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Relationship between solitonic solutions of five-dimensional Einstein equations

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We give the relation between the solutions generated by the inverse scattering method and the Bäcklund transformation applied to the vacuum five-dimensional Einstein equations. In particular, we show that the two-solitonic solutions generated from an arbitrary diagonal seed by the Bäcklund transformation are contained within those generated from the same seed by the inverse scattering method.

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I. INTRODUCTION

Recently, the studies on higher-dimensional black holes have received much attention, since it has been predicted that they would be produced in a future linear collider [1]. In particular, a stationary black hole solution of the Einstein equation has an important role in that it is expected to describe the classical equilibrium state and Hawking emission from it is considered to give us the signal of black hole production in the linear collider. By a lot of studies on higher-dimensional black holes, it has been clarified that they have a more complicated structure than in four dimensions [2].

Some higher-dimensional stationary black hole solutions were found. As higher-dimensional generalization of the Kerr black hole solution, the Myers-Perry black hole solution, which has an event horizon with spherical topology, was found [3]. Emparan and Reall found a black ring solution as the five-dimensional vacuum Einstein equation, which has an event horizon diffeomorphic to $S^1 \times S^2$ [4] and describes a black hole rotating in the S^1 direction.

The black ring solution which is rotating only in an S^2 direction was also found by two of the present authors [5] by using one of the solitonic solution-generating techniques [6], the so-called Bäcklund transformation. This is essentially the technique to generate new solutions of the Ernst equation from a known solution. In addition, it was shown by the same authors that the black ring with S^1 rotation can be generated by the same solution-generating technique [7].

As another solitonic technique, the inverse scattering technique [8] developed by Belinski and Zakharov is well known. This technique is essentially based on the fact that the Einstein's second-order nonlinear partial differential equations can be replaced with a pair of first-order linear partial differential equations called Lax pair. This method produces vacuum solutions from a certain known vacuum solution called a seed and succeeded in generation of a lot of four-dimensional solutions. In fact, as four-dimensional solutions the Kerr black hole solution, the multi-Kerr black hole solutions, and the Tomimastu-Sato solutions can be generated from the Minkowski seed and physically interesting various solutions were also generated [9,10].

Recently, some higher-dimensional black hole/ring solutions have been generated by using the inverse scattering method. As an infinite number of static solutions of the five-dimensional vacuum Einstein equations with axial symmetry, the five-dimensional Schwarzschild solution and the static black ring solution were reproduced [11], which gave the first example of the generation of a higher-dimensional asymptotically flat black hole solution by the inverse scattering method. The Myers-Perry solution with single and double angular momenta were regenerated from the Minkowski [11,12] and some unphysical seed [13], respectively. The black ring solutions with S^2 rotation was also reproduced by using this method from the Minkowski seed [12]. The S^1-rotating black ring solution was also reproduced from the Levi-Civita solution via the inverse scattering method by one of the authors [14].

Thus, the inverse scattering method is a powerful formalism for solving systems of nonlinear partial differential equations such as the Einstein equation. When we try to find a further new higher-dimensional black hole/ring solution, it is important to know the relationship between solutions generated by the Bäcklund transformation used in Ref. [5] and ones generated by the inverse scattering method used in Refs. [12,14] in that we can avoid the overlap of obtained solutions. Though the relationship between the four-dimensional solutions generated by the inverse scattering method and some Bäcklund transformations were considered in Refs. [15,16] (see also Refs. [10,17] about the relationship between the other generation techniques), the relationship between higher-dimensional solutions generated by these techniques discussed here may be nontrivial and useful. In usual cases, in these solitonic generation techniques a diagonal metric is often used as a seed since such a seed simplifies the analysis. These techniques have a merit in that more complex solutions can be generated from simple solutions. This is why in this article, we investigate the relation between the solutions generated from a diagonal seed by both solitonic generation techniques applied to five dimensions, and show that the two-solitonic solutions generated by Bäcklund transformation in Ref. [5] from an arbitrary diagonal seed coincide with two-solitonic solutions generated from the same seed by the inverse scattering method under the special normalization.

This article is organized as follows: In Sec. II, we review the Bäcklund transformation developed by two of the authors in Ref. [5] and the inverse scattering method applied to five dimensions by one of the authors [12]. In Sec. III, we show that the two-solitonic solutions generated by the former technique from a diagonal seed coincide with the special solutions generated by the latter technique from the same seed under the special normalization.

II. PRELIMINARY

In this article, we consider the spacetimes which satisfy the following conditions: (1) five dimensions, (2) asymptotically flat spacetimes, (3) the solutions of vacuum Einstein equations, (4) having three commuting Killing vectors including one time-translational Killing vector and two axial Killing vectors, (5) having a single nonzero angular momentum component.

A. Bäcklund transformation

Here we review the generation technique of the five-dimensional solution established by Ref. [5]. Under the conditions (1)–(5), we can start the analysis from the following form of the metric

$$ds^2 = e^{-T}[-e^S(dt-\omega d\phi)^2 + e^{T+2U_1}\rho^2(d\phi)^2 + e^{2(\gamma+U_1)+T}(d\rho^2+dz^2)] + e^{2T}(d\psi)^2. \quad (1)$$

Using this metric form the Einstein equations are reduced to the following set of equations,

(i)
$$\nabla^2 T = 0,$$

(ii)
$$\begin{cases} \partial_\rho \gamma_T = \frac{3}{4}\rho[(\partial_\rho T)^2 - (\partial_z T)^2] \\ \partial_z \gamma_T = \frac{3}{2}\rho[\partial_\rho T \partial_z T], \end{cases}$$

(iii)
$$\nabla^2 \mathcal{E}_S = \frac{2}{\mathcal{E}_S + \bar{\mathcal{E}}_S}\nabla\mathcal{E}_S \cdot \nabla\mathcal{E}_S,$$

(iv)
$$\begin{cases} \partial_\rho \gamma_S = \frac{\rho}{2(\mathcal{E}_S+\bar{\mathcal{E}}_S)}(\partial_\rho\mathcal{E}_S\partial_\rho\bar{\mathcal{E}}_S - \partial_z\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S) \\ \partial_z \gamma_S = \frac{\rho}{2(\mathcal{E}_S+\bar{\mathcal{E}}_S)}(\partial_\rho\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S + \partial_\rho\mathcal{E}_S\partial_z\bar{\mathcal{E}}_S), \end{cases}$$

(v)
$$(\partial_\rho\Phi, \partial_z\Phi) = \rho^{-1}e^{2S}(-\partial_z\omega, \partial_\rho\omega),$$

(vi)
$$\gamma = \gamma_S + \gamma_T,$$

(vii)
$$U_1 = -\frac{S+T}{2},$$

where the function $\Phi(\rho, z)$ is defined through the equation (v) and the function $\mathcal{E}_S$ is defined by $\mathcal{E}_S := e^S + i\Phi$. It should be noted that e^S and Φ correspond to a gravitational potential and a twist potential. The equation (iii) is exactly the same as the Ernst equation in four dimensions [18]. The most nontrivial task to obtain new metrics is to solve the equation (iii) because of its nonlinearity. Here the method similar to the Neugebauer's Bäcklund transformation [19] or the Hoenselaers-Kinnersley-Xanthopoulos transformation [20] is used.

Following the procedure given by Castejon-Amenedo and Manko [21], for a static seed solution $e^{S^{(0)}}$ a new Ernst potential can be written in the form

$$\mathcal{E}_S = e^{S^{(0)}}\frac{x(1+ab)+iy(b-a)-(1-ia)(1-ib)}{x(1+ab)+iy(b-a)+(1-ia)(1-ib)},$$

where x and y are the prolate-spheroidal coordinates: $\rho = \sigma\sqrt{x^2-1}\sqrt{1-y^2}$, $z=\sigma xy$ with the ranges $1 \le x$ and $-1 \le y \le 1$, and the functions a and b satisfy the following simple first-order differential equations

$$\begin{aligned} (\ln a)_{,x} &= \frac{1}{x-y}[(xy-1)S^{(0)}_{,x} + (1-y^2)S^{(0)}_{,y}], \\ (\ln a)_{,y} &= \frac{1}{x-y}[-(x^2-1)S^{(0)}_{,x} + (xy-1)S^{(0)}_{,y}], \\ (\ln b)_{,x} &= -\frac{1}{x+y}[(xy+1)S^{(0)}_{,x} + (1-y^2)S^{(0)}_{,y}], \\ (\ln b)_{,y} &= -\frac{1}{x+y}[-(x^2-1)S^{(0)}_{,x} + (xy+1)S^{(0)}_{,y}]. \end{aligned} \quad (2)$$

The corresponding expressions for the metric functions can be obtained by using the formulas shown by [21]. For the seed,

$$ds^2 = e^{-T^{(0)}}[-e^{S^{(0)}}dt^2 + e^{-S^{(0)}}\rho^2(d\phi)^2 + e^{2\gamma_{(0)}-S^{(0)}}(d\rho^2+dz^2)] + e^{2T^{(0)}}(d\psi)^2, \quad (3)$$

a new solution is given by

$$\begin{aligned} ds^2 = &-e^{S^{(0)}-T^{(0)}}\frac{A}{B}\left[dt - \left(2\sigma e^{-S^{(0)}}\frac{C}{A} + C_1\right)d\phi\right]^2 \\ &+ \frac{B}{A}e^{-S^{(0)}-T^{(0)}}\sigma^2(x^2-1)(1-y^2)d\phi^2 + e^{2T^{(0)}}d\psi^2 \\ &+ e^{2\gamma-S^{(0)}-T^{(0)}}\frac{B}{A}\sigma^2(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right), \end{aligned} \quad (4)$$

where A, B, and C are defined by

$$\begin{aligned} A &:= (x^2-1)(1+ab)^2 - (1-y^2)(b-a)^2, \\ B &:= [(x+1)+(x-1)ab]^2 + [(1+y)a+(1-y)b]^2, \\ C &:= (x^2-1)(1+ab)[(1-y)b-(1+y)a] \\ &\quad + (1-y^2)(b-a)[(x+1)-(x-1)ab]. \end{aligned} \quad (5)$$

To assure that the spacetime does not have global rotation, the constant C_1 is given by

$$C_1 = \frac{2\sigma^{1/2}\alpha}{1+\alpha\beta}. \tag{6}$$

B. Inverse scattering techniques

We will give a summary of the inverse scattering method developed by Belinski and Zakharov [8] which is applied to five dimensions.

As in the previous subsection, we consider the asymptotically flat, five-dimensional stationary and axisymmetric vacuum spacetime with three commuting Killing vector fields $V_{(i)}$ ($i = 1, 2, 3$) following the argument in [12,14]. The commutativity of Killing vectors $[V_{(i)}, V_{(j)}] = 0$ enables us to find a coordinate system such that $V_{(i)} = \partial/\partial x^i$ ($i = 1, 2, 3$) and the metric is independent of the coordinates x^i, where $(\partial/\partial x^1)$ is the Killing vector field associated with time translation and $(\partial/\partial x^2)$, $(\partial/\partial x^3)$ denote the spacelike Killing vector fields with closed orbits. We put $x^1 = t$, $x^2 = \phi$, and $x^3 = \psi$. From the theorem in Ref. [22], in such a spacetime, the metric can be written in the canonical form [22] as

$$ds^2 = f(d\rho^2 + dz^2) + g_{ij}dx^i dx^j, \tag{7}$$

where $f = f(\rho, z)$ and $g_{ij} = g_{ij}(\rho, z)$ are a function and an induced metric on the three-dimensional space, respectively. Both of them depend only on the coordinates ρ and z. Here it is the most convenient to choose the 3×3 matrix $g = (g)_{ij}$ as to satisfy the condition

$$\det g = -\rho^2. \tag{8}$$

This is compatible with the vacuum Einstein equations $g^{ij}R_{ij} = 0$, which reduces to $(\partial_\rho^2 + \partial_z^2)(-\det g)^{1/2} = 0$. It follows from $R_{ij} = 0$ that the matrix g satisfies the solitonic equation

$$(\rho g_{,\rho} g^{-1})_{,\rho} + (\rho g_{,z} g^{-1})_{,z} = 0. \tag{9}$$

From the other components of the Einstein equations $R_{\rho\rho} - R_{zz} = 0$ and $R_{\rho z} = 0$, we obtain the equations which determine the function $f(\rho, z)$ for a given solution of the solitonic Eq. (9)

$$(\ln f)_{,\rho} = -\frac{1}{\rho} + \frac{1}{4\rho}\mathrm{Tr}(U^2 - V^2), \tag{10}$$

$$(\ln f)_{,z} = \frac{1}{2\rho}\mathrm{Tr}(UV), \tag{11}$$

where the 3×3 matrices $U(\rho, z)$ and $V(\rho, z)$ are defined by

$$U := \rho g_{,\rho} g^{-1}, \qquad V := \rho g_{,z} g^{-1}. \tag{12}$$

The integrability condition with respect to f is automatically satisfied for the solution g of Eq. (9). Note also that $R_{\rho\rho} + R_{zz} = 0$ is consistent with the solution (9)–(11).

Although our immediate goal is to solve the differential equations (9), it cannot be generally solved due to its nonlinearity. But in analogy with the soliton technique, we can find the Lax pair for the matrix equations (9). We consider Schrödinger-type equations for the 3×3 matrix $\psi(\lambda, \rho, z)$ as in four dimensions;

$$D_1\psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\psi, \qquad D_2\psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\psi, \tag{13}$$

where λ is a complex spectral parameter independent of ρ and z. The differential operators D_1 and D_2 are defined as

$$D_1 := \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\partial_\lambda, \qquad D_2 := \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\partial_\lambda, \tag{14}$$

which can be shown to commute $[D_1, D_2] = 0$. Note that Eq. (14) is invariant under the transformation $\lambda \to -\rho^2/\lambda$. Then the compatibility condition $[D_1, D_2]\psi = 0$ reduces to the Einstein equations (9) with

$$g(\rho, z) = \psi(0, \rho, z). \tag{15}$$

It is worth noting that the Einstein's second-order nonlinear partial differential equations (9) are reduced to a pair of first-order linear partial differential equations (13).

Let g_0, U_0, V_0, and ψ_0 be particular solutions of Eqs. (9) and (13). We shall call the known solution g_0 the seed solution. We are going to seek a new solution of the form

$$\psi = \chi\psi_0, \tag{16}$$

which leads the following equations that the dressing matrix $\chi(\lambda, \rho, z)$ must satisfy

$$D_1\chi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2},$$
$$D_2\chi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2}. \tag{17}$$

In order for the solutions $g(\rho, z)$ to be real and symmetric, we impose the following conditions on the dressing matrix χ,

$$\bar{\chi}(\bar{\lambda}, \rho, z) = \chi(\lambda, \rho, z), \qquad \bar{\psi}(\bar{\lambda}, \rho, z) = \psi(\lambda, \rho, z), \tag{18}$$

and

$$g = \chi(-\rho^2/\lambda, \rho, z) g_0^T \chi(\lambda, \rho, z), \tag{19}$$

where $\bar{\chi}$ and ${}^T\chi$ denote complex conjugation and the transposition of χ. From Eqs. (16) and (19), the dressing matrix χ asymptotes to a unit matrix $\chi \to I$ as $\lambda \to \infty$.

The general n-soliton solutions for the matrix g are generated due to the presence of the simple poles of the dressing matrix on the complex λ-plane:

$$\chi = I + \sum_{k=1}^{n} \frac{R_k}{\lambda - \mu_k}, \tag{20}$$

where the matrices R_k and the position of the pole μ_k depend only on the variables ρ and z. Here and hereafter, the subscript k, l counts the number of solitons. It is the characteristic feature of solitons that the dressing matrix χ is represented as the meromorphic function on the complex λ-plane. Pole trajectories $\mu_k(\rho, z)$ are determined by the condition that the left-hand side of Eq. (17) has no poles of second order at $\lambda = \mu_k$, which yields the following two differential equations for $\mu_k(\rho, z)$:

$$\mu_{k,z} = -\frac{2\mu_k^2}{\mu_k^2 + \rho^2}, \qquad \mu_{k,\rho} = \frac{2\rho\mu_k}{\mu_k^2 + \rho^2}, \tag{21}$$

which are expressed by the solutions of the following quadratic equations

$$\mu_k^2 + 2(z - w_k)\mu_k - \rho^2 = 0, \tag{22}$$

where w_k are arbitrary constants. Solving Eq. (22), one can easily see

$$\mu_k = w_k - z \pm \sqrt{(z - w_k)^2 + \rho^2}, \tag{23}$$

where w_k are arbitrary constants. Since the matrices R_k are degenerate at the poles $R_k\chi^{-1}(\mu_k) = 0$, which follows from the condition $\chi\chi^{-1} = I$ at $\lambda = \mu_k$, it is possible to write down the matrix elements of R_k in the form

$$(R_k)_{ij} = n_i^{(k)} m_j^{(k)}. \tag{24}$$

The fact that Eq. (17) has no residues at the poles $\lambda = \mu_k$ leads to obtain the vectors $m_i^{(k)}$ as

$$m^{i(k)} = m_{0j}^{(k)} [\psi_0^{-1}(\mu_k, \rho, z)]^{ji}, \tag{25}$$

where $m_{0i}^{(k)}$ are arbitrary constants. The vectors $n_i^{(k)}$, on the other hand, are determined by the condition that Eq. (19) is regular at $\lambda = \mu_k$ as

$$n_i^{(k)} = \sum_{l=1}^{n} \mu_k^{-1} (\Gamma^{-1})_{kl} L_i^{(l)}, \tag{26}$$

where the vectors $L_i^{(k)}$ and the symmetric matrix Γ_{kl} are given by

$$L_i^{(k)} = m^{j(k)} (g_0)_{ij}, \tag{27}$$

$$\Gamma_{kl} = \frac{m^{i(k)} (g_0)_{ij} m^{j(l)}}{\rho^2 + \mu_k \mu_l}, \tag{28}$$

respectively. Therefore one can now find from Eqs. (15), (16), and (20) that the matrix g becomes

$$g_{ij}^{(\mathrm{unphys})} = \psi(0, \rho, z)_{ij} = (g_0)_{ij} - \sum_{k,l=1}^{n} (\Gamma^{-1})_{kl} \mu_k^{-1} \mu_l^{-1} L_i^{(k)} L_j^{(l)}. \tag{29}$$

This metric does not meet the condition $\det g = -\rho^2$, which we have denoted $g^{(\mathrm{unphys})}$. In order to satisfy the gauge condition $\det g = -\rho^2$, the metric should be appropriately normalized. One example is to normalize all the metric components by the same weight as

$$g^{(\mathrm{phys})} = (-1)^{n/3} \rho^{-2n/3} \left(\prod_{k=1}^{n} \mu_k^{2/3} \right) g^{(\mathrm{unphys})}, \tag{30}$$

where $g^{(\mathrm{phys})}$ is the metric which fulfills the condition $\det g = -\rho^2$. Actually, the four-dimensional Kerr solution is obtained similarly by the overall normalization as Eq. (30). Substituting the physical metric solution $g^{(\mathrm{phys})}$ given by Eq. (30) into Eqs. (10) and (11), we obtain a physical value of f as

$$f = C_0 f_0 \rho^{-n(n-1)/3} \det(\Gamma_{kl}) \prod_{k=1}^{n} [\mu_k^{2(n+2)/3} (\mu_k^2 + \rho^2)^{-1/3}] \cdot \prod_{k>l}^{n} (\mu_k - \mu_l)^{-4/3}, \tag{31}$$

where C_0 is an arbitrary constant, and f_0 is a value of f corresponding to the seed g_0. In general, these solutions have two angular momentum components. In this article, we study the relationship between the solutions generated by the inverse scattering method applied to five dimensions and those generated by the Bäcklund transformation in Ref. [5] which generates the only solutions with a single angular momentum component. So, we should compare the solutions which have a single angular momentum component generated by them with each other. As discussed in Ref. [12], since the two-solitonic solution (30) with $n = 2$ would not be regular on a certain part of an axis (as far as we choose a seed regular on it), it is suitable to normalize the metric so that $(g_0)_{33}$ is unchanged:

$$g^{(\mathrm{phys})} = \begin{pmatrix} \left(\prod_{k=1}^{n} \frac{\mu_k}{\rho} \right) g_{AB}^{(\mathrm{unphys})} & 0 \\ 0 & (g_0)_{33} \end{pmatrix}, \tag{32}$$

where $A, B = 1, 2$. Here, we consider the two-soliton solution. We choose the sign of plus in Eq. (23) and take the constants $w_1 = -w_2 = -\sigma$.

III. RELATION BETWEEN TWO-SOLITONIC SOLUTIONS

In this section, we show that for a general diagonal seed solution which takes the form of

$$ds^2 = g_1' dt^2 + g_2' d\phi^2 + g_3' d\psi^2 + f'(d\rho^2 + dz^2), \tag{33}$$

where g_1', g_2', and g_3' are functions of ρ and z, and satisfy the constraint $g_1' g_2' g_3' = -\rho^2$, the two-solitonic solutions generated by the inverse scattering method under the special normalization (32) coincide with ones generated by the Bäcklund transformation explained in Sec. II A. To do so, as a diagonal seed, instead of (33) it is sufficient to consider the following metric form,

$$ds^2 = -dt^2 + g_2 d\phi^2 + g_3 d\psi^2 + f(d\rho^2 + dz^2), \quad (34)$$

where g_2 and g_3 are functions of ρ and z, and satisfy the constraint $g_2 g_3 = \rho^2$ (in fact, starting with this form of the seed metric simplifies the proof).

The reason for this is explained as follows. Let us consider the conformal transformation of the two-dimensional metric $g_{AB}(A, B = t, \phi)$ and the rescale of the $\psi\psi$-component with the determinant $\det g$ invariant ;

$$g_0 = \mathrm{diag}(-1, g_2, g_3) \to g_0' = \mathrm{diag}(-\Omega, \Omega g_2, \Omega^{-2} g_3), \quad (35)$$

where $\ln\Omega$ must be a harmonic function on the three-dimensional Euclid space in order to assure that the transformed metric is the solution of Eq. (9). (Since the metric function f or f' is determined by only the three-dimensional metric g_0 or g_0', we need not consider this for the present purpose.) Under this transformation, the physical metric (32) is transformed as

$$g = \begin{pmatrix} g_{AB} & 0 \\ 0 & g_3 \end{pmatrix} \to g' = \begin{pmatrix} \Omega g_{AB} & 0 \\ 0 & \Omega^{-2} g_3 \end{pmatrix}. \quad (36)$$

From this, we see that the transformation (35) of a seed commutes with the operation of putting two solitons on the background. Therefore we can obtain the two-solitonic solution generated from a diagonal seed $g_0' = \mathrm{diag}(g_1', g_2', g_3')$ such that the tt-component is not -1 by the transformation (36) with $\Omega = g_1'$ for the two-solitonic solution generated from the seed (34).

For the Bäcklund transformation, the same fact also holds, i.e. by the transformation (35) of the seed, the solution generated is transformed as Eq. (36). Note that the seed functions for the metric (34) can be written in terms of g_2 as

$$S^{(0)} = T^{(0)} = -\frac{1}{2}\ln\left(\frac{g_2}{\rho^2}\right). \quad (37)$$

The two-solitonic solution for the general seed metric with seed functions $S^{(0)} = -1/2\ln(g_2/\rho^2)$ and $T'_{(0)} \neq -1/2\ln(g_2/\rho^2)$ can be obtained from the solitonic solution of the seed (34) with the subsequent transformation (35) with $\Omega = e^{S^{(0)} - T'_{(0)}}$. As a result, we can conclude that it is sufficient to assume the form of the diagonal seed as Eq. (34), i.e., the seed such that $(g_0)_{tt} = -1$ as far as we consider a diagonal seed solution.

To begin with, we show that for an arbitrary diagonal seed (34), the solutions of Eqs. (2) are given by

$$a = -\alpha(\kappa_1 + 1)\sigma^{3/2}\frac{g_2^{1/2}(x+1)(1-y)}{\rho\psi_2[\rho, z, \mu_2]},$$
$$b = -\beta\frac{\rho\psi_2[\rho, z, \mu_1]}{(\kappa_2 - 1)\sigma^{3/2}(x-1)(1-y)g_2^{1/2}}, \quad (38)$$

where $\psi_2[\rho, z, \lambda]$ is the $\phi\phi$-components of the solution ψ of Eqs. (13) (we may assume the generating matrix $\psi_0[\rho, z, \lambda]$ to be diagonal $\psi_0[\rho, z, \lambda] = \mathrm{diag}(\psi_1[\rho, z, \lambda], \psi_2[\rho, z, \lambda], \psi_3[\rho, z, \lambda])$ for a diagonal seed). α, β, κ_1, and κ_2 are arbitrary constants.

From Eq. (37), the right-hand side in the first equation of (2) is reduced to

$$\frac{1}{x-y}[(xy-1)S^{(0)}_{,x} + (1-y^2)S^{(0)}_{,y}]$$
$$= -\frac{1}{2(x-y)}\left[(xy-1)\left(\ln\frac{g_2}{\rho^2}\right)_{,x} + (1-y^2)\left(\ln\frac{g_2}{\rho^2}\right)_{,y}\right]. \quad (39)$$

On the other hand, using the first equation of (38), the left-hand side in Eq. (2) becomes

$$(\ln a)_{,x} = \frac{1}{2}(\ln g_2)_{,x} + \frac{1}{x+1} - (\ln\rho)_{,x} - (\ln\psi_2[\rho, z, \mu_2])_{,x}$$
$$= \frac{1}{2}(\ln g_2)_{,x} + \frac{1}{x+1} - (\ln\rho)_{,x} - \frac{\sigma^2}{\rho}x(1-y^2)$$
$$\times (\ln\psi_2[\rho, z, \mu_2])_{,\rho} - \sigma y(\ln\psi_2[\rho, z, \mu_2])_{,z}. \quad (40)$$

Let us note that the term containing $(\ln\psi[\rho, z, \mu_2])_{,\rho}$ in the above equation is computed as

$$(\ln\psi_2[\rho, z, \mu_2])_{,\rho} = (\ln\psi_2[\rho, z, \lambda])_{,\rho}|_{\lambda=\mu_2}$$
$$+ (\ln\psi_2[\rho, z, \lambda])_{,\lambda}|_{\lambda=\mu_2}\cdot\mu_{2,\rho}$$
$$= -\frac{2\mu_2\rho}{\rho^2 + \mu_2^2}(\ln\psi_2[\rho, z, \lambda])_{,\lambda}|_{\lambda=\mu_2}$$
$$+ \frac{\rho^2(\ln g_2)_{,\rho} + \rho\mu_2(\ln g_2)_{,z}}{\rho^2 + \mu_2^2}$$
$$+ (\ln\psi_2[\rho, z, \lambda])_{,\lambda}|_{\lambda=\mu_2}\cdot\frac{2\mu_2\rho}{\rho^2 + \mu_2^2}$$
$$= \frac{\rho^2(\ln g_2)_{,\rho} + \rho\mu_2(\ln g_2)_{,z}}{\rho^2 + \mu_2^2}, \quad (41)$$

where we used Eqs. (13) and (21). Similarly, the term containing $(\ln\psi[\rho, z, \mu_2])_{,z}$ in Eq. (40) can be computed as

$$(\ln\psi_2[\rho, z, \mu_2])_{,z} = \frac{\rho^2(\ln g_2)_{,z} - \rho\mu_2(\ln g_2)_{,\rho}}{\rho^2 + \mu_2^2}. \quad (42)$$

Therefore, using Eqs. (41) and (42), Eq. (40) becomes

$$(\ln a)_{,x} = -\frac{1}{2(x-y)}\left[(xy-1)\left(\ln\frac{g_2}{\rho^2}\right)_{,x} + (1-y^2)\left(\ln\frac{g_2}{\rho^2}\right)_{,y}\right]. \tag{43}$$

This coincides with the right-hand side of Eq. (39), which implies that a in Eq. (38) is a solution of the first equation in Eq. (2). Similarly, we can show a satisfies the second equation of (2) and that b also satisfies the third and fourth equations of (2). As a result we see that the solutions (2) are given by Eqs. (38) for a diagonal seed whose tt-component is -1.

Substituting Eq. (38) into Eqs. (4) and (5), we obtain the following general solution generated from a static seed solution,

$$g_{tt} = -\frac{\tilde{A}}{\tilde{B}}, \qquad g_{t\phi} = 2\sigma^{1/2}g_2\frac{\tilde{C}}{\tilde{B}} + C_1\frac{\tilde{A}}{\tilde{B}}, \qquad g_{\phi\phi} = \frac{g_{t\phi}^2 - \rho^2}{g_{tt}}. \tag{44}$$

Here we have introduced new functions $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ defined as

$$\begin{aligned}\tilde{A} = {} & -\beta^2\psi_2[\mu_1]^2\psi_2[\mu_2]^2(1+y)^2 \\ & + \sigma\alpha^2\beta^2(\kappa_1+1)^2 g_2\psi_2[\mu_1]^2(x+1)^2 \\ & + \sigma(\kappa_2-1)^2 g_2\psi_2[\mu_2]^2(x-1)^2 \\ & - \sigma^2\alpha^2(\kappa_1+1)^2(\kappa_2-1)^2 g_2^2(1-y)^2 \\ & + 2\sigma\alpha\beta(\kappa_1+1)(\kappa_2-1)g_2\psi_2[\mu_1]\psi_2[\mu_2](x^2-y^2),\end{aligned} \tag{45}$$

$$\begin{aligned}\tilde{B} = {} & \beta^2\psi_2[\mu_1]^2\psi_2[\mu_2]^2(1-y^2) \\ & + \sigma^2\alpha^2(\kappa_1+1)^2(\kappa_2-1)^2 g_2^2(1-y^2) \\ & + \sigma\alpha^2\beta^2(\kappa_1+1)^2\psi_2[\mu_1]^2 g_2(x^2-1) \\ & + \sigma(\kappa_2-1)^2\psi_2[\mu_2]^2 g_2(x^2-1) \\ & + 2\sigma\alpha\beta(\kappa_1+1)(\kappa_2-1)g_2\psi_2[\mu_1]\psi_2[\mu_2](x^2-y^2),\end{aligned} \tag{46}$$

$$\begin{aligned}\tilde{C} = {} & -\beta(\kappa_2-1)\psi_2[\mu_1]\psi_2[\mu_2]^2(x+y) \\ & - \alpha\beta^2(\kappa_1+1)\psi_2[\mu_1]^2\psi_2[\mu_2](x-y) \\ & + \sigma\alpha^2\beta(\kappa_2-1)(\kappa_1+1)^2 g_2\psi_2[\mu_1](x+y) \\ & + \sigma\alpha(\kappa_1+1)(\kappa_2-1)^2 g_2\psi_2[\mu_2](x-y).\end{aligned} \tag{47}$$

Next, let us consider the solutions generated from the inverse scattering method. Under the special normalization (32), the two-soliton solution can be written in the following form:

$$\begin{aligned} g_{tt}^{(\mathrm{phys})} &= -\frac{G_{tt}}{\mu_1\mu_2\Sigma}, \\ g_{t\phi}^{(\mathrm{phys})} &= -g_2\frac{(\rho^2+\mu_1\mu_2)G_{t\phi}}{\mu_1\mu_2\Sigma}, \\ g_{\phi\phi}^{(\mathrm{phys})} &= -g_2\frac{G_{\phi\phi}}{\mu_1\mu_2\Sigma},\end{aligned} \tag{48}$$

$$g_{\psi\psi}^{(\mathrm{phys})} = g_3, \qquad g_{\phi\psi}^{(\mathrm{phys})} = g_{t\psi}^{(\mathrm{phys})} = 0, \tag{49}$$

where the functions G_{tt}, $G_{t\phi}$, $G_{\phi\phi}$, and Σ are defined as

$$\begin{aligned} G_{tt} = {} & -m_{01}^{(1)2}m_{01}^{(2)2}\psi_2[\mu_1]^2\psi_2[\mu_2]^2(\mu_1-\mu_2)^2\rho^4 \\ & + m_{01}^{(1)2}m_{02}^{(2)2}g_2\mu_2^2(\rho^2+\mu_1\mu_2)^2\psi_2[\mu_1]^2 \\ & + m_{01}^{(2)2}m_{02}^{(1)2}g_2\mu_1^2(\rho^2+\mu_1\mu_2)^2\psi_2[\mu_2]^2 \\ & - m_{02}^{(1)2}m_{02}^{(2)2}g_2^2\mu_1^2\mu_2^2(\mu_1-\mu_2)^2 \\ & - 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\psi_2[\mu_1]\psi_2[\mu_2] \\ & \times(\rho^2+\mu_1^2)(\rho^2+\mu_2^2)\mu_1\mu_2,\end{aligned} \tag{50}$$

$$\begin{aligned} G_{\phi\phi} = {} & m_{01}^{(1)2}m_{01}^{(2)2}\mu_1^2\mu_2^2(\mu_1-\mu_2)^2\psi_2[\mu_1]^2\psi_2[\mu_2]^2 \\ & + m_{02}^{(1)2}m_{02}^{(2)2}g_2^2(\mu_1-\mu_2)^2\rho^4 \\ & - m_{01}^{(1)2}m_{02}^{(2)2}g_2\mu_1^2\psi_2[\mu_1]^2(\rho^2+\mu_1\mu_2)^2 \\ & - m_{01}^{(2)2}m_{02}^{(1)2}g_2\mu_2^2(g_2-\mu_2)^2(\rho^2+\mu_1\mu_2)^2 \\ & + 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\mu_1\mu_2\psi_2[\mu_2]\psi_2[\mu_1] \\ & \times(\rho^2+\mu_1^2)(\rho^2+\mu_2^2),\end{aligned} \tag{51}$$

$$\begin{aligned} G_{t\phi} = {} & m_{01}^{(1)}m_{01}^{(2)2}m_{02}^{(1)}\mu_2(\mu_1-\mu_2)\psi_2[\mu_2]^2\psi_2[\mu_1](\rho^2+\mu_1^2) \\ & + m_{01}^{(1)}m_{02}^{(1)}m_{02}^{(2)2}g_2\mu_2(\mu_2-\mu_1)\psi_2[\mu_1](\rho^2+\mu_1^2) \\ & + m_{01}^{(1)2}m_{01}^{(2)}m_{02}^{(2)}\mu_1(\mu_2-\mu_1)\psi_2[\mu_1]^2\psi_2[\mu_2](\rho^2+\mu_2^2) \\ & + m_{01}^{(2)}m_{02}^{(1)2}m_{02}^{(2)}\mu_1 g_2\psi_2[\mu_2](\rho^2+\mu_2^2)(\mu_1-\mu_2),\end{aligned} \tag{52}$$

$$\begin{aligned} \Sigma = {} & m_{01}^{(1)2}m_{01}^{(2)2}\psi_2[\mu_1]^2\psi_2[\mu_2]^2(\mu_1-\mu_2)^2\rho^2 \\ & + m_{02}^{(1)2}m_{02}^{(2)2}g_2^2(\mu_1-\mu_2)^2\rho^2 \\ & + m_{01}^{(1)2}m_{02}^{(2)2}g_2\psi_2[\mu_1]^2(\rho^2+\mu_1\mu_2)^2 \\ & + m_{02}^{(1)2}m_{01}^{(2)2}g_2\psi_2[\mu_2]^2(\rho^2+\mu_1\mu_2)^2 \\ & - 2m_{01}^{(1)}m_{01}^{(2)}m_{02}^{(1)}m_{02}^{(2)}g_2\psi_2[\mu_1]\psi_2[\mu_2] \\ & \times(\rho^2+\mu_1^2)(\rho^2+\mu_2^2),\end{aligned} \tag{53}$$

where the two functions g_2 and g_3 are given by Eq. (34).

In order for the metric to approach the Minkowski spacetime asymptotically, let us consider the coordinate transformation of the physical metric such that

$$t \to t' = t - C_1\phi, \qquad \phi \to \phi' = \phi, \tag{54}$$

where C_1 is a constant chosen to ensure the asymptotic flatness. We should note that the transformed metric also satisfies the supplementary condition $\det g = -\rho^2$. Under this transformation, the physical metric components become

$$\begin{aligned} g_{tt}^{(\rm phys)} &\to g_{t't'}^{(\rm phys)} = g_{tt}^{(\rm phys)}, \\ g_{t\phi}^{(\rm phys)} &\to g_{t'\phi'}^{(\rm phys)} = g_{t\phi}^{(\rm phys)} + C_1 g_{tt}^{(\rm phys)}, \\ g_{\phi\phi}^{(\rm phys)} &\to g_{\phi'\phi'}^{(\rm phys)} = g_{\phi\phi}^{(\rm phys)} + 2C_1 g_{t\phi}^{(\rm phys)} + C_1^2 g_{tt}^{(\rm phys)}. \end{aligned} \tag{55}$$

If we choose the parameters such that

$$m_{01}^{(1)} m_{01}^{(2)} = \beta, \tag{56}$$

$$m_{01}^{(2)} m_{02}^{(1)} = \sigma^{1/2}(\kappa_2 - 1), \tag{57}$$

$$m_{01}^{(1)} m_{02}^{(2)} = -\sigma^{1/2}\alpha\beta(\kappa_1 + 1), \tag{58}$$

$$m_{02}^{(1)} m_{02}^{(2)} = -\sigma\alpha(\kappa_1 + 1)(\kappa_2 - 1), \tag{59}$$

$$C_1 = \frac{2\sigma^{1/2}\alpha}{1 + \alpha\beta}, \tag{60}$$

and use the prolate spherical coordinate (x, y), we can confirm that the transformed metric coincides with the metric (44) generated by the technique used in Ref. [5] from a diagonal seed. In order to show the coincidence of the metrics, it is sufficient to check only two components g_{tt} and $g_{t\phi}$ due to the supplementary condition $\det g = -\rho^2$ and the fact that the metric components $g_{\rho\rho}$ and g_{zz} (or, g_{xx} and g_{yy}) are determined by the three-dimensional metric g_{ij}.

IV. SUMMARY AND DISCUSSION

In this article, we studied the relation between the inverse scattering method and the Bäcklund transformation applied to five dimensions. We showed that the two-solitonic solution generated from an arbitrary diagonal seed by the Bäcklund transformation coincides with one generated from the same diagonal seed by the inverse scattering method under the special normalization (32). This implies that the five-dimensional solutions generated by the inverse scattering method contain the ones generated by the Bäcklund transformation used in Ref. [5] as concerned with the two-solitonic solutions generated from a diagonal seed. As clarified in the previous works [5,7,12,14], if we choose the five-dimensional Minkowski or the Euclidean C-metric as a diagonal seed, we can obtain the black ring solution with a rotating two-sphere [5] or the black ring solution found by Emparan and Reall [4] as the two-solitonic solution, respectively. Therefore, we see that the previous works [12,14] correspond to the special cases of the present result.

However, while the Bäcklund transformation used in Ref. [5] can generate solutions with a single angular momentum component at the most, the inverse scattering method can generate five-dimensional solutions with two angular momentum components, as discussed in Refs. [12,14]. In fact, the Myers-Perry black hole solution with two angular momentum components was generated by the inverse scattering method [13]. It is expected that a new black ring solution with two angular momentum components may be generated by this method.

Though in this article, we focus on the two-solitonic solution generated from a diagonal seed by these techniques, we also expect solutions generated from nondiagonal seeds by both generation techniques or a multisolitonic solution (more than two) generated by them to be the same if we choose the special normalization (32) in the inverse scattering method. It would be also interesting to deal with the generation of solutions in the five-dimensional Einstein-Maxwell equations with the same symmetries [23].

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