\phi|^2 + |\chi|^2) + ee'B_\mu |\chi|^2 = 0, \\
& -\partial_v \partial_v B_\mu + i \frac{e'}{2} \chi^* \overleftrightarrow{\partial}_\mu \chi + e'^2 B_\mu |\chi|^2 + e'eA_\mu |\chi|^2 = 0.
\end{aligned} \tag{A1}$$

We are looking for a solution of the type (24). As for the Nielsen-Olesen vortex, we impose the asymptotic behavior of the functions A and f :

$$\begin{aligned}
f(r) & \xrightarrow{r \rightarrow 0} f_1 r, & f(r) & \xrightarrow{r \rightarrow \infty} 1, \\
A(r) & \xrightarrow{r \rightarrow 0} a_1 r & A(r) & \xrightarrow{r \rightarrow \infty} \frac{1}{er}.
\end{aligned} \tag{A2}$$

For the finiteness of the action, the function $g(r)$, $B(r)$ should respect the following boundary conditions:

$$\begin{aligned}
B(r) & \xrightarrow{r \rightarrow 0} b_1 r, & g(r) & \xrightarrow{r \rightarrow 0} g_0, \\
B(r) + rB'(r) & \xrightarrow{r \rightarrow \infty} 0, & g(r) & \xrightarrow{r \rightarrow \infty} 0.
\end{aligned} \tag{A3}$$

We also introduce dimensionless variables with the substitutions

$$\begin{aligned}
A & = \tilde{A} \frac{\sqrt{\lambda v^2}}{e}, & f & = \tilde{f} \frac{\sqrt{\lambda v^2}}{e}, & g & = \sqrt{\frac{\lambda}{h}} \tilde{g}, \\
B & = \tilde{B} \frac{\sqrt{\lambda v^2}}{e'}, & r & = \frac{\tilde{r}}{\sqrt{\lambda v^2}}.
\end{aligned} \tag{A4}$$

The remaining parameters are

$$\begin{aligned}
\mu & = \frac{\lambda}{e^2}, & \mu' & = \frac{\lambda}{e'^2}, & \rho & = \frac{\Lambda}{h}, \\
H & = \frac{h}{\lambda}, & m^2 & = \frac{M^2}{\lambda v^2}.
\end{aligned} \tag{A5}$$

The equations of motion (A2) in polar coordinates and with the ansatz (24) reads

$$\begin{aligned}
& -\tilde{f}''(r) - \frac{1}{\tilde{r}} \tilde{f}'(r) + \left[\tilde{A}(r) - \frac{1}{\tilde{r}} \right]^2 \tilde{f}(r) + \mu \tilde{f}(r)^3 - \tilde{f}(r) + \tilde{g}(r)^2 \tilde{f}(r) = 0, \\
& -\tilde{g}''(r) - \frac{1}{\tilde{r}} \tilde{g}'(r) + (\tilde{A}(r) + \tilde{B}(r))^2 \tilde{g}(r) + \rho \tilde{g}(r)^3 + m^2 \tilde{g}(r) + H \tilde{g}(r) (\mu \tilde{f}(r)^2 - 1) = 0, \\
& -\tilde{A}''(r) - \frac{\tilde{A}'(r)}{\tilde{r}} + \frac{\tilde{A}(\tilde{r})}{\tilde{r}^2} + \tilde{f}(r)^2 \left[\tilde{A}(r) - \frac{1}{\tilde{r}} \right] + \frac{1}{H\mu} (\tilde{B}(r) + \tilde{A}(r)) \tilde{g}^2(r) = 0, \\
& -\tilde{B}''(r) - \frac{\tilde{B}'(r)}{\tilde{r}} + \frac{\tilde{B}(r)}{\tilde{r}^2} + \frac{1}{H\mu'} (\tilde{B}(r) + \tilde{A}(r)) \tilde{g}^2(r) = 0,
\end{aligned} \tag{A6}$$

where the prime means derivative with respect to \tilde{r} .

APPENDIX B: ANALYTICAL APPROXIMATIONS FOR THE FERMIONIC ZERO-MODES

From the system of Eqs. (28), we neglect the field B_μ , eliminate the field A_μ by the variable change $\Psi \rightarrow \exp(-\frac{\xi}{2} \int dr A(r)) \Psi$ and contract the four first order differential equations into two second order ones. One obtains the following equations for the new variable Ψ :

$$\begin{aligned}
& \left(-\frac{m(m-1)}{r^2} + \frac{f'(r)m}{rf(r)} - f(r)^2 f_1^2 \right) \Psi_2(r) - \frac{f'(r) \Psi_2'(r)}{f(r)} \\
& + \Psi_2''(r) = f(r)g(r)f_1 f_3 \Psi_4(r),
\end{aligned} \tag{B1}$$

$$\begin{aligned}
& \left(-\frac{m(m-1)}{r^2} - \frac{f'(r)(m-1)}{rf(r)} - f(r)^2 f_2^2 \right) \Psi_3(r) \\
& - \frac{f'(r) \Psi_3'(r)}{f(r)} + \Psi_3''(r) = f(r)g(r)f_2 f_3 \Psi_1(r),
\end{aligned} \tag{B2}$$

with

$$\begin{aligned}
\Psi_1(r) & = \frac{1}{f(r)f_1} \left(\Psi_2'(r) - \frac{m\Psi_2(r)}{r} \right), \\
\Psi_4(r) & = \frac{1}{f(r)f_2} \left(\frac{(m-1)\Psi_3(r)}{r} + \Psi_3'(r) \right).
\end{aligned} \tag{B3}$$

We discuss the case of the zero-mode ψ_{cl}^1 in the $m=0$ partial wave, the case of ψ_{cl}^2 in the $m=1$ partial wave is treated analogously. At zero-order of perturbation we have the two first components ($\psi_{1,2}^1$) given by (29) and the two last ones ($\psi_{3,4}^1$) vanish. If we consider now a nonvanishing $f_3 g(r)$ in (B2), the function ψ_3^1 is given at first order perturbation theory by

$$\Psi_3(r) = f_2 f_3 \int dr' G(r, r') f(r') g(r') \psi_1^1(r'), \tag{B4}$$

where $G(r, r')$ is the Green's function of the differential operator in the left hand side of Eq. (B2). We were not able to find a general expression for $G(r, r')$ for an arbitrary function $f(r)$, but satisfactory results are obtained using the

Green's function $G(r, r')$ for constant¹¹ $f(r) = v$. In this particular case

$$G(r, r') = -\frac{1}{f_2} \sinh(f_2 r_{<}) \exp(-f_2 r_{>}), \quad (\text{B5})$$

with $r_{>} = \max(r, r')$, $r_{<} = \min(r, r')$. Form (B4), we get, for $r \gg 1$:

$$\Psi_3(r) \cong -f_3 v \exp(-f_2 r) \int dr' \sinh(f_2 r') g(r') \Psi_1(r'), \quad (\text{B6})$$

From this relation, we can read the factor β_1 in Eq. (31).

APPENDIX C: FOURIER TRANSFORMS

For computing cross sections, the unitary gauge is best suited. It is however known to be singular, which may lead to discontinuities in the fermionic wave functions. This can be easily cured using the following regularized gauge condition:

$$\alpha(r, \theta) = \theta - 2\pi \Theta^\varepsilon(\theta - \pi), \quad (\text{C1})$$

where $\Theta^\varepsilon(\theta - \pi)$ is continuous and goes to the step function as $\varepsilon \rightarrow 0$ (see Ref. [11] for more details). The Fourier transforms¹² of the fields are (we consider here only the case $f_3 = 0$ for fermions):

¹¹This approximation is exact in the limit of small instanton size and precise for light fermions, because they do not probe the instanton center.

¹²We retain only the pole term here, the rest do not contribute to the final amplitude [20]

$$\begin{aligned} \tilde{\phi}(p) &= \frac{f_\infty}{p^2 + m_H^2}, \\ \tilde{\chi}(p) &= \frac{g_\infty}{p^2 + m_\chi^2} \frac{p e^{i\theta_p}}{m}, \end{aligned} \quad (\text{C2})$$

$$\begin{aligned} \tilde{A}_\mu(p) &= \frac{i a_\infty}{m_W} \frac{\varepsilon_{\mu\nu} p_\nu}{m_W^2 + p^2}, \\ \psi_{R,L}^j(p) &= -i c_\infty \sqrt{\frac{2}{\pi p}} e^{i/2\gamma_s \theta_p} \frac{F_j + p}{F_j^2 + p^2}, \end{aligned}$$

where $p = \sqrt{p_\mu p_\mu}$ and θ_p the angle between the spacial axis and the vector p . If $f_3 \neq 0$ but the field B is neglected, the Fourier transforms of the two fermionic zero modes read:

$$\begin{aligned} \tilde{\psi}_{\text{cl}}^1(p) &= \begin{pmatrix} -i\alpha_1 \sqrt{\frac{2}{\pi p}} e^{i/2\theta_p} \frac{F_1 + p}{F_1^2 + p^2} \\ -i\alpha_1 \sqrt{\frac{2}{\pi p}} e^{-(i/2)\theta_p} \frac{F_1 + p}{F_1^2 + p^2} \\ -i\alpha_2 \sqrt{\frac{2}{\pi p}} e^{-(i/2)\theta_p} \frac{F_2 + p}{F_2^2 + p^2} \\ -i\alpha_2 \sqrt{\frac{2}{\pi p}} e^{-(3i/2)\theta_p} \frac{F_2 + p}{F_2^2 + p^2} \end{pmatrix}, \\ \tilde{\psi}_{\text{cl}}^2(p) &= \begin{pmatrix} -i\beta_1 \sqrt{\frac{2}{\pi p}} e^{3i/2\theta_p} \frac{F_1 + p}{F_1^2 + p^2} \\ -i\beta_1 \sqrt{\frac{2}{\pi p}} e^{i/2\theta_p} \frac{F_1 + p}{F_1^2 + p^2} \\ -i\beta_2 \sqrt{\frac{2}{\pi p}} e^{i/2\theta_p} \frac{F_2 + p}{F_2^2 + p^2} \\ -i\beta_2 \sqrt{\frac{2}{\pi p}} e^{-(i/2)\theta_p} \frac{F_2 + p}{F_2^2 + p^2} \end{pmatrix}. \end{aligned} \quad (\text{C3})$$

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