Thermodynamic route to field equations in Lanczos-Lovelock gravity

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Spacetimes with horizons show a resemblance to thermodynamic systems and one can associate the notions of temperature and entropy with them. In the case of Einstein-Hilbert gravity, it is possible to interpret Einstein's equations as the thermodynamic identity $TdS = dE + PdV$ for a spherically symmetric spacetime and thus provide a thermodynamic route to understand the dynamics of gravity. We study this approach further and show that the field equations for the Lanczos-Lovelock action in a spherically symmetric spacetime can also be expressed as $TdS = dE + PdV$ with S and E given by expressions previously derived in the literature by other approaches. The Lanczos-Lovelock Lagrangians are of the form $\mathcal{L} = Q_a{}^{bcd}R^a{}_{bcd}$ with $\nabla_b Q_a{}^{bcd} = 0$. In such models, the expansion of $Q_a{}^{bcd}$ in terms of the derivatives of the metric tensor determines the structure of the theory and higher order terms can be interpreted as quantum corrections to Einstein gravity. Our result indicates a deep connection between the thermodynamics of horizons and the allowed quantum corrections to standard Einstein gravity, and shows that the relation $TdS = dE + PdV$ has a greater domain of validity than Einstein's field equations.

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I. INTRODUCTION

There is an intriguing analogy between the gravitational dynamics of horizons and thermodynamics, which is not yet understood at a deeper level [1]. One possible way of interpreting these results is to assume that spacetime is analogous to an elastic solid and equations describing its dynamics are similar to those of elasticity (the "Sakharov paradigm"; see, e.g., Ref. [2]). The unknown, microscopic degrees of freedom of spacetime (which should be analogous to the atoms in the case of solids) will play a role only when spacetime is probed at Planck scales (which would be analogous to the lattice spacing of a solid). The exception to this general rule arises when we consider horizons [3] which have finite temperature and block information from a family of observers. In a manner which is not fully understood, the horizons link certain aspects of microscopic physics with the bulk dynamics just as thermodynamics can provide a link between statistical mechanics and (zero temperature) dynamics of a solid. If this picture is correct, then one should be able to link the equations describing bulk spacetime dynamics with horizon thermodynamics. There have been several approaches which have attempted to do this with different levels of success [1,2,4]. The most explicit example occurs [5] in the case of spherically symmetric horizons in 4-D. In this case, Einstein's equations can be interpreted as a thermodynamic relation $TdS = dE + PdV$ arising out of virtual displacements of the horizon.

This result was derived in the context of Einstein-Hilbert gravity arising from the Lagrangian $L_{\rm EH} \propto R\sqrt{-g}$. But if gravity is a long wavelength, emergent phenomenon, then the Einstein-Hilbert action is just the first term in the expansion for the low energy effective action. It is natural to expect *quantum corrections* to the Einstein-Hilbert action functional, which will, of course, depend on the nature of the microscopic theory but will generally involve higher derivative correction terms in the Einstein-Hilbert action [6]. In particular, such terms also arise in the effective low energy actions of string theories [7]. One such higher derivative term which has attracted a fair amount of attention is Lanczos-Lovelock gravity [8] of which the lowest order correction appears as a Gauss-Bonnet term in $D(>4)$ dimensions. We study the structure of a general Lanczos-Lovelock–type Lagrangian and the resulting equation of motion for a static and spherically symmetric spacetime, near a Killing horizon. We find that the equivalence of the equation of motion and the thermodynamic identity *$TdS = dE + PdV$ transcends Einstein gravity and is applicable even in this more general case.* This remarkable result indicates that there is a deep connection between the thermodynamics of gravitational horizons and the structure of the quantum corrections to Einstein gravity.

Our result requires fairly involved combinatorial arguments and detailed algebra. In order not to lose the physical picture, we have structured the paper as follows: in the next section, we will briefly review the case of Einstein-Hilbert gravity to set the stage [5]. In Sec. III we will display an explicit calculation relating the equation of motion for gravity with a Gauss-Bonnet correction term (the simplest nontrivial example of Lanczos-Lovelock gravity) with thermodynamic quantities. In Sec. IV we will generalize

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the result to Lanczos-Lovelock actions with all the terms allowed for a given number of dimensions D, and discuss the implications in Sec. V.

II. THE EINSTEIN-HILBERT CASE

Consider a static, spherically symmetric spacetime with a horizon, described by the metric

$$ds^2 = -f(r)c^2dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2. \quad (1)$$

We will assume that the function $f(r)$ has a simple zero at $r = a$ and that $f'(a)$ is finite, so that spacetime has a horizon at $r = a$ with nonvanishing surface gravity $\kappa = f'(a)/2$. Periodicity in Euclidean time allows us to associate a temperature with the horizon as $k_B T = \hbar c \kappa / 2\pi = \hbar c f'(a)/4\pi$, where we have introduced normal units. (Even for spacetimes with multihorizons, this prescription is locally valid for each horizon surface.) Einstein's equation for this metric, $rf'(r) - (1 - f) = (8\pi G/c^4)Pr^2$ (where P is the radial pressure), evaluated at $r = a$ gives

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a - \frac{1}{2}\right] = 4\pi P a^2. \quad (2)$$

If we now consider two solutions with two different radii a and $a + da$ for the horizon, then multiplying Eq. (2) by da, and introducing a $\hbar$ factor *by hand* into an otherwise classical equation, we can rewrite it as

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T}\underbrace{\frac{c^3}{G\hbar}d\left(\frac{1}{4}4\pi a^2\right)}_{dS}\underbrace{-\frac{1}{2}\frac{c^4 da}{G}}_{-dE} = \underbrace{Pd\left(\frac{4\pi}{3}a^3\right)}_{PdV} \quad (3)$$

and read off the expressions:

$$S = \frac{1}{4L_P^2}(4\pi a^2) = \frac{1}{4}\frac{A_H}{L_P^2}, \qquad E = \frac{c^4}{2G}a = \frac{c^4}{G}\left(\frac{A_H}{16\pi}\right)^{1/2}, \quad (4)$$

where A_H is the horizon area and $L_P^2 = G\hbar/c^3$. Thus Einstein's equations can be cast as a thermodynamic identity. Three comments are relevant regarding this result, especially since these comments are valid for our generalization discussed in the rest of the paper as well:

(a) The combination TdS is completely classical and is independent of $\hbar$, but $T \propto \hbar$ and $S \propto 1/\hbar$. This is analogous to the situation in classical thermodynamics when compared to statistical mechanics. The TdS in thermodynamics is independent of Boltzmann's constant while statistical mechanics will lead to $S \propto k_B$ and $T \propto 1/k_B$.

(b) In spite of superficial similarity, Eq. (3) is different from the conventional first law of black hole thermodynamics (as well as some previous attempts to relate thermodynamics and gravity, like e.g. the second paper in Ref. [2]), due to the presence of the PdV term. This relation is more in tune with the membrane paradigm [9] for the black holes. This is easily seen, for example, in the case of Reissner-Nordstrom black holes for which $P \neq 0$. If a *chargeless* particle of mass dM is dropped into a Reissner-Nordstrom black hole, then an elementary calculation shows that the energy defined above as $E \equiv a/2$ changes by $dE = (da/2) = (1/2)[a/(a - M)]dM \neq dM$, while it is $dE + PdV$ which is precisely equal to dM, making sure that $TdS = dM$. So we need the PdV term to get $TdS = dM$ when a *chargeless* particle is dropped into a Reissner-Nordstrom black hole. More generally, if da arises due to changes dM and dQ, it is easy to show that Eq. (3) gives $TdS = dM - (Q/a)dQ$, where the second term arises from the electrostatic contribution from the horizon surface charge as expected in the membrane paradigm.

(c) In standard thermodynamics, we consider two equilibrium states of a system differing infinitesimally in the extensive variables like entropy, energy, and volume by dS, dE, and dV, while having the same values for the intensive variables like temperature (T) and pressure (P). Then, the first law of thermodynamics asserts that $TdS = PdV + dE$ for these states. In a similar way, Eq. (3) can be interpreted as a connection between two quasistatic equilibrium states, where both of them are spherically symmetric solutions of Einstein equations with the radius of horizon differing by da while having the same source T_{ij} and temperature $T = \kappa/2\pi$. This formalism does not depend upon what causes the change of the horizon radius and is therefore very generally applicable. Note that the structure of Eq. (3) itself allows us to "read off" the expressions for entropy and energy. The validity of this approach as well as the uniqueness of the resulting expressions for S and E are discussed at length in Ref. [5] and will not be repeated here.

III. A FIRST CORRECTION: GAUSS-BONNET GRAVITY

We shall now turn our attention to the more general case. (Hereafter, we shall adopt natural units, in which $\hbar = c = G = 1$.) A natural generalization of the Einstein-Hilbert Lagrangian is provided by the Lanczos-Lovelock Lagrangian, which is the sum of dimensionally extended Euler densities,

$$\mathcal{L}^{(D)} = \sum_{m=1}^{K} c_m \mathcal{L}_m^{(D)}, \quad (5)$$

where the c_m are arbitrary constants and $\mathcal{L}_m^{(D)}$ is the mth order Lanczos-Lovelock term given by

$$\mathcal{L}_m^{(D)} = \frac{1}{16\pi}2^{-m}\delta^{a_1 b_1 \ldots a_m b_m}_{c_1 d_1 \ldots c_m d_m} R^{c_1 d_1}_{a_1 b_1} \ldots R^{c_m d_m}_{a_m b_m}, \quad (6)$$

where R^{ab}_{cd} is the D-dimensional Riemann tensor, and the generalized alternating ("determinant") tensor $\delta^{\cdots}_{\cdots}$ is totally antisymmetric in both sets of indices. For $D = 2m$, $16\pi\mathcal{L}^{(2m)}_m$ is the Euler density of the $2m$-dimensional manifold. We set $\mathcal{L}_0 = 1/16\pi$, and hence c_0 is proportional to the cosmological constant. The Einstein-Hilbert Lagrangian is a special case of Eq. (5) when only c_1 is nonzero. These Lagrangians are free from ghosts and are quasilinear in nature (see the first reference in [7]).

In this section we will concentrate on the first correction term, namely $\mathcal{L}_2$, which is the Gauss-Bonnet Lagrangian. In four dimensions, this term is a total derivative, while higher order interactions are simply zero; we will work with spacetimes for which $D > 4$. Then the relevant action functional of the theory is given by

$$\mathcal{A} = \int d^D x\sqrt{-g}\left[\frac{1}{16\pi}(R + \alpha\mathcal{L}_{\rm GB})\right] + \mathcal{A}_{\rm matter}, \quad (7)$$

where R is the D-dimensional Ricci scalar, and $\mathcal{L}_{\rm GB}$ is the Gauss-Bonnet Lagrangian which has the form

$$\mathcal{L}_{\rm GB} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}. \quad (8)$$

This type of action can be derived from superstring theory in the low energy limit. In that case, α is regarded as the inverse string tension and is positive definite. At least in this context, it makes sense to think of the second term as a correction to Einstein gravity. (We have not added a cosmological constant to the action for simplicity; all our results below trivially generalize in the presence of a bulk cosmological constant.) The equation of motion for this semiclassical action in Eq. (7) is given by

$$G_{ab} + \alpha H_{ab} = 8\pi T_{ab}, \quad (9)$$

where

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2}g_{ab}R, \quad (10)$$

$$H_{ab} \equiv 2[RR_{ab} - 2R_{aj}R^j{}_b - 2R^{ij}R_{aibj} + R_a{}^{ijk}R_{bijk}] - \tfrac{1}{2}g_{ab}\mathcal{L}_{\rm GB}. \quad (11)$$

Consider again a static spherically symmetric solution of the form [10]

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2_{D-2}, \quad (12)$$

where $d\Omega^2_{D-2}$ is the metric of the $(D-2)$-dimensional space of constant curvature k. Here and below, we will work with the case of spherical geometry with $k = 1$, but all the results can easily be generalized for $k \neq 1$. Spherical symmetry allows us to write, in the energy momentum tensor [11], $T^t_t = T^r_r \equiv \epsilon(r)/8\pi$. Then the equation of motion which determines the only nontrivial metric component $f(r)$ is given by [12]

$$rf' - (D-3)(1-f) + \frac{\bar{\alpha}}{r^2}(1-f) \times [2rf' - (D-5)(1-f)] = \frac{2\epsilon(r)}{D-2}r^2, \quad (13)$$

where $\bar{\alpha} = (D-3)(D-4)\alpha$. Note that $D = 4$ and $\alpha = 0$ refer to Einstein-Hilbert gravity, and in this limit one can recover Eq. (3). The horizon is obtained from the location of zeros of the function $f(r)$. In general, $f(r)$ may have several zeros but we will concentrate locally on any one of them. Let $r = a$ be a horizon for this spacetime with $f(r = a) = 0$, the temperature associated with this horizon being $T = \kappa/2\pi = f'(a)/4\pi$. As before, we evaluate Eq. (13) at $r = a$ to obtain

$$f'(a)\left[a + \frac{2\bar{\alpha}}{a}\right] - (D-3) - \frac{\bar{\alpha}(D-5)}{a^2} = \frac{2\epsilon(a)}{D-2}a^2. \quad (14)$$

Our aim is to introduce in this equation a factor dV and see whether one can read off entropy S and energy E from an equation of the form $TdS = dE + PdV$. Knowing the volume element in the D-dimensional space, we multiply both sides of the above equation by the factor ${}^{(D-2)}A_{D-2}a^{D-4}da/16\pi$, where $A_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the area of a unit $(D-1)$ sphere. Identifying the pressure $P = T^r_r$ and the relevant volume $V = A_{D-2}a^{D-1}/(D-1)$, we can rewrite this equation (after some straightforward algebra) in the form

$$\frac{\kappa}{2\pi}d\left(\frac{A_{D-2}}{4}a^{D-2}\left[1 + \left(\frac{D-2}{D-4}\right)\frac{2\bar{\alpha}}{a^2}\right]\right) - d\left[\frac{(D-2)A_{D-2}a^{D-3}}{16\pi}\left(1 + \frac{\bar{\alpha}}{a^2}\right)\right] = PdV. \quad (15)$$

The first term in the left-hand side is in the form TdS and our analysis allows us to read off the expression of entropy S for the horizon as

$$S = \frac{A_{D-2}}{4}a^{D-2}\left[1 + \left(\frac{D-2}{D-4}\right)\frac{2\bar{\alpha}}{a^2}\right]. \quad (16)$$

This is *precisely* the expression for the entropy in Gauss-Bonnet gravity calculated by several authors [13] by more sophisticated methods. Further, we can interpret the second term on the left-hand side of Eq. (15) as dE, where E is the energy of the system defined as

$$E = \frac{(D-2)A_{D-2}a^{D-3}}{16\pi}\left(1 + \frac{\bar{\alpha}}{a^2}\right), \quad (17)$$

which also matches with the correct expression of energy E for Gauss-Bonnet gravity without a cosmological constant [13,14].

This shows that our result for the Einstein-Hilbert action generalizes to the Gauss-Bonnet case as well, and precisely reproduces the expressions for entropy and energy (obtained in the literature by other methods). It is clear that, at least in this context, the thermodynamic relation transcends the Einstein field equations. We will now show that the same result holds for the general Lanczos-Lovelock action.

IV. GRAVITY WITH THE COMPLETE LANCZOS-LOVELOCK ACTION

We now turn to the more general case of Lanczos-Lovelock gravity in D dimensions with a Lagrangian given by $\mathcal{L}^{(D)} = \sum_{m=1}^{K} c_m \mathcal{L}_m^{(D)}$, where

$$\mathcal{L}_m^{(D)} = \frac{1}{16\pi} 2^{-m} \delta^{a_1 a_2 \ldots a_{2m}}_{b_1 b_2 \ldots b_{2m}} R^{b_1 b_2}_{a_1 a_2} \ldots R^{b_{2m-1} b_{2m}}_{a_{2m-1} a_{2m}}. \quad (18)$$

We assume that $D \geq 2K + 1$ and ignore the cosmological constant for simplicity. These Lagrangians have the peculiar property [15] that their variation leads to equations of motion that are equivalent to the ordinary *partial* derivatives of the Lagrangian density with respect to the metric components g^{ab},

$$E^a_b \equiv \sum_{m=1}^{K} c_m E^a_{b(m)} = \frac{1}{2} T^a_b, \qquad E^a_{b(m)} \equiv \frac{1}{\sqrt{-g}} g^{ai} \frac{\partial}{\partial g^{ib}} (\mathcal{L}_m^{(D)} \sqrt{-g}). \quad (19)$$

The factor $1/2$ with T^a_b appears since we have normalized $\mathcal{L}_m^{(D)}$ to contain a factor of $1/(16\pi)$.

We are interested in the near-horizon structure of the E^t_t equation, for a spherically symmetric metric of the form Eq. (12), and will demonstrate that this structure can (*also*) be represented as the thermodynamic identity $TdS = dE + PdV$. To this end, we will consider the Rindler limit (see the first reference in [1]) of such a metric, by which we mean that we will study the metric (12) near the horizon at $r = a$ and bring it to the Rindler form

$$ds^2 = -N^2 dt^2 + \frac{dN^2}{\kappa^2 + \mathcal{O}(N)} + \sigma_{AB} dy^A dy^B. \quad (20)$$

This form essentially arises by using a coordinate system in which the level surfaces of the metric component g_{00} (which vanishes on the horizon) define the spatial coordinate N. The constant κ appearing in the g_{11} term above can be shown to coincide with the surface gravity of the horizon. In this section, capitalized Latin indices correspond to the transverse coordinates on the $t =$ constant, $N =$ constant surfaces of dimension $D - 2$, and σ_{AB} is the metric on these surfaces. Denoting the extrinsic curvature of these $(D - 2)$ surfaces by K_{AB}, it is easy to show that, for the metric (12), the Rindler limit gives

$$\sigma_{AB} = \sigma_{(1)AB} + \frac{N^2}{\kappa a} \sigma_{(1)AB} + \mathcal{O}(N^4), \quad (21a)$$

$$\sigma^{AB} = \sigma^{AB}_{(1)} - \frac{N^2}{\kappa a} \sigma^{AB}_{(1)} + \mathcal{O}(N^4), \quad (21b)$$

$$K_{AB} = -\frac{N}{a} \sigma_{(1)AB} + \mathcal{O}(N^2), \quad (21c)$$

where $\sigma_{(1)AB} = a^2 \tilde{\sigma}_{(1)AB}$, $\tilde{\sigma}_{(1)AB}$ being the metric on a unit $(D - 2)$ sphere, and $\sigma^{AC}_{(1)} \sigma_{(1)CB} = \delta^A_B$.

Next we display the (near-horizon) structure of the D-dimensional Riemann tensor. We will drop the superscript D when considering D-dimensional quantities, but retain the superscript for $(D - 2)$-dimensional quantities. It will turn out, for reasons that will become apparent shortly, that the Riemann tensor components of the form R^{ti}_{jk} and R^{jk}_{ti} will not contribute to the E^t_t equation of motion. The remaining components of R^{ij}_{kl} are

$$R^{NA}_{NB} = \kappa \partial_N K^A_B + \mathcal{O}(N^2) = -\frac{\kappa}{a} \delta^A_B + \mathcal{O}(N^2), \quad (22a)$$

$$R^{NA}_{BC} = K^A_{C:B} - K^A_{B:C} = \mathcal{O}(N), \quad (22b)$$

$$R^{BC}_{NA} = g_{NN}(K^{C:B}_A - K^{B:C}_A) = \mathcal{O}(N), \quad (22c)$$

$$R^{AB}_{CD} = {}^{(D-2)}R^{AB}_{CD} + \mathcal{O}(N^2), \quad (22d)$$

where the colon denotes a covariant derivative using the $(D - 2)$-dimensional metric σ_{AB}. Also, since the $(D - 2)$-dimensional hypersurfaces are maximally symmetric, their Riemann tensor ${}^{(D-2)}R^{AB}_{CD}$ takes on the particularly simple form

$${}^{(D-2)}R^{AB}_{CD} = \frac{1}{a^2} (\delta^A_C \delta^B_D - \delta^B_C \delta^A_D). \quad (23)$$

With these results, we can begin analyzing the near-horizon structure of the E^t_t equation in Lanczos-Lovelock gravity. Since the equation depends linearly on the terms $E^t_{t(m)}$, it is sufficient to analyze these terms individually,

$$E^t_{t(m)} = \frac{1}{\sqrt{-g}} g^{tt} \frac{\partial}{\partial g^{tt}} (\mathcal{L}_m^{(D)} \sqrt{-g}). \quad (24)$$

On writing $R^{ij}_{kl} = g^{ja} R^i_{akl}$, the derivative with respect to g^{tt} can be performed. Using the symmetries of the alternating tensor, together with the relation $(\partial\sqrt{-g})/(\partial g^{tt}) = -(1/2)\sqrt{-g} g_{tt}$ and the fact that $g_{0N} = g_{0A} = 0$ for the static Rindler metric, we find

$$E^t_{t(m)} = \frac{1}{16\pi} \frac{m}{2^m} \delta^{a_1 a_2 \ldots a_{2m}}_{t b_2 \ldots b_{2m}} R^{t b_2}_{a_1 a_2} \ldots R^{b_{2m-1} b_{2m}}_{a_{2m-1} a_{2m}} - \frac{1}{2} \mathcal{L}_m^{(D)}. \quad (25)$$

We will now show that the summations involved in the first term of Eq. (25) are cancelled by terms in $\mathcal{L}_m^{(D)}$. Let us categorize the terms that appear in $\mathcal{L}_m^{(D)}$ into those in which the index value t appears at least once, which we denote by $\{T\}$, and those in which t does not appear, which we denote by $\{\bar{T}\}$. Symbolically then, $\mathcal{L}_m^{(D)} = \{T\} + \{\bar{T}\}$. In the case of standard Einstein gravity, we have $16\pi\mathcal{L}_1^{(D)} = R = 2R_t^t + R_{\alpha\beta}^{\alpha\beta}$ and one clearly recognizes $2R_t^t$ as $\{T\}$. Since the Einstein tensor is $G_t^t = R_t^t - (1/2)R$, the terms $\{T\}$ in R are precisely cancelled in G_t^t. We will now show that exactly the same feature occurs in the mth Lanczos-Lovelock case. To see this, we construct the set $\{T\}$ as follows. Focusing on the lower row of the alternating tensor in the expression (18) for $\mathcal{L}_m^{(D)}$, we have $2m$ choices for the location of the index value t. Because of the symmetries of the alternating tensor and the Riemann tensor, each choice results in the same term, and we can write

$$\{T\} = \frac{2}{16\pi}\frac{m}{2^m}\delta^{a_1a_2\ldots a_{2m}}_{tb_2\ldots b_{2m}} R^{tb_2}_{a_1a_2}\ldots R^{b_{2m-1}b_{2m}}_{a_{2m-1}a_{2m}}. \tag{26}$$

A comparison with Eq. (25) shows that the first term in that equation is simply $(1/2)\{T\}$, and we are left with

$$E^t_{t(m)} = -\tfrac{1}{2}\{\bar{T}\}. \tag{27}$$

Note that the set $\{\bar{T}\}$ is not *a priori* a null set since we have assumed $D \geq 2m + 1$. To simplify the contribution of $\{\bar{T}\}$, we further split this set as follows. This set contains terms with exactly one occurrence of the index value N, denoted $\{\bar{T}, 1N\}$, terms with two occurrences of N, denoted $\{\bar{T}, 2N\}$, and terms with no occurrences of N, denoted $\{\bar{T}, \bar{N}\}$. (The total antisymmetry of the alternating tensor forbids more than one occurrence of N in any row.) Each term in the set $\{\bar{T}, 1N\}$ contains one factor of the type R^{BC}_{NA} or R^{NA}_{BC}, and Eq. (22) shows that these terms are $\mathcal{O}(N)$ and do not contribute on the horizon. (In fact, these terms can be shown to vanish.) Similarly, the set $\{\bar{T}, 2N\}$ contains one type of term in which the two N's appear in *different* factors of $R^{\alpha\beta}_{\mu\nu}$. These terms contain two factors each of R^{BC}_{NA} or R^{NA}_{BC}, rendering these terms $\mathcal{O}(N^2)$. The contribution from $\{\bar{T}, 2N\}$ reduces to the $4m$ identical terms in which both the N's appear in the *same* factor of $R^{\alpha\beta}_{\mu\nu}$, and is given by

$$\{\bar{T}, 2N\} = \frac{4}{16\pi}\frac{m}{2^m}\delta^{NA_2\ldots A_{2m}}_{NB_2\ldots B_{2m}} R^{NB_2}_{NA_2}\ldots R^{B_{2m-1}B_{2m}}_{A_{2m-1}A_{2m}} + \mathcal{O}(N^2), \tag{28}$$

where $A_2, B_2, \ldots = y^A$. The set $\{\bar{T}, \bar{N}\}$ will not be *a priori* a null set whenever $D \geq 2m + 2$, and its contribution is

$$\{\bar{T}, \bar{N}\} = \frac{1}{16\pi}\frac{1}{2^m}\delta^{A_1A_2\ldots A_{2m}}_{B_1B_2\ldots B_{2m}} R^{B_1B_2}_{A_1A_2}\ldots R^{B_{2m-1}B_{2m}}_{A_{2m-1}A_{2m}} = \mathcal{L}_m^{(D-2)} + \mathcal{O}(N^2), \tag{29}$$

where we have used Eq. (22d) and recognized the structure of $\mathcal{L}_m^{(D-2)}$ in the resulting term. Finally, using Eqs. (28) and (29), substituting for the near-horizon structure of R^{NB}_{NA} from Eq. (22) and relabeling some indices, we find

$$E^t_{t(m)} = \frac{\kappa m}{16\pi}\frac{1}{2^{m-1}}\left(\frac{1}{a}\delta^{B_1}_{A_1}\right)\delta^{NA_1\ldots A_{2m-1}}_{NB_1\ldots B_{2m-1}}\,{}^{(D-2)}R^{B_2B_3}_{A_2A_3}\ldots - \frac{1}{2}\mathcal{L}_m^{(D-2)} + \mathcal{O}(N), \tag{30}$$

where $\mathcal{L}_m^{(D-2)}$ will contribute only when $D \geq 2m + 2$. The first term of Eq. (30) can be simplified by noting the following. The alternating tensor $\delta^{NA_1\ldots A_{2m-1}}_{NB_1\ldots B_{2m-1}}$ can be replaced by $\delta^{A_1A_2\ldots A_{2m-1}}_{B_1B_2\ldots B_{2m-1}}$ since $\delta^N_N = 1$ and $\delta^N_A = 0$. Further, due to the total antisymmetry of the alternating tensor, each factor of ${}^{(D-2)}R^{AB}_{CD}$ can be replaced by $(2/a^2)\delta^A_C\delta^B_D$, and there are $(m-1)$ such factors. Putting everything together, we find

$$E^t_{t(m)} = \frac{\kappa m}{16\pi}\frac{1}{a^{2m-1}}(\delta^{A_1A_2\ldots A_{2m-1}}_{B_1B_2\ldots B_{2m-1}})\delta^{B_1}_{A_1}\delta^{B_2}_{A_2}\ldots\delta^{B_{2m-1}}_{A_{2m-1}} - \frac{1}{2}\mathcal{L}_m^{(D-2)} + \mathcal{O}(N). \tag{31}$$

We can further perform the summations over A_1 and B_1 and rearrange terms to obtain

$$\frac{\kappa m}{8\pi}\frac{D-2m}{a^{2m-1}}(\delta^{A_2A_3\ldots A_{2m-1}}_{B_2B_3\ldots B_{2m-1}})\delta^{B_2}_{A_2}\delta^{B_3}_{A_3}\ldots\delta^{B_{2m-1}}_{A_{2m-1}} = 2E^t_{t(m)} + \mathcal{L}_m^{(D-2)} + \mathcal{O}(N). \tag{32}$$

We have relegated the proof of Eq. (32) to the Appendix, since it involves combinatorial arguments and is rather involved.

We are now ready to make the connection with the thermodynamic identity by a procedure which is *identical* to that used in the Einstein-Hilbert and Gauss-Bonnet cases. We wish to multiply the E^t_t equation of motion $E^t_t = (1/2)T^t_t$ evaluated *on the horizon* by the volume differential $dV = A_{D-2}a^{D-2}da$ and try to "read off" expressions for the entropy, energy, etc. We note that multiplying Eq. (32) by the coupling constant c_m and summing over m will give $2E^t_t$ as the first term on the right-hand side, which we can replace by T^t_t. We also know that in the spherically symmetric case we have $T^t_t = T^r_r = T^N_N = P$ with P the radial pressure. The equation obtained after these replacements will be equivalent to the equation of motion, and multiplying it with dV will result in the following,

$$\frac{\kappa}{2\pi}d\left(\sum_{m=1}^{K}\frac{m}{4}c_m A_{D-2}a^{D-2m}(\delta^{A_2\ldots A_{2m-1}}_{B_2\ldots B_{2m-1}})\delta^{B_2}_{A_2}\ldots\delta^{B_{2m-1}}_{A_{2m-1}}\right) = PdV + \sum_{m=1}^{K}c_m A_{D-2}a^{D-2}\mathcal{L}_m^{(D-2)}da. \tag{33}$$

Recognizing $\kappa/2\pi$ as the temperature T, we are forced to identify the quantity inside parentheses on the left-hand side above as the entropy S. Noting that the alternating tensor that appears here contains $2m-2$ indices per row, and recalling the simple structure of the $(D-2)$-dimensional Riemann tensor from Eq. (23), we can rewrite our entropy as $S = \sum_{m=1}^{K} S^{(m)}$, with $S^{(m)}$ given by

$$S^{(m)} = 4\pi m c_m A_{D-2} a^{D-2} \mathcal{L}_{m-1}^{(D-2)} = 4\pi m c_m \int_{\mathcal{H}} \mathcal{L}_{m-1}^{(D-2)} \sqrt{\sigma} d^{D-2} y. \qquad (34)$$

Note that, in our approach, which is identical to what we followed in the Einstein-Hilbert and Gauss-Bonnet cases, we have no choice in the expression for S. Remarkably enough, this is *precisely* the entropy of the horizon in Lanczos-Lovelock gravity which has been computed by several authors (see, e.g., the first reference in [13]).

Having identified the TdS and PdV terms in Eq. (33), we ask whether the remaining quantity can be interpreted as the differential of some function. We find that this is indeed the case and we have

$$\sum_{m=1}^{K} c_m A_{D-2} a^{D-2} \mathcal{L}_m^{(D-2)} da = d\left(\sum_{m=1}^{K} c_m E_{(m)}\right), \qquad (35)$$

$$E_{(m)} = \frac{1}{16\pi} A_{D-2} a^{D-(2m+1)} \prod_{j=2}^{2m} (D-j). \qquad (36)$$

The proof of Eq. (35) can be found in the Appendix. This requires us (again we have no choice in the matter) to interpret the quantity $E = \sum_{m=1}^{K} c_m E_{(m)}$ as the energy associated with the horizon; incredibly enough, we find that this expression *exactly* has been computed by other authors [16] as the energy of the horizon in spherically symmetric Lanczos-Lovelock gravity.

Incidentally, the expression for the differential of the energy dE in all the cases presented here shows that this contribution arises from the term $\mathcal{L}_m^{(D-2)}$ (which, for the Einstein-Hilbert case in $D = 4$ for example, is simply ${}^{(2)}R$). Thus the energy associated with the horizon originates in the transverse geometry of the horizon.

We have therefore proved that, for the spherically symmetric case, the equation of motion $E_t^t = (1/2)T_t^t$ can be recast in the form

$$\left(\frac{\kappa}{2\pi}\right) dS = dE + PdV, \qquad (37)$$

with the differentials being interpreted as arising due to a change in the radius of the horizon. In principle, the corrections to the entropy and the energy coming from the higher order Lanczos-Lovelock terms need not have preserved the structure of the first law of thermodynamics apparent above in the gravitational field equations.

We find it rather far-fetched to believe that this precise analogy of the field equations with the first law of thermodynamics (albeit for the spherically symmetric case) is a mere coincidence. This feature of the field equations seems to point towards a deeper principle which is yet to be understood.

V. DISCUSSION

The fact that the expression for entropy (S) and energy (E) obtained from this approach, by casting the equation in the form $TdS = dE + PdV$, *matches exactly* with the standard quantum field theory calculations, as in the case of Einstein-Hilbert gravity, is nontrivial and intriguing. However, it resonates well with an alternative perspective on gravity which was developed in a series of recent papers [15,17]. This alternative paradigm views semiclassical gravity as based on a generic Lagrangian of the form $L = Q_a{}^{bcd} R^a{}_{bcd}$ with $\nabla_b Q_a{}^{bcd} = 0$. The expansion of $Q_a{}^{bcd}$ in terms of the derivatives of the metric tensor determines the structure of the theory uniquely. The zeroth order term gives the Einstein-Hilbert action, and the first order correction is given by the Gauss-Bonnet action. More importantly, *any* such Lagrangian can be decomposed into surface and bulk terms as $\sqrt{-g}L = \sqrt{-g}L_{\rm bulk} + L_{\rm sur}$, where

$$L_{\rm bulk} = 2Q_a{}^{bcd}\Gamma^a_{dk}\Gamma^k_{bc}, \qquad L_{\rm sur} = \partial_c[\sqrt{-g}V^c], \qquad V^c = 2Q_a{}^{bcd}\Gamma^a_{bd}. \qquad (38)$$

Obviously, both $L_{\rm sur}$ and $L_{\rm bulk}$ contain the same information in terms of $Q_a{}^{bcd}$ and hence can *always* be related to each other [15,18]. It is easy to verify, for example [19], that

$$L = \frac{1}{2} R^a_{bcd}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right); \qquad L_{\rm bulk} = \sqrt{-g}\left(\frac{\partial V^c}{\partial \Gamma^a_{bd}}\right)\Gamma^a_{dk}\Gamma^k_{bc}. \qquad (39)$$

Thus the knowledge of the functional form of $L_{\rm sur}$ or—equivalently—V^c allows us to determine $L_{\rm bulk}$ and even L. (The first relation also shows that $(\partial V^c/\partial \Gamma^a_{bd})$ is generally covariant in spite of the appearance.) These relations make the actions based on $L = Q_a{}^{bcd} R^a{}_{bcd}$ with $\nabla_b Q_a{}^{bcd} = 0$ intrinsically "holographic," with the surface term containing equivalent information to the bulk. What is more, one can show that the surface term leads to the Wald entropy in spacetimes with horizon [4,18]. Since Lanczos-Lovelock Lagrangians have this structure, it is quite understandable that the semiclassical equations of motion have a thermodynamic interpretation.

We can summarize the broader picture as follows: Any geometrical description of gravity that obeys the principle of equivalence and is based on a nontrivial metric will allow for the propagation of light rays to be affected by gravity. This, in turn, leads to regions of spacetime which are causally inaccessible to classes of observers. (These two features are reasonably independent of the precise field equations which determine the metric.) The inaccessibility of regions of spacetime leads to association of entropy with spacetime horizons. Such a point of view suggests that there will exist a thermodynamic route to the description of gravitational dynamics in *any* metric theory which satisfies the principle of equivalence. So, the thermodynamic interpretation of gravity, encoded in the identity $TdS = PdV + dE$, should be fairly generic and the semi-classical corrections to gravity—arising from the correct microscopic theory—should preserve the form of this identity (with only the expressions for S and E getting quantum corrections). We have shown that this is indeed the case for spherically symmetric horizons in the Lanczos-Lovelock Lagrangian. Such an interpretation offers a new outlook towards the dynamics of gravity and might provide valuable clues regarding the nature of quantum gravity.

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APPENDIX

In this appendix we shall prove Eqs. (32) and (35). In order to prove Eq. (32), it is sufficient to show that

$$\begin{aligned}\Delta_{(k)} &\equiv (\delta^{T_1T_2\ldots T_k}_{B_1B_2\ldots B_k})\delta^{B_1}_{T_1}\delta^{B_2}_{T_2}\ldots\delta^{B_k}_{T_k}\\ &= (D-(k+1))(\delta^{T_2T_3\ldots T_k}_{B_2B_3\ldots B_k})\delta^{B_2}_{T_2}\delta^{B_3}_{T_3}\ldots\delta^{B_k}_{T_k}\\ &= (D-(k+1))\Delta_{(k-1)},\end{aligned} \tag{A1}$$

with $k = 2m - 1$, and the indices T_1, B_1, etc. ranging over the $(D - 2)$ values $2, 3 \ldots (D - 1)$. Having proved this, a simple rearrangement of terms in Eq. (31) leads to Eq. (32). We will prove this result for a general k since it will come in handy when proving Eq. (35). To simplify notation, we introduce the following symbol for a single product of Kronecker deltas,

$$\mathcal{D}\begin{bmatrix} T_1T_2\ldots T_k\\ B_1B_2\ldots B_k\end{bmatrix} \equiv \delta^{T_1}_{B_1}\delta^{T_2}_{B_2}\ldots\delta^{T_k}_{B_k}. \tag{A2}$$

The alternating tensor is normalized to take values 0, 1, and -1. In practice, this can be done by antisymmetrizing only the upper row of indices, and we can write

$$\delta^{T_1T_2\ldots T_k}_{B_1B_2\ldots B_k} = \sum_{\pi\in S_{(k)}}\mathrm{sgn}(\pi)\mathcal{D}\begin{bmatrix}\pi(T_1)\pi(T_2)\ldots\pi(T_k)\\ B_1B_2\ldots B_k\end{bmatrix}, \tag{A3}$$

where $S_{(k)}$ is the set of permutations of k objects and $\mathrm{sgn}(\pi)$ denotes the signature of the permutation π. Our goal is to perform the summations over the indices T_1 and B_1 in the quantity $\Delta_{(k)}$ defined in (A1). To simplify this computation, we can split up the set $S_{(k)}$ as the union of sets $S^j_{(k)}$ with $1 \le j \le k$, where $S^j_{(k)}$ is the set of permutations π which map T_j to T_1, i.e.,

$$S^j_{(k)} = \{\pi \in S_{(k)} \mid \pi(T_j) = T_1\}. \tag{A4}$$

Noting that for $\pi \in S^1_{(k)}$, $\pi(T_1) = T_1$, we can write

$$\begin{aligned}\Delta_{(k)} &= \delta^{T_1}_{B_1}(\delta^{T_2\ldots T_k}_{B_2\ldots B_k})\mathcal{D}\begin{bmatrix}B_1B_2\ldots B_k\\ T_1T_2\ldots T_k\end{bmatrix}\\ &\quad+\sum_{j=2}^{k}\sum_{\pi\in S^j_{(k)}}\mathrm{sgn}(\pi)\mathcal{D}\begin{pmatrix}\pi(T_1)\pi(T_2)\ldots\pi(T_k)\\ B_1B_2\ldots B_k\end{pmatrix}\\ &\quad\times\mathcal{D}\begin{bmatrix}B_1B_2\ldots B_k\\ T_1T_2\ldots T_k\end{bmatrix}\\ &= (D-2)\Delta_{(k-1)} + \sum_{j=2}^{k}\mathcal{M}^j_{(k)},\end{aligned} \tag{A5}$$

where the last line defines the quantities $\mathcal{M}^j_{(k)}$ for $2 \le j \le k$. For a particular value of j, we get

$$\begin{aligned}\mathcal{M}^j_{(k)} &= \sum_{\pi\in S^j_{(k)}}\mathrm{sgn}(\pi)\mathcal{D}\begin{bmatrix}\pi(T_1)\ldots T_1\pi(T_{j+1})\ldots\pi(T_k)\\ B_1\ldots B_jB_{j+1}\ldots B_k\end{bmatrix}\\ &\quad\times\mathcal{D}\begin{bmatrix}B_1B_2\ldots B_k\\ T_1T_2\ldots T_k\end{bmatrix},\end{aligned} \tag{A6}$$

where we have set $\pi(T_j) = T_1$. We now have a simple product of Kronecker deltas for each $\pi \in S^j_{(k)}$, with T_1 and B_1 appearing explicitly. Performing the summations over T_1 and B_1 reduces this to

$$\begin{aligned}\mathcal{M}^j_{(k)} &= \sum_{\pi\in S^j_{(k)}}\mathrm{sgn}(\pi)\mathcal{D}\begin{bmatrix}\pi(T_1)\ldots\pi(T_{j-1})\pi(T_{j+1})\ldots\pi(T_k)\\ B_j\ldots B_{j-1}B_{j+1}\ldots B_k\end{bmatrix}\\ &\quad\times\mathcal{D}\begin{bmatrix}B_2\ldots B_k\\ T_2\ldots T_k\end{bmatrix}.\end{aligned} \tag{A7}$$

From the definition of $S^j_{(k)}$, for each $\pi \in S^j_{(k)}$, the ordered set $\mathcal{P}_\pi = \{\pi(T_1), \pi(T_2), \ldots, \pi(T_{j-1}), \pi(T_{j+1}), \ldots, \pi(T_k)\}$ is simply a rearrangement of the ordered set $\mathcal{P} = \{T_2, T_3, \ldots, T_j, \ldots, T_k\}$. Hence there exists a one-to-one mapping between $S^j_{(k)}$ and the set $S_{(k-1)}$ of permutations of $(k - 1)$ objects. We would like to replace the summation $\sum_{\pi\in S^j_{(k)}}$ by the summation $\sum_{\tilde{\pi}\in S_{(k-1)}}$. To ensure that each term in the summation retains its correct signature after this replacement, we must introduce an overall factor of $\mathrm{sgn}(C_j)$, which is the signature of the permutation $C_j \in S^j_{(k)}$ that is mapped to the identity of $S_{(k-1)}$. It is easy to see that this permutation is the semicyclic rearrangement given by $\{T_1, T_2, \ldots, T_j, \ldots, T_k\} \to \{T_2, T_3, \ldots, T_j, T_1, T_{j+1}, \ldots, T_k\}$, which has signature $(-1)^{j+1}$. We can write

$$\mathcal{M}^{j}_{(k)} = (-1)^{j+1} \times \sum_{\tilde{\pi}\in S_{(k-1)}} \mathrm{sgn}(\tilde{\pi}) \times \mathcal{D}\begin{bmatrix} \tilde{\pi}(T_2)\ldots\tilde{\pi}(T_j)\tilde{\pi}(T_{j+1})\ldots\tilde{\pi}(T_k) \\ B_j\ldots B_{j-1}B_{j+1}\ldots B_k \end{bmatrix} \times \mathcal{D}\begin{bmatrix} B_2\ldots B_k \\ T_2\ldots T_k \end{bmatrix}. \tag{A8}$$

The order of the first j indices in the lower row of the first factor of $\mathcal{D}$ in (A8) is not in the standard form. To get B_2 below $\tilde{\pi}(T_2)$ and so on, we simply perform the cyclic permutation $\{B_j, B_2, \ldots, B_{j-1}\} \to \{B_2, B_3, \ldots, B_{j-1}, B_j\}$, with the other indices left untouched. This can be done since permutations of the upper indices in the alternating tensor are equivalent to those of the lower indices [20]. The permutation introduces a factor of $(-1)^j$, which combines with the $(-1)^{j+1}$ in (A8) to give an overall factor of (-1). We now find that

$$\mathcal{M}^{j}_{(k)} = -\sum_{\tilde{\pi}\in S_{(k-1)}} \mathrm{sgn}(\tilde{\pi})\mathcal{D}\begin{bmatrix} \tilde{\pi}(T_2)\ldots\tilde{\pi}(T_k) \\ B_2\ldots B_k \end{bmatrix}\mathcal{D}\begin{bmatrix} B_2\ldots B_k \\ T_2\ldots T_k \end{bmatrix} = -\Delta_{(k-1)} \tag{A9}$$

independent of j. Since there are $(k-1)$ such terms, (A5) gives us the required result, namely,

$$\Delta_{(k)} = (D-(k+1))\Delta_{(k-1)}. \tag{A10}$$

Setting $k = 2m - 1$ completes the proof of Eq. (32). The result in (A10) also allows us to prove Eq. (35) in the following way. Using arguments similar to those presented below Eq. (30) and evaluating all quantities on the horizon, the left-hand side of Eq. (35) for a single value of m can be expanded to give

$$\begin{aligned} c_m A_{D-2} a^{D-2} \mathcal{L}^{(D-2)}_m &= \frac{c_m A_{D-2}}{16\pi}\frac{a^{D-2}}{2^m}(\delta^{A_1A_2\ldots A_{2m}}_{B_1B_2\ldots B_{2m}})^{(D-2)} R^{B_1B_2}_{A_1A_2}\ldots \\ &= \frac{c_m}{16\pi}A_{D-2}a^{D-(2m+2)}(\delta^{A_1A_2\ldots A_{2m}}_{B_1B_2\ldots B_{2m}})\mathcal{D}\begin{bmatrix} B_1B_2\ldots B_{2m} \\ A_1A_2\ldots A_{2m} \end{bmatrix} \\ &= \frac{c_m}{16\pi}A_{D-2}a^{D-(2m+2)}\Delta_{(2m)} \\ &= \frac{c_m}{16\pi}A_{D-2}a^{D-(2m+2)}(D-(2m+1))\Delta_{(2m-1)} \\ &= \frac{c_m}{16\pi}A_{D-2}a^{D-(2m+2)}(D-(2m+1))\prod_{j=2}^{2m}(D-j) = \frac{dE_{(m)}}{da}, \end{aligned} \tag{A11}$$

where we have recursively used (A10) to obtain the fourth equality, and used Eq. (36) to write the last equality. This completes the proof of Eq. (35), and consequently of the result in Eq. (37).

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