Stability of the evaporating Schwarzschild-de Sitter black hole final state

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Time-independent backreaction corrections of the renormalized expectation value of the stress tensor operator of a massless quantum scalar field, coupled in a two-dimensional spherically symmetric Schwarzschild-de Sitter static black hole metric, are used to obtain its final state. According to the work by Christensen and Fulling, the renormalized stress tensor is found to be determined by the nonlocal contribution of the trace anomaly and some additional parameters. Mathematical derivations of the backreaction equations, in close relation to the work by Bousso and Hawking, show that the scenario of the black hole Hawking radiation is reduced to a remnant, stable, static, Schwarzschild-de Sitter *mini* black hole which has still new black hole and cosmological shrunk horizons.

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I. INTRODUCTION

Hawking's remarkable discovery [1,2] of black hole evaporation has raised several puzzles. For instance, a black hole defined in a suitable vacuum state decays quantum mechanically. In other words, it indeed indicates that the black holes are quantum mechanically unstable, but they do not allow directly one to study the time evolution of the black holes geometry. So one maybe asks a question: *What is the final state of the evaporating quantum black holes*? Properly a detailed picture of the evaporation process and its final state can be given only within the framework of a complete and self-consistent pure quantum gravity theory, which has not yet been found. It will be valid for the Planck scale of the Universe [3,4] in which:

$$M_p = (\hbar c/G)^{1/2} = 2.18 \times 10^{-8}\ \text{kg}; \quad \text{Planck mass}, \tag{1.1}$$

$$D_p = (\hbar G/c^3)^{1/2} = 1.62 \times 10^{-35}\ \text{m}; \quad \text{Planck distance}, \tag{1.2}$$

$$T_p = \hbar/M_p c^2 = 5.31 \times 10^{-44}\ \text{s}; \quad \text{Planck time}. \tag{1.3}$$

In the absence of a pure quantum gravity theory, the answer of backreaction corrections on the background metric of the evaporating black holes should be obtained in the spirit of semiclassical gravity theories by solving the set of equations:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = -8\pi \langle \hat{T}_{\alpha\beta} \rangle_{\text{ren}}, \tag{1.4}$$

where the units are chosen so that, $G = K = c = \hbar = 1$. G, K, and Λ are the Newtonian, Boltzmann, and cosmological constants, respectively. Λ as a vacuum energy density, should lead to curvature of space-time in absence of all matter and radiation[1] in which $\langle \hat{T}_{\alpha\beta} \rangle_{\text{ren}} = 0$. $G_{\alpha\beta}$ is the Einstein's tensor and $\langle \hat{T}_{\alpha\beta} \rangle_{\text{ren}}$ is the renormalized expectation value of the stress-energy tensor operator of a quantum matter field described in its suitable vacuum state. The main problem in this approach is to find $\langle \hat{T}_{\alpha\beta} \rangle_{\text{ren}}$ for a sufficiently arbitrary nonstatic and nonspherically symmetric four-dimensional black hole space-time. This conjecture is hard to check in details because of the many degrees of freedom and inherent complexity of the process in four-dimensional space-times. So as an exactly solvable toy model in close relation to the work by *Callan-Giddings-Harvey-Strominger* [5], the final state of the asymptotically flat, two-dimensional, evaporating, dilatonic, black holes (i.e. coupled to dilaton and conformal matter fields) is predicted as a flat space-time with no naked singularity [6–9]. The final state of the two-dimensional, evaporating, dilatonic, Reissner-Nordström black hole reduces to a remnant, stable, nonsingular space-time [10]. Also the evaporating dilatonic Schwarzschild-de Sitter black holes final state, whose size is comparable to that of the cosmological horizon, are not evaporated [11] and they are in thermal equilibrium. Whereas in this paper we will apply another approach for studying the end point of the nondilatonic Schwarzschild-de Sitter static black hole Hawking radiation. The plan of this paper is as follows. In Sec. II, we review the Hawking radiation of a nondilatonic, Schwarzschild-de Sitter, static black hole, minimally coupled to a two-dimensional massless quantum scalar field [12], by using the results of the Hadamard renormalization prescriptions [12–17]. In this approach, the stress tensor of the quantum scalar field is found to be determined by the nonlocal contribution of the trace anomaly and some additional parameters in close relation to the work by Christensen and Fulling [18]. In Sec. III, we use the Hawking radiation stress-energy tensor of the Schwarzschild-de Sitter static black hole, for derivation of the linear-order, time-independent, backreaction Eqs. (1.4). In this section we follow the ideas proposed by York [19] and Hochberg *et al.* [20] in which time-

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[1]Quantum field theory predicts that the vacuum is not a featureless void but is continuously disturbed by particle-antiparticle pair creation and annihilation. So it should be contained by energy. Also we note that the experimental limit on Λ is obtained as $|\Lambda| \leq 10^{-54}\ \text{cm}^{-2}$ [3].

independent sources of the backreaction Eqs. (1.4) reduce to some suitable static solutions. In Sec. IV, we discuss the results of the backreaction corrections of the quantum scalar field on the Schwarzschild-de Sitter black hole event horizons. Mathematical derivations predict that the time-independent stress-energy tensor of the black hole Hawking radiation leads to a remnant, stable, static Schwarzschild-de Sitter mini black hole which has still black hole and cosmological shrunk horizons. Finally in Sec. V, we presented a summary and concluding remarks.

II. SCHWARZSCHILD-DE SITTER STATIC BLACK HOLE AND ITS THERMAL RADIATION

We consider a linear, two-dimensional, massless quantum scalar field ϕ, propagating on a curved space-time with the action of the standard form as

$$S[\phi] = -\frac{1}{2}\int d^2x\sqrt{g(x)}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \qquad (2.1)$$

where ϕ is a dimensionless quantity. This model can be at most regarded as a toy model. Because in two-dimensional analog, the minimal coupling reduces to the conformal coupling [21]. Also four-dimensional quantum scalar fields with the inverse of length dimension will have a kinetic term different from (2.1), with a coupling involving $\frac{1}{r^2}$, which follows by dimensional reduction of a standard kinetic term in four dimensions. ∂_α and g indicate partial differentiation and absolute value of the determinant of the metric $g_{\alpha\beta}$, respectively. The corresponding scalar field equation ϕ is obtained such as

$$\Box\phi(x) = 0, \qquad (2.2)$$

where $\Box = \frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta)$. The stress-energy tensor of ϕ which has a dimension of (length)$^{-2}$, is defined by the singular expression

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \tfrac{1}{2}g_{\alpha\beta}\partial^\xi\phi\partial_\xi\phi. \qquad (2.3)$$

The renormalization theory, such as the Hadamard renormalization method for the expectation value of a two-dimensional stress-energy tensor operator $\hat{T}_{\alpha\beta}$, defined in terms of a massless quantum scalar field operator $\hat{\phi}$, leads to a suitable regularized stress tensor $\langle\hat{T}_{\alpha\beta}\rangle_{\rm ren}$, satisfying the following constraints conditions

$$\nabla_\alpha\langle\hat{T}^{\alpha\beta}\rangle_{\rm ren} = 0 \qquad (2.4)$$

and

$$\langle\hat{T}^\xi_\xi\rangle_{\rm ren} = \frac{1}{24\pi}R, \qquad (2.5)$$

in which ∇_α is a covariant differentiation and R is the *Ricci* scalar of a suitable two-dimensional curved space-time [12,18,22]. So the renormalization theory of quantum fields, propagated on an arbitrary curved space-time leads to that $\langle\hat{T}_{\alpha\beta}\rangle_{\rm ren}$ should be a purely geometrical object. One of suitable vacuum solutions of the Einstein's equation defined by $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$, is static and spherically symmetric,*initial* Schwarzschild-de Sitter black hole metric [23]

$$dS_i^2 = \Omega(r;\xi)(-dt^2 + dr^{*2}) + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (2.6)$$

where the index i denotes to the word *initial* (unperturbed metric). Also

$$r^* = \int\frac{dr}{\Omega(r;\xi)}, \qquad \Omega(r;\xi) = 1 - \frac{2M}{r} - \xi\left(\frac{2M}{r}\right)^{-2}, \qquad (2.7)$$

and $\xi = \frac{\Lambda(2M)^2}{3}$ is a suitable cosmological coupling constant of the black hole. The parameters defined by M and Λ are black hole mass and cosmological constants, respectively. Using the assumption $0 < \xi < 1$, the equation $\Omega(r;\xi) = 0$ has real roots such that $r_b^i \approx 2M(1+\xi)$, and $r_c^i \approx \frac{2M}{\sqrt{\xi(1+\sqrt{\xi})}}$ (see Fig. 1), named as the black hole and cosmological horizons, respectively. The size of the black hole horizon varies between zero and the size of the cosmological horizon. If the black hole horizon is much smaller than the cosmological horizon (i.e. $\xi \sim 0.79;\ 1$), the effect of the radiation coming from the cosmological horizon is negligible [11]. In this case it is named as a degenerate Schwarzschild-de Sitter black hole. The degenerate solution, in which the black hole has the maximum size, is called the Nariai solution [24]. In this solution the two horizons have the same size and so the same temperature. Therefore it shall be in thermal equilibrium when we choose $0 \ll \xi < 1$. Intuitively, one would expect any slight perturbation of the geometry to cause the black hole to become hotter than the background. Thus, one may suspect

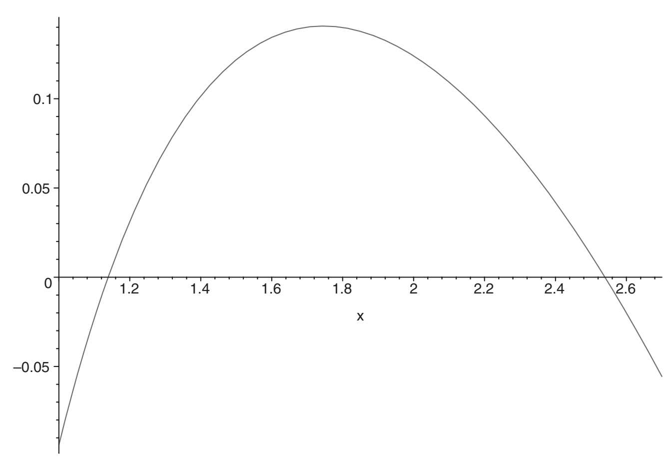

FIG. 1. Diagram of the function $\Omega(x) = 1 - \frac{1}{x} - 0.094x^2$ (see Eqs. (2.7) and (3.36)), in which $x = \frac{r}{2M}$ shows that the unperturbed Schwarzschild-de Sitter static black hole has two event horizons at $x_b \simeq 1.14$ and $x_c \simeq 2.54$, which are named unperturbed black hole and cosmological event horizons, respectively.

the thermal equilibrium of the Nariai solution (large scale black hole) to be unstable and its final state should be obtained by the solutions of the backreaction Eqs. (1.4). In order to make statements about the final state of the evaporating quantum black holes, one should really study the dynamical, time-dependent process of black hole formation and evaporation such as the work by Russo *et al.* [6–10]. But according to the ideas by York [19] and Hochberg *et al.* [20], the time-independent solutions of the backreaction Eqs. (1.4) may be still predict some suitable physical situations for the evaporating black hole final state. In such case the choice of vacuum (i.e., Boulware, Hartle-Hawking, or Unruh) is crucial for the calculation of the stress-energy tensor components, such as its expectation value, evaluated in a suitable vacuum state of an observer (i.e., Unruh state for a freely falling frame), become finite [25]. According to the work by Christensen and Fulling [18], the two-dimensional ($\theta, \varphi =$ constant). The spherically symmetric Schwarzschild-de Sitter static black hole metric (2.6), is used to derive a time-independent solution of the Eqs. (2.4) and (2.5) such that [12]

$$\langle \hat{T}_{\alpha\beta}(t, r^*)\rangle_{\rm ren} \sim \frac{1}{12\pi(2M)^2}\begin{pmatrix} -P(r;\xi) & \frac{\xi}{2} \\ \frac{\xi}{2} & Q(r;\xi)\end{pmatrix}, \quad (2.8)$$

where

$$P(r;\xi) = -\frac{1}{8} + 2\xi + O(\xi^2) - \frac{\xi^2}{2}\left(\frac{2M}{r}\right)^{-2} - \frac{5\xi}{2}\left(\frac{2M}{r}\right) + \left(\frac{2M}{r}\right)^3 - \frac{7}{8}\left(\frac{2M}{r}\right)^4 \quad (2.9)$$

and

$$Q(r;\xi) = -\frac{1}{8} - \frac{\xi}{2} + O(\xi^2) - \frac{\xi^2}{2}\left(\frac{2M}{r}\right)^{-2} + \frac{\xi}{2}\left(\frac{2M}{r}\right) - \frac{1}{8}\left(\frac{2M}{r}\right)^4. \quad (2.10)$$

The stress-energy tensor (2.8), defined in quasiflat regions of the black hole in which

$$Q(r_{q.f};\xi) = P(r_{q.f};\xi); \qquad r_{q.f} \sim \frac{2M}{\sqrt{3\xi}}, \quad (2.11)$$

can be decomposed in terms of thermal equilibrium and radiating gas stress-energy tensors, respectively, as:

$$\frac{\pi}{12}T_c^2\begin{pmatrix} -2 & 0 \\ 0 & 2\end{pmatrix} \quad (2.12)$$

and

$$\frac{\pi}{12}T_b^2\begin{pmatrix} -1 & 1 \\ 1 & 1\end{pmatrix}, \quad (2.13)$$

where

$$T_b(\xi) \simeq \frac{\sqrt{8\xi}}{8\pi M} \quad \text{and} \quad T_c(\xi) \simeq \frac{\sqrt{4\xi\sqrt{3\xi} - \frac{28}{3}\xi - 1}}{8\pi M} \quad (2.14)$$

are defined, respectively, as cosmological thermal equilibrium and black hole radiation temperatures related to the cosmological and black hole event horizons [12]. The relation defined by (2.14) shows that $T_c(\xi \simeq 2) = 0$. Hawking radiation stress tensor (2.8), as a spherically symmetric static source of the backreaction Eqs. (1.4), provides the radial variations of the black hole mass and cosmological constant, in close relation to the ideas by York and Hochberg *et al.* [23,24]. In the next section we try to solve the linear-order time-independent solution of the backreaction Eqs. (1.4) by using (2.8).

III. TIME-INDEPENDENT BACKREACTION EQUATIONS

A physically general line element describing a spherically symmetric, evaporating Schwarzschild-de Sitter *final* static black hole may be proposed by the form [19,20]:

$$dS_f^2 = -\left[1 - \frac{2m(r)}{r} - \frac{\lambda(r)}{3}r^2\right]e^{2\psi(r)}dv^2 + 2e^{\psi(r)}dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (3.1)$$

where the index f denotes to the word *final* state of evaporating black hole. Also we assume that ψ, m, and λ are some suitable functions defined explicitly in terms of the radius r. Because they are related to the time-independent black hole Hawking radiation stress-energy tensor (2.8), by the backreaction Eqs. (1.4). Eddington-Finkelestein advanced time of a suitable freely falling observer 'v' is described by relation of $v = t + r^*$ in which

$$r^* = \int \frac{dr}{[1 - \frac{2m(r)}{r} - \frac{\lambda(r)}{3}r^2]}. \quad (3.2)$$

Applying (3.2), the metric (3.1) reduces to the following line element

$$dS_f^2 = -\left[1 - \frac{2m(r)}{r} - \frac{\lambda(r)}{3}r^2\right]e^{2\psi(r)}dt^2 + \frac{[2 - e^{\psi(r)}]e^{\psi(r)}}{[1 - \frac{2m(r)}{r} - \frac{\lambda(r)}{3}r^2]}dr^2 + 2e^{\psi(r)}(1 - e^{\psi(r)})drdt + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (3.3)$$

It leads to the static Schwarzchild de Sitter metric (2.6) under the boundary conditions

$$\psi(r_b) = 0; \qquad m(r_b) = M; \qquad \lambda(r_b) = \Lambda, \quad (3.4)$$

where $M > 0$ and $\Lambda > 0$ are, respectively, constants of the initial black hole mass and cosmology. Time-independent Hawking radiation stress tensor (2.8) described in the

Eddington-Finkelestein freely falling coordinate system (v, r) is obtained as (See the appendix)

$$\langle \hat{T}_{\alpha\beta}(v, r)\rangle_{\rm ren} \sim \frac{1}{12\pi(2M)^2} \times \begin{pmatrix} -P(r;\xi) + Q(r;\xi) + \xi & \frac{\frac{\xi}{2}+Q(r;\xi)}{\Omega(r;\xi)} \\ \frac{\frac{\xi}{2}+Q(r;\xi)}{\Omega(r;\xi)} & \frac{Q(r;\xi)}{\Omega^2(r;\xi)} \end{pmatrix}, \tag{3.5}$$

where $\Omega(r;\xi)$, $P(r;\xi)$, and $Q(r;\xi)$ are given by the relations (2.7), (2.9), and (2.10), respectively. In order to make statements about the final state of the evaporating black holes, one should really study the dynamical, time-dependent process of black hole formation and evaporation such as the work by Russo *et al.* [6–10]. But according to the ideas by York [19] and Hochberg *et al.* [20], in which the time-independent spherically symmetric source of the backreaction Eqs. (1.4) perturbs the black hole metric in terms of radial variable r. So it may be predicting some suitable physical situations for the evaporating black hole final state. Using the two-dimensional analog of the metric defined by (3.1), nonzero v, r components of the Einstein's tensor are obtained such as

$$G_{vv} = -\left[1 - \frac{2m(r)}{r} - \frac{1}{3}\lambda(r)r^2\right] \times \left[\frac{2m'(r)}{r^2} + \frac{\lambda'(r)}{3}r + \lambda(r)\right]e^{2\psi(r)}, \tag{3.6}$$

$$G_{vr} = G_{rv} = \left[\frac{2m'(r)}{r^2} + \frac{\lambda'(r)}{3}r + \lambda(r)\right]e^{\psi(r)}, \tag{3.7}$$

$$G_{rr} = -\frac{2\psi'(r)}{r}, \tag{3.8}$$

where prime $'$ indicates differentiation with respect to r and $G^\theta_\theta = G^\varphi_\varphi$ which follow from the Bianchi identity $\nabla_\alpha G^\alpha_\beta = 0$. Using suitable dimensionless functions

$$\frac{m(x)}{M} = \mu(x) \quad \text{and} \quad \frac{\lambda(x)}{\Lambda} = \omega(x); \qquad x = \frac{r}{2M}, \tag{3.9}$$

the set of relations defined by (3.5), (3.6), (3.7), (3.8), and (3.9) yield the time-independent backreaction Eqs. (1.4), respectively, as

$$\left\{1 - \frac{\mu(x)}{x} - \xi x^2\omega(x)\right\}\left\{\frac{\mu'(x)}{x^2} + \frac{\xi}{2}[x\omega'(x) + 6\omega(x)]\right\} + \frac{1}{3}\{P(x;\xi) - Q(x;\xi) - \xi\}e^{-2\psi(x;\xi)} = 0, \tag{3.10}$$

$$\left\{\frac{\mu'(x)}{x^2} + \frac{\xi}{2}[x\omega'(x) + 6\omega(x)]\right\} = -\frac{[\frac{\xi}{2} + Q(x;\xi)]e^{-\psi(x;\xi)}}{3\Omega(x;\xi)}, \tag{3.11}$$

and

$$\psi(x;\xi) = C_\psi + \frac{1}{3}\int \frac{xQ(x;\xi)dx}{\Omega^2(x;\xi)}, \tag{3.12}$$

in which $\Omega(x;\xi)$, $P(x;\xi)$, and $Q(x;\xi)$ are given by (2.7), (2.9), and (2.10). Furthermore C_ψ is an integral constant and is determined by using (3.4). Eliminating $\mu'(x)$ and $\omega'(x)$ defined by (3.10), the relation (3.11) leads to the following identity, between $\mu(x)$ and $\omega(x)$,

$$\mu(x,\xi) = x(1 + \Gamma\Omega e^{-\psi}) - \xi x^3\omega; \qquad \Gamma(x,\xi) = \frac{\xi + Q - P}{\frac{\xi}{2} + Q}. \tag{3.13}$$

Applying (3.13), the relation defined by (3.11) becomes an integral equation for $\omega(x;\xi)$ such as follows,

$$\omega(x;\xi) = C_\omega + \frac{2}{\xi}\int \frac{dxe^{-\psi}}{x^3}\left\{e^\psi + x\Gamma\Omega' + \Omega(\Gamma + x\Gamma' - x\Gamma\psi') + \frac{x^2(\frac{\xi}{2} + Q)}{3\Omega}\right\}, \tag{3.14}$$

where C_ω is a constant of integration and it is determined by boundary conditions (3.4). The explicit form of the relations (3.12), (3.13), and (3.14) can be obtained by perturbation methods in which the dimensionless cosmological coupling constant ξ is a suitable order parameter. Applying the series expansions

$$\Omega(x;\xi) = \sum_{n=0}^{\infty}\xi^n\Omega_n(x); \qquad P(x;\xi) = \sum_{n=0}^{\infty}\xi^n P_n(x), \tag{3.15}$$

$$Q(x;\xi) = \sum_{n=0}^{\infty}\xi^n Q_n(x); \qquad \psi(x;\xi) = \sum_{n=0}^{\infty}\psi_n(x), \tag{3.16}$$

and

$$\Gamma(x;\xi) = \sum_{n=0}^{\infty}\xi^n\Gamma_n(x), \tag{3.17}$$

in which $0 < \xi < 1$ and the terms $\Omega_0, \ldots, P_0, \ldots, \Gamma_0, \ldots$ up to the order of (ξ^2) are obtained, respectively, by the relations (2.7), (2.9), (2.10), (3.12), and (3.13) such as follows,

$$\Omega_0(x) = 1 - \frac{1}{x}; \qquad \Omega_1(x) = x^2, \tag{3.18}$$

$$P_0(x) = -\frac{1}{8} + \frac{1}{x^3} - \frac{7}{8x^4}; \qquad P_1(x) = 2 - \frac{5}{2x}, \tag{3.19}$$

$$Q_0(x) = -\frac{1}{8} - \frac{1}{8x^4}; \qquad Q_1(x) = -1 + \frac{1}{x}, \tag{3.20}$$

$$\psi_0(x) = \frac{1}{3}\int \frac{xQ_0(x)}{\Omega_0^2(x)}dx = \frac{1}{12(x-1)} - \frac{1}{24}\ln(x) - \frac{1}{12}\ln(x-1) - \frac{x}{12} - \frac{x^2}{48}, \tag{3.21}$$

$$\psi_1(x) = \frac{1}{3}\int \frac{x[\Omega_0(x)Q_1(x) - 2Q_0(x)\Omega_1(x)]}{\Omega_0^3(x)}dx = \frac{x^4}{48} + \frac{x^3}{12} + \frac{x^2}{12} + \frac{x}{2} - \frac{1}{12(x-1)^2} + \ln(x-1) - \frac{2}{3(x-1)}, \tag{3.22}$$

$$\Gamma_0(x) = \frac{8}{x^3} - \frac{6}{x^4} + O\left(\frac{1}{x^5}\right), \tag{3.23}$$

and

$$\Gamma_1(x) = -16 + \frac{12}{x} - \frac{64}{x^3} + \frac{40}{x^4} + O\left(\frac{1}{x^5}\right). \tag{3.24}$$

Using the relations (3.15), (3.16), and (3.17), we can derive $\mu(x;\xi)$ and $\omega(x;\xi)$ defined by (3.13) and (3.14), respectively, as

$$\omega(x;\xi) = C_\omega + \frac{\omega_{-1}(x)}{\xi} + \omega_0(x) + O(\xi); \quad \mu(x;\xi) = \mu_0(x) + \xi\mu_1(x) + O(\xi^2), \tag{3.25}$$

where

$$\omega_{-1}(x) = -\frac{1}{x^2} + \frac{2}{3}\int \frac{Q_0(x)e^{-\psi_0(x)}}{x\Omega_0(x)}dx + 2\int\left\{\frac{\Gamma_0(x)\Omega_0(x)}{x^3} + \frac{[\Gamma_0(x)\Omega_0'(x) + \Gamma_0'(x)\Omega_0(x) - \Gamma_0(x)\psi_0'(x)]}{x^2}\right\} \times e^{-\psi_0(x)}dx, \tag{3.26}$$

$$\omega_0 = 2\int \frac{dxe^{-\psi_0(x)}}{x^3}\{\Omega_1(x)\Gamma_0(x) + \Omega_0(x)\Gamma_1(x) - \Omega_0(x)\Gamma_0(x)\} + 2\int \frac{dxe^{-\psi_0(x)}\Gamma_0(x)}{x^2}\{\Omega_1'(x) + \psi_0'(x) - \Omega_0'(x) - \psi_1'(x)\} + 2\int \frac{dxe^{-\psi_0(x)}\Gamma_1(x)}{x^2}\{\Omega_0'(x) - \psi_0'(x)\} + 2\int \frac{dxe^{-\psi_0(x)}}{x^2}\{\Gamma_0'(x)[\Omega_1(x) - \Omega_0(x)] + \Gamma_1'(x)\Omega_0(x)\} + \frac{1}{3}\int \frac{dxe^{-\psi_0(x)}}{x\Omega_0^2(x)}\{\Omega_0(x) + 2\Omega_0(x)[Q_1(x) - Q_0(x)] - 2Q_0(x)\Omega_1(x)\}, \tag{3.27}$$

$$\mu_0(x) = x[1 + \Gamma_0(x)\Omega_0(x)e^{-\psi_0(x)}] - x^3\omega_{-1}(x), \tag{3.28}$$

and

$$\mu_1(x) = xe^{-\psi_0(x)}[\Gamma_1(x)\Omega_0(x) + \Gamma_0(x)\Omega_1(x) - \psi_1(x)\Gamma_0(x)\Omega_0(x)] - x^3\omega_0(x). \tag{3.29}$$

Applying the relations (3.18), (3.19), (3.20), (3.21), (3.22), (3.23), (3.24), (3.25), (3.26), (3.27), (3.28), and (3.29), the explicit form of the solutions (3.25) reduces to the following relations, respectively,

$$\omega(x\to 0) \sim C_\omega + \frac{7}{\xi x^{167/24}} - \frac{6}{x^{191/24}} + \ldots \tag{3.30}$$

and

$$\omega(x\to 1) \sim C_\omega + \frac{(97 - 24\xi^{-1})}{(x-1)^{11/12}} + \frac{(1 + 3\xi^{-1})}{(x-1)^{23/12}} + \ldots \tag{3.31}$$

Using the solutions (3.18), (3.19), (3.20), (3.21), (3.22), (3.23), (3.24), (3.25), (3.30), and (3.31), the explicit form of the function $\mu(x)$ is obtained by the relations

$$\mu(x\to 0) \sim \frac{16\xi}{x^{119/24}} - \frac{(13 + 43.5\xi)}{x^{95/24}} + \frac{8}{x^{71/24}} + \ldots, \tag{3.32}$$

$$\mu(x\to 1) \sim (x-1)^{1/12}\left\{\frac{\xi}{6}\ln(x-1) + \frac{(24 - 97\xi)}{(x-1)} - \frac{(3+\xi)}{(x-1)^2} + \ldots\right\}. \tag{3.33}$$

We apply the solutions (3.21) and (3.22) and obtain the series expansion (3.16) such as follows,

$$\psi(x\to 0) \sim C_\psi + \frac{7\xi}{12} - \frac{1}{24}\ln(x) + \ldots \tag{3.34}$$

and

$$\psi(x\to 1) \sim C_\psi + \frac{(12\xi - 1)}{12}\ln(x-1) + \frac{(1 - 8\xi)}{12(x-1)} - \frac{\xi}{12(x-1)^2} + \ldots \tag{3.35}$$

The boundary conditions (3.4) defined at $x_b = \frac{r_b}{2M} = 1 + \xi$; $0 < \xi \ll 1$, reduces to the following relations

$$\frac{1}{\xi^{1/12}} - \frac{\xi}{6}\ln(\xi) - \frac{23}{\xi} + \frac{3}{\xi^2} \approx 0; \qquad \xi \cong 0.094, \tag{3.36}$$

$$C_\omega(\xi) = 1 - \frac{97}{\xi^{11/12}} + \frac{23}{\xi^{23/12}} - \frac{13}{\xi^{35/12}} \simeq -11\,561.39, \tag{3.37}$$

and

$$C_\psi(\xi) = \frac{4}{3} + \left(\frac{1}{12} - \xi\right)\ln(\xi) \simeq 1.36, \quad (3.38)$$

in which we use (3.31), (3.33), and (3.35), respectively. In the next section we try to obtain the backreaction effects of the quantum scalar field, on the black hole and cosmological event horizons.

IV. BACKREACTION EFFECTS AND SHRUNK HORIZONS

In Sec. II, we saw that the Schwarzschild-de Sitter static black hole metric (2.6), with positive cosmological coupling constant $0 < \xi < 1$, has, respectively, black hole and cosmological event horizons which are obtained by Eq. (2.7) such that

$$1 - \frac{2M}{r} - \xi\left(\frac{r}{2M}\right)^2 = 0; \qquad \xi = \frac{1}{3}\Lambda(2M)^2. \quad (4.1)$$

In Sec. III we determined the value of ξ such as $\xi = 0.094$, by solving the time-independent backreaction equations of the perturbed black hole metric. Applying $\xi = 0.094$, Eq. (4.1) leads us to the black hole and cosmological event horizons of the unperturbed Schwarzschild-de Sitter spacetime (2.6), respectively, at distances (see Fig. 1)

$$x_b^i \approx 1.14 \quad \text{and} \quad x_c^i \approx 2.54. \quad (4.2)$$

Using the solutions (3.30), (3.31), (3.32), (3.33), (3.34), (3.35), (3.36), (3.37), and (3.38), the explicit form of the remnant, stable, static, evaporating Schwarzschild-de Sitter *mini* black hole metric (3.3) is obtained as

$$dS_f^2 = -\Sigma(x)d\tau^2 + \Delta(x)dx^2 + \Pi(x)d\tau dx + x^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (4.3)$$

where we defined $\tau = \frac{t}{2M}$ and $x = \frac{r}{2M}$. Also we obtained approximated forms of the metric components defined by $\Sigma(x)$, $\Delta(x)$, and $\Pi(x)$ such as follows,

$$\Sigma(x \to 0) \sim \frac{17}{x^{2/24}} + \frac{10}{x^{73/24}} - \frac{272}{x^{97/24}} + \frac{459}{x^{121/24}} - \frac{51}{x^{145/24}}, \quad (4.4)$$

$$\Delta(x \to 0) \sim \frac{19}{3}x^{141/24}, \quad (4.5)$$

$$\Pi(x \to 0) \sim -\frac{37.8}{x^{2/24}}, \quad (4.6)$$

$$\Sigma(x \to 1) \sim \frac{1.97}{(x-1)^{2.9}} + \frac{45.54}{(x-1)^{1.9}} + \frac{0.62}{(x-1)^{0.98}} - \frac{225.88}{(x-1)^{0.9}} - 0.46(x-1)^{0.1}\ln(x-1), \quad (4.7)$$

$$\Delta(x \to 1) \sim -\frac{0.05}{(x-1)^{1.07}\ln(x-1)}, \quad (4.8)$$

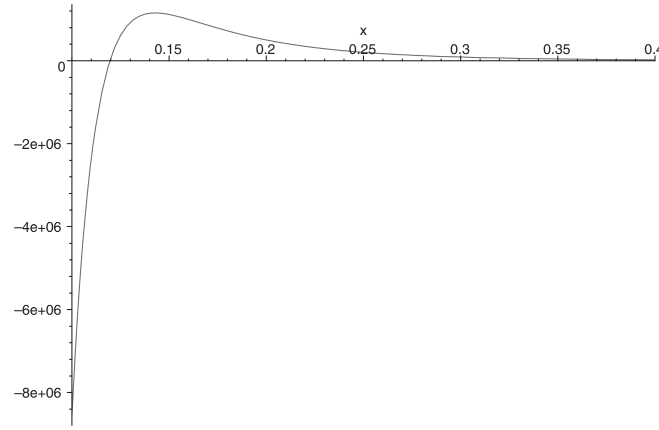

FIG. 2. Diagram of the function $\Sigma(x \to 0)$ (see Eq. (4.4)) shows that the perturbed black hole event horizon is made at $x_b = \frac{r_b}{2M} \approx 0.12$.

$$\Pi(x \to 1) \sim \frac{0.16}{(x-1)^{0.9}} - \frac{1.2}{(x-1)^{0.98}} + \frac{1.04}{(x-1)^{1.9}}. \quad (4.9)$$

The event horizons of the final Schwarzschild-de Sitter mini black hole, reacted by the quantum scalar field, are determined by the equation $\Sigma(x) = 0$, respectively, at distances (see Eqs. (4.3) and (4.6) and their diagrams 2 and 3)

$$x_b^f \approx 0.12 \quad \text{and} \quad x_c^f \approx 1.25. \quad (4.10)$$

This result predicts the end point of the Schwarzschild-de Sitter black hole evaporation such as, it reduces to a remnant *stable* static Schwarzschild-de Sitter *mini* black hole. It will be in accord with the work by Bousso and Hawking [11] in which an evaporating Schwarzschild-de Sitter dilatonic black hole whose size is comparable to that of the cosmological horizon is stable (i.e., it is in thermal equilibrium). Comparing (4.2) and (4.10), not only does the scale of mini black hole horizons become smaller than its first horizons but also the final shrunk horizons come near to each other.

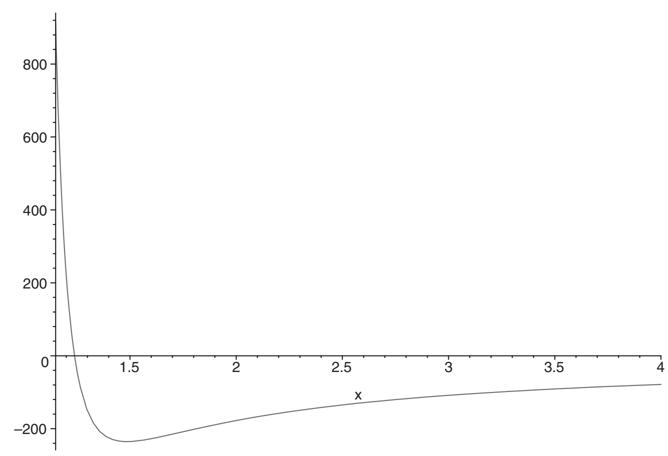

FIG. 3. Diagram of the function $\Sigma(x \to 1)$ (see Eq. (4.7)) shows that the perturbed cosmological horizon is made at $x_c = \frac{r_c}{2M} \approx 1.25$.

V. CONCLUDING REMARKS

According to the work by Christensen and Fulling we obtained the Hawking radiation stress-energy tensor of the Schwarzschild-de Sitter static black hole. Then it is used to solve the time-independent backreaction equations. The linear-order solution of the backreaction equations gave perturbed metric of the black hole which again has two black hole and cosmological shrunk event horizons. In other words, the backreaction effects of the Hawking radiation stress-energy tensor provide that the evaporating Schwarzschild-de Sitter static black hole reduces to a remnant, stable, static, mini black hole. Its singularity $r = 0$ is still covered by a spacelike hypersurface as a final event horizon of the black hole. Furthermore the scale of mini black hole horizons becomes smaller than of its first horizons and the final shrunk horizons come near to each other.

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APPENDIX

Using the relation

$$\langle \hat{T}_{\alpha\beta} \rangle_{\rm ren} = \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} \langle \hat{T}_{\mu\nu} \rangle_{\rm ren} \tag{A1}$$

and assumptions

$$v = t + r^*; \qquad dr^* = \frac{dr}{\Omega(r)}, \tag{A2}$$

we can obtain

$$\langle \hat{T}_{vv} \rangle_{\rm ren} = \langle \hat{T}_{tt} \rangle_{\rm ren} + \langle \hat{T}_{r^*r^*} \rangle_{\rm ren} + \langle \hat{T}_{tr^*} \rangle_{\rm ren} + \langle \hat{T}_{r^*t} \rangle_{\rm ren}, \tag{A3}$$

$$\langle \hat{T}_{vr} \rangle_{\rm ren} = \langle \hat{T}_{rv} \rangle_{\rm ren} = \frac{\langle \hat{T}_{tr^*} \rangle_{\rm ren} + \langle \hat{T}_{r^*r^*} \rangle_{\rm ren}}{\Omega(r)}, \tag{A4}$$

and

$$\langle \hat{T}_{rr} \rangle_{\rm ren} = \frac{\langle \hat{T}_{r^*r^*} \rangle_{\rm ren}}{\Omega^2(r)}. \tag{A5}$$

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