Thermodynamics of charged rotating black branes in Brans-Dicke theory with quadratic scalar field potential

M. H. Dehghani,[1,2,*] J. Pakravan,[1] and S. H. Hendi[1]

[1]*Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran*
[2]*Research Institute for Astrophysics and Astronomy of Maragha (RIAAM), Maragha, Iran*
(Received 4 September 2006; published 10 November 2006)

We construct a class of charged rotating solutions in $(n + 1)$-dimensional Maxwell-Brans-Dicke theory with flat horizon in the presence of a quadratic potential and investigate their properties. These solutions are neither asymptotically flat nor (anti)-de Sitter. We find that these solutions can present black brane, with inner and outer event horizons, an extreme black brane or a naked singularity provided the parameters of the solutions are chosen suitably. We compute the finite Euclidean action through the use of counterterm method, and obtain the conserved and thermodynamic quantities by using the relation between the action and free energy in grand-canonical ensemble. We find that these quantities satisfy the first law of thermodynamics, and the entropy does not follow the area law.

DOI: 10.1103/PhysRevD.74.104014 PACS numbers: 04.50.+h, 04.20.Jb, 04.70.Bw, 04.70.Dy

I. INTRODUCTION

There has been much more interest in alternative theories of gravity in recent years. This is due to the fact that at the present epoch the Universe expands with acceleration instead of deceleration along the scheme of standard Friedmann models [1]. One of the alternative theory of gravity is the scalar-tensor gravity pioneered by Jordan, Brans and Dicke (JBD) [2]. In recent years this theory has attracted a great deal of attention, in particular, in the strong field domains, as it arises naturally as the low energy limit of many theories of quantum gravity such as the supersymmetric string theory or the Kaluza-Klein theory. It has been shown that the JBD theory seems to be better than the Einstein gravity for solving the graceful exit problem in the inflation model [3]. This is because the scalar field in the BD theory provided a natural termination of the inflationary era via bubble nucleation without the need for finely tuned cosmological parameters. This theory contains an adjustable parameter ω that represents the strength of coupling between scalar field and the matter.

Because of highly nonlinear character of JBD theory, a desirable prerequisite for studying strong field situation is to have knowledge of exact explicit solutions of the field equations. Four forms of static spherically symmetric vacuum solution of the BD theory in four dimensions are available in the literature which are constructed by Brans himself [4]. However, it has been shown that among these four classes of the static spherically symmetric solutions of the vacuum Brans-Dicke theory of gravity only two are really independent [5], and only one of them is permitted for all values of ω. Although this class of solutions, in general, gives rise to naked singularity, for some particular choices of the solution's parameters it represents a black hole different from Schwarzschild one [6]. The other class of solutions is valid only for $\omega < -3/2$ which implies nonpositive contribution of matter to effective gravitational constant and thus a violation of weak energy condition [4]. Static charged solutions of Brans-Dicke-Maxwell gravity have been investigated in [7], and the nontrivial Kerr-Newman type black hole solutions different from general relativistic solutions have been constructed in JBD for $-5/2 < \omega < -3/2$ [8]. Black hole solutions with minimally and conformally coupled self-interacting potential have been found in three [9] and four [10] dimensions in the presence of cosmological constant. Constructing new exact solutions of JBD theory from the known solution has been also considered in [11]. Till now, charged rotating black hole solutions for an arbitrary value of ω has not been constructed. In this paper, we want to construct exact charged rotating black hole solutions in Brans-Dicke theory for an arbitrary value of ω and investigate their properties.

The outline of our paper is as follows. In Sec. II, we give a brief review of the field equations of Brans-Dicke theory in Jordan (or string) and Einstein frames. In Sec. III, we obtain charge rotating solution in $(n + 1)$-dimensions with k rotation parameters. In Sec. IV, we obtain the finite action, and compute the conserved and thermodynamic quantities of the $(n + 1)$-dimensional black brane solutions with a complete set of rotational parameters. We also show that these quantities satisfy the first law of thermodynamics. We finish our paper with some concluding remarks.

II. FIELD EQUATION AND CONFORMAL TRANSFORMATION

Long-range forces are known to be transmitted by the tensor gravitational field $g_{\mu\nu}$ and the vector electromagnetic field A_μ. It is natural then to suspect that other long-range forces may be produced by scalar fields. Such theories have been suggested since before relativity. The simplest theory in which a scalar field shares the stage with gravitation is that of Brans-Dicke theory. In n dimensions,

*Electronic address: mhd@shirazu.ac.ir

the action of the Brans-Dicke-Maxwell theory with one scalar field Φ and a self-interacting potential $V(\Phi)$ can be written as

$$I_G = -\frac{1}{16\pi}\int_{\mathcal{M}} d^{n+1}x\sqrt{-g}\Big(\Phi\mathcal{R} - \frac{\omega}{\Phi}(\nabla\Phi)^2 - V(\Phi) - F_{\mu\nu}F^{\mu\nu}\Big), \tag{1}$$

where $\mathcal{R}$ is the Ricci scalar, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field, A_μ is the vector potential, ω is the coupling constant, Φ denotes the BD scalar field and $V(\Phi)$ is a self-interacting potential for Φ. Varying the action (1) with respect to the metric, scalar and vector fields give the field equations as

$$G_{\mu\nu} = \frac{\omega}{\Phi^2}\Big(\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}(\nabla\Phi)^2\Big) - \frac{V(\Phi)}{2\Phi}g_{\mu\nu} + \frac{1}{\Phi}(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi) + \frac{2}{\Phi}\Big(F_{\mu\lambda}F_\nu^\lambda - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}\Big), \tag{2}$$

$$\nabla^2\Phi = -\frac{n-3}{2[(n-1)\omega+n]}F^2 + \frac{1}{2[(n-1)\omega+n)]}\times\Big[(n-1)\Phi\frac{dV(\Phi)}{d\Phi} - (n+1)V(\Phi)\Big], \tag{3}$$

$$\nabla_\mu F^{\mu\nu} = 0, \tag{4}$$

where $G_{\mu\nu}$ and ∇_μ are the Einstein tensor and covariant differentiation corresponding to the metric $g_{\mu\nu}$ respectively. Solving the field Eqs. (2)–(4) directly is a nontrivial task because the right hand side of (2) includes the second derivatives of the scalar. We can remove this difficulty by the conformal transformation

$$\bar{g}_{\mu\nu} = \Phi^{2/(n-1)}g_{\mu\nu}, \qquad \bar{\Phi} = \frac{n-3}{4\alpha}\ln\Phi, \tag{5}$$

where

$$\alpha = (n-3)/\sqrt{4(n-1)\omega + 4n} \tag{6}$$

One may note that α goes to zero as ω goes to infinity and the BD theory reduces to Einstein theory. By this transformation, the action (1) transforms to

$$\bar{I}_G = -\frac{1}{16\pi}\int_{\mathcal{M}} d^{n+1}x\sqrt{-\bar{g}}\Big\{\bar{\mathcal{R}} - \frac{4}{n-1}(\bar{\nabla}\bar{\Phi})^2 - \bar{V}(\bar{\Phi}) - \exp\Big(-\frac{4\alpha\bar{\Phi}}{(n-1)}\Big)\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}\Big\}, \tag{7}$$

where $\bar{\mathcal{R}}$ and $\bar{\nabla}$ are the Ricci scalar and covariant differentiation corresponding to the metric $\bar{g}_{\mu\nu}$, and $\bar{V}(\bar{\Phi})$ is

$$\bar{V}(\bar{\Phi}) = \Phi^{-(n+1)/(n-1)}V(\Phi)$$

Varying the action (7) with respect to $\bar{g}_{\mu\nu}$, $\bar{\Phi}$ and $\bar{F}_{\mu\nu}$, we obtain equations of motion as

$$\bar{\mathcal{R}}_{\mu\nu} = \frac{4}{n-1}\Big(\bar{\nabla}_\mu\bar{\Phi}\bar{\nabla}_\nu\bar{\Phi} + \frac{1}{4}\bar{V}\bar{g}_{\mu\nu}\Big) + 2e^{-4\alpha\bar{\Phi}/(n-1)}\times\Big(\bar{F}_{\mu\lambda}\bar{F}_\nu^\lambda - \frac{1}{2(n-1)}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}\bar{g}_{\mu\nu}\Big), \tag{8}$$

$$\bar{\nabla}^2\bar{\Phi} = \frac{n-1}{8}\frac{\partial\bar{V}}{\partial\bar{\Phi}} - \frac{\alpha}{2}e^{-4\alpha\bar{\Phi}/(n-1)}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}, \tag{9}$$

$$\partial_\mu[\sqrt{-\bar{g}}e^{-4\alpha\bar{\Phi}/(n-1)}\bar{F}^{\mu\nu}] = 0 \tag{10}$$

Therefore, if $(\bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \bar{\Phi})$ is the solution of Eqs. (8)–(10) with potential $\bar{V}(\bar{\Phi})$, then

$$[g_{\mu\nu}, F_{\mu\nu}, \Phi] = \Big[\exp\Big(-\frac{8\alpha\bar{\Phi}}{(n-1)(n-3)}\Big)\times\bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \exp\Big(\frac{4\alpha\bar{\Phi}}{n-3}\Big)\Big] \tag{11}$$

is the solution of Eqs. (2)–(4) with potential $V(\Phi)$.

III. CHARGED ROTATING SOLUTIONS IN $n+1$ DIMENSIONS WITH k ROTATION PARAMETERS

Here we construct the $(n+1)$-dimensional solutions of BD theory with $n \geq 4$ and the quadratic potential

$$V(\Phi) = 2\Lambda\Phi^2$$

Applying the conformal transformation (5), the potential $\bar{V}(\bar{\Phi})$ becomes

$$\bar{V}(\bar{\Phi}) = 2\Lambda\exp\Big(\frac{4\alpha\bar{\Phi}}{n-1}\Big), \tag{12}$$

which is a Liouville-type potential. Thus, the problem of solving Eqs. (2)–(4) with quadratic potential reduces to the problem of solving Eqs. (8)–(10) with Liouville-type potential.

The rotation group in $n+1$ dimensions is $SO(n)$ and therefore the number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is $[n/2] \equiv k$, where $[n/2]$ is the integer part of $n/2$. The solutions of the field Eqs. (8)–(10) with k rotation parameter a_i, and Liouville-type potential is [12]

$$
\begin{aligned}
d\bar{s}^2 &= -f(r)\left(\Xi dt - \sum_{i=1}^{k} a_i d\varphi_i\right)^2 \\
&\quad + \frac{r^2}{l^4} R^2(r) \sum_{i=1}^{k} (a_i dt - \Xi l^2 d\varphi_i)^2 \\
&\quad - \frac{r^2}{l^2} R^2(r) \sum_{i=1}^{k} (a_i d\varphi_j - a_j d\varphi_i)^2 + \frac{dr^2}{f(r)} \\
&\quad + \frac{r^2}{l^2} R^2(r) dX^2, \\
\Xi^2 &= 1 + \sum_{i=1}^{k} \frac{a_i^2}{l^2}, \\
\bar{F}_{tr} &= \frac{q\Xi}{(rR)^{n-1}} \exp\left(\frac{4\alpha\bar{\Phi}}{n-1}\right) \\
\bar{F}_{\varphi r} &= -\frac{a_i}{\Xi} \bar{F}_{tr}.
\end{aligned} \tag{13}
$$

where dX^2 is the Euclidean metric on $(n-k-1)$-dimensional submanifold with volume ω_{n-k-1}. Here $f(r)$, $R(r)$ and $\bar{\Phi}(r)$ are

$$
\begin{aligned}
f(r) &= \frac{2\Lambda(\alpha^2+1)^2 c^{2\gamma}}{(n-1)(\alpha^2-n)} r^{2(1-\gamma)} - \frac{m}{r^{(n-2)}} r^{(n-1)\gamma} \\
&\quad + \frac{2q^2(\alpha^2+1)^2 c^{-2(n-2)\gamma}}{(n-1)(\alpha^2+n-2) r^{2(n-2)(1-\gamma)}},
\end{aligned} \tag{14}
$$

$$
R(r) = \exp\left(\frac{2\alpha\bar{\Phi}}{n-1}\right) = \left(\frac{c}{r}\right)^{\gamma}, \tag{15}
$$

$$
\bar{\Phi}(r) = \frac{(n-1)\alpha}{2(1+\alpha^2)} \ln\left(\frac{c}{r}\right), \tag{16}
$$

where c is an arbitrary constant and $\gamma = \alpha^2/(\alpha^2+1)$. Using the conformal transformation (11), the $(n+1)$-dimensional rotating solutions of BD theory with k rotation parameters can be obtained as

$$
\begin{aligned}
ds^2 &= -U(r)\left(\Xi dt - \sum_{i=1}^{k} a_i d\varphi_i\right)^2 \\
&\quad + \frac{r^2}{l^4} H^2(r) \sum_{i=1}^{k} (a_i dt - \Xi l^2 d\varphi_i)^2 \\
&\quad - \frac{r^2}{l^2} H^2(r) \sum_{i=1}^{k} (a_i d\varphi_j - a_j d\varphi_i)^2 + \frac{dr^2}{V(r)} \\
&\quad + \frac{r^2}{l^2} H^2(r) dX^2,
\end{aligned} \tag{17}
$$

where $U(r)$, $V(r)$, $H(r)$ and $\Phi(r)$ are

$$
\begin{aligned}
U(r) &= \frac{2\Lambda(\alpha^2+1)^2 c^{2\gamma(n-5)/(n-3)}}{(n-1)(\alpha^2-n)} r^{2[1-\gamma(n-5))/(n-3)]} \\
&\quad - \frac{mc^{-4\gamma/(n-3)}}{r^{(n-2)}} r^{\gamma[n-1+4/(n-3)]} \\
&\quad + \frac{2q^2(\alpha^2+1)^2 c^{-2\gamma[n-2+2/(n-3)]}}{(n-1)(\alpha^2+n-2) r^{2[(n-2)(1-\gamma)-2\gamma/(n-3)]}},
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
V(r) &= \frac{2\Lambda(\alpha^2+1)^2 c^{2\gamma(n-2)/(n-3)}}{(n-1)(\alpha^2-n)} r^{2[1-\gamma(n-1)/(n-3)]} \\
&\quad - \frac{mc^{4\gamma/(n-3)}}{r^{(n-2)}} r^{\gamma[n-1-4/(n-3)]} \\
&\quad + \frac{2q^2(\alpha^2+1)^2 c^{-2\gamma[n-2-2/(n-3)]}}{(n-1)(\alpha^2+n-2) r^{2[(n-2)(1-\gamma)+2\gamma/(n-3)]}},
\end{aligned} \tag{19}
$$

$$
H(r) = \left(\frac{c}{r}\right)^{\gamma(n-5)/(n-3)}, \tag{20}
$$

$$
\Phi(r) = \left(\frac{c}{r}\right)^{2\gamma(n-1)/(n-3)}. \tag{21}
$$

The electromagnetic field becomes:

$$
F_{tr} = \frac{qc^{(3-n)\gamma}}{r^{(n-3)(1-\gamma)+2}} \qquad F_{\varphi r} = -\frac{a_i}{\Xi} F_{tr}. \tag{22}
$$

It is worth to note that the scalar field $\Phi(r)$ and electromagnetic field $F_{\mu\nu}$ become zero as r goes to infinity. These solutions reduce to the charged rotating solutions of Einstein gravity as ω goes to infinity (α vanishes) [13,14]. It is also notable to mention that these solutions are valid for all values of ω.

Properties of the solutions

One can show that the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $r=0$, and therefore there is a curvature singularity located at $r=0$. Seeking possible black hole solutions, we turn to look for the existence of horizons. As in the case of rotating black hole solutions of the Einstein gravity, the above metric given by (17)–(21) has both Killing and event horizons. The Killing horizon is a null surface whose null generators are tangent to a Killing field. It is easy to see that the Killing vector

$$
\chi = \partial_t + \sum_{i=1}^{k} \Omega_i \partial_{\phi_i}, \tag{23}
$$

is the null generator of the event horizon, where k denotes the number of rotation parameters. Setting $a_i \to ia_i$ yields the Euclidean section of (17), whose regularity at $r=r_+$ requires that we should identify $\phi_i \sim \phi_i + \beta_+\Omega_i$, where β_+ and Ω_i's are the inverse Hawking temperature and the angular velocities of the outer event horizon. One obtains:

$$\Omega_i = \frac{a_i}{\Xi l^2}. \tag{24}$$

The temperature may be obtained through the use of definition of surface gravity,

$$T_+ = \frac{1}{2\pi}\sqrt{-\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)}, \tag{25}$$

where χ is the Killing vector (23). One obtains

$$\begin{aligned} T_+ &= \frac{f'(r_+)}{4\pi\Xi} \\ &= \frac{1}{4\pi\Xi}\left(\frac{(n-\alpha^2)m}{\alpha^2+1} r_+^{(n-1)(\gamma-1)} \right. \\ &\quad \left. - \frac{4q^2(\alpha^2+1)c^{-2(n-2)\gamma}}{(\alpha^2+n-2)r_+^\gamma} r_+^{(2n-3)(\gamma-1)}\right) \\ &= -\frac{2(1+\alpha^2)}{4\pi\Xi(n-1)}\left(\Lambda c^{2\gamma} r_+^{1-2\gamma} \right. \\ &\quad \left. + \frac{q^2 c^{-2(n-2)\gamma}}{r_+^\gamma} r_+^{(2n-3)(\gamma-1)}\right), \end{aligned} \tag{26}$$

which shows that the temperature of the solution is invariant under the conformal transformation (5). This is due to the fact that the conformal parameter is regular at the horizon.

As one can see from Eq. (18), the solution is ill-defined for $\alpha^2 = n$ with a quadratic potential ($\Lambda \neq 0$). The cases with $\alpha^2 > n$ and $\alpha^2 < n$ should be considered separately. In the first case where $\alpha^2 > n$, as r goes to infinity the dominant term in Eq. (18) is the second term, and therefore the spacetime has a cosmological horizon for positive values of the mass parameter, despite the sign of the cosmological constant Λ. In the second case where $\alpha^2 < n$, as r goes to infinity the dominant term is the first term, and therefore there exist a cosmological horizon for $\Lambda > 0$, while there is no cosmological horizons if $\Lambda < 0$. Indeed, in the latter case ($\alpha^2 < n$ and $\Lambda < 0$) the spacetimes associated with the solution (18)–(21) exhibit a variety of possible causal structures depending on the values of the metric parameters α, m, q, and Λ. One can obtain the causal structure by finding the roots of $V(r) = 0$. Unfortunately, because of the nature of the exponents in (19), it is not possible to find explicitly the location of horizons for an arbitrary value of α (ω). But, we can obtain some information by considering the temperature of the horizons.

Equation (26) shows that the temperature is negative for the two cases of (i) $\alpha > \sqrt{n}$ despite the sign of Λ, and (ii) positive Λ despite the value of α. As we argued above in these two cases we encounter with cosmological horizons, and therefore the cosmological horizons have negative temperature. Numerical calculations show that the temperature of the event horizon goes to zero as the black brane approaches the extreme case. Thus, one can see from Eq. (26) that there exists an extreme black brane only for negative Λ and $\alpha < \sqrt{n}$, if

$$r_{\rm ext}^{(3-n)\gamma+n-2} = \frac{4q^2(1+\alpha^2)^2 c^{-2(n-2)\gamma}}{m_{\rm ext}(n-\alpha^2)(\alpha^2+n-2)} \tag{27}$$

where $m_{\rm ext}$ is the extremal mass parameter of black brane. If one substitutes this $r_{\rm ext}$ into the equation $f(r_{\rm ext}) = 0$, then one obtains the condition for extreme black brane as:

$$\begin{aligned} m_{\rm ext} &= \frac{4q^2(1+\alpha^2)^2 c^{-2\gamma(n-2)}}{(n-\alpha^2)(\alpha^2+n-2)} \\ &\quad \times \left(\frac{-\Lambda c^{2\gamma(n-1)}}{q_{\rm ext}^2}\right)^{[(3-n)\gamma+n-2]/[2(\gamma-1)(1-n)]} \end{aligned} \tag{28}$$

Indeed, the metric of Eqs. (17)–(21) has two inner and outer horizons located at r_- and r_+, provided the mass parameter m is greater than $m_{\rm ext}$, an extreme black brane in the case of $m = m_{\rm ext}$, and a naked singularity if $m < m_{\rm ext}$. Note that in the absence of scalar field ($\alpha = \gamma = 0$) $m_{\rm ext}$ reduces to that obtained in [14].

Before going to the calculations of other thermodynamic and conserved quantities, we draw the Penrose diagram to show that the casual structure is asymptotically well behaved. For reason of economy, we draw the Penrose diagram only for the solution that presents a black brane with inner and outer horizons (negative Λ and $\alpha < \sqrt{n}$). The causal structure can be constructed following the general prescriptions indicated in [15]. The Penrose diagram is shown in Figs. 1 and 2 for $\alpha < 1$ and $1 \leq \alpha < \sqrt{n}$ respectively. Also it is worth to write down the asymptotic behavior of the Ricci scalar. Indeed, the form of the Ricci scalar for large values of r is:

$$\begin{aligned} R &= -\frac{n^2}{(n-3)^2 l^2}\frac{(2\alpha^2+n-3)[4\alpha^2+(n+1)(n-3)]}{n-\alpha^2} \\ &\quad \times \left(\frac{c}{r}\right)^{2\gamma(n-1)/(n-3)} \end{aligned} \tag{29}$$

which does not approach a nonzero constant as in the case of asymptotically AdS spacetimes. It is worth to mention that the Ricci scalar of the solution (17)–(20) goes to zero as r goes to infinity, but with a slower rate than that of an asymptotically flat spacetimes in the absence of the scalar field.

Next, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces is

$$u^0 = \frac{1}{N}, \qquad u^r = 0, \qquad u^i = -\frac{V^i}{N},$$

and the electric field is $E^\mu = g^{\mu\rho}F_{\rho\nu}u^\nu$, where N and V^i are the lapse and shift function. Denoting the volume of the

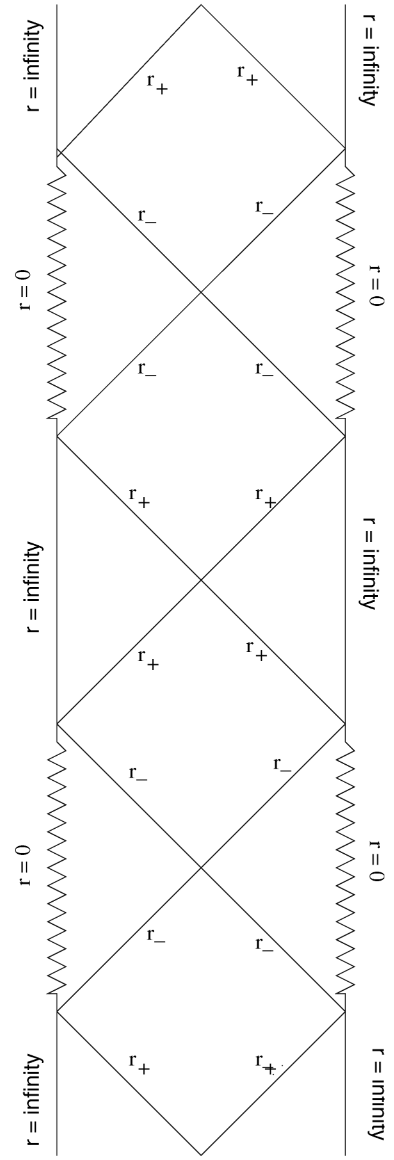

FIG. 1. Penrose diagram for negative Λ and $\alpha < 1$.

hypersurface boundary at constant t and r by $V_{n-1} = (2\pi)^k \omega_{n-k-1}$, the electric charge per unit volume V_{n-1} can be found by calculating the flux of the electric field at infinity, yielding

$$Q = \frac{\Xi q}{4\pi l^{n-2}} \tag{30}$$

Comparing the above charge with the charge of black brane solutions of Einstein-Maxwell-dilaton gravity obtained in [12], one finds that charge is invariant under the conformal transformation (5). The electric potential U, measured at infinity with respect to the horizon, is defined by [16]

$$U = A_\mu \chi^\mu|_{r\to\infty} - A_\mu \chi^\mu|_{r=r_+}, \tag{31}$$

where χ is the null generators of the event horizon. One can easily show that the vector potential A_μ corresponding

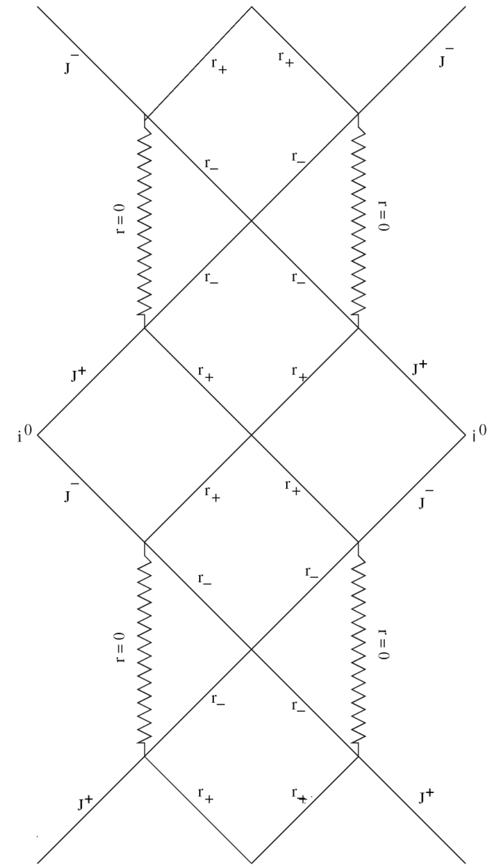

FIG. 2. Penrose diagram for negative Λ and $1 \leq \alpha < \sqrt{n}$.

to electromagnetic tensor (22) can be written as

$$A_\mu = \frac{qc^{(3-n)\gamma}}{\Gamma r^\Gamma}(\Xi \delta_\mu^t - a_i \delta_\mu^i) \qquad \text{(no sum on } i\text{)}, \tag{32}$$

where $\Gamma = \gamma(3-n) + n - 2$. Therefore the electric potential is

$$U = \frac{qc^{(3-n)\gamma}}{\Xi \Gamma r_+^\Gamma} \tag{33}$$

IV. ACTION AND CONSERVED QUANTITIES

The action (1) does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivative of $\delta g_{\mu\nu}$ normal to the boundary. These normal derivative terms do not vanish by themselves, but are canceled by the variation of the surface term

$$I_b = -\frac{1}{8\pi}\int_{\partial \mathcal{M}} d^n x \sqrt{-\gamma} K \Phi \tag{34}$$

where γ and K are the determinant of the induced metric and the trace of extrinsic curvature of boundary. In general the action $I_G + I_b$, is divergent when evaluated on the solutions, as is the Hamiltonian and other associated conserved quantities. Rather than eliminating these divergences by incorporating reference term, a counterterm $I_{\rm ct}$ may be added to the action which is functional only of the boundary curvature invariants. For asymptotically (A)dS solutions of Einstein gravity, the way that one deals with these divergences is through the use of counterterm method inspired by (A)dS/CFT correspondence [17]. However, in the presence of a nontrivial BD scalar field with potential $V(\Phi) = 2\Lambda\Phi^2$, the spacetime may not behave as either dS ($\Lambda > 0$) or AdS ($\Lambda < 0$). In fact, it has been shown that with the exception of a pure cosmological constant potential, where $\alpha = 0$, no AdS or dS static spherically symmetric solution exist for Liouville-type potential [18]. But, as in the case of asymptotically AdS spacetimes, according to the domain-wall/QFT (quantum field theory) correspondence [19], there may be a suitable counterterm for the action which removes the divergences. In this paper, we deal with the spacetimes with zero curvature boundary, and therefore all the counterterm containing the curvature invariants of the boundary are zero. Thus, the counterterm reduces to a volume term as

$$I_{\rm ct} = -\frac{(n-1)}{8\pi l_{\rm eff}}\int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma}, \tag{35}$$

where $l_{\rm eff}$ is given by

$$l_{\rm eff}^2 = \frac{(n-1)(\alpha^2-n)}{2\Lambda\Phi^3} \tag{36}$$

As α goes to zero, the effective $l_{\rm eff}^2$ of Eq. (36) reduces to $l^2 = -n(n-1)/2\Lambda$ of the (A)dS spacetimes. One may note that the counterterm has the same form as in the case of asymptotically AdS solutions with zero curvature boundary, where l is replaced by $l_{\rm eff}$. The total action, I, can be written as

$$I = I_G + I_b + I_{\rm ct}. \tag{37}$$

The Euclidean actions per unit volume V_{n-1} can then be obtained through the use of Eqs. (1), (34), and (35) as

$$I = \beta\frac{(\alpha^4-1)}{16\pi l^n}\left(-\frac{nc^{(n+1)\gamma}}{\alpha^2-n}r_+^{n-(n+1)\gamma} + \frac{2q^2l^2c^{(3-n)\gamma}}{(\alpha^2+n-2)(n-1)}r_+^{(n-3)\gamma-n+2}\right) \tag{38}$$

It is a matter of calculation to obtain the action as a function of the intensive quantities β, Ω and U by using the expression for the temperature, the angular velocity and the potential given in Eqs. (24), (26), and (33) as

$$I = \frac{\beta(\alpha^4-1)c^{(n-3)\gamma}}{16\pi l^n}\left(-\frac{nc^{4\gamma}r_+^{n-(n+1)\gamma}}{\alpha^2-n} - \frac{2U^2(\alpha^2+n-2)l^2r_+^{(n-2)(1-\gamma)+\gamma}}{(n-1)(\Omega^2l^2-1)(\alpha^2+1)^2}\right), \tag{39}$$

where r_+ is

$$r_+ = \left[-\frac{(\pi(n-1)+\sqrt{[\pi^2(n-1)^2-\Lambda U^2\beta^2(\alpha^2+n-2)^2]})}{\beta\Lambda(1+\alpha^2)\sqrt{1-\Omega^2l^2}}c^{-2\gamma}\right]^{1/(1-2\gamma)} \tag{40}$$

Since the Euclidean action is related to the free energy in the grand-canonical ensemble, the electric charge Q, the angular momentum J_i, the entropy S and the mass M can be found using the familiar thermodynamics relations:

$$\begin{aligned}
Q &= -\beta^{-1}\frac{\partial I}{\partial U} = \frac{\Xi q}{4\pi l^{n-2}},\\
J_i &= -\beta^{-1}\frac{\partial I}{\partial \Omega_i} = \frac{c^{(n-1)\gamma}}{16\pi l^{n-2}}\left(\frac{n-\alpha^2}{1+\alpha^2}\right)\Xi m a_i,\\
S &= \left(\beta\frac{\partial}{\partial\beta}-1\right)I = \frac{\Xi c^{(n-1)\gamma}}{4l^{(n-2)}}r_+^{(n-1)(1-\gamma)},\\
M &= \left(\frac{\partial}{\partial\beta} - \beta^{-1}U\frac{\partial}{\partial U} - \beta^{-1}\sum\Omega_i\frac{\partial}{\partial\Omega_i}\right)\\
I &= \frac{c^{(n-1)\gamma}}{16\pi l^{n-2}}\left(\frac{(n-\alpha^2)\Xi^2+\alpha^2-1}{1+\alpha^2}\right)m
\end{aligned}$$

For $a_i = 0$ ($\Xi = 1$), the angular momentum per unit length vanishes, and therefore a_i is the ith rotational parameter of the spacetime. One may note that the charge Q calculated above coincides with Eq. (31) It is worth to note that the area law is no longer valid in Brans-Dicke theory [9,20]. Nevertheless, the entropy remains unchanged under conformal transformations. Comparing the conserved and thermodynamic quantities calculated in this section with those obtained in Ref. [12], one finds that they are invariant under the conformal transformation (11). Straightforward calculations show that these quantities calculated satisfy the first law of thermodynamics,

$$dM = TdS + \sum_{i=1}^{k}\Omega_i dJ_i + UdQ \tag{41}$$

V. CLOSING REMARKS

Till now, no charged rotating black hole solutions has been constructed for an arbitrary value of coupling constant ω. In this paper, we presented a class of exact charged rotating black brane solutions in Brans-Dicke theory with a quadratic scalar field potential for an arbitrary value of ω and investigated their properties. We found that these solutions are neither asymptotically flat nor (A)dS. These solutions which exist only for $\alpha^2 \neq n$ have a cosmological horizon for (i) $\alpha^2 > n$ despite the sign of Λ, and (*ii*) positive values of Λ, despite the magnitude of α. For $\alpha^2 < n$, the solutions present black branes with outer and inner horizons if $m > m_{\rm ext}$, an extreme black hole if $m = m_{\rm ext}$, and a naked singularity if $m < m_{\rm ext}$. The Hawking temperature is negative for inner and cosmological horizons, and it is positive for outer horizons. We computed the finite action through the use of counterterm method and obtained the thermodynamic and conserved quantities of the solutions by using the relation between the action and free energy. We found that the entropy does not follow the area law. We also found that the conserved and thermodynamic quantities are invariant under the conformal transformation (11) and satisfy the first law of thermodynamics.

ACKNOWLEDGMENTS

This work has been supported by Research Institute for Astronomy and Astrophysics of Maragha, Iran

[1] A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998); S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); J. L. Tonry *et al.*, *ibid.* **594**, 1 (2003); A. T. Lee *et al.*, *ibid.* **561**, L1 (2001); C. B. Netterfield *et al.*, *ibid.* **571**, 604 (2002);N. W. Halverson *et al.*, *ibid.* **568**, 38 (2002); D. N. Spergel *et al.*, Astrophys. J. Suppl. Ser. **148**, 175 (2003).

[2] P. Jordan, Z. Phys. **157**, 112 (1959); C. H. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[3] D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989); P. J. Steinhardt and F. S. Accetta, *ibid.* **64**, 2740 (1990); O. Bertolami and P. J. Martins, Phys. Rev. D **61**, 064007 (2000); N. Banerjee and D. Pavon *ibid.* **63**, 043504 (2001); A. A. Sen, S. Sen, and S. Sethi *ibid.* **63**, 107501 (2001); Li-e Qiang, Y. Ma, M. Han, and D. Yu, *ibid.* **71**, 061501(R) (2005).

[4] C. H. Brans, Phys. Rev. **125**, 2194 (1962).

[5] A. Bhadra and K. Sarkar, Gen. Relativ. Gravit. **37**, 2189 (2005).

[6] M. Campanelli and C. O. Lousto, Int. J. Mod. Phys. D **2**, 451 (1993).

[7] R. G. Cai and Y. S. Myung, Phys. Rev. D **56**, 3466 (1997).

[8] H. Kim, Phys. Rev. D **60**, 024001 (1999).

[9] C. Martinez and J. Zanelli, Phys. Rev. D **54**, 3830 (1996); M. Henneaux, C. Martinez, R. Troncoso, and J. Zanelli, *ibid.* **65**, 104007 (2002).

[10] C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **67**, 024008 (2003); **70**, 084035 (2004); C. Martinez and R. Troncoso, *ibid.* **74**, 064007 (2006).

[11] M. H. Dehghani and M. Shojania, Can. J. Phys. **80**, 951 (2002).

[12] A. Sheykhi, M. H. Dehghani, N. Riazi, and J. Pakravan, Phys. Rev. D **74**, 084016 (2006).

[13] J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D **54**, 3840 (1996).

[14] M. H. Dehghani, Phys. Rev. D **66**, 044006 (2002); M. H. Dehghani and A. Khodam-Mohammadi, *ibid.* **67**, 084006 (2003).

[15] M. Walker, J. Math. Phys. (N.Y.) **11**, 2280 (1970).

[16] M. Cvetic and S. S. Gubser, J. High Energy Phys. 04 (1999) 024; M. M. Caldarelli, G. Cognola, and D. Klemm, Class. Quant. Grav. **17**, 399 (2000).

[17] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); E. Witten, *ibid.* **2**, 253 (1998); O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000); V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).

[18] S. J. Poletti and D. L. Wiltshire, Phys. Rev. D **50**, 7260 (1994).

[19] H. J. Boonstra, K. Skenderis, and P. K. Townsend, J. High Energy Phys. 01 (1999) 003; K. Behrndt, E. Bergshoeff, R. Hallbersma, and J. P. Van der Scharr, Class. Quant. Grav. **16**, 3517 (1999); R. G. Cai and N. Ohta, Phys. Rev. D **62**, 024006 (2000).

[20] M. Visser, Phys. Rev. D **48**, 5697 (1993); F. Englert, L. Houart, and P. Windey, Phys. Lett. B **372**, 29 (1996); G. Kang, Phys. Rev. D **54**, 7483 (1996); A. Ashtekar, A. Corichi, and D. Sudarsky, Class. Quant. Grav. **20**, 3413 (2003); C. Martinez, R. Troncoso, and J. P. Staforelli, Phys. Rev. D **74**, 044028 (2006).