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Angular momentum conservation for dynamical black holes

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Angular momentum can be defined by rearranging the Komar surface integral in terms of a twist form, encoding the twisting around of space-time due to a rotating mass, and an axial vector. If the axial vector is a coordinate vector and has vanishing transverse divergence, it can be uniquely specified under certain generic conditions. Along a trapping horizon, a conservation law expresses the rate of change of angular momentum of a general black hole in terms of angular momentum densities of matter and gravitational radiation. This identifies the transverse-normal block of an effective gravitational-radiation energy tensor, whose normal-normal block was recently identified in a corresponding energy conservation law. Angular momentum and energy are dual, respectively, to the axial vector and a previously identified vector, the conservation equations taking the same form. Including charge conservation, the three conserved quantities yield definitions of an effective energy, electric potential, angular velocity and surface gravity, satisfying a dynamical version of the so-called first law of black-hole mechanics. A corresponding zeroth law holds for null trapping horizons, resolving an ambiguity in taking the null limit.

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I. INTRODUCTION

The theory of black holes appears finally to be reaching a stage of maturity in which it can be applied in the most interesting, distorted, dynamic situations, with appropriate definitions of relevant physical quantities. This article mainly concerns angular momentum and its conservation, which is the last major piece of what seems to be an essentially complete new paradigm for black holes. It therefore seems timely to review below the key ideas and results of what might be called the heroic, classical and modern eras.

The first solution which would nowadays be called a black hole was discovered by Schwarzschild [1] within a few weeks of the final formulation of the field equations of general relativity by Einstein [2], as the external gravitational field of a point with mass M. Charge Q was soon added by Reissner [3] and Nordström [4], but even the Schwarzschild solution was not properly understood for decades. Schwarzschild described the mass point as located at what is now understood as the horizon, despite its nonzero area A. Einstein & Rosen [5] realized that the spatial geometry had a wormhole structure, extending through a minimal surface. Oppenheimer & Snyder [6] constructed a model of stellar collapse in which the star collapses through the horizon. Finally Kruskal [7], as reported in a paper actually written by Wheeler [8], described the entire space-time, whence it became clear that there was a trapped region inside the horizon, from which nothing could escape to the exterior. Angular momentum J was added by Kerr [9] and, including Q, by Newman *et al.* [10]. Uniqueness theorems identify these as the only black holes which are stationary, asymptotically flat, electrovac solutions.

Wheeler [11] is credited with coining the term "black hole" and Penrose [12] with defining event horizons, which became the accepted definition of black holes. Hawking [13] showed that the area of an event horizon was nondecreasing, $A' \geq 0$. The result became known as a "second law", due to inaccurate analogies with the laws of thermodynamics and the results summarized by Bardeen *et al.* [14]: a "zeroth law" that surface gravity κ is constant on a stationary black hole; a "first law"

$$\delta E = \kappa \delta A / 8\pi + \Omega \delta J + \Phi \delta Q \tag{1}$$

for perturbations of stationary black holes, where Ω is the angular speed and Φ the electric potential of the horizon, and the ADM energy E measures the total mass of the space-time; and a "third law" that $\kappa \not\to 0$ by positive-energy perturbations of stationary black holes. This summarizes the classical theory of black holes as described in textbooks [15–17].

The last results above are perhaps best described as black-hole statics, being properties of stationary black holes, specifically of Killing horizons, rather than of general event horizons. While adequate in some astrophysical situations, this classical theory is inapplicable to general dynamical processes, for instance black-hole formation, rapid evolution and binary mergers. A theory of black-hole dynamics is needed, with generalizations of all the above-mentioned quantities. Event horizons are not an appropriate platform, since they cannot be located by mortals, let alone admit physical measurements. A more practical way to locate a black hole is by a *marginal surface*, an extremal surface of a null hypersurface, where light rays are momentarily caught by the gravitational field. Marginal surfaces are used extensively in numerical simulations, where they have historically been called apparent horizons, though the textbook definition of the latter is different [15–17]. Here a hypersurface foliated by marginal surfaces will be called a *trapping horizon*.

A systematic treatment of trapping horizons [18] distinguished four subclasses, called future or past, outer or inner trapping horizons, with the future outer type proposed as the practical location of a black hole. Such a horizon was shown to have several expected properties of a black hole, assuming the Einstein equation and positive-energy conditions: the horizon is achronal, being null in a special case of quasistationarity, but otherwise being spatial; the area form $*1$ of the marginal surfaces is constant in the null case and increasing in the spatial case; and the marginal surfaces have spherical topology, if compact. The area law implies that the area $A = \oint_S *1$ of the marginal surfaces S is nondecreasing,

$$L_\xi A \geq 0, \tag{2}$$

where L denotes the Lie derivative and ξ the generating vector of the marginal surfaces. So a black hole grows if something falls into it, otherwise staying the same size.

Comprehensive treatments were subsequently given for spherical symmetry [19–21], cylindrical symmetry [22] and a quasispherical approximation [23–26]. In each case, definitions were found for all the nonzero physical quantities mentioned above, providing prototypes of all except J and Ω. In addition, an effective energy tensor Θ for gravitational radiation was found, entering equations additively to the matter energy tensor T. The Einstein equations were decomposed into forms with manifest physical meaning, such as a quasi-Newtonian gravitational law, a wave equation for the gravitational radiation, and an energy conservation law which can be written in the form

$$L_\xi M = \oint_S *(T_{\alpha\beta} + \Theta_{\alpha\beta})k^\alpha \tau^\beta, \tag{3}$$

where τ in the normal dual of ξ and k is a certain vector, playing the role of a Killing vector, which is null on the horizon. Such equations actually hold not just on a trapping horizon, but anywhere in the space-time, energy conservation reducing at null infinity to the Bondi energy equation.

Contemporaneously, Ashtekar *et al.* and others [27–35] developed a theory of null trapping horizons with various additional conditions, under the names nonrotating isolated horizons, nonexpanding horizons, weakly isolated horizons, rigidly rotating horizons and (strongly) isolated horizons. Each is intended to capture the idea that the black hole is quasistationary in some sense. They gave definitions of all the relevant physical quantities and derived a generalized version of the so-called first law.

Subsequently, Ashtekar & Krishnan [36–38] studied future spatial trapping horizons under the name dynamical horizons, giving classes of definitions of energy and angular momentum, deriving corresponding flux equations and obtaining a version of the so-called first law for $Q = 0$. However, the "3 + 1" formalism used to describe spatial trapping horizons breaks down when the horizon becomes null, so that the isolated-horizon and dynamical-horizon frameworks were essentially distinct. Some connections were drawn, particularly for slowly evolving horizons by Booth & Fairhurst [39–41]. Recently, Andersson *et al.* [42] showed that a stable trapping horizon is, on any one marginal surface, either spatial or null everywhere, so that transitions between the two types happen simultaneously on a marginal surface. They and Ashtekar & Galloway [43] also obtained some existence and uniqueness results for trapping horizons.

A unified framework for any trapping horizon is provided by a dual-null formalism [44,45], which was used throughout the earlier studies [18–26]. The energy flux equation was then cast in a surface-integral form where the null limit could be taken [46,47]. Moreover, it was cast in the form of a conservation law (3), by identifying an effective energy tensor Θ. The mass M, which might take any value on a given S by choice of scaling of k, was chosen to be the irreducible mass or Hawking mass [48] for consistency with the earlier studies. This corresponds to the simplest general definition of k, such that it becomes a unit vector for round spheres near infinity.

The main task here is to make similar refinements for angular momentum, as briefly described in shorter articles [49,50]. In particular, one desires not just a flux equation but a conservation law of the form

$$L_\xi J = -\oint_S *(T_{\alpha\beta} + \Theta_{\alpha\beta})\psi^\alpha \tau^\beta. \tag{4}$$

Here ψ should be an axial vector in some sense, playing the role of an axial Killing vector, with J being the angular momentum about that axis. It turns out that natural restrictions on ψ allow it to be uniquely specified under certain generic conditions. The angular momentum, initially a functional $J[\psi]$, is obtained directly from the Komar integral [51] in terms of a 1-form ω known as the twist [45], which reduces to the 1-form used for dynamical horizons [36–38]. It encodes the rotational frame-dragging predicted in the Lense-Thirring effect, thereby giving a precise meaning to the twisting around of space-time due to a rotating mass.

The null limit is more subtle for angular momentum than for energy, where the irreducible mass is uniquely defined for a null trapping horizon, energy conservation (3) reducing correctly to $L_\xi M = 0$. The dual-null foliation becomes nonunique for a null hypersurface, with ω becoming nonunique. This is reflected in the fact that different 1-forms were used for isolated horizons [27–35] and dynamical horizons [36–38]. Neither 1-form is necessarily preserved along a null trapping horizon, but a certain linear combination is so preserved. However, all three 1-forms coincide if the nonuniqueness of the dual-null foliation for a null hypersurface is fixed in a certain way. Then $J[\psi]$ becomes unique for a null trapping horizon and conservation (4) reduces as desired to $L_\xi J = 0$. Thus a consistent treatment of angular momentum naturally resolves the issue of the degeneracy of the null limit. It follows that a black hole cannot change its angular momentum without increasing its area.

The article is organized as follows. Section II summarizes the underlying geometry. Section III reviews trapping horizons and conservation of energy. Section IV derives angular momentum from the Komar integral and shows how restrictions on the axial vector can be used to construct a unique definition. Section V derives and discusses the conservation law. Section VI includes charge conservation and discusses local versus quasilocal conservation. Section VII describes the state space, defining the remaining physical quantities and deriving a dynamical version of the so-called first law. Section VIII considers null trapping horizons, deriving a zeroth law. Appendices concern (A) weak fields, (B) normal fundamental forms and (C) a Kerr example. The above discussion serves as a summary.

II. GEOMETRY

General relativity will be assumed, with space-time metric g. The geometrical object of interest is a one-parameter family $\{S\}$ of spatial surfaces S, locally generating a foliated hypersurface H. Labelling the surfaces by a coordinate x, they are generated by a vector $\xi = \partial/\partial x$, which can be taken to be normal to the surfaces, $\perp\xi = 0$, where $\perp$ denotes projection onto S. A Hodge duality operation on normal vectors η, $\perp\eta = 0$, yields a dual normal vector η^* satisfying

$$\perp\eta^* = 0, \qquad g(\eta^*, \eta) = 0, \qquad g(\eta^*, \eta^*) = -g(\eta, \eta). \tag{5}$$

In particular,

$$\tau = \xi^* \tag{6}$$

is normal to H, with the same scaling (Fig. 1). The coordinate freedom here is just $x \mapsto \tilde{x}(x)$ and choice of transverse coordinates on S, under which all the key formulas will be invariant. The generating vector ξ may have any causal character, at each point. For instance, a future outer trapping horizon is spatial while growing, becomes null when quasistationary, and would become temporal if shrinking during evaporation [18,46,47].

A dual-null formalism [44,45] describes two families of null hypersurfaces $\Sigma_\pm$, intersecting in a two-parameter family of spatial surfaces, including the desired one-parameter family. Some merits of the dual-null approach, apart from comparative ease of calculation, are that it is adapted both to marginal surfaces, defined as extremal surfaces of null hypersurfaces, and to radiation propagation, which makes it easier to identify terms arising due to gravitational radiation [46,47,50]. Relevant aspects of the formalism are summarized as follows.

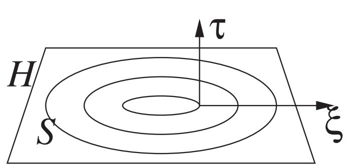

FIG. 1. A non-null hypersurface H foliated by spatial surfaces S, with generating vector ξ and its normal dual $\tau = \xi^*$. If H becomes null, ξ and τ coincide.

Labelling $\Sigma_\mp$ by coordinates $x^\pm$ which increase to the future, one may take transverse coordinates x^a on S, which for a sphere would normally be angular coordinates $x^a = (\vartheta, \varphi)$. Writing space-time coordinates $x^\alpha = (x^+, x^-, x^a)$ indicates how one may use Greek letters $(\alpha, \beta, \ldots)$ for space-time indices and corresponding Latin letters $(a, b, \ldots)$ for transverse indices. The coordinate basis vectors are $\partial_\alpha = \partial/\partial x^\alpha$ and the dual 1-forms are dx^α, satisfying $\partial_\beta(dx^\alpha) = \delta^\alpha_\beta$. Coordinate vectors commute, $[\partial_\alpha, \partial_\beta] = 0$, where the brackets denote the Lie bracket or commutator. Two coordinate vectors have a special role, the evolution vectors $\partial_\pm = \partial/\partial x^\pm$ which generate the dynamics, spanning an integrable evolution space. The corresponding normal 1-forms $dx^\pm$ are null by assumption:

$$g^{-1}(dx^\pm, dx^\pm) = 0. \tag{7}$$

The relative normalization of the null normals may be encoded in a function f defined by

$$e^f = -g^{-1}(dx^+, dx^-) \tag{8}$$

where the metric sign convention is that spatial metrics are positive definite. The transverse metric, or the induced metric on S, is found to be

$$h = g + 2e^{-f} dx^+ \otimes dx^-, \tag{9}$$

where $\otimes$ denotes the symmetric tensor product. There are two shift vectors

$$s_\pm = \perp\partial_\pm, \tag{10}$$

where $\perp$ is generalized to indicate projection by h. The null normal vectors

$$l_\pm = \partial_\pm - s_\pm = -e^{-f} g^{-1}(dx^\mp) \tag{11}$$

are future-null and satisfy

$$g(l_\pm, l_\pm) = 0, \qquad g(l_+, l_-) = -e^{-f}, \qquad l_\pm(dx^\pm) = 1,$$
$$l_\pm(dx^\mp) = 0, \qquad \perp l_\pm = 0. \tag{12}$$

The metric takes the form

$$g = h_{ab}(dx^a + s^a_+ dx^+ + s^a_- dx^-) \otimes (dx^b + s^b_+ dx^+ + s^b_- dx^-) - 2e^{-f} dx^+ \otimes dx^-. \tag{13}$$

Then $(h, f, s_\pm)$ are configuration fields and the independent momentum fields are found to be linear combinations of the following transverse tensors:

$$\theta_\pm = *L_\pm * 1 \tag{14}$$

$$\sigma_\pm = \perp L_\pm h - \theta_\pm h \quad (15)$$

$$\nu_\pm = L_\pm f \quad (16)$$

$$\omega = \tfrac{1}{2} e^f h([l_-, l_+]), \quad (17)$$

where $*$ is the Hodge operator of h and $L_\pm$ is shorthand for the Lie derivative along $l_\pm$. Then the functions $\theta_\pm$ are the null expansions, the traceless bilinear forms $\sigma_\pm$ are the null shears, the 1-form ω is the twist, measuring the lack of integrability of the normal space, and the functions $\nu_\pm$ are the inaffinities, measuring the failure of the null normals to be affine. The fields $(\theta_\pm, \sigma_\pm, \nu_\pm, \omega)$ encode the extrinsic curvature of the dual-null foliation. These extrinsic fields are unique up to interchange $\pm \mapsto \mp$ and diffeomorphisms $x^\pm \mapsto \tilde{x}^\pm(x^\pm)$ which relabel the null hypersurfaces. Further description of the geometry was given recently [47].

As described, the dual-null formalism is manifestly covariant on S, with transverse indices not explicitly denoted, while $\pm$ indices indicate the chosen normal basis [45,47]. Conversely, one can use a formalism which is manifestly covariant on the normal space, with transverse but not normal indices explicitly denoted [52]. Both types of formalism can seem obscure to the uninitiated, so indices will be explicitly denoted in longer formulas in this article, nevertheless being omitted where the meaning should be clear. Capital Latin letters $(A, B, \ldots)$ will be used for normal indices, when not denoted by $\pm$ in the dual-null basis. Then the configuration fields are (h_{ab}, f, s_A^b), the momentum fields are $(\theta_A, \sigma_{Abc}, \nu_A, \omega_a)$ and the derivative operators are $(\perp L_A, D_a)$, where D is the covariant derivative operator of h.

Since the normal space is not integrable unless $\omega = 0$, it generally does not admit a coordinate basis. However, one may still take $dx^\pm$ as basis 1-forms, in which case the dual basis vectors are $l_\pm$, as follows from (12), implying $l_A(dx^B) = \delta_A^B$. In this basis, the normal metric

$$\gamma = g - h \quad (18)$$

has components which follow from (9) as

$$\gamma_{AB} = -e^{-f}(dx_A^+ dx_B^- + dx_A^- dx_B^+), \quad (19)$$

and its inverse has components

$$\gamma^{AB} = -e^f(l_+^A l_-^B + l_-^A l_+^B), \quad (20)$$

which can be used to lower and raise normal indices. Also useful is the binormal

$$\epsilon_{AB} = e^{-f}(dx_A^+ dx_B^- - dx_A^- dx_B^+) \quad (21)$$

or its inverse

$$\epsilon^{AB} = e^f(l_-^A l_+^B - l_+^A l_-^B). \quad (22)$$

The mixed form

$$\epsilon_B^A = l_+^A dx_B^+ - l_-^A dx_B^- \quad (23)$$

has components $\epsilon_\pm^\pm = \pm 1$, $\epsilon_\mp^\pm = 0$, so can be used to express the duality operation (5) on normal vectors, extended to the dual-null foliation, as

$$(\eta^*)^A = \epsilon_B^A \eta^B. \quad (24)$$

The dual-null Hamilton equations and integrability conditions for vacuum Einstein gravity were derived previously [45], with matter terms added subsequently [25]. The components of the field equations which are relevant to angular momentum turn out to be the twisting equations

$$\perp L_\pm \omega_a = -\theta_\pm \omega_a \pm \tfrac{1}{2} D_a \nu_\pm \mp \tfrac{1}{2} D_a \theta_\pm \mp \tfrac{1}{2} \theta_\pm D_a f \pm \tfrac{1}{2} h^{cd} D_d \sigma_{\pm ac} \mp 8\pi T_{a\pm}, \quad (25)$$

where $T_{a\pm} = h_a^\gamma T_{\gamma\beta} l_\pm^\beta$ is the transverse-normal projection of the energy tensor T, and units are such that Newton's gravitational constant is unity. The corresponding all-index version can be written using the binormal as

$$\perp L_B \omega_a = -\theta_B \omega_a + \tfrac{1}{2} \epsilon_B^E (D_a \nu_E - D_a \theta_E - \theta_E D_a f + h^{cd} D_d \sigma_{Eac} - 16\pi T_{aE}). \quad (26)$$

III. TRAPPING HORIZONS AND CONSERVATION OF ENERGY

Returning to a general foliated hypersurface H, a normal vector η has components $\eta^\pm$ along $l_\pm$, so that $\eta = \eta^+ l_+ + \eta^- l_-$, and its normal dual is $\eta^* = \eta^+ l_+ - \eta^- l_-$. In particular, the generating vector is

$$\xi = \xi^+ l_+ + \xi^- l_- \quad (27)$$

and its dual is

$$\tau = \xi^+ l_+ - \xi^- l_-. \quad (28)$$

Since the horizon is given parametrically by functions $x^\pm(x)$, the components $\xi^\pm = \partial x^\pm / \partial x$ are independent of transverse coordinates:

$$D\xi^\pm = 0. \quad (29)$$

It is also useful to introduce the expansion

$$\theta_\eta = L_\eta \log(*1) = \theta_A \eta^A \quad (30)$$

along a normal vector η, particularly the expansion θ_ξ along the generating vector.

A trapping horizon [18,20,46,47] is a hypersurface H foliated by marginal surfaces, where S is marginal if one of the null expansions, θ_+ or θ_-, vanishes everywhere on S. Then S is an extremal surface of the null hypersurface Σ_+ or Σ_-.

The recently derived energy conservation law [46,47] will be stated here for later comparison, modifying some notation. Assuming compact S henceforth, the transverse surfaces have area

$$A = \oint_S *1, \tag{31}$$

and the area radius

$$R = \sqrt{A/4\pi} \tag{32}$$

is often more convenient. The Hawking mass [48]

$$M = \frac{R}{2}\left(1 - \frac{1}{16\pi}\oint_S *\gamma^{AB}\theta_A\theta_B\right) \tag{33}$$

can be used as a measure of the active gravitational mass on a transverse surface. Assuming the null energy condition, M is the irreducible mass $R/2$ of a future outer trapping horizon, $L_\xi M \geq 0$, by the area law (2). On a stationary black-hole horizon, M reduces to the usual definition of irreducible mass for a Kerr-Newman black hole, namely, the mass which must remain even if rotational or electrical energy is extracted. It is generally not the ADM energy, but an effective energy E is defined in Sec. VII which does recover the ADM energy in this case. Equality on a trapping horizon will be denoted by $\cong$, e.g. $R \cong 2M$.

Mass or energy has a certain duality with time, e.g. there is a standard formula for energy if a stationary Killing vector exists. For a general compact surface, the simplest definition of such a vector which applies correctly for a Schwarzschild black hole is [46,47]

$$k = (g^{-1}(dR))^* \tag{34}$$

or $k^A = \epsilon^{AB}L_B R$. This vector actually was found to be the appropriate dual of M, in the sense of conservation laws for trapping horizons [46,47] and for uniformly expanding flows [52,53]. In either case, the energy conservation law can be written as

$$L_\xi M \cong \oint_S *(T_{AB} + \Theta_{AB})k^A\tau^B, \tag{35}$$

where Θ is an effective energy tensor for gravitational radiation. This determines only the normal-normal components of Θ, as

$$\Theta_{\pm\pm} = \|\sigma_\pm\|^2/32\pi \tag{36}$$

$$\Theta_{\pm\mp} = e^{-f}\left|\omega \mp \frac{1}{2}Df\right|^2/8\pi, \tag{37}$$

where $|\zeta|^2 = h^{ab}\zeta_a\zeta_b$ and $\|\sigma\|^2 = h^{ac}h^{bd}\sigma_{ab}\sigma_{cd}$. Further discussion is referred to [46,47,52,53].

IV. ANGULAR MOMENTUM

The standard definition of angular momentum for an axial Killing vector ψ and at spatial infinity is the Komar integral [51]

$$J[\psi] = -\frac{1}{16\pi}\oint_S *\epsilon_{\alpha\beta}\nabla^\alpha\psi^\beta. \tag{38}$$

Now consider ψ to be a general transverse vector, $\perp\psi = \psi$

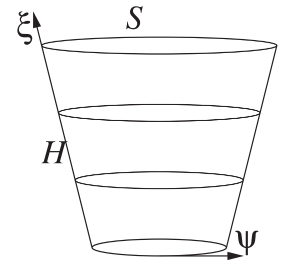

FIG. 2. A transverse vector ψ.

(Fig. 2). Since $\epsilon_{\alpha\beta}\psi^\beta = 0$, the Komar integral can be rewritten via (22) as

$$J[\psi] = \frac{1}{8\pi}\oint_S *\psi^a\omega_a, \tag{39}$$

where ω is the twist (17). Since the twist encodes the nonintegrability of the normal space, it provides a geometrical measure of rotational frame-dragging. It is an invariant of a dual-null foliation and therefore of a non-null foliated hypersurface H, so the twist expression for $J[\psi]$ is also an invariant. Appendix A shows that $J[\psi]$ recovers the standard definition of angular momentum for a weak-field metric [16], with the twist being directly related to the precessional angular velocity of a gyroscope due to the Lense-Thirring effect. Thus the twist does indeed encode the twisting around of space-time caused by a rotating mass.

There are several definitions of angular momentum which are similar surface integrals of an axial vector contracted with a 1-form [30,36,54], the situation being clarified by Gourgoulhon [55] and in Appendix B. Ashtekar & Krishnan [36] gave a definition for a dynamical horizon which involves a 1-form coinciding with ω. Brown & York [54] gave a definition which was stated only for an axial Killing vector ψ, involving a 1-form which is generally inequivalent to ω, but can be made to coincide if adapted to a trapping horizon. Ashtekar *et al.* [30,31] gave a definition for a type II (rigidly rotating) isolated horizon, using a 1-form which is generally inequivalent to ω. However, it can be made to coincide with ω if the dual-null foliation is fixed in a natural way, as described in the penultimate section.

The above properties suggest (39) as a general quasilocal definition of angular momentum. However, if the transverse vector ψ does not have properties expected of an axial vector, the physical interpretation as angular momentum is questionable. For instance, it would be natural to expect an axial vector to have integral curves which form a smooth foliation of circles, apart from two poles, assuming spherical topology for S. In the following, two conditions on ψ with various motivations will be considered, which, taken together, determine ψ uniquely in a certain generic situation. These conditions then yield a conservation law with the desired form (4), as described in the next section.

Ashtekar & Krishnan [37] proposed that ψ has vanishing transverse divergence:

$$D_a\psi^a \cong 0. \tag{40}$$

This condition holds if ψ is an axial Killing vector, and can be understood as a weaker condition, equivalent to ψ generating a symmetry of the area form rather than of the whole metric, since $L_\psi(*1) = *D_a\psi^a$. Alternatively, assuming that the integral curves of ψ are closed, it can always be satisfied by choice of scaling of ψ, as discussed by Booth & Fairhurst [40]. The original motivation was that the different 1-forms used for dynamical and isolated horizons, denoted here by ω and $\omega + \frac{1}{2}Df$, will then give the same angular momentum $J[\psi]$, by the Gauss divergence theorem.

Spherical topology will be assumed henceforth for S, which follows from the topology law [18] for outer trapping horizons, assuming the dominant energy condition. If there exist angular coordinates (ϑ, φ) on S with $\psi = \partial/\partial\varphi$, completing coordinates (x, ϑ, φ) on H, then since coordinate vectors commute,

$$L_\xi\psi \cong 0. \tag{41}$$

This condition was proposed as a natural way to propagate ψ along H by Gourgoulhon [55]. Now there is a commutator identity [45]

$$L_\xi(D_a\psi^a) - D_a(L_\xi\psi)^a = \psi^a D_a\theta_\xi. \tag{42}$$

Therefore assuming both conditions (40) and (41) forces

$$\psi^a D_a\theta_\xi \cong 0. \tag{43}$$

This is automatically satisfied if $D\theta_\xi \cong 0$, as in spherical symmetry or along a null trapping horizon. However, generically one expects $D\theta_\xi \not\cong 0$ almost everywhere. It must vanish somewhere on a sphere, by the hairy ball theorem, but the simplest generic situation is that there are curves γ of constant θ_ξ which form a smooth foliation of circles, covering the surface except for two poles (Fig. 3). Assuming so, since ψ is tangent to γ, one can find a unique ψ, up to sign, in terms of the unit tangent vector $\hat\psi$ and arc length ds along γ:

$$\psi \cong \hat\psi \oint_\gamma ds/2\pi, \tag{44}$$

where the scaling ensures that the axial coordinate φ is identified at 0 and 2π. Then the angular momentum becomes unique up to sign, $J[\psi] = J$, the sign being naturally fixed by $J \geq 0$ and continuity of ψ, corresponding to a choice of orientation.

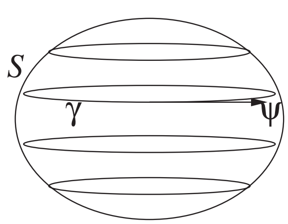

FIG. 3. Curves γ of constant expansion θ_ξ.

For an axisymmetric space-time with axial Killing vector ψ, (40) holds, while (41) holds if ξ respects the symmetry, $L_\psi\xi = 0$, so the above construction, if unique as assumed, yields the correct axial vector. In particular, the construction does work for a Kerr space-time, as described in Appendix C.

To summarize this section, the definition (39) of angular momentum can be made generically unique if the axial vector is a coordinate vector, (41), and generates a symmetry of the area form, (40). The construction can be applied in any situation where $D\theta_\xi \not\cong 0$ almost everywhere, though the physical interpretation as angular momentum seems to be safest in the case of two poles, which locate the axis of rotation. Then J is proposed to measure the angular momentum about that axis.

V. CONSERVATION OF ANGULAR MOMENTUM

The main result of this paper is that

$$L_\xi J \cong -\oint_S *(T_{aB} + \Theta_{aB})\psi^a\tau^B \tag{45}$$

holds along a trapping horizon under the conditions (40) and (41), where

$$\Theta_{aB} = -\frac{1}{16\pi}h^{cd}D_d\sigma_{Bac} \tag{46}$$

is thereby determined to be the transverse-normal block of the effective energy tensor for gravitational radiation. It can be shown by differentiating the angular momentum (39) using (41) to give

$$L_\xi J \cong \frac{1}{8\pi}\oint_S *(\theta_\xi\psi^a\omega_a + \psi^a L_\xi\omega_a), \tag{47}$$

then expanding ξ by (27) and using the twisting Eqs. (25) to express $L_\xi\omega$. The term in $\theta_\xi = \xi^+\theta_+ + \xi^-\theta_-$ cancels with the first term from (25), while the D gradients may all be removed as total divergences due to (29) and (40) and the fact that (43) reduces to $\psi^a D_a\theta_\mp \cong 0$ on a trapping horizon with $\theta_\pm \cong 0$. This leaves just

$$L_\xi J \cong \frac{1}{8\pi}\oint_S *\psi^a\bigg(\xi^+\bigg(\frac{1}{2}h^{cd}D_d\sigma_{+ac} - 8\pi T_{a+}\bigg) - \xi^-\bigg(\frac{1}{2}h^{cd}D_d\sigma_{-ac} - 8\pi T_{a-}\bigg)\bigg), \tag{48}$$

which is an expanded form of (45), noting (28) and thereby identifying (46).

Apart from the inclusion of Θ, the conservation law (45) is the standard surface-integral form of conservation of angular momentum, were ψ an axial Killing vector. It thereby describes the increase or decrease of angular momentum of a black hole due to infall of corotating or counter-rotating matter, respectively. The corresponding volume-integral form for a spatial horizon H, expressing the change $[J]_{\partial H}$ in J between two marginal surfaces, follows as

$$[J]_{\partial H} \cong -\int_H \hat{*}(T_{aB} + \Theta_{aB})\psi^a \hat{\tau}^B, \tag{49}$$

where $\hat{\tau} = \tau/\sqrt{g(\xi, \xi)}$ is the unit normal vector and $\hat{*}1 = *\sqrt{g(\xi, \xi)} \wedge dx$ is the proper volume element. Although more familiar, as for the energy conservation law [46,47], this form becomes degenerate as the horizon becomes null, since $\hat{*}1 \to 0$ while $\hat{\tau}$ ceases to exist. Since this is a physically important limit, where a black hole is not growing, the surface-integral form (45) is preferred.

The null shears $\sigma_{\pm bc}$ have previously been identified in the energy conservation law (35) as encoding the ingoing and outgoing transverse gravitational radiation, via the energy densities (36), which agree with expressions in other limits, such as the Bondi energy density at null infinity and the Isaacson energy density of high-frequency linearized gravitational waves [46,47]. So the expression (46) implies that gravitational radiation with a transversely differential waveform will generally possess angular momentum density. One can see corresponding terms in the linearized approximation [16], but they are set to zero by the transverse-traceless gauge conditions, which are "transverse" in a different sense to that used here. In any case, the conservation law shows that a black hole can spin up or spin down even in vacuum, at a rate related to ingoing and outgoing gravitational radiation.

The identification of the transverse-normal block (46) of Θ appears to be new. Previous versions of angular momentum flux laws for dynamical black holes [36–41,55] contain different terms, which are not in energy-tensor form, i.e. some tensor contracted with ψ and τ. They can be recovered by removing a transverse divergence from $\Theta_{aB}\psi^a\tau^B$, yielding $\sigma_\tau^{ad}D_d\psi_a/16\pi = \sigma_\tau^{ad}L_\psi h_{ad}/32\pi$, where $\sigma_\tau^{ad} = \tau^B h^{ae}h^{cd}\sigma_{Bce}$ encodes the shear along τ. Such terms have been described by analogy with viscosity [55,56].

The conservation laws (35) and (45) take a similar form, expressing rate of change of mass M and angular momentum J as surface integrals of densities of energy and angular momentum, with respect to preferred vectors k and ψ which play the role of stationary and axial Killing vectors, even if there are no symmetries. Of the ten conservation laws in flat-space physics, they are the two independent laws expected for an astrophysical black hole, which defines its own spin axis and center-of-mass frame, in which its momentum vanishes.

VI. QUASILOCAL CONSERVATION LAWS

For an electromagnetic field with charge-current density vector j, the total electric charge Q in a region H of a spatial hypersurface is defined as

$$[Q]_{\partial H} = -\int_H \hat{*}g(j, \hat{\tau}). \tag{50}$$

The surface-integral form of conservation of charge follows by the same arguments relating (45) and (49):

$$L_\xi Q = -\oint_S *g(j, \tau). \tag{51}$$

As before, this is more general, since H may have any signature. The conservation laws for energy (35) and angular momentum (45) evidently take the same form

$$L_\xi M \cong -\oint_S *g(\tilde{\jmath}, \tau), \qquad L_\xi J \cong -\oint_S *g(\bar{\jmath}, \tau) \tag{52}$$

by identifying current vectors

$$\tilde{\jmath}^B = -k_A(T^{AB} + \Theta^{AB}), \qquad \bar{\jmath}^B = \psi_a(T^{aB} + \Theta^{aB}). \tag{53}$$

The local differential form of charge conservation,

$$\nabla_\alpha j^\alpha = 0, \tag{54}$$

where ∇ is the covariant derivative of g, notably does not hold for $\tilde{\jmath}$ or $\bar{\jmath}$ in general. A weaker property holds, obtained as follows. First note that in any of the three conservation laws (35), (45), and (51), ξ and τ may be interchanged. Thus there are really two independent laws in each case. This can be understood from special relativity: if ξ were causal, one would interpret them as expressing rate of change of energy, angular momentum or charge as, respectively, power, torque or current; while if ξ were spatial, one would normally convert to volume-integral form and regard them as defining the energy, angular momentum or charge in a region. One can make either interpretation for a black hole, since H would be generically spatial, but the marginal surfaces S locate the black hole in a family of time slices.

Given two independent equations in the normal space, it follows that

$$\epsilon^A_B L_A M \cong -\oint_S *1 \wedge \tilde{\jmath}_B, \qquad \epsilon^A_B L_A J \cong -\oint_S *1 \wedge \bar{\jmath}_B. \tag{55}$$

Expressed in terms of the curl and divergence of the normal space,

$$\mathrm{curl}\, M \cong -\oint_S *1 \wedge \tilde{\jmath}, \qquad \mathrm{curl} J \cong -\oint_S *1 \wedge \bar{\jmath}, \tag{56}$$

whereas

$$\nabla_\alpha \jmath = *\mathrm{div}(*1 \wedge \jmath)\alpha \tag{57}$$

for any normal vector $\jmath$. Then divcurl $= 0$ yields

$$\oint_S *\nabla_\alpha \tilde{\jmath}^\alpha \cong \oint_S *\nabla_\alpha \bar{\jmath}^\alpha \cong 0. \tag{58}$$

This subtly confirms the view that energy and angular momentum in General Relativity cannot be localized [16], but might be quasilocalized, as surface integrals [57]. The corresponding conservation laws have indeed been obtained in surface-integral but not local form.

VII. STATE SPACE

There are now three conserved quantities (M, J, Q), forming a state space for dynamical black holes. Following various authors [31,32,36–39], related quantities may then be defined by formulas satisfied by Kerr-Newman black holes, specifically those for the ADM energy

$$E \cong \frac{\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}{4M}, \tag{59}$$

the surface gravity

$$\kappa \cong \frac{(2M)^4 - (2J)^2 - Q^4}{2(2M)^3\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}, \tag{60}$$

the angular speed

$$\Omega \cong \frac{J}{M\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}, \tag{61}$$

and the electric potential

$$\Phi \cong \frac{((2M)^2 + Q^2)Q}{2M\sqrt{((2M)^2 + Q^2)^2 + (2J)^2}}. \tag{62}$$

It would be preferable to have independently motivated definitions of these quantities, but so far this has been done only in spherical symmetry [19–21], where there are natural definitions of E, κ and $\Phi = Q/R$ which can be applied anywhere in the space-time, coinciding with the above expressions on the outer horizons of a Reissner-Nordström black hole.

In the dynamical context, $E \geq M$ is generally not the ADM energy, since there may be matter or gravitational radiation outside the black hole. Rather, it can be interpreted as the effective energy of the black ole, as follows. Defining the moment of inertia I by the usual formula $J \cong I\Omega$ yields

$$I \cong M\sqrt{((2M)^2 + Q^2)^2 + (2J)^2} \cong ER^2. \tag{63}$$

Expanding E for $J \ll M^2$ and $Q \ll M$ yields

$$E \approx M + \frac{1}{2}I\Omega^2 + \frac{1}{2}Q^2/R. \tag{64}$$

The second and third terms are standard expressions for rotational kinetic energy and electrostatic energy. Thus the irreducible mass M plays the role of a rest mass, with E including contributions from rotational and electrical energy.

Returning to the general case, the above definitions satisfy the state-space formulas

$$\kappa \cong 8\pi\frac{\partial E}{\partial A} \cong \frac{1}{4M}\frac{\partial E}{\partial M}, \qquad \Omega \cong \frac{\partial E}{\partial J}, \qquad \Phi \cong \frac{\partial E}{\partial Q}. \tag{65}$$

There follows a dynamic version of the so-called first law of black-hole mechanics [14]:

$$L_\xi E \cong \frac{\kappa}{8\pi}L_\xi A + \Omega L_\xi J + \Phi L_\xi Q. \tag{66}$$

As desired, the state-space perturbations in the classical law for Killing horizons [14], or the version for isolated horizons [30–32], have been replaced by the derivatives along the trapping horizon, thereby promoting it to a genuine dynamical law.

The rate of change of effective energy can also be written in energy-tensor form,

$$L_\xi E = \oint_S *((T_{\alpha\beta} + \Theta_{\alpha\beta})K^\alpha\tau^\beta - \Phi j_\beta\tau^\beta), \tag{67}$$

where

$$K = 4M\kappa k - \Omega\psi \tag{68}$$

plays the role of the stationary Killing vector. Note that in the classical theory of stationary black holes, the state variables are usually taken to be (E, J, Q), with the irreducible mass M defined as a dependent variable. The dynamical theory reveals that (M, J, Q) are the more basic variables, since they each satisfy a simple conservation law. Then the effective energy E is defined as a dependent variable and therefore satisfies the above conservation law. This reflects a shift in emphasis from the classical to the dynamical theory: the so-called first law is a dependent result, obtained from more fundamental conservation laws for energy, angular momentum and charge.

VIII. NULL TRAPPING HORIZONS AND ZEROTH LAW

A zeroth law for trapping horizons follows from the above, if one defines local equilibrium by the absence of relevant fluxes:

$$g(\tilde{j}, \tau) \cong g(\bar{j}, \tau) \cong g(j, \tau) \cong 0. \tag{69}$$

Then (M, J, Q) are constant on the horizon and so is κ. In fact, these conditions do hold on a null trapping horizon under the dominant energy condition, as shown below. This treatment also turns out to be compatible with the definition of weakly isolated horizon [29,31–35].

Consider a null trapping horizon, assumed henceforth in this section to be given by $\theta_+ \cong 0$. The null focussing equation yields $T_{++} + ||\sigma_+||^2/32\pi \cong 0$, so the null energy condition, which implies $T_{++} \geq 0$, yields [18]

$$T_{++} \cong 0, \qquad \sigma_+ \cong 0. \tag{70}$$

Thus the degenerate metric of the horizon is preserved along the generating vector. The dominant energy condition, which implies that the energy-momentum $P_\alpha = -T_{\alpha\beta} l_+^\beta$ is causal, further yields [29]

$$T_{+a} \cong 0, \tag{71}$$

since $P = -T_{++} dx^+ - T_{+-} dx^- - T_{+a} dx^a$ would otherwise be spatial.

On a null hypersurface H, one can take the null coordinate to be the generating coordinate, $x^+ \cong x$, meaning that the shift vector vanishes, $s_+ \cong 0$, so that $\xi \cong \tau \cong l_+$. Since $k \cong -e^f L_- R l_+$ (34), one finds

$$\begin{aligned} g(\tilde{j}, \tau) &\cong e^f L_- R (T_{++} + \Theta_{++}), \\ g(\bar{j}, \tau) &\cong \psi^a (T_{a+} + \Theta_{a+}). \end{aligned} \tag{72}$$

These fluxes vanish by the above results (70) and (71) and the expressions (36) and (46) for components of Θ. The other flux in (69) vanishes due to the Maxwell equations [29].

For a null trapping horizon, the dual-null foliation is not unique, so the question arises whether there is a natural way to fix it. An affirmative answer is given by noting that the above results also imply, via the twisting Eqs. (25),

$$L_+ (\omega - \tfrac{1}{2} Df) \cong 0. \tag{73}$$

This restriction on the dual-null geometry is suggestive of a proto-conservation law for angular momentum. Now the only normal fundamental form intrinsic to a null hypersurface is $\zeta_{-+} = \frac{1}{2} Df + \omega$ of Appendix B, which was therefore used by Ashtekar *et al.* [29] to define angular momentum for an isolated horizon. However, it is the other null normal fundamental form $\zeta_{+-} = \frac{1}{2} Df - \omega$, depending on the dual-null foliation, which is preserved as above. Given that the general definition (39) of angular momentum involves ω, it seems best to fix the unwanted freedom by

$$Df \cong 0. \tag{74}$$

Recalling the definition (8) or (12) of f, this condition fixes the normalization of the extrinsic null normal l_- with respect to the intrinsic null normal l_+, which is always possible on a null hypersurface H. In fact, it is common simply to fix $f \cong 0$. A similar normalization is also used in the context of null infinity.

Then the definition (39) of angular momentum becomes unambiguous on a null trapping horizon, coincides with the definition for isolated horizons [30–32], and is preserved along the horizon, assuming only that ψ is a coordinate vector field (41):

$$L_\xi J \cong 0. \tag{75}$$

Since the area law [18] shows that A is increasing unless H is null everywhere on a given S, this answers, in the negative, a simply stated physical question: whether a black hole can change its angular momentum without increasing its area.

The above reasoning has largely recovered the notion of a weakly isolated horizon introduced by Ashtekar *et al.* [29], except that the scaling freedom in l_+ has not been fixed. In more detail, Ashtekar *et al.* [29] introduced a 1-form which will here be denoted by ϖ, defined by

$$\hat{\nabla}_\alpha l_+^\beta = \varpi_\alpha l_+^\beta, \tag{76}$$

where $\hat{\nabla}$ is the covariant derivative operator of H. The transverse and normal components are found as

$$\perp \varpi = -\omega - \tfrac{1}{2} Df, \qquad l_+^\alpha \varpi_\alpha = -\nu_+. \tag{77}$$

Since ϖ is an invariant of H and l_+, Ashtekar *et al.* [29] demanded

$$L_+ \varpi \cong 0 \tag{78}$$

in order to define a weakly isolated horizon. The transverse part agrees with the above results, which also imply $D\nu_+ \cong 0$, while the normal part further fixes the inaffinity ν_+ to be constant on H. This fixes the scaling of l_+ up to a constant multiple. Ashtekar *et al.* [29] defined the surface gravity to be

$$\hat{\kappa} \cong -\nu_+, \tag{79}$$

which recovers the standard surface gravity of a Killing horizon if l_+ is the null Killing vector [17]. Then the constancy of $\hat{\kappa}$ can also be interpreted as a zeroth law. This still leaves nonzero $\hat{\kappa}$ ambiguous up to a constant multiple, not necessarily agreeing with the definition (60), which therefore fixes that freedom.

The above considerations appear to have a closed a gap in the general paradigm, concerning how a growing black hole ceases to grow. It seems that the generically spatial trapping horizon simply becomes null. It is difficult to find a practical formalism describing all cases without some degeneracy arising in the null case, but the dual-null formalism appears to be adequate; one fixes the additional freedom in the null case by (74). In particular, no additional conditions need be imposed on the horizon itself, as compared with the variety of definitions of isolated horizons [27–35]. Numerical evidence that such horizons exist in practice has been given by Dreyer *et al.* [34], who looked for and found approximately null trapping horizons, under the name nonexpanding horizons.

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APPENDIX A: TWIST AND WEAK FIELDS

The twist may be calculated by first finding the shift vectors $s_\pm$, due to the form [45]

$$\omega = \tfrac{1}{2}e^f h([\partial_+, s_-] - [\partial_-, s_+] + [s_-, s_+]). \tag{A1}$$

If it is more convenient to use an orthonormal basis $\{l_0, l_1\}$ of the normal space,

$$\perp l_0 = \perp l_1 = 0 = g(l_0, l_1), \quad g(l_0, l_0) = -1 = -g(l_1, l_1), \tag{A2}$$

then

$$\omega = \tfrac{1}{2}h([l_0, l_1]) \tag{A3}$$

follows by linear combinations from (39), or directly from the Komar integral (38). If the basis is adapted to a coordinate basis via

$$l_A = \partial_A - s_A, \qquad s_A = \perp \partial_A, \tag{A4}$$

then

$$\omega = \tfrac{1}{2}h([\partial_1, s_0] - [\partial_0, s_1] + [s_0, s_1]). \tag{A5}$$

In either case, the first step is to find the shift vectors.

The weak-field metric [16], in standard spherical polar coordinates $(t, r, \vartheta, \varphi)$ adapted to the axis of rotation, is

$$g \sim -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4J}{r}\sin^2\vartheta dt d\varphi + \left(1 + \frac{2M}{r}\right)(dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)), \tag{A6}$$

where, in this appendix only, M and J denote the mass and angular momentum as defined in this approximation, obtained by linearizing the metric and neglecting higher powers of $1/r$. The inverse metric is

$$g^{-1} \sim -\left(1 + \frac{2M}{r}\right)\partial_t^2 - \frac{4J}{r^3}\partial_t\partial_\varphi + \left(1 - \frac{2M}{r}\right)\left(\partial_r^2 + \frac{1}{r^2}\left(\partial_\vartheta^2 + \frac{\partial_\varphi^2}{\sin^2\vartheta}\right)\right). \tag{A7}$$

Taking the transverse surfaces S as those of constant (t, r), one can read off the nonzero component of the shift 1-forms s_{Ab} as

$$s_{t\varphi} = g_{t\varphi} \sim -\frac{2J}{r}\sin^2\vartheta, \qquad s_t^\varphi \sim -\frac{2J}{r^3}. \tag{A8}$$

Then the nonzero component of the twist is given by

$$\omega^\varphi \sim \frac{1}{2}\partial_r s_t^\varphi \sim \frac{3J}{r^4}, \qquad \omega_\varphi \sim \frac{3J}{r^2}\sin^2\vartheta. \tag{A9}$$

The area form is

$$* 1 \sim r^2 \sin\vartheta d\vartheta \wedge d\varphi, \tag{A10}$$

so that

$$* \omega_\varphi \sim 3J\sin^3\vartheta d\vartheta \wedge d\varphi. \tag{A11}$$

Standard integrals yield

$$\oint_S *\omega_\varphi \sim 8\pi J. \tag{A12}$$

Since $\omega_a\psi^a = \omega_\varphi$ if $\psi = \partial/\partial\varphi$, this agrees with the general definition (39) of angular momentum.

A directly measurable quantity due to rotational frame-dragging is the precessional angular velocity [16]

$$\vec{\Omega}_{LT} \sim \frac{J}{r^3}(3(\hat{z}\cdot\hat{r})\hat{r} - \hat{z}) \tag{A13}$$

of a gyroscope due to the Lense-Thirring effect, where $\hat{r}$ is a unit vector in the direction of the gyroscope and $\hat{z}$ is a unit vector along the axis of rotation. Results of measurements of the effect due to the Earth by Gravity Probe B are expected soon. If the twist ω is formally converted to an angular velocity $\vec{\Omega}$ by

$$\omega \sim \vec{\Omega} \times \hat{r}, \tag{A14}$$

then

$$\vec{\Omega} \sim \left|\frac{\omega}{\sin\vartheta}\right|\hat{z} \sim \frac{3J}{r^3}\hat{z} \tag{A15}$$

does have the direction and relativistic dimensions of angular velocity. Then

$$\vec{\Omega}_{LT} \sim (\vec{\Omega}\cdot\hat{r})\hat{r} - \tfrac{1}{3}\vec{\Omega}. \tag{A16}$$

Curiously, this is a linear transformation of $\vec{\Omega}$, by the same traceless tensor used in defining quadrupole moments [16]. The puckish role of the factor of 3 in the above calculations is also noteworthy. In any case, it confirms that the twist does indeed encode the twisting around of space-time due to a rotating mass, in a directly measurable way.

For completeness, the agreement of the weak-field mass with the Hawking mass can also be checked, as follows. One needs to keep track of an extra power of $1/r$ in the area form

$$* 1 \sim \left(1 + \frac{2M}{r}\right)r^2 \sin\vartheta d\vartheta \wedge d\varphi. \tag{A17}$$

Then the expansion 1-form θ_A has nonzero component

$$\theta_r \sim \partial_r \log\left(\left(1 + \frac{2M}{r}\right)r^2\right) \sim \frac{2}{r}\left(1 - \frac{M}{r}\right). \tag{A18}$$

Then

$$\gamma^{AB}\theta_A\theta_B \sim \gamma^{rr}\theta_r\theta_r \sim \frac{4}{r^2}\left(1 - \frac{4M}{r}\right), \tag{A19}$$

and

$$\oint_S *\gamma^{AB}\theta_A\theta_B \sim \oint_S 4\left(1 - \frac{2M}{r}\right)\sin\vartheta d\vartheta \wedge d\varphi \sim 16\pi\left(1 - \frac{2M}{r}\right). \tag{A20}$$

Since $R \sim r$, this agrees with the Hawking mass (33).

APPENDIX B: NORMAL FUNDAMENTAL FORMS

Various definitions of angular momentum [30,36,54] are similar to (39), with ω replaced by a 1-form which is, implicitly or explicitly, a normal fundamental form. To clarify the situation, normal fundamental forms are reviewed below, referring to previous treatments [45,55].

Writing the twist (17) explicitly in components,

$$2\omega_\alpha = e^f h_{\alpha\beta}(l_-^\gamma \nabla_\gamma l_+^\beta - l_+^\gamma \nabla_\gamma l_-^\beta). \quad \text{(B1)}$$

If $l_\pm$ are adapted to a coordinate basis via (11), then commutativity $\nabla_{[\beta}\nabla_{\gamma]}x^\pm = 0$ allows it to be written as

$$2\omega_\alpha = e^f h_\alpha^\beta (l_-^\gamma \nabla_\beta l_{+\gamma} - l_+^\gamma \nabla_\beta l_{-\gamma}). \quad \text{(B2)}$$

This is the difference of two normal fundamental forms $\zeta_{\mp\pm}$ with components

$$\zeta_{\mp\pm\alpha} = e^f l_\mp^\gamma h_\alpha^\beta \nabla_\beta l_{\pm\gamma}. \quad \text{(B3)}$$

They are the independent normal fundamental forms, since the corresponding $\zeta_{\pm\pm}$ vanish. Their sum is Df, since the normalization in (12) yields

$$D_\alpha f = e^f h_\alpha^\beta (l_-^\gamma \nabla_\beta l_{+\gamma} + l_+^\gamma \nabla_\beta l_{-\gamma}). \quad \text{(B4)}$$

Then

$$\zeta_{\mp\pm} = \tfrac{1}{2} Df \pm \omega, \quad \text{(B5)}$$

and the normal fundamental forms are thereby encoded in ω and Df.

For an orthonormal basis (A2), there is just one independent normal fundamental form $\hat{\zeta}_{01} = -\hat{\zeta}_{10}$, given by

$$\hat{\zeta}_{01\alpha} = l_0^\gamma h_\alpha^\beta \nabla_\beta l_{1\gamma}, \qquad \hat{\zeta}_{10\alpha} = l_1^\gamma h_\alpha^\beta \nabla_\beta l_{0\gamma}, \quad \text{(B6)}$$

with corresponding $\hat{\zeta}_{00}$, $\hat{\zeta}_{11}$ vanishing. Under a boost transformation

$$l_0 \mapsto l_0 \cosh\rho + l_1 \sinh\rho, \qquad l_1 \mapsto l_0 \sinh\rho + l_1 \cosh\rho, \quad \text{(B7)}$$

which preserves the orthonormal conditions (A2), the normal fundamental form is generally not invariant:

$$\hat{\zeta}_{01} \mapsto \hat{\zeta}_{10} - D\rho, \qquad \hat{\zeta}_{10} \mapsto \hat{\zeta}_{10} + D\rho. \quad \text{(B8)}$$

However, if the orthonormal basis is adapted to the dual-null basis by, e.g. $\sqrt{2}l_\pm = l_0 + l_1$, then

$$\hat{\zeta}_{01} = -\hat{\zeta}_{10} = \omega. \quad \text{(B9)}$$

For spatial H, the same result is obtained by adapting the orthonormal basis by choosing

$$l_0 = \tau/\sqrt{g(\xi, \xi)}, \qquad l_1 = \xi/\sqrt{g(\xi, \xi)}. \quad \text{(B10)}$$

The missing information in Df can be recovered by instead defining

$$\zeta_{01\alpha} = \frac{\tau^\gamma h_\alpha^\beta \nabla_\beta \xi_\gamma}{g(\xi, \xi)}, \qquad \zeta_{10\alpha} = \frac{\xi^\gamma h_\alpha^\beta \nabla_\beta \tau_\gamma}{g(\xi, \xi)},$$
$$\zeta_{00\alpha} = \frac{\tau^\gamma h_\alpha^\beta \nabla_\beta \tau_\gamma}{g(\xi, \xi)}, \qquad \zeta_{11\alpha} = \frac{\xi^\gamma h_\alpha^\beta \nabla_\beta \xi_\gamma}{g(\xi, \xi)}. \quad \text{(B11)}$$

Then

$$\zeta_{01} = -\zeta_{10} = \omega, \qquad \zeta_{00} = -\zeta_{11} = \tfrac{1}{2} Df. \quad \text{(B12)}$$

These 1-forms nevertheless become degenerate when ξ becomes null.

As shown explicitly by Gourgoulhon [55], the 1-form used by Brown & York [54] to define angular momentum is, in this notation, $-\hat{\zeta}_{10}$. Therefore it coincides with ω if the orthonormal basis is adapted to a trapping horizon, but generally not if it is adapted to a foliation of spatial hypersurfaces intersecting the trapping horizon in the marginal surfaces. The 1-form used by Ashtekar & Krishnan [36] to define angular momentum for dynamical horizons is also $-\hat{\zeta}_{10}$, this time explicitly adapted to the horizon by (B10), therefore coinciding with ω in this case. The 1-form used by Ashtekar *et al.* [30–32] to define angular momentum for isolated horizons is ζ_{-+}, which generally does not coincide with ω. However, it does coincide if the gauge freedom is fixed by (74).

APPENDIX C: KERR EXAMPLE

Consider a Kerr space-time in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$, with S given by constant (t, r) and $\xi = \partial/\partial r$. Then

$$*1 = \sqrt{(r^2 + a^2)^2 - \Delta a^2 \sin^2\vartheta}\, \sin\vartheta d\vartheta \wedge d\varphi, \quad \text{(C1)}$$

where $\Delta = r^2 - 2mr + a^2$,

$$\theta_\xi = \frac{2r(r^2 + a^2) - (r - m)a^2 \sin^2\vartheta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\vartheta}, \quad \text{(C2)}$$

and

$$D\theta_\xi = \frac{2a^2(r^2 + a^2)(r^3 - 3mr^2 + a^2 r + ma^2)\sin\vartheta\cos\vartheta}{((r^2 + a^2)^2 - \Delta a^2 \sin^2\vartheta)^2} d\vartheta. \quad \text{(C3)}$$

If $a \neq 0$, $D\theta_\xi$ is nonzero except at the poles, equator and isolated values of r, so the construction yields a unique continuous ψ, $\psi = \partial/\partial\varphi$.

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