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Modifying the Einstein equations off the constraint hypersurface

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A new technique is presented for modifying the Einstein evolution equations off the constraint hypersurface. With this approach the evolution equations for the constraints can be specified freely. The equations of motion for the gravitational field variables are modified by the addition of terms that are linear and nonlocal in the constraints. These terms are obtained from solutions of the linearized Einstein constraints.

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The Einstein equations separate into a set of evolution equations and a set of constraints. The evolution equations are partial differential equations (PDE's) that determine how the gravitational field variables g_{ab} (the spatial metric) and K_{ab} (the extrinsic curvature) evolve forward in time. The constraint equations are PDE's that the field variables must satisfy at each instant of time. From a Hamiltonian point of view, the evolution equations define solution trajectories in phase space with coordinates g_{ab} and momenta K_{ab}. Physical trajectories are those that lie in the constraint hypersurface, or subspace, of the gravitational phase space.

Einstein's theory of gravity is a "first class" theory, that is, the time derivatives of the constraints are linear combinations of the constraints. This property implies that, analytically, the constraints will hold at each instant of time if they hold at the initial time. However, for numerically generated solutions of the theory the initial data will not satisfy the constraints precisely and numerical errors will kick the phase space trajectory away from the constraint hypersurface. This is a critical problem for numerical modeling because the Einstein evolution equations, as they are usually written, admit solutions that rapidly diverge away from the constraint hypersurface [1,2]. Any numerical scheme that evolves the gravitational field data using the evolution equations in one of their traditional forms will eventually fail to produce physically meaningful results. Inevitably the numerical solution will choose to follow a trajectory that violates the constraints.

A number of strategies have been devised to address this problem. One approach is to modify the theory off the constraint hypersurface by adding linear combinations of constraints to the evolution equations [1,3–7]. In this way, one hopes to alter the solution trajectories so that they are better behaved away from the constraint hypersurface. We will use the terminology "off-shell" to refer to solution trajectories that lie off the constraint hypersurface.

The strategy discussed in this paper is of this sort. We add terms proportional to the constraints to the Einstein evolution equations in such a way that the evolution equations for the constraints can be freely specified. In principle, we can eliminate all constraint violating modes by demanding, for example, that the time derivatives of the constraints should vanish. The price we pay for this degree of control over the unphysical, off-shell solutions is that the terms added to the evolution equations are nonlocal. They are determined through the solution of an elliptic system of PDE's.

Another strategy for keeping a numerically generated solution from diverging away from the constraint hypersurface is constrained evolution. In this scheme the constraints are used in place of certain evolution equations to update some of the gravitational field variables in time. This approach has worked well for spherically and axisymmetric problems [8–12]. A closely related idea is constraint projection [5,13,14]. With constraint projection, one evolves the full set of field variables using the evolution equations, then periodically (perhaps every time step) solves the constraints to project the solution back to the constraint hypersurface. Both constrained evolution and constraint projection require the solution of the constraint equations during the course of evolution. For these approaches to be viable, the constraints must be expressed as an elliptic system of PDE's. From a computational perspective, our strategy is closely related to constraint projection since we also solve an elliptic system of PDE's at every (or nearly every) time step. In fact, the PDE's that we solve are the linearized Einstein constraints.

It will be useful to give an overview of our procedure for modifying the off-shell solutions in a formal, general context. Consider a theory, like general relativity, described by a set of first class constraints $\mathcal{C}_A$. Let ψ_μ denote the basic field variables. These variables satisfy first order in time differential equations of motion, $\dot{\psi}_\mu = (\dot{\psi}_\mu)_{\text{old rhs}}$, where the "old right-hand sides" $(\dot{\psi}_\mu)_{\text{old rhs}}$ are functions of ψ_μ and their spatial derivatives. We have included the descriptor "old" since we will soon create "new" right-hand sides by adding functions of the constraints. The evolution equations for the constraints are obtained from the evolution equations for the ψ's by differentiating the constraints in time. This yields $\dot{\mathcal{C}}_A = (\dot{\mathcal{C}}_A)_{\text{old rhs}}$ where the right-hand sides (rhs's) are given by

$$(\dot{\mathcal{C}}_A)_{\text{old rhs}} = \frac{\delta \mathcal{C}_A}{\delta \psi_\mu} (\dot{\psi}_\mu)_{\text{old rhs}}. \tag{1}$$

The expression $\delta\mathcal{C}_A/\delta\psi_\mu$ is the Fréchet derivative of the constraints with respect to the field variables [15]. It satisfies $\mathcal{C}_A(\psi+\sigma)-\mathcal{C}_A(\psi)=(\delta\mathcal{C}_A/\delta\psi_\mu)\sigma_\mu$, in the limit as the norm of σ_μ goes to zero. (ψ_μ and σ_μ are defined as vectors in a suitable Banach space.) If $\mathcal{C}_A$ depends on spatial derivatives of ψ_μ, then the Fréchet derivative is a differential operator.

Now let us split the basic variables ψ_μ into two sets, ϕ_A and χ_i. Note that there are as many ϕ's as there are constraints. With this splitting, Eq. (1) becomes

$$(\dot{\mathcal{C}}_A)_{\text{old rhs}}=\frac{\delta\mathcal{C}_A}{\delta\phi_B}(\dot\phi_B)_{\text{old rhs}}+\frac{\delta\mathcal{C}_A}{\delta\chi_i}(\dot\chi_i)_{\text{old rhs}}. \quad (2)$$

Next, we replace the old equations of motion for the ϕ's with new equations of motion. This leads to new equations of motion for the constraints,

$$(\dot{\mathcal{C}}_A)_{\text{new rhs}}=\frac{\delta\mathcal{C}_A}{\delta\phi_B}(\dot\phi_B)_{\text{new rhs}}+\frac{\delta\mathcal{C}_A}{\delta\chi_i}(\dot\chi_i)_{\text{old rhs}}. \quad (3)$$

By subtracting the previous two results, we find

$$\Lambda_A=\frac{\partial\mathcal{C}_A}{\partial\phi_B}\Phi_B, \quad (4)$$

where Φ_A is the difference between the new and old rhs's for the variables ϕ_A, $\Phi_A\equiv(\dot\phi_A)_{\text{new rhs}}-(\dot\phi_A)_{\text{old rhs}}$, and Λ_A is the difference between the new and old rhs's for the constraints, $\Lambda_A\equiv(\dot{\mathcal{C}}_A)_{\text{new rhs}}-(\dot{\mathcal{C}}_A)_{\text{old rhs}}$. In terms of the original field variables ψ_μ, the new equations of motion are

$$(\dot\psi_\mu)_{\text{new rhs}}=(\dot\psi_\mu)_{\text{old rhs}}+\frac{\delta\psi_\mu}{\delta\phi_A}\Phi_A. \quad (5)$$

Now we turn the reasoning around. We do not actually choose new equations of motion for the ϕ's. Instead, we specify new evolution equations for the constraints by freely choosing the expressions $(\dot{\mathcal{C}}_A)_{\text{new rhs}}$. The functions Λ_A are then determined, and Eqs. (4) are solved for Φ_A. The new equations of motion for the original field variables are given by Eqs. (5).

Because the theory is first class, $(\dot{\mathcal{C}}_A)_{\text{old rhs}}$ is a linear combination of constraints. Let us choose $(\dot{\mathcal{C}}_A)_{\text{new rhs}}$ to be a linear combination of constraints as well. Then Λ_A is a linear combination of constraints and, according to Eq. (4), Φ_A is a (possibly nonlocal) linear combination of constraints. It follows that the new equations of motion for ψ_μ differ from the old equations by a linear combination of constraints.

Equations (4) are the linearized constraints. To be precise, consider a field configuration ψ_μ that does not satisfy the constraints and let $\bar\psi_\mu=\psi_\mu-\Psi_\mu$. If $\bar\psi_\mu$ satisfies the constraints then to linear order, Ψ_μ satisfies $\mathcal{C}_A=(\delta\mathcal{C}_A/\delta\psi_\mu)\Psi_\mu$. This is Eq. (4) with $\Psi_\mu=(\delta\psi_\mu/\delta\phi_A)\Phi_A$ and $\mathcal{C}_A=\Lambda_A$.

With the procedure outlined above, we can freely specify the rhs's of the constraint evolution equations, as long as they are a linear combination of constraints. We then solve Eq. (4) for Φ_A and modify the equations of motion for the ψ's to Eq. (5). In this way we leave the equations of motion for the basic field variables unchanged on the constraint hypersurface, but we can modify their off-shell form to eliminate the constraint violating modes.

Let us apply this formalism to general relativity. The basic field variables (the ψ's) are the spatial metric g_{ab} and the extrinsic curvature K_{ab}. The equations of motion as written by York [16] are $\partial_\perp g_{ab}=(\partial_\perp g_{ab})_{\text{old rhs}}$ and $\partial_\perp K_{ab}=(\partial_\perp K_{ab})_{\text{old rhs}}$, where

$$(\partial_\perp g_{ab})_{\text{old rhs}}=-2\alpha K_{ab}, \quad (6a)$$

$$(\partial_\perp K_{ab})_{\text{old rhs}}=\alpha(KK_{ab}-2K_{ac}K^c_b+R_{ab})-D_aD_b\alpha. \quad (6b)$$

Here, α is the lapse function, D_a is the spatial covariant derivative, and R_{ab} is the spatial Ricci tensor. The operator $\partial_\perp\equiv\partial_t-\mathcal{L}_\beta$ is the difference between the time derivative and the Lie derivative along the shift vector β^a. This operator plays the role of the "dot" in the formal analysis.

The constraints for general relativity are

$$\mathcal{H}\equiv K^2-K_{ab}K^{ab}+R, \quad (7a)$$

$$\mathcal{M}_a\equiv D_bK^b_a-D_aK. \quad (7b)$$

With the equations of motion (6), we find

$$(\partial_\perp\mathcal{H})_{\text{old rhs}}=2\alpha K\mathcal{H}-2\alpha D_a\mathcal{M}^a-4\mathcal{M}^aD_a\alpha, \quad (8a)$$

$$(\partial_\perp\mathcal{M}_a)_{\text{old rhs}}=\alpha K\mathcal{M}_a-\mathcal{H}D_a\alpha-\alpha D_a\mathcal{H}/2 \quad (8b)$$

for the rhs's of the constraint evolution equations.

In order to proceed, we must select a subset of the variables g_{ab}, K_{ab} to play the role of the ϕ's. We will use the conformal transverse-traceless decomposition developed by Lichnerowicz and York for solving the initial value problem (see, for example, Ref. [17]). To begin, we split the metric and extrinsic curvature into

$$g_{ab}=\varphi^4\tilde g_{ab}, \quad (9a)$$

$$K_{ab}=\varphi^{-2}A_{ab}+\tfrac{1}{3}\varphi^4\tilde g_{ab}\tau, \quad (9b)$$

where φ is the conformal factor, $\tilde g_{ab}$ is the conformal metric, τ is the trace of the extrinsic curvature, and A_{ab} is symmetric and trace free. Note that these definitions are invariant under the conformal transformation [18,19] $\tilde g_{ab}\to\xi^4\tilde g_{ab}$, $\varphi\to\xi^{-1}\varphi$, $A_{ab}\to\xi^{-2}A_{ab}$, $\tau\to\tau$.

The tensor A_{ab} can be decomposed in terms of a symmetric, transverse, traceless tensor B_{ab} and a vector w_a,

$$A_{ab}=(\tilde{\mathbb{L}}w)_{ab}+B_{ab}. \quad (10)$$

The operator $\tilde{\mathbb{L}}$ is defined by $(\tilde{\mathbb{L}}w)_{ab} \equiv \tilde{D}_a w_b + \tilde{D}_b w_a - 2\tilde{g}_{ab}\tilde{D}_c w^c/3$. It follows that w_a satisfies the elliptic equation $\tilde{D}^b(\tilde{\mathbb{L}}w)_{ab} = \tilde{D}^b A_{ab}$. The conformal transformation rule for B_{ab} is $B_{ab} \to \xi^{-2}B_{ab}$ and the rule for w_a is defined by the relation $(\tilde{\mathbb{L}}w)_{ab} \to \xi^{-2}(\tilde{\mathbb{L}}w)_{ab}$.

Let φ and w_a play the role of the ϕ's in Eqs. (4) and (5). The Fréchet derivatives that appear in those equations are somewhat tedious but straightforward to compute. If we let Φ and W_a denote the unknowns (the Φ_A's) in Eq. (4), we find the following results:

$$\Lambda_0 = -8D^2(\Phi/\varphi) - 2K^{ab}(\tilde{\mathbb{L}}W)_{ab}/\varphi^2 - [4K^2 - 12K_{ab}K^{ab} + 4R]\Phi/\varphi, \tag{11a}$$

$$\Lambda_a = D^b[(\tilde{\mathbb{L}}W)_{ab}/\varphi^2] - 6(D^b K_{ab} - D_a K/3)\Phi/\varphi. \tag{11b}$$

Here, Λ_0 and Λ_a are the differences between the new and old rhs's of the evolution equations for the constraints:

$$\Lambda_0 \equiv (\partial_\perp \mathcal{H})_{\text{new rhs}} - (\partial_\perp \mathcal{H})_{\text{old rhs}}, \tag{12a}$$

$$\Lambda_a \equiv (\partial_\perp \mathcal{M}_a)_{\text{new rhs}} - (\partial_\perp \mathcal{M}_a)_{\text{old rhs}}. \tag{12b}$$

The unknowns in Eqs. (11) are the differences between the new and old rhs's for the φ and w_a equations of motion. From the formal Eq. (5), we find the new equations of motion:

$$(\partial_\perp g_{ab})_{\text{new rhs}} = (\partial_\perp g_{ab})_{\text{old rhs}} + 4g_{ab}\Phi/\varphi, \tag{13a}$$

$$(\partial_\perp K_{ab})_{\text{new rhs}} = (\partial_\perp K_{ab})_{\text{old rhs}} + (\tilde{\mathbb{L}}W)_{ab}/\varphi^2 - 2(K_{ab} - Kg_{ab})\Phi/\varphi \tag{13b}$$

for the metric and extrinsic curvature.

Let us assume that the new equations of motion for φ and w_a, like the old, are conformally invariant. Then we see that Φ and W_a inherit the conformal transformation properties $\Phi \to \xi^{-1}\Phi$ and $(\tilde{\mathbb{L}}W)_{ab} \to \xi^{-2}(\tilde{\mathbb{L}}W)_{ab}$. It follows that Eqs. (11) and (13) are invariant under conformal transformations. In other words, these equations hold for any choice of splitting of the physical metric g_{ab} into a conformal factor φ^4 and conformal metric $\tilde{g}_{ab}$. For simplicity, we can choose the conformal factor to be unity, $\varphi = 1$, and the conformal metric to coincide with the physical metric, $\tilde{g}_{ab} = g_{ab}$. Then Eqs. (11) become

$$\Lambda_0 = -8D^2\Phi - 2K^{ab}(\mathbb{L}W)_{ab} - [4K^2 - 12K_{ab}K^{ab} + 4R]\Phi, \tag{14a}$$

$$\Lambda_a = D^b(\mathbb{L}W)_{ab} - 6(D^b K_{ab} - D_a K/3)\Phi. \tag{14b}$$

This is an elliptic system of linear PDE's for Φ and W_a. The new equations of motion (13) become

$$(\partial_\perp g_{ab})_{\text{new rhs}} = (\partial_\perp g_{ab})_{\text{old rhs}} + 4g_{ab}\Phi, \tag{15a}$$

$$(\partial_\perp K_{ab})_{\text{new rhs}} = (\partial_\perp K_{ab})_{\text{old rhs}} + (\mathbb{L}W)_{ab} - 2(K_{ab} - Kg_{ab})\Phi. \tag{15b}$$

This is our main result. We can now specify new rhs's for the Hamiltonian and momentum constraint equations of motion. The Λ's are found from Eqs. (12), and Eqs. (14) are solved for Φ and W_a. These results are used in the new equations of motion, Eqs. (15). Note that these equations contain only the physical metric and extrinsic curvature—all references to the conformal transverse-traceless splitting have disappeared.

What follows is a numerical demonstration that Eqs. (14) and (15) allow us the freedom to prescribe the evolution of the constraints by altering the Einstein equations off-shell. For this test we use periodic identification in the x, y, and z coordinate directions with periods of 2π. Thus, for now, we intentionally avoid facing the very important issue of boundary conditions. We use initial data that violates the constraints. The spatial metric is flat with Cartesian coordinates, $g_{ab} = \delta_{ab}$. The diagonal components of the extrinsic curvature are given by $K_{xx} = K_{yy} = K_{zz} = A/3$. The off-diagonal component K_{xy} is

$$K_{xy} = \varepsilon_1 \cos^2(z) + \varepsilon_2 \cos(x)\cos(y). \tag{16}$$

The components K_{yz} and K_{zx} are obtained from K_{xy} by cyclic permutations of x, y, and z. The initial data is evolved with unit lapse $\alpha = 1$ and vanishing shift $\beta^a = 0$. For the test results shown here, we use the values $A = 0.02$, $\varepsilon_1 = 0.01$, and $\varepsilon_2 = 0.0005$.

Figure 1 shows the common logarithm of the L_2 norm of the constraints as a function of time, with and without the off-shell modification terms in Eqs. (15). For the new constraint equations we have chosen

$$(\partial_\perp \mathcal{H})_{\text{new rhs}} = -0.4\mathcal{H} - \mathcal{L}_\beta \mathcal{H}, \tag{17a}$$

$$(\partial_\perp \mathcal{M}_a)_{\text{new rhs}} = -0.4\mathcal{M}_a - \mathcal{L}_\beta \mathcal{M}_a, \tag{17b}$$

so that at each point in space $\mathcal{H}$ and $\mathcal{M}_a$ have time

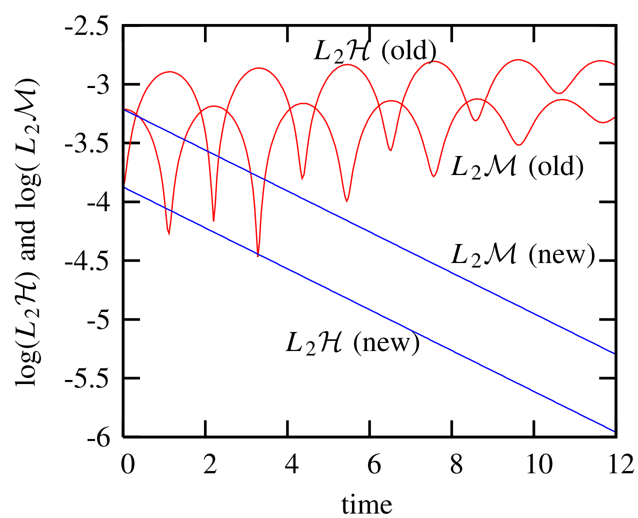

FIG. 1 (color online). Common logarithm of the L_2 norms of the Hamiltonian and momentum constraints, using the old and new equations of motion.

dependence $\sim \exp(-0.4t)$. The L_2 norms are defined without any factors of g_{ab} or $\sqrt{g}$. That is, we define $L_2\mathcal{H} = \sqrt{\sum \mathcal{H}^2/N^3}$ where the sum extends over the N^3 grid points. Similarly, we define $L_2\mathcal{M} = \sqrt{\sum \mathcal{M}_a\mathcal{M}_a/N^3}$ and include a sum over the index a. With these definitions, the only time dependence that should appear in the L_2 norms is the exponential decay $\sim \exp(-0.4t)$. This is precisely what we see in Fig. 1 with the new equations of motion. For the old equations of motion, we see strong oscillations on a short time scale and exponential growth on a longer time scale.

Our numerical code uses pseudospectral collocation [20] with a Fourier basis in each of the coordinate directions. Fourth-order Runge-Kutta is used for time stepping. The elliptic equations (14) are solved with the iterative method GMRES [21]. We use a left preconditioner consisting of the inverse of the diagonal part of the elliptic operator. One of the numerical issues that we face is spectral blocking [20]. This is the phenomenon in which aliasing causes an unphysical increase in power in the highest wave number modes that can be supported on the grid. Filtering can help alleviate this problem. For the simulations shown in Fig. 1, we use $N = 20$ collocation points in each dimension and a filter that sets the two highest frequencies to zero at the end of each time step. The time step is 0.04, compared to a light-crossing time of approximately 2π.

The results displayed in Fig. 1 show that we have indeed modified the equations of motion off-shell in such a way that unwanted growth in the constraints is eliminated. Ultimately, what we would like to show is the ability to prevent constraint growth in the first place. Our preliminary attempts to demonstrate this ability have not been completely successful, for reasons that we suspect are purely numerical. Although we cannot rule out the possibility that the combined system Eqs. (14) and (15) is mathematically ill-defined in some sense, the problems that we have encountered appear to be caused by numerical issues. One issue is spectral blocking, mentioned above. Another issue is the failure of our elliptic solver to converge to a solution under circumstances that we do not yet understand. We suspect that a better preconditioner will make our elliptic solver more robust and dependable.

For the simulation shown in Fig. 1, with the new equations of motion, the constraints continue to drop exponentially until $t \approx 15$. Beyond this time the constraints begin to suffer from high wave number variations whose growth counteracts the exponential drop of the longer wavelength modes. This breakdown is sensitive to the spatial resolution and amount of filtering and appears to be related to spectral blocking. With the old equations of motion, the Hamiltonian and momentum constraints continue to grow exponentially until $t \approx 45$. At that time $L_2\mathcal{H}$ has a value of $\sim 10^{-1}$ and the simulation breaks down. Again, this appears to be related to spectral blocking. We plan to study these issues in more detail and present further numerical tests in a future publication.

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