Tensor-vector-scalar cosmology: Covariant formalism for the background evolution and linear perturbation theory

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A relativistic theory of gravity has recently been proposed by Bekenstein, where gravity is mediated by a tensor, a vector, and a scalar field, thus called TeVeS. The theory aims at modifying gravity in such a way as to reproduce Milgrom's modified Newtonian dynamics (MOND) in the weak field, nonrelativistic limit, which provides a framework to solve the missing mass problem in galaxies without invoking dark matter. In this paper I apply a covariant approach to formulate the cosmological equations for this theory, for both the background and linear perturbations. I derive the necessary perturbed equations for scalar, vector, and tensor modes without adhering to a particular gauge. Special gauges are considered in the appendixes.

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I. INTRODUCTION

Bekenstein has recently proposed a relativistic theory of gravity where gravity is mediated by a tensor, a vector, and a scalar field, thus called TeVeS [1], aiming at explaining the missing mass problem.

The missing mass problem is the longest standing problem of modern cosmology. It spans a wide range of scales, from galaxies to the cosmic microwave background. The problem is easy to state: the observed mass coming from all visible matter at the scales of interest cannot account for the Newtonian (or general relativistic) gravitational force observed acting on the same objects. The problem has a long history [2], and manifests in the rotation in the rotation curves of galaxies, motions of clusters of galaxies, gravitational lensing, and the absence of strong damping of linear perturbations on very large scales, to name a few.

One could imagine that this missing mass is composed of baryons in objects other than stars, for example, Jupiter-size planets or brown dwarves, collectively called MACHOS, or baryonic dark matter. These objects cannot be seen because they do not emit light of their own. However, microlensing studies did not detect the abundance needed for these objects to make up for the missing mass [3]. Moreover, the abundances of elements predicted by big bang nucleosynthesis (BBN) give a matter density far below the needed mass density [4].

The popular approach to solving the missing mass problem is to posit a matter component which does not interact with electromagnetic radiation and therefore cannot be detected by observing photons at various frequencies. Even though it cannot be seen directly, its presence is evident from the pull of gravity. Thus, one attributes the extra gravitational force observed to a "dark matter" component whose abundance is required to greatly exceed the visible matter abundance. Dark matter candidates have been traditionally split [5] into "hot dark matter" and "cold dark matter," although an intermediate possibility, namely, "warm dark matter," is sometimes considered.

The earliest possibility considered for a dark matter candidate was a massive neutrino [6,7], since neutrinos are particles which are known to exist as well as being very weakly interacting. However, massive neutrinos cannot be the dominant form of the dark matter. If the dark matter is composed of massive neutrinos, then their mass must be at most 30–70 eV for reasonable values of the Hubble constant, if they are not to overclose the universe [6]. On the other hand, the Tremaine-Gunn inequality [8] gives a lower bound on the neutrino mass if neutrinos are to be bounded gravitationally within some radius. For example, for dwarf spheroidal galaxies, their mass should be greater than ∼300–400 eV which is well above the cosmologically allowed mass range. Finally, the recent Mainz and Troisk experiments from tritium beta decay, combined with neutrino oscillation experiments, give an upper limit for the neutrino mass of around 2.2–2.5 eV [9]. Massive neutrinos are therefore ruled out as dark matter candidates capable of solving the missing mass problem.

Cold dark matter is composed of very massive slowly moving and weakly interacting particles. A plethora of such particles generically arises in particle physics models beyond the standard model quite naturally. The list of candidates is very long, ranging from light particles [10] and supersymmetric particles [11], to Kaluza-Klein modes [12] and many more exotic objects. This subject (see [13] for a nice recent review) has been studied in great depth and has been shown to agree with observations to a very good degree. Still, the actual nature of cold dark matter is left to speculation at the present time.

The alternative approach is to point a finger at the gravitational field, or the laws of motion. This path was initially followed by Milgrom [14] who proposed that, for accelerations smaller than some acceleration scale a_0, gravity departs from Newtonian gravity (which is still valid for large accelerations), in such a way as to explain the flat rotation curves of galaxies. It was thus dubbed modified

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Newtonian dynamics (MOND). Alternatively, it was shown that one can cast a modified dynamics theory into a modified gravity theory which provides essentially the same phenomenology [15], although it might be possible that the two are not entirely equivalent [16]. Other modified gravity theories beyond the MOND paradigm have also been proposed [17] but I do not consider them further in this paper.

While MOND was successful as a simple phenomenological model in describing rotation curves, it had other serious problems. Taken at face value, it violates conservation of momentum, energy and angular momentum [14]. This, though, stems from the fact that it is not a theory but rather an empirical law. Bekenstein and Milgrom have found a nonrelativistic self-consistent realization of MOND, based on an aquadratic Lagrangian, thus called AQUAL [15]. However, being nonrelativistic, the theory could not make clear predictions about cosmological scales, for example, the cosmic microwave background (CMB) or the formation of linear structure (the relativistic AQUAL proposed in the same paper suffered from acausal propagation of small perturbations).

Nevertheless, even in the absence of a consistent relativistic MOND theory, several authors have tried to squeeze a cosmology out of MOND. Some authors based their calculations on general relativity (GR) to conclude that MOND is not compatible with cosmological observations [18], a not so robust conclusion; the correct conclusion would have been that a baryonic, dark matter free universe, evolving under Einstein gravity, cannot fit the cosmological observations. In other words, they have shown the missing mass problem on cosmological scales. Nevertheless, those studies showed that relativistic MOND theories would have to give spectra very similar to dark matter cosmologies, if they are to be considered as serious competitors. Given that relativistic MOND theories such as TeVeS have generically more parameters, this would tend to favor dark matter cosmologies when tests such as Bayesian evidence are performed, unless their likelihood is strongly peaked about a very small region in parameter space, which is still an open question.

Other authors [19] used GR to make predictions for early universe cosmology (for example, the CMB), where MOND effects were argued to be negligible, and then used heuristic arguments to make predictions about the growth of a linear/nonlinear structure. As the authors of [20] have shown, their CMB predictions are indeed compatible with the robust TeVeS calculations for some range of parameters, but can be quite different in other cases. The matter power spectrum predictions exhibit a similar behavior: the baryonic oscillations, which were thought to be a MOND prediction, can be absent in TeVeS, just like in dark matter cosmology. The growth of structure in TeVeS cosmology has also been verified subsequently with analytic calculations [21].

Lue and Starkman [22] assumed that Birkhoff's theorem would be satisfied in a relativistic MOND theory. If that is true, one can then build a cosmology out of MOND, in the same way one can derive the Friedmann equation in the matter era out of Newtonian gravity. What they found was that the growth rate of perturbations in the matter era in this MOND-like cosmology is slower than in GR cosmology with dark matter. Therefore, if MOND is to be the limit of a relativistic theory which can successfully fit cosmological observations, it *must* violate Birkhoff's theorem. Indeed, TeVeS theory does violate Birkhoff's theorem; not only are spherically symmetric solutions not necessarily static, even the static solutions are not unique [23]. The bottom line is that, to make robust cosmological predictions, consistent relativistic MOND theories are needed.

Relativistic MOND realizations were constructed to overcome the acausal propagation of relativistic AQUAL typically based on two metrics which are conformally related via a scalar field. Phase coupling gravitation (PCG) [24–26] is one such example. However, the PCG parameters that could provide good MOND phenomenology were ruled out by solar system tests. Moreover, just like relativistic AQUAL, PCG cannot provide the observed gravitational lensing from visible matter alone. Part of the problem is the conformal relation between the two metrics [27,28]. One can generalize this relation to a disformal one [29] by including an additive tensor in the transformation, not related to the two metrics, for example, built out of the gradient of a scalar field. However, it was soon realized that any generalized scalar-tensor gravitation theory, even with a disformal relation between the two metrics in the theory, would produce less bending of light than GR and thus could not be used as a basis for relativistic MOND [30].

Sanders' stratified theory [31] manages to solve the lensing problem. Instead of using just a scalar field to disformally relate the two metrics in the theory, a vector field is used. The vector field in Sanders' stratified theory is, however, nondynamical, which contradicts the spirit of general covariance. This was solved by Bekenstein, who made the Sanders field dynamical by including an action for it (which is a special case of the Jacobson-Mattingly [32] Einstein-Ether theory action). The resulting theory was called tensor-vector-scalar (TeVeS) [1] gravitational theory, and was shown to provide MOND and Newtonian limits in the weak field nonrelativistic limit, and was devoid of acausal propagation of perturbations.

Ultimately, as with every physical theory, TeVeS has to face astrophysical and cosmological observations on every scale. Some strong lensing studies have already been performed [33]. The theory was also confronted with observations from our own Galaxy [34]. Alternatively, one can test the theory with large scale structure observations or the cosmic microwave background, both of which require the propagation of linear perturbations about a Friedmann-Lemaître-Robertson-Walker (FLRW) background. A first study in this direction has already been carried out [20]. The subject of the present work deals with the formulation of linear perturbation theory in TeVeS, generalizing the equations given in [20] to include curved spatial hypersurfaces, multiple gauges, and all perturbation modes (scalar, vector, and tensor modes). This opens up a whole new series of observational tests that can be performed on TeVeS, involving the largest scales in the universe.

This paper is organized as follows. In Sec. II, I give a short overview of the TeVeS theory, focusing on the action and the field equations. At the same time, I introduce a somewhat simpler notation than the original TeVeS paper [1] which can simplify both the actions and the field equations. Relativistic fluids are then introduced which are of particular importance to cosmology. The section concludes with the transformation of connections between the two frames associated with the theory.

In Sec. III, I lay down a covariant formulation of the FLRW cosmology. The covariant equations are derived on the basis of the symmetries of the FLRW spacetime. A short discussion of the effective Friedmann equation follows, focusing on the time variation of the effective gravitational coupling strength, and the definition of the relative fluid densities. A specific choice of a coordinate system relevant to calculations for CMB anisotropies and a large scale structure power spectrum is given in the end.

Section IV takes on linear perturbations about the FLRW cosmology. The relevant tensors are perturbed covariantly, without adhering to a particular gauge or perturbation mode. Thus the final perturbed equations contain scalar, vector, and tensor perturbations. The use of the covariant approach is important due to the nonuniqueness of connections, which depend on which metric is used. Since the transformation of connections is derived in a covariant fashion, the final perturbed equations can be derived unambiguously. This section also gives relations of the perturbed metrics in the two frames.

In Sec. V, I take the perturbed field equations derived in the third section, and split them into irreducible parts by separating out the scalar, vector, and tensor perturbation modes. The resulting equations are still in a gauge nonfixed form, which makes it straightforward to check that they are indeed gauge invariant as expected.

Finally, the paper concludes in Sec. VI, where a summary is given of the results.

The reader will also find the appendixes useful. In particular, Appendix B gives the perturbed equations for scalar modes in three different gauges, while Appendix C gives a lot of intermediate steps in the derivation of the perturbed Einstein tensor. These steps were not included in the main part of the paper, to make it more readable, but are very useful to have when following the calculations. The reader will also find useful the two tables of symbols. Table I comprises all symbols except the ones related to perturbations which are tabulated in Table II.

Throughout the paper, I will use a signature $+2$ metric, and the curvature conventions of Misner, Thorne, and Wheeler [35]. Greek indices are abstract tensor indices with no respect to any coordinate system. When writing tensor components in a particular coordinate system, Latin indices are used for the spatial part of the tensor with a "hat" on the index, while $\hat{0}$ is used for the temporal part of the tensor. I will also use units such that the speed of light, Planck's constant divided by 2π, and Boltzmann's constant are all equal to unity.

II. FUNDAMENTALS OF TEVES

A. Preliminaries

TeVeS theory is a bimetric theory where gravity is mediated by a tensor field $\tilde{g}_{\mu\nu}$ with an associated metric-compatible connection $\tilde{\nabla}_\mu$ and well-defined inverse $\tilde{g}^{\mu\nu}$ such that $\tilde{g}^{\mu\rho}\tilde{g}_{\rho\nu} = \delta^\mu{}_\nu$, a timelike (dual) vector field A_μ such that $\tilde{g}^{\mu\nu}A_\mu A_\nu = -1$, and a scalar field ϕ. Matter is required to obey the weak equivalence principle, which means that there is a metric $g_{\mu\nu}$ with an associated metric-compatible connection ∇_μ, universal to all matter fields, such that test particles follow its geodesics. The tensor field $\tilde{g}_{\mu\nu}$ will be called the *Einstein-Hilbert frame metric* (see below), while $g_{\mu\nu}$ will be called the *matter-frame metric*.

The relation between the above four tensor fields (when the field equations are satisfied) is

$$g_{\mu\nu} = e^{-2\phi}\tilde{g}_{\mu\nu} - 2\sinh(2\phi)A_\mu A_\nu \tag{1}$$

with inverse

$$g^{\mu\nu} = e^{2\phi}\tilde{g}^{\mu\nu} + 2\sinh(2\phi)A^\mu A^\nu, \tag{2}$$

where $A^\mu = \tilde{g}^{\mu\nu}A_\nu$.

B. The action principle

The theory is based on an action S, which splits as $S = S_g + S_A + S_\phi + S_m$, where S_g, S_A, S_ϕ, and S_m are the actions for $\tilde{g}_{\mu\nu}$, vector field A_μ, scalar field ϕ, and matter, respectively.

The action for $\tilde{g}_{\mu\nu}$, A_μ, and ϕ is most easily written in the Einstein-Hilbert frame, and is such that S_g is of the Einstein-Hilbert form

$$S_g = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\tilde{R}, \tag{3}$$

where $\tilde{g}$ and $\tilde{R}$ are the determinant and scalar curvature of $\tilde{g}_{\mu\nu}$, respectively, and G is the bare gravitational constant. Because of the complicated nature of the equations, the numerical value of G will not be the measured value of Newton's constant as measured on Earth. The precise relation between them depends on the spherically symmetric solution which, apart from depending on the arbitrary function V (see below), is also not unique [1,23].

The action for the vector field A_μ is given by

$$S_A = -\frac{1}{32\pi G}\int d^4x\sqrt{-\tilde{g}}[K_B F_{\mu\nu}F^{\mu\nu} - 2\lambda(A_\mu A^\mu + 1)], \tag{4}$$

where $F_{\mu\nu} = 2\tilde{\nabla}_{[\mu}A_{\nu]}$, $F^{\mu\nu} = \tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}F_{\alpha\beta}$, λ is a Lagrange multiplier ensuring the timelike constraint on A_μ, and K_B is a dimensionless constant.

The action for the scalar field ϕ is given by

$$S_\phi = -\frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}[\mu(\tilde{g}^{\mu\nu} - A^\mu A^\nu)\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + V(\mu)], \tag{5}$$

where μ is a nondynamical dimensionless scalar field, and $V(\mu)$ is an arbitrary function. The arbitrary function V is related to Bekenstein's function F as

$$V(\mu) = \frac{1}{16\pi\ell^2}\mu^2 F(\mu) = \frac{4\pi G^2}{\ell^2}\sigma_B^4 F(G\sigma_B^2), \tag{6}$$

where $\mu = 8\pi G\sigma_B^2$, σ_B being Bekenstein's auxiliary scalar field and ℓ a scale. Note that I have absorbed Bekenstein's constant k_B into my definition of the function V.

The matter is coupled only to the matter-frame metric $g_{\mu\nu}$ and thus its action is of the form

$$S_m[g, \chi^A, \partial\chi^A] = \int d^4x\sqrt{-g}L[g, \chi^A, \partial\chi^A] \tag{7}$$

for some generic collection of matter fields χ^A.

C. The field equations

Variation of the action with respect to (w.r.t.) the matter-frame metric gives the matter stress-energy tensor as usual by

$$\delta S_m = -\frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}, \tag{8}$$

where $T_{\mu\nu}$ is the standard matter stress-energy tensor.

Variation of the action with respect to the three gravitational fields gives the field equations in the Einstein-Hilbert frame.

The field equations for $\tilde{g}_{\mu\nu}$ are given by [36]

$$\tilde{G}_{\mu\nu} = Y_{\mu\nu} + 8\pi G S_{\mu\nu}, \tag{9}$$

where $\tilde{G}_{\mu\nu}$ is the Einstein tensor of $\tilde{g}_{\mu\nu}$; the tensors $Y_{\mu\nu}$ and $S_{\mu\nu}$ are given by

$$\begin{aligned} Y_{\mu\nu} = {} & \mu[\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - 2A^\alpha\tilde{\nabla}_\alpha\phi A_{(\mu}\tilde{\nabla}_{\nu)}\phi] \\ & + \tfrac{1}{2}(\mu V' - V)\tilde{g}_{\mu\nu} - \lambda A_\mu A_\nu \\ & + K_B[F^\alpha{}_\mu F_{\alpha\nu} - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\tilde{g}_{\mu\nu}] \end{aligned} \tag{10}$$

and

$$S_{\mu\nu} = T_{\mu\nu} + 2(1 - e^{-4\phi})A^\lambda T_{\lambda(\mu}A_{\nu)}, \tag{11}$$

respectively, where $V' \equiv \frac{dV}{d\mu}$.

The field equations for the vector field A_μ are

$$K_B\tilde{\nabla}_\mu F^\mu{}_\nu = -\lambda A_\nu - \mu A^\mu\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + 8\pi G j_\nu, \tag{12}$$

where the current j_μ is given by

$$j_\mu = (1 - e^{-4\phi})A^\lambda T_{\lambda\mu}. \tag{13}$$

The Lagrange multiplier is not arbitrary but can be calculated by contracting (12) with A^μ and is given by

$$\lambda = K_B A_\nu\tilde{\nabla}_\mu F^{\mu\nu} + \mu A^\mu A^\nu\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - 8\pi G A^\mu j_\mu. \tag{14}$$

Inserting the above equation in (12), one gets alternative field equations for the vector field as

$$[\delta^\alpha{}_\nu + A^\alpha A_\nu][K_B\tilde{\nabla}_\mu F^\mu{}_\alpha + \mu A^\mu\tilde{\nabla}_\mu\phi\tilde{\nabla}_\alpha\phi - 8\pi G j_\alpha] = 0 \tag{15}$$

which do not explicitly include the Lagrange multiplier.

The field equation for the scalar field ϕ is

$$\tilde{\nabla}_\mu\Gamma^\mu = 8\pi G J, \tag{16}$$

where

$$\Gamma^\mu = \mu(\tilde{g}^{\mu\nu} - A^\mu A^\nu)\tilde{\nabla}_\nu\phi \tag{17}$$

and where the scalar source J is given by

$$J = e^{-2\phi}[g^{\mu\nu} + 2e^{-2\phi}A^\mu A^\nu]T_{\mu\nu}. \tag{18}$$

Apart from the field equations above, TeVeS theory has two constraint equations which are as follows. The first constraint is nothing but the timelike constraint on the vector field,

$$\tilde{g}^{\mu\nu}A_\mu A_\mu = -1, \tag{19}$$

which is found by varying the action with respect to the Lagrange multiplier λ. The second constraint fixes the nondynamical scalar field μ in terms of the other fields of the theory. It is given by

$$(\tilde{g}^{\mu\nu} - A^\mu A^\nu)\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi = -V' \tag{20}$$

which is found by varying the action with respect to μ. The second constraint equation must be inverted to find μ as a function of $\tilde{g}^{\mu\nu}$, A^μ, and ϕ. Therefore, the arbitrary function V and its derivatives are nothing but functions of kinetic terms for ϕ, contracted with $\tilde{g}^{\mu\nu}$ and A^μ.

D. Fluids

The stress-energy tensor of a fluid with density ρ, pressure P, velocity $u^\mu = g^{\mu\nu}u_\nu$, and shear $\Sigma_{\mu\nu}$ is

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu} + \Sigma_{\mu\nu}. \tag{21}$$

The velocity vector field is normalized with respect to the matter-frame metric as $g^{\mu\nu}u_\mu u_\nu = u^\mu u_\mu = -1$, while the shear obeys the two conditions $g^{\mu\nu}\Sigma_{\mu\nu} = 0$ and $u^\mu \Sigma_{\mu\nu} = 0$.

Using (11) one obtains the contribution of the fluid stress-energy tensor to the generalized Einstein equations as

$$\begin{aligned} S_{\mu\nu} = (\rho + P)[u_\mu u_\nu + 2(1 - e^{-4\phi})A^\alpha u_\alpha u_{(\mu}A_{\nu)}] \\ + P[g_{\mu\nu} + 4\sinh(2\phi)A_\mu A_\nu] + \Sigma_{\mu\nu} \\ + 2(1 - e^{-4\phi})A^\lambda \Sigma_{\lambda(\mu}A_{\nu)}. \end{aligned} \tag{22}$$

Similarly, Eq. (13) gives the contribution of the fluid stress-energy tensor to the vector field equations as

$$\begin{aligned} j_\nu = (1 - e^{-4\phi})[(\rho + P)A^\mu u_\mu u_\nu + A^\mu \Sigma_{\mu\nu}] \\ + 2\sinh(2\phi)PA_\nu. \end{aligned} \tag{23}$$

Finally, the contribution of the fluid stress-energy tensor to the scalar field equation is obtained from (18) as

$$\begin{aligned} J = e^{-2\phi}[P - \rho + 2e^{-2\phi}(\rho + P)(A^\mu u_\mu)^2 \\ + 2e^{-2\phi}A^\mu A^\nu \Sigma_{\mu\nu}]. \end{aligned} \tag{24}$$

The fluid evolution equations are obtained as usual from $\nabla_\mu T^\mu{}_\nu = 0$, where $T^\mu{}_\nu = g^{\mu\rho}T_{\rho\nu}$. Using (21), and expanding, gives

$$\begin{aligned} u_\nu u^\mu \nabla_\mu(\rho + P) + (\rho + P)u_\nu \nabla_\mu u^\mu + (\rho + P)u^\mu \nabla_\mu u_\nu \\ + \nabla_\nu P + \nabla_\mu \Sigma^\mu{}_\nu = 0, \end{aligned} \tag{25}$$

where $\Sigma^\mu{}_\nu = g^{\mu\rho}\Sigma_{\rho\nu}$. Contracting (25) with u^ν gives the energy "conservation" equation as

$$u^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu u^\mu = 0, \tag{26}$$

and subtracting (26) from (25) yields the momentum transfer equation as

$$(\rho + P)u^\mu \nabla_\mu u_\nu + (\delta^\mu{}_\nu + u_\nu u^\mu)\nabla_\mu P + \nabla_\mu \Sigma^\mu{}_\nu = 0. \tag{27}$$

E. Transformation of connections

I conclude this section by considering the transformation of connections from $\tilde{\nabla}_\mu$ to ∇_μ and vice versa. This will come out to be very useful below, particularly in linear perturbation theory.

Consider two metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ on a manifold M, with connections ∇_μ and $\tilde{\nabla}_\mu$, respectively (not necessarily the two metrics of the TeVeS theory). The connections are required to agree on scalars, i.e. $\nabla_\mu f = \tilde{\nabla}_\mu f$ for any $f \in C^\infty(M)$. Acting on any form $u_\mu \in T^*M$, the connections are related by

$$\tilde{\nabla}_\mu u_\nu = \nabla_\mu u_\nu - D^\lambda{}_{\mu\nu}u_\lambda, \tag{28}$$

where the connection tensor $D^\lambda{}_{\mu\nu}$ is given by

$$D^\lambda{}_{\mu\nu} = \tfrac{1}{2}\tilde{g}^{\lambda\rho}(\nabla_\mu \tilde{g}_{\rho\nu} + \nabla_\nu \tilde{g}_{\rho\mu} - \nabla_\rho \tilde{g}_{\mu\nu}). \tag{29}$$

Using the metric relations (1) and (2) in (29), one gets for TeVeS

$$\begin{aligned} D^\alpha{}_{\mu\nu} = 2\delta^\alpha{}_{(\mu}\tilde{\nabla}_{\nu)}\phi - [g_{\mu\nu} + 2e^{2\phi}A_\mu A_\nu]g^{\alpha\beta}\tilde{\nabla}_\beta\phi \\ + 4A^\alpha A_{(\mu}\tilde{\nabla}_{\nu)}\phi + (1 - e^{-4\phi})A^\alpha \tilde{\nabla}_{(\mu}A_{\nu)} \\ + (e^{4\phi} - 1)A_{(\mu}F_{\nu)}{}^\alpha + 4\sinh^2(2\phi)A_{(\mu}F_{\nu)}{}^\beta A_\beta A^\alpha, \end{aligned} \tag{30}$$

whereas, with respect to ∇_μ,

$$\begin{aligned} D^\alpha{}_{\mu\nu} = 2\delta^\alpha{}_{(\mu}\nabla_{\nu)}\phi - [\tilde{g}_{\mu\nu} + (e^{4\phi} + 1)A_\mu A_\nu]\tilde{g}^{\alpha\beta}\nabla_\beta\phi \\ + 2(e^{4\phi} + 1)A^\alpha A_{(\mu}\nabla_{\nu)}\phi + (e^{4\phi} - 1)A^\alpha \nabla_{(\mu}A_{\nu)} \\ + (e^{4\phi} - 1)A_{(\mu}F_{\nu)}{}^\alpha. \end{aligned} \tag{31}$$

The above transformations can be useful in writing the field equations in purely matter-frame form, or Einstein-frame form, although the result could be very complicated.

III. ROBERTSON-WALKER COSMOLOGY

In this section, I consider the evolution of the background cosmology. The field equations are first found in covariant form, to facilitate an easier transition to the inclusion of linear perturbations. A special coordinate system common to calculations in cosmology, particularly in cosmological perturbations, is introduced at the end of the section.

A. Preliminaries

1. Covariant description of Robertson-Walker geometry: Geodesic congruences, metrics, and projectors

The background spacetime is assumed to be homogeneous and isotropic, meaning that both of the metrics are of Robertson-Walker form. This assumption permits one to construct a smooth congruence of timelike geodesics, which are normal to a hypersurface of spatial homogeneity and isotropy. The words timelike and geodesic imply a metric and a compatible connection. Even though in this theory there are two different metrics (and compatible connections), this is not a problem. The homogeneity and isotropy is a property of the manifold and not the metric, and any of the two metrics may be used to construct such a congruence.

Let the pair $(\tilde{\gamma}_{\mu\nu}, {}^{(\tilde{\gamma})}\tilde{\nabla}_\mu)$ be the Robertson-Walker metric and associated metric-compatible connection in the Einstein-Hilbert frame. One can then identify the vector field A^μ as the unit vector field tangent to the congruence of timelike geodesics mentioned above, for the pair $(\tilde{\gamma}_{\mu\nu}, {}^{(\tilde{\gamma})}\tilde{\nabla}_\mu)$ (see Appendix A). Since by construction A^μ is hypersurface orthogonal, the Frobenius theorem, geodesic equation and unit-timelike condition give

$$F_{\mu\nu} = 0. \tag{32}$$

Now the Robertson-Walker metric is conformal to the metric of a static spacetime foliated by spaces of constant curvature of radius r_c which will be called the conformal static metric. Minkowski space (which has an infinite radius of curvature) and Einstein-static space (with positive radius of curvature) are two cases of conformal static metrics. This means that one can write $\tilde{\gamma}_{\mu\nu}$ as

$$\tilde{\gamma}_{\mu\nu} = b^2 \tilde{\eta}_{\mu\nu}, \tag{33}$$

where b is the scale factor in the Einstein-Hilbert frame and where $\tilde{\eta}_{\mu\nu}$ is the conformal static metric mentioned above with $\bar{\nabla}_\mu$ its associated metric-compatible connection. Similarly, the matter-frame Robertson-Walker metric $\gamma_{\mu\nu}$, with associated metric-compatible connection ${}^{(\gamma)}\nabla_\mu$, can be written as

$$\gamma_{\mu\nu} = a^2 \eta_{\mu\nu}, \tag{34}$$

where a is the scale factor in the matter frame and $\eta_{\mu\nu}$ is also a conformal static metric, with $\hat{\nabla}_\mu$ being its metric-compatible connection.

Consider now a different unit-timelike geodesic congruence of curves with respect to $(\eta_{\mu\nu}, \hat{\nabla}_\mu)$. Let t^μ be the tangent vector field of this congruence assumed to be Killing (this can be accommodated by the symmetries of the static spacetime), which by construction obeys

$$\eta_{\mu\nu} t^\mu t^\nu = -1 \tag{35}$$

and

$$t^\nu \hat{\nabla}_\nu t_\mu = 0, \tag{36}$$

where $t_\mu = \eta_{\mu\nu} t^\nu$. A further property of t^μ is that it is covariantly constant,

$$\hat{\nabla}_\mu t_\nu = 0, \tag{37}$$

which follows from the fact that it is Killing, geodesic and hypersurface orthogonal. Transforming $\eta_{\mu\nu}$ to $\tilde{\gamma}_{\mu\nu}$, one finds that t^μ is related to A^μ by

$$A^\mu = \frac{1}{a} e^{\bar{\phi}} t^\mu \tag{38}$$

and

$$A_\mu = a e^{-\bar{\phi}} t_\mu. \tag{39}$$

With the help of A^μ or equivalently t^μ, one can construct two projection tensors, given by $s^\mu{}_\nu = -A^\mu A_\nu$ which projects tensors along A^μ and $q^\mu{}_\nu = \delta^\mu{}_\nu + A^\mu A_\nu$ which projects tensors on the hypersurface of homogeneity and isotropy. The two projectors have the property that $s^\mu{}_\nu A^\nu = A^\mu$, $q^\mu{}_\alpha q^\alpha{}_\nu = q^\mu{}_\nu$, and $q^\mu{}_\nu A^\nu = 0$. Of course, in the case of a FLRW background, $q^\mu{}_\nu = \bar{q}^\mu{}_\nu$ where

$$\bar{q}^\mu{}_\nu = \delta^\mu{}_\nu + t^\mu t_\nu, \tag{40}$$

whereas when perturbations are included, $q^\mu{}_\nu$ will acquire a perturbed part due to the perturbations coming from A^μ (see the perturbation section), while $\bar{q}^\mu{}_\nu$ is by definition unperturbed, and the two projectors will not be equal.

2. Relations between the scale factors and between the conformal static metrics

A further property of the FLRW spacetime is that any gradient of a scalar function will be in the direction A^μ or equivalently t^μ. Thus, letting the background value of the scalar field be $\bar{\phi}$, its gradient is ${}^{(\tilde{\gamma})}\tilde{\nabla}_\mu \bar{\phi} = -(\mathcal{L}_A \bar{\phi}) A_\mu = -(t^\nu {}^{(\tilde{\gamma})}\tilde{\nabla}_\nu \bar{\phi}) t_\mu$. The same holds for any other scalar field, e.g. the scale factors in the two frames, or the background density and pressure of the fluids.

The spatial metric in either frame is obtained using the spatial projector (40) acting on the corresponding metric. Thus, in the Einstein-Hilbert frame the spatial metric is $\tilde{q}_{\mu\nu} = \tilde{\gamma}_{\mu\nu} + A_\mu A_\nu$, whereas in the matter frame it is $q_{\mu\nu} = \gamma_{\mu\nu} + e^{2\bar{\phi}} A_\mu A_\nu$. Equation (1) then implies that the two are conformally related as $q_{\mu\nu} = e^{-2\bar{\phi}} \tilde{q}_{\mu\nu}$, which prompts the relation

$$a = b e^{-\bar{\phi}} \tag{41}$$

between the two scale factors.

Using (1), (33), (34), and (39) the relation between the two conformal static metrics and t_μ is obtained as

$$\eta_{\mu\nu} = \tilde{\eta}_{\mu\nu} - (1 - e^{-4\bar{\phi}}) t_\mu t_\nu, \tag{42}$$

with inverse

$$\eta^{\mu\nu} = \tilde{\eta}^{\mu\nu} + (e^{4\bar{\phi}} - 1) t^\mu t^\nu. \tag{43}$$

3. Connections

Let $\tilde{C}^\alpha{}_{\mu\nu}$ be the connection tensor for the connection transformation ${}^{(\tilde{\gamma})}\tilde{\nabla}_\mu \to \bar{\nabla}_\mu$. Using (29) one gets

$$\tilde{C}^\alpha{}_{\mu\nu} = [-2\delta^\alpha{}_{(\mu} t_{\nu)} + e^{4\bar{\phi}} \tilde{\eta}_{\mu\nu} t^\alpha] \mathcal{L}_t \ln b. \tag{44}$$

Let also $C^\alpha{}_{\mu\nu}$ be the connection tensor which performs the transformation ${}^{(\gamma)}\nabla_\mu \to \hat{\nabla}_\mu$. Using (29) one gets

$$C^\alpha{}_{\mu\nu} = [-2\delta^\alpha{}_{(\mu} t_{\nu)} + \eta_{\mu\nu} t^\alpha] \mathcal{L}_t \ln a. \tag{45}$$

Finally, consider the connection tensor $E^\alpha{}_{\mu\nu}$ for the connection transformation $\bar{\nabla}_\mu \to \hat{\nabla}_\mu$. This is the same as $\bar{\nabla}_\mu \to {}^{(\tilde{\gamma})}\tilde{\nabla}_\mu \to {}^{(\gamma)}\nabla_\mu \to \hat{\nabla}_\mu$, and therefore $E^\alpha{}_{\mu\nu} = -\tilde{C}^\alpha{}_{\mu\nu} + D^\alpha{}_{\mu\nu} + C^\alpha{}_{\mu\nu}$, where $D^\alpha{}_{\mu\nu}$ is the connection tensor in (31) adapted to the Robertson-Walker geometry. Alternatively, $E^\alpha{}_{\mu\nu}$ is obtained directly from (29), (37), (42), and (43). The final expression is

$$E^{\alpha}{}_{\mu\nu} = -2(t^{\beta}\hat{\nabla}_{\beta}\bar{\phi})t^{\alpha}t_{\mu}t_{\nu}. \tag{46}$$

B. The field equations

Here I consider the field equations adapted to the symmetries of the FLRW spacetime. I do not explicitly consider the vector field equation, as it is trivially satisfied.

1. The fluid tensors and evolution equations

Let us first consider the fluid related variables, J, j_{μ}, and $S_{\mu\nu}$. The fluid velocity can be expressed in terms of t^{μ} as $u_{\mu} = at_{\mu}$ and $u^{\mu} = \frac{1}{a}t^{\mu}$ which give $A^{\mu}u_{\mu} = -e^{\bar{\phi}}$. For the same reasons as above (see Appendix A), the fluid velocity is geodesic, i.e. $u^{\mu}{}^{(\gamma)}\nabla_{\mu}u_{\nu} = 0$. The scalar source is then given by (24) as

$$\bar{J} = e^{-2\bar{\phi}}(\bar{\rho} + 3\bar{P}), \tag{47}$$

where $\bar{\rho}$ and $\bar{P}$ are the FLRW background density and pressure of the fluid. The fluid current is given by (23) as

$$\bar{j}_{\mu} = -2\sinh(2\bar{\phi})e^{-\phi}a\bar{\rho}t_{\mu} \tag{48}$$

and the fluid stress-energy tensor in the Einstein-Hilbert frame is given as

$$\bar{S}_{\mu\nu} = a^2[-(1-2e^{-4\phi})\bar{\rho}t_{\mu}t_{\nu} + \bar{P}\bar{q}_{\mu\nu}], \tag{49}$$

where $\bar{q}_{\mu\nu} = \eta_{\mu\lambda}\bar{q}^{\lambda}{}_{\nu}$.

Changing connection from ${}^{(\gamma)}\nabla_{\mu}$ to $\hat{\nabla}_{\mu}$, the energy conservation equation for the fluid (26) becomes

$$t^{\mu}\hat{\nabla}_{\mu}\bar{\rho} + 3\bar{\rho}(1+w)t^{\mu}\hat{\nabla}_{\mu}\ln a = 0, \tag{50}$$

where w is the equation of state parameter such that $\bar{P} = w\bar{\rho}$. The momentum transfer equation (27) is trivially satisfied.

2. The constraint equation

Consider now the constraint (20) which when adapted to the symmetries of FLRW spacetime gives

$$(t^{\mu}\hat{\nabla}_{\mu}\bar{\phi})^2 = \tfrac{1}{2}a^2e^{-2\bar{\phi}}V'. \tag{51}$$

This equation is then inverted to get $\bar{\mu} = \bar{\mu}(a, \bar{\phi}, t^{\mu}\hat{\nabla}_{\mu}\bar{\phi})$.

3. The scalar field equation

The vector field (17) adapted to the FLRW symmetries can be rewritten as $\Gamma^{\mu} = -\bar{\Gamma}A^{\mu}$, where $\bar{\Gamma} = A_{\mu}\Gamma^{\mu} = 2(\bar{\mu}/a)e^{\bar{\phi}}(t^{\rho}\hat{\nabla}_{\rho}\bar{\phi})$. Using the above equations in the scalar field equation gives a system of two first-order equations, which are

$$t^{\mu}\hat{\nabla}_{\mu}\bar{\Gamma} = -3(t^{\rho}\hat{\nabla}_{\rho}\ln b)\bar{\Gamma} - 8\pi Gae^{-\bar{\phi}}J \tag{52}$$

and

$$t^{\rho}\hat{\nabla}_{\rho}\bar{\phi} = \frac{1}{2\bar{\mu}}ae^{-\bar{\phi}}\bar{\Gamma}. \tag{53}$$

The two first-order equations can then be combined into a single second-order equation which is

$$\hat{\nabla}^2\bar{\phi} = \frac{1}{U}[3\bar{\mu}(t^{\mu}\hat{\nabla}_{\mu}\ln b)(t^{\nu}\hat{\nabla}_{\nu}\bar{\phi}) + 4\pi Ga^2e^{-2\bar{\phi}}\bar{J}] + (t^{\mu}\hat{\nabla}_{\mu}\bar{\phi})(t^{\mu}\hat{\nabla}_{\mu}\bar{\phi} - t^{\mu}\hat{\nabla}_{\mu}\ln a), \tag{54}$$

where

$$U(\mu) = \bar{\mu} + 2\frac{V'(\mu)}{V''(\mu)}. \tag{55}$$

4. The Lagrange multiplier

Next in line comes the Lagrange multiplier which is given by

$$\bar{\lambda} = \tfrac{1}{2}\bar{\mu}V' - 16\pi G\sinh(2\bar{\phi})\bar{\rho}. \tag{56}$$

5. The generalized Einstein equations

First let us compute the tensor $\bar{Y}_{\mu\nu}$. Using (56) in (10) one obtains

$$\bar{Y}_{\mu\nu} = a^2\{\tfrac{1}{2}e^{2\bar{\phi}}(\mu V' - V)\eta_{\mu\nu} + [\cosh(2\bar{\phi})\bar{\mu}V' - \sinh(2\bar{\phi})V + 8\pi G(1-e^{-4\bar{\phi}})\bar{\rho}]t_{\mu}t_{\nu}\} \tag{57}$$

which when combined with (49) gives the right-hand side of the generalized Einstein equations as

$$\bar{Y}_{\mu\nu} + 8\pi G\bar{S}_{\mu\nu} = a^2\{[\tfrac{1}{2}e^{2\bar{\phi}}(\mu V' - V) + 8\pi G\bar{P}]\eta_{\mu\nu} + [\cosh(2\bar{\phi})\bar{\mu}V' - \sinh(2\bar{\phi})V + 8\pi G(e^{-4\bar{\phi}}\bar{\rho} + \bar{P})]t_{\mu}t_{\nu}\}. \tag{58}$$

The conformal relation of the Robertson-Walker metric to the conformal static metric makes it convenient to use conformal transformations to calculate the Einstein tensor $\bar{G}_{\mu\nu}$ for the Einstein-Hilbert frame metric (33). Now the Ricci tensor of the conformal static metric is simply $2(K/r_c^2)\bar{q}_{\mu\nu}\bar{q}_{\mu\nu}$ where K is an integer taking the value $K = 0$ for a spatially flat hypersurface (the conformal static space is Minkowski spacetime), $K = 1$ for a positively curved spatial hypersurface (the conformal static metric is Einstein-static spacetime), and $K = -1$ for a negatively curved spatial hypersurface.

Then, after performing the conformal transformation the Einstein tensor is found to be

$$\bar{G}_{\mu\nu} = \frac{K}{r_c^2}[3e^{-4\bar{\phi}}t_{\mu}t_{\nu} - \bar{q}_{\mu\nu}] - 2[\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\ln b - (\bar{\nabla}_{\mu}\ln b)(\bar{\nabla}_{\nu}\ln b) - (\tilde{\eta}^{\alpha\beta}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\ln b)\tilde{\eta}_{\mu\nu}] + \tilde{\eta}^{\alpha\beta}(\bar{\nabla}_{\alpha}\ln b)(\bar{\nabla}_{\beta}\ln b)\tilde{\eta}_{\mu\nu}. \tag{59}$$

Changing connection to $\hat{\nabla}_\mu$ and using (42) and (43) gives

$$\bar{G}_{\mu\nu} = \frac{K}{r_c^2}[3e^{-4\bar{\phi}}t_\mu t_\nu - \bar{q}_{\mu\nu}] + 3(t^\alpha \hat{\nabla}_\alpha \ln b)^2 t_\mu t_\nu + 2e^{4\phi}\bar{q}_{\mu\nu}\Big[\hat{\nabla}^2 \ln b - (t^\beta \hat{\nabla}_\beta \ln b)\Big(2t^\alpha\hat{\nabla}_\alpha\phi - \frac{1}{2}t^\alpha\hat{\nabla}_\alpha \ln b\Big)\Big]. \tag{60}$$

Contracting (58) and (60) with $t^\mu t^\nu$ gives the generalized Friedmann equation as

$$3(t^\mu \bar{\nabla}_\mu \ln b)^2 = a^2\Big[\frac{1}{2}e^{-2\bar{\phi}}(\mu V' + V) + e^{-4\bar{\phi}}\Big(8\pi G\bar{\rho} - \frac{3K}{r_c^2 a^2}\Big)\Big], \tag{61}$$

while contracting the same equations with $\bar{q}^{\mu\nu}$ gives the generalized Raychandhuri equation as

$$2\hat{\nabla}^2 \ln b - 4t^\alpha\hat{\nabla}_\alpha\phi t^\beta\hat{\nabla}_\beta \ln b - (t^\alpha\hat{\nabla}_\alpha \ln b)^2 = a^2\Big[\frac{1}{2}e^{-2\bar{\phi}}(\mu V' - V) + e^{-4\bar{\phi}}\Big(8\pi G\bar{P} + \frac{K}{r_c^2 a^2}\Big)\Big]. \tag{62}$$

Other contractions give trivially zero.

C. Choosing a coordinate system

A most convenient coordinate system that is commonly used in cosmological perturbation theory is the conformal synchronous coordinate system with t denoting conformal time and $x^{\hat{a}}$ the spatial coordinates. There are two choices regarding the frame for which the conformal static metric used above takes the standard form. Since the connection to matter observables is through matter-frame variables, it is more convenient to put the matter-frame conformal static metric in standard form, i.e. $\eta_{\hat{0}\hat{0}} = -1$, $\eta_{\hat{0}\hat{a}} = 0$, and $\eta_{\hat{a}\hat{b}} = \bar{q}_{\hat{a}\hat{b}}$. This gives the matter-frame metric as

$$ds^2 = a^2[-dt^2 + \bar{q}_{\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}}] \tag{63}$$

in this coordinate system. The vanishing of the Lie derivative with respect to all the Killing vectors of the background spacetime gives $\phi = \bar{\phi}(t)$ only, as well as $t_{\hat{a}} = 0$ and $t^{\hat{a}} = 0$. The $t_{\hat{0}}$ component is found using (35) which gives $t_{\hat{0}} = 1$ and $t^{\hat{0}} = -1$. Finally, the components of $\tilde{\eta}_{\mu\nu}$ are $\tilde{\eta}_{\hat{0}\hat{0}} = -e^{-4\bar{\phi}}$, $\tilde{\eta}_{\hat{0}\hat{a}} = 0$, and $\tilde{\eta}_{\hat{a}\hat{b}} = \bar{q}_{\hat{a}\hat{b}}$.

Adopting the covariant equations of the previous subsection in this coordinate system gives the constraint equation as

$$\dot{\bar{\phi}}^2 = \tfrac{1}{2}a^2 e^{-2\phi}V' \tag{64}$$

which must be inverted to get $\bar{\mu}(a, \bar{\phi}, \dot{\bar{\phi}})$. Similarly, the scalar field equation becomes

$$\ddot{\bar{\phi}} = \dot{\bar{\phi}}\Big(\frac{\dot{a}}{a} - \dot{\bar{\phi}}\Big) - \frac{1}{U}\Big[3\bar{\mu}\frac{\dot{b}}{b}\dot{\bar{\phi}} + 4\pi G a^2 e^{-2\bar{\phi}}\bar{J}\Big], \tag{65}$$

the Friedmann equation gives

$$3\frac{\dot{b}^2}{b^2} = a^2\Big[\frac{1}{2}e^{-2\bar{\phi}}(\mu V' + V) + e^{-4\bar{\phi}}\Big(8\pi G\bar{\rho} - \frac{3K}{r_c^2 a^2}\Big)\Big], \tag{66}$$

and the Raychandhuri equation gives

$$-2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} - 4\frac{\dot{b}}{b}\dot{\bar{\phi}} = a^2\Big[\frac{1}{2}e^{-2\bar{\phi}}(\mu V' - V) + e^{-4\bar{\phi}}\Big(8\pi G\bar{P} + \frac{K}{r_c^2 a^2}\Big)\Big]. \tag{67}$$

Finally, the fluid evolves according to

$$\dot{\bar{\rho}} + 3\frac{\dot{a}}{a}(1 + w)\bar{\rho} = 0. \tag{68}$$

D. Effective Friedmann equation and relative densities

The physical Hubble parameter is, as usual, $H = \frac{\dot{a}}{a^2}$ and, after transforming the scalar field time derivative as $\dot{\bar{\phi}} = a\frac{d\bar{\phi}}{d\ln a}H$, the effective Friedmann equation reads

$$3H^2 = 8\pi G_{\rm eff}\Big(\bar{\rho}_\phi + \bar{\rho} - \frac{3K}{8\pi G r_c^2}\Big), \tag{69}$$

where the effective gravitational coupling strength is

$$G_{\rm eff} = G\frac{e^{-4\bar{\phi}}}{(1 + \frac{d\bar{\phi}}{d\ln a})^2} \tag{70}$$

and the scalar field density $\bar{\rho}_\phi$ is

$$\bar{\rho}_\phi = \frac{1}{16\pi G}e^{2\bar{\phi}}(\bar{\mu}V' + V). \tag{71}$$

Note that the effective gravitational strength is time varying. The relative densities Ω_i for fluid i are then defined as

$$\Omega_i = 8\pi G_{\rm eff}\frac{\bar{\rho}_i}{3H^2} = \frac{\bar{\rho}_i}{\rho_i + \bar{\rho}_\phi}. \tag{72}$$

The above relation can also be used to define the relative density for the scalar field, Ω_ϕ.

IV. PERTURBATION THEORY

Cosmological perturbation theory dates back to the work of Lifshitz [37], who used a coordinate based approach and worked with the synchronous gauge. Many subsequent studies also adopted the same approach [38]. The synchronous gauge was found to contain spurious gauge modes [39], causing confusion in some earlier studies as to what was the physical growing mode. Indeed, some early studies identified these residual gauge modes and had to carefully remove them from the solutions. The existence of residual gauge freedom in the synchronous gauge led Gerlach and Sengupta, and Bardeen and others [40] to construct gauge invariant variables which, as the name implies, were devoid of unphysical gauge modes. Covariant [41] studies of perturbation theory were initiated by Hawking and later developed by Ellis and Bruni and others into a fully covariant and gauge invariant theory. Gauge-ready approaches, where one can always choose a gauge at will, depending on what is more appropriate numerically, were also studied by Hwang and Noh [42].

A. Perturbations of the gravitational variables

1. Scalar field perturbation

The scalar field is perturbed as

$$\phi = \bar{\phi} + \varphi, \tag{73}$$

where φ is the scalar field perturbation.

2. Metric perturbations

The Einstein-Hilbert frame metric is perturbed as

$$\tilde{g}_{\mu\nu} = b^2(\tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}), \tag{74}$$

where $\tilde{h}_{\mu\nu}$ is the Einstein-Hilbert frame metric perturbation. The inverse metric is given by $\tilde{g}^{\mu\nu} = \frac{1}{b^2}(\tilde{\eta}^{\mu\nu} - \tilde{h}^{\mu\nu})$, where $\tilde{h}^{\mu\nu} = \tilde{\eta}^{\mu\alpha}\tilde{\eta}^{\nu\beta}\tilde{h}_{\alpha\beta}$. One changes connection from $\tilde{\nabla}_\mu$ to $\bar{\nabla}_\mu$ via $\tilde{\nabla}_\mu u_\nu = \bar{\nabla}_\mu u_\nu - (\tilde{C}^\lambda_{\mu\nu} + \tilde{f}^\lambda_{\mu\nu})u_\lambda$ for some form u_μ, where the connection tensor $\tilde{f}^\lambda_{\mu\nu}$ is given by

$$\tilde{f}^{\lambda}_{\mu\nu} = \tfrac{1}{2}\tilde{\eta}^{\lambda\rho}(\bar{\nabla}_\mu\tilde{h}_{\nu\rho} + \bar{\nabla}_\nu\tilde{h}_{\mu\rho} - \bar{\nabla}_\rho\tilde{h}_{\mu\nu}). \tag{75}$$

Similarly, the matter-frame metric is perturbed as

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}), \tag{76}$$

where $h_{\mu\nu}$ is the matter-frame metric perturbation. The inverse metric is given by $g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} - h^{\mu\nu})$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. A connection change from ∇_μ to $\hat{\nabla}_\mu$ on any form u_μ is as $\nabla_\mu u_\nu = \hat{\nabla}_\mu u_\nu - (C^\lambda_{\mu\nu} + f^\lambda_{\mu\nu})u_\lambda$, where the connection tensor $f^\lambda_{\mu\nu}$ is given by

$$f^{\lambda}_{\mu\nu} = \tfrac{1}{2}\eta^{\lambda\rho}(\hat{\nabla}_\mu h_{\nu\rho} + \hat{\nabla}_\nu h_{\mu\rho} - \hat{\nabla}_\rho h_{\mu\nu}). \tag{77}$$

3. Vector field perturbations

Let the vector field perturbation be α_μ, defined by perturbing the vector field as

$$A_\mu = ae^{-\bar{\phi}}(t_\mu + \alpha_\mu) \tag{78}$$

and

$$A^\mu = \frac{1}{a}e^{\bar{\phi}}(t^\mu + \alpha^\mu), \tag{79}$$

where

$$\alpha^\mu = e^{-4\bar{\phi}}(\tilde{\eta}^{\mu\nu}\alpha_\nu - \tilde{h}^{\mu\nu}t_\nu). \tag{80}$$

The field strength tensor $F_{\mu\nu}$ then takes the form

$$F_{\mu\nu} = 2ae^{-\bar{\phi}}[\hat{\nabla}_{[\mu}\alpha_{\nu]} + t^\alpha(\hat{\nabla}_\alpha\bar{\phi} - \hat{\nabla}_\alpha \ln a)t_{[\mu}\alpha_{\nu]}]. \tag{81}$$

Now define the "electric" and "magnetic" field parts of $F_{\mu\nu}$ as E_μ and $B_{\mu\nu}$ which are given by

$$E_\mu = \frac{e^{\bar{\phi}}}{a}t^\beta F_{\alpha\beta} = \frac{e^{\bar{\phi}}}{a}\bar{q}^\alpha{}_\mu t^\beta F_{\alpha\beta} \tag{82}$$

and

$$B_{\mu\nu} = \frac{e^{\bar{\phi}}}{a}\bar{q}^\alpha{}_\mu\bar{q}^\beta{}_\nu F_{\alpha\beta} = -B_{\nu\mu}. \tag{83}$$

The two tensors E_μ and $B_{\mu\nu}$ obey $t^\mu E_\mu = t^\mu B_{\mu\nu} = 0$, meaning that they are purely spatial.

Explicitly, they are given by

$$E_\mu = \hat{\nabla}_\mu(t^\beta\alpha_\beta) - t^\beta\hat{\nabla}_\beta\alpha_\mu + t^\alpha(\hat{\nabla}_\alpha\bar{\phi} - \hat{\nabla}_\alpha \ln a)(\alpha_\mu + t_\mu t^\beta\alpha_\beta) \tag{84}$$

and

$$B_{\mu\nu} = 2\bar{q}^\alpha{}_\mu\bar{q}^\beta{}_\nu\hat{\nabla}_{[\alpha}\alpha_{\beta]} \tag{85}$$

which means that $F_{\mu\nu} = ae^{-\bar{\phi}}(t_\mu E_\nu - E_\mu t_\nu + B_{\mu\nu})$.

4. Perturbation of the timelike vector constraint

Let us consider the timelike constraint on A_μ which must be preserved even after the metric and the vector field are perturbed. This gives various relations between the metric perturbations and the vector field perturbations. Perturbing the constraint gives

$$\tilde{g}_{\mu\nu}A^\mu A^\nu = e^{4\bar{\phi}}(\tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu})(t^\mu + \alpha^\mu)(t^\nu + \alpha^\nu). \tag{86}$$

Expanding the right-hand side, transforming $\tilde{\eta}_{\mu\nu}$ to $\eta_{\mu\nu}$ with (42), and then imposing the constraint with (19) and (35) gives

$$\tilde{h}_{\mu\nu}t^\mu t^\nu = -2e^{-4\bar{\phi}}t_\mu\alpha^\mu. \tag{87}$$

Two more relations are

$$t^\mu\alpha_\mu = -t_\mu\alpha^\mu \tag{88}$$

and

$$\tilde{h}^{\mu\nu}t_\mu t_\nu = e^{8\bar{\phi}}\tilde{h}_{\mu\nu}t^\mu t^\nu. \tag{89}$$

5. Relating Einstein-Hilbert and matter-frame metric perturbations

Now let us find a relation between $\tilde{h}_{\mu\nu}$ and $h_{\mu\nu}$. Start by perturbing the metric transformation (1), which gives

$$\delta g_{\mu\nu} = e^{-2\bar{\phi}}\delta\tilde{g}_{\mu\nu} - 2[e^{-2\bar{\phi}}\tilde{g}_{\mu\nu} + 2\cosh(2\bar{\phi})A_\mu A_\nu]\varphi - 4\sinh(2\bar{\phi})A_{(\mu}\delta A_{\nu)}. \quad (90)$$

Using (42), (74), (76), and (78) gives the required relation, which is

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + 2(1 - e^{-4\bar{\phi}})t_{(\mu}\alpha_{\nu)} + 2[\bar{q}_{\mu\nu} + t_\mu t_\nu]\varphi, \quad (91)$$

while the inverse relation is

$$\tilde{h}^{\mu\nu} = h^{\mu\nu} + 2(e^{4\bar{\phi}} - 1)t^{(\mu}\alpha^{\nu)} + 2[\bar{q}^{\mu\nu} + t^\mu t^\nu]\varphi. \quad (92)$$

B. Perturbations of the fluid

1. Perturbations of the fluid tensors

The fluid density and pressure are perturbed as $\rho = \bar{\rho} + \delta\rho$ and $P = \bar{P} + \delta P$, respectively. The fluid velocity is perturbed as

$$u_\mu = a(t_\mu + \theta_\mu) \quad (93)$$

and

$$u^\mu = \frac{1}{a}(t^\mu + \theta^\mu), \quad (94)$$

with

$$\theta^\mu = \eta^{\mu\nu}\theta_\mu - h^{\mu\nu}t_\nu. \quad (95)$$

The shear $\Sigma_{\mu\nu}$ is already a perturbation and obeys the identities $\eta^{\mu\nu}\Sigma_{\mu\nu} = 0$ and $t^\mu\Sigma_{\mu\nu} = 0$.

2. Perturbing the timelike constraint on the fluid velocity

This is similar to the timelike constraint of the vector field. It gives

$$h^{\mu\nu}t_\mu t_\nu = h_{\mu\nu}t^\mu t^\nu = 2t^\mu\theta_\mu \quad (96)$$

and

$$t^\mu\theta_\mu = -t_\mu\theta^\mu. \quad (97)$$

Using (91), one finds a relation between the fluid velocity perturbation and the scalar and vector field perturbations as

$$t^\mu\theta_\mu = t^\mu\alpha_\mu - \varphi, \quad (98)$$

which also gives

$$A^\mu u_\mu = -e^{\bar{\phi}}(1 + \varphi). \quad (99)$$

3. Perturbing the fluid source tensors

Using the above relations the scalar source perturbation δJ is found to be

$$\delta J = e^{-2\bar{\phi}}[\delta\rho + 3\delta P - 2(\bar{\rho} + 3\bar{P})\varphi]. \quad (100)$$

Perturbing the current gives

$$\delta j_\mu = ae^{-\bar{\phi}}\{-2\sinh(2\bar{\phi})[(\bar{\rho} + \bar{P})\theta_\mu - \bar{P}\alpha_\mu + \delta\rho t_\mu] + [-(e^{2\bar{\phi}} + 3e^{-2\bar{\phi}})\bar{\rho} + 2\sinh(2\bar{\phi})\bar{P}]t_\mu\varphi\}. \quad (101)$$

Finally, perturbing the Einstein-frame stress-energy tensor yields

$$\delta S_{\mu\nu} = a^2\{\bar{P}\tilde{h}_{\mu\nu} - 2\bar{P}\bar{q}_{\mu\nu}\varphi - 2(1 - e^{-4\bar{\phi}})\bar{\rho}t_{(\mu}\alpha_{\nu)} - 2[(1 + 3e^{-4\bar{\phi}})\bar{\rho} + e^{-4\bar{\phi}}\bar{P}]t_\mu t_\nu\varphi - (1 - 2e^{-4\bar{\phi}})\delta\rho t_\mu t_\nu + \delta P\bar{q}_{\mu\nu} + 2e^{-4\bar{\phi}}(\bar{\rho} + \bar{P})t_{(\mu}\theta_{\nu)} + \hat{\Sigma}_{\mu\nu}\}, \quad (102)$$

where $\hat{\Sigma}_{\mu\nu} = \frac{1}{a^2}\Sigma_{\mu\nu}$.

C. Perturbed field equations

1. The perturbed fluid equations of motion

Let the density contrast and sound speed be given as usual by $\delta = \frac{\delta\rho}{\bar{\rho}}$ and $C_s^2 = \frac{\delta P}{\delta\rho}$, respectively. Then the perturbed fluid equations become

$$t^\mu\hat{\nabla}_\mu\delta + 3(t^\mu\hat{\nabla}_\mu \ln a)(C_s^2 - w)\delta + (1 + w)(\hat{\nabla}_\mu\theta^\mu + \tfrac{1}{2}t^\nu\hat{\nabla}_\nu h^\mu{}_\mu) = 0 \quad (103)$$

and

$$\bar{q}^\nu{}_\mu\left[t^\rho\hat{\nabla}_\rho\theta_\nu + (1 - 3w)(t^\rho\hat{\nabla}_\rho \ln a)\theta_\nu - \frac{1}{2}\hat{\nabla}_\nu(t^\alpha t^\beta h_{\alpha\beta}) + \hat{\nabla}_\nu\left(\frac{C_s^2\delta}{1 + w}\right) + \frac{t^\rho\hat{\nabla}_\rho w}{1 + w}\theta_\nu + \frac{1}{\bar{\rho} + \bar{P}}\hat{\nabla}_\rho\hat{\Sigma}^\rho{}_\nu\right] = 0. \quad (104)$$

2. Perturbed constraint equation

The constraint equation (20) yields

$$\delta\mu = 2\frac{V'}{V''}t^\nu\alpha_\nu + \frac{4e^{2\bar{\phi}}}{a^2V''}(t^\mu\hat{\nabla}_\mu\bar{\phi})(t^\nu\hat{\nabla}_\nu\varphi). \quad (105)$$

Unlike the unperturbed case, this need not be inverted. Rather, it gives $\delta\mu$ directly in terms of the other variables which is then used in the relevant places in the other perturbed equations.

3. Perturbed scalar field equation

Following the same approach as for the background, one can start from (17), which is then split into the background part $\bar{\Gamma}^\mu$ found in a previous section, and a perturbation γ^μ, as $\Gamma^\mu = \bar{\Gamma}^\mu + \gamma^\mu$. Then one performs the projection $\Gamma^\mu = -\Gamma A^\mu + q^\mu{}_\nu\Gamma^\nu$, where $\Gamma = A_\mu\Gamma^\mu$. The scalar field Γ is again split into a background and a perturbed part as $\Gamma = \bar{\Gamma} + \gamma$, where $\gamma = ae^{-\bar{\phi}}(\alpha_\mu\bar{\Gamma}^\mu + t_\mu\gamma^\mu)$ is the perturbed part which makes $\gamma^\mu = -\frac{e^{\bar{\phi}}}{a}(\gamma + \bar{\Gamma}t^\nu\alpha_\nu)t^\mu + \bar{q}^\mu{}_\nu\gamma^\nu$. After some calculations, one gets the perturbation

γ^μ to be

$$\gamma^\mu = \frac{\bar{\mu}}{a^2} e^{2\bar{\phi}} (t^\lambda \hat{\nabla}_\lambda \bar{\phi})[t^\nu \tilde{h}^\mu{}_\nu - t^\nu \alpha_\nu t^\mu - \alpha^\mu] + \frac{\mu}{a^2}(e^{-2\bar{\phi}} \bar{q}^{\mu\nu} - 2e^{2\bar{\phi}} t^\mu t^\nu)\hat{\nabla}_\nu \varphi - 4\frac{e^{2\bar{\phi}}}{a^2}\frac{V'}{V''}[(t^\rho \hat{\nabla}_\rho \bar{\phi}) t^\lambda \alpha_\lambda + t^\beta \hat{\nabla}_\beta \varphi] t^\mu, \quad (106)$$

while its projection on the hypersurface is

$$\bar{q}^\mu{}_\nu \gamma^\nu = \frac{\bar{\mu}}{a^2} e^{2\bar{\phi}} (t^\lambda \hat{\nabla}_\lambda \bar{\phi})[\bar{q}^\mu{}_\nu t^\rho \tilde{h}^\nu{}_\rho - \bar{q}^\mu{}_\nu \alpha^\nu] + \frac{\bar{\mu}}{a^2} e^{-2\bar{\phi}} \bar{q}^{\lambda\nu} \hat{\nabla}_\lambda \varphi. \quad (107)$$

The scalar field equation is then split into two first-order equations given by

$$t^\mu \hat{\nabla}_\mu \gamma = -3(t^\rho \hat{\nabla}_\rho \ln b)\gamma + \frac{\bar{\mu}}{a} e^{-3\bar{\phi}} \Delta\varphi + \bar{\Gamma}\left[\hat{\nabla}_\mu(\bar{q}^{\mu\nu} t^\rho \tilde{h}_{\rho\nu}) - \frac{1}{2} e^{-4\bar{\phi}} \bar{q}^{\mu\nu} \hat{\nabla}_\mu \alpha_\nu - \frac{1}{2} t^\lambda \hat{\nabla}_\lambda (\bar{q}^{\mu\nu} \tilde{h}_{\mu\nu})\right] - 8\pi G a e^{-3\bar{\phi}} \bar{\rho}[(1 + 3C_s^2)\delta - (1 + 3w)(t^\nu \alpha_\nu + 2\varphi)], \quad (108)$$

where $\Delta = \bar{q}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu$ is the Laplace-Beltrami operator and

$$t^\nu \hat{\nabla}_\nu \varphi = \frac{1}{2U} a e^{-\bar{\phi}} \gamma - (t^\mu \hat{\nabla}_\mu \bar{\phi}) t^\nu \alpha_\nu. \quad (109)$$

4. Perturbed vector field equation

The divergence of $F^\mu{}_\nu$ in terms of E_μ and $B_{\mu\nu}$ is

$$\tilde{\nabla}_\mu F^\mu{}_\nu = \frac{e^{\bar{\phi}}}{a}[t^\mu \hat{\nabla}_\mu E_\nu - e^{-4\bar{\phi}} \bar{q}^{\mu\alpha} t_\nu \hat{\nabla}_\mu E_\alpha + e^{-4\bar{\phi}} \bar{q}^{\alpha\mu} \hat{\nabla}_\alpha B_{\mu\nu} + (t^\mu \hat{\nabla}_\mu \ln b) E_\nu]. \quad (110)$$

It is easier to perturb the vector field equation (15) which does not include the Lagrange multiplier. It gives

$$K_B[t^\mu \hat{\nabla}_\mu E_\alpha + e^{-4\bar{\phi}} \bar{q}^{\mu\nu} \hat{\nabla}_\mu B_{\nu\alpha} + (t^\mu \hat{\nabla}_\mu \ln b) E_\alpha] + \bar{q}^\nu{}_\alpha \{\bar{\mu}(t^\beta \hat{\nabla}_\beta \bar{\phi})[\hat{\nabla}_\nu \varphi + (t^\beta \hat{\nabla}_\beta \bar{\phi})\alpha_\nu] + 8\pi G a^2 (1 - e^{-4\bar{\phi}})(\bar{\rho} + \bar{P})(\theta_\nu - \alpha_\nu)\} = 0 \quad (111)$$

which is a first-order equation for E_μ. The other equation needed is a rearrangement of the definition of E_μ as a first-order equation for α_μ,

$$t^\beta \hat{\nabla}_\beta (\bar{q}^\nu{}_\mu \alpha_\nu) = -E_\mu + \bar{q}^\nu{}_\mu \{\hat{\nabla}_\nu (t^\beta \alpha_\beta) + t^\alpha (\hat{\nabla}_\alpha \bar{\phi} - \hat{\nabla}_\alpha \ln a)\alpha_\nu\}. \quad (112)$$

5. Perturbed Lagrange multiplier

The perturbed Lagrange multiplier is given by

$$\delta\lambda = K_B \frac{e^{-2\bar{\phi}}}{a^2} \bar{q}^{\mu\nu} \hat{\nabla}_\mu E_\nu + \frac{1}{2} V' \delta\mu + 2\bar{\mu} \frac{e^{2\bar{\phi}}}{a^2} \times (t^\mu \hat{\nabla}_\mu \bar{\phi}) t^\nu \hat{\nabla}_\nu \varphi + \bar{\mu} V' t^\mu \alpha_\mu - 8\pi G \delta(A^\mu j_\mu), \quad (113)$$

where

$$\delta(A^\mu j_\mu) = 4\cosh(2\bar{\phi})\bar{\rho}\varphi + 2\sinh(2\bar{\phi})\delta\rho, \quad (114)$$

giving

$$\delta\lambda = K_B \frac{e^{-2\bar{\phi}}}{a^2} \bar{q}^{\mu\nu} \hat{\nabla}_\mu E_\nu + \frac{1}{2} V' \delta\mu + 2\bar{\mu} \frac{e^{2\bar{\phi}}}{a^2} (t^\mu \hat{\nabla}_\mu \bar{\phi}) t^\nu \hat{\nabla}_\nu \varphi + \bar{\mu} V' t^\mu \alpha_\mu - 8\pi G \bar{\rho}[4\cosh(2\bar{\phi})\varphi + 2\sinh(2\bar{\phi})\delta]. \quad (115)$$

6. Perturbed generalized Einstein equations

The perturbed tensor $Y_{\mu\nu}$ yields

$$\delta Y_{\mu\nu} = \frac{1}{2} b^2 (\bar{\mu} \bar{V}' - \bar{V}) \tilde{h}_{\mu\nu} - 4\bar{\mu}(t^\alpha \hat{\nabla}_\alpha \bar{\phi}) t_{(\mu} \hat{\nabla}_{\nu)} \varphi - K_B e^{-4\bar{\phi}} \bar{q}^{\alpha\beta} \hat{\nabla}_\alpha E_\beta t_\mu t_\nu + \left[e^{2\bar{\phi}} \bar{\mu} - \left(\bar{\mu} - 2\frac{V'}{V''}\right) e^{-2\bar{\phi}}\right][a^2 V' t^\alpha \alpha_\alpha + 2e^{2\bar{\phi}} (t^\alpha \hat{\nabla}_\alpha \bar{\phi})(t^\beta \hat{\nabla}_\beta \varphi)] t_\mu t_\nu + e^{2\bar{\phi}} \bar{\mu}[a^2 V' t^\beta \alpha_\beta + 2e^{2\bar{\phi}} (t^\beta \hat{\nabla}_\beta \bar{\phi})(t^\alpha \hat{\nabla}_\alpha \varphi)]\eta_{\mu\nu} + 8\pi G a^2 \bar{\rho}\{(1 - e^{-4\bar{\phi}})[2t_{(\mu}\alpha_{\nu)} + t_\mu t_\nu \delta] + 2(1 + e^{-4\bar{\phi}})\varphi t_\mu t_\nu\} \quad (116)$$

which, when combined with $S_{\mu\nu}$, gives the right-hand side of the generalized Einstein equations as

$$\delta Y_{\mu\nu} + 8\pi G\delta S_{\mu\nu} = \frac{1}{2}b^2(\bar{\mu}\bar{V}' - \bar{V})\tilde{h}_{\mu\nu} - 4\bar{\mu}(t^\alpha\hat{\nabla}_\alpha\bar{\phi})t_{(\mu}\hat{\nabla}_{\nu)}\varphi - K_B e^{-4\bar{\phi}}\bar{q}^{\alpha\beta}\hat{\nabla}_\alpha E_\beta t_\mu t_\nu + \left[e^{2\bar{\phi}}\bar{\mu} - \left(\bar{\mu} - 2\frac{V'}{V''}\right)e^{-2\bar{\phi}}\right][a^2V't^\alpha\alpha_\alpha + 2e^{2\bar{\phi}}(t^\alpha\hat{\nabla}_\alpha\bar{\phi})(t^\beta\hat{\nabla}_\beta\varphi)]t_\mu t_\nu + e^{2\bar{\phi}}\bar{\mu}[a^2V't^\beta\alpha_\beta + 2e^{2\bar{\phi}}(t^\beta\hat{\nabla}_\beta\bar{\phi})(t^\alpha\hat{\nabla}_\alpha\varphi)]\eta_{\mu\nu} + 8\pi Ga^2\bar{\rho}\{w\tilde{h}_{\mu\nu} + (C_s^2\delta - 2w\varphi)\bar{q}_{\mu\nu} + e^{-4\bar{\phi}}[\delta - 2(2+w)\varphi]t_\mu t_\nu + 2e^{-4\bar{\phi}}(1+w)t_{(\mu}\theta_{\nu)} + \hat{\Sigma}_{\mu\nu}\}. \tag{117}$$

Contracting with $t^\mu t^\nu$ gives

$$t^\mu t^\nu[\delta Y_{\mu\nu} + 8\pi G\delta S_{\mu\nu}] = a^2e^{-2\bar{\phi}}\left[2\frac{V'}{V''}V' - V\right]t^\mu\alpha_\mu + 2U(t^\alpha\hat{\nabla}_\alpha\bar{\phi})(t^\beta\hat{\nabla}_\beta\varphi) - K_Be^{-4\bar{\phi}}\bar{q}^{\alpha\beta}\hat{\nabla}_\alpha E_\beta + 8\pi Ga^2e^{-4\bar{\phi}}\bar{\rho}[\delta - 2\varphi - 2t^\mu\alpha_\mu], \tag{118}$$

contracting with $\bar{q}^\mu{}_\alpha t^\nu$ yields

$$\bar{q}^\mu{}_\alpha t^\nu[\delta Y_{\mu\nu} + 8\pi G\delta S_{\mu\nu}] = \tfrac{1}{2}b^2(\bar{\mu}\bar{V}' - \bar{V})\bar{q}^\mu{}_\alpha t^\nu\tilde{h}_{\mu\nu} + 2\bar{\mu}(t^\beta\hat{\nabla}_\beta\bar{\phi})\bar{q}^\mu{}_\alpha\hat{\nabla}_\mu\varphi + 8\pi Ga^2\bar{\rho}\{w\bar{q}^\mu{}_\alpha t^\nu\tilde{h}_{\mu\nu} - e^{-4\bar{\phi}}(1+w)\bar{q}^\mu{}_\alpha\theta_\mu\}, \tag{119}$$

while contracting with $\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta$ gives

$$\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta[\delta Y_{\mu\nu} + 8\pi G\delta S_{\mu\nu}] = a^2[\tfrac{1}{2}e^{2\bar{\phi}}(\bar{\mu}\bar{V}' - \bar{V}) + 8\pi G\bar{P}]\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta\tilde{h}_{\mu\nu} + 8\pi Ga^2\bar{\rho}[(C_s^2\delta - 2w\varphi)\bar{q}_{\alpha\beta} + \bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta\hat{\Sigma}_{\mu\nu}] + \bar{\mu}e^{2\bar{\phi}}[a^2V't^\rho\alpha_\rho + 2e^{2\bar{\phi}}(t^\rho\hat{\nabla}_\rho\bar{\phi})(t^\lambda\hat{\nabla}_\lambda\varphi)]\bar{q}_{\alpha\beta}. \tag{120}$$

The above equation can be further simplified by separating it into trace and traceless parts. The trace part is found by contracting with $\bar{q}^{\mu\nu}$ and is given by

$$\bar{q}^{\mu\nu}[\delta Y_{\mu\nu} + 8\pi G\delta S_{\mu\nu}] = a^2[\tfrac{1}{2}e^{2\bar{\phi}}(\bar{\mu}\bar{V}' - \bar{V}) + 8\pi G\bar{P}]\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu} + 24\pi Ga^2\bar{\rho}(C_s^2\delta - 2w\varphi) + 3\bar{\mu}e^{2\bar{\phi}}[a^2V't^\rho\alpha_\rho + 2e^{2\bar{\phi}}(t^\rho\hat{\nabla}_\rho\bar{\phi})(t^\lambda\hat{\nabla}_\lambda\varphi)], \tag{121}$$

while the traceless part is

$$[\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta - \tfrac{1}{3}\bar{q}^{\mu\nu}\bar{q}_{\alpha\beta}][\delta Y_{\mu\nu} + 8\pi G\delta S_{\mu\nu}] = a^2[\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta - \tfrac{1}{3}\bar{q}^{\mu\nu}\bar{q}_{\alpha\beta}]\{8\pi G\hat{\Sigma}_{\mu\nu} + [\tfrac{1}{2}e^{2\bar{\phi}}(\bar{\mu}\bar{V}' - \bar{V}) + 8\pi G\bar{P}]\tilde{h}_{\mu\nu}\}. \tag{122}$$

Now let us turn to the left-hand side of the generalized Einstein equations. The perturbed Ricci tensor of $\tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}$ is simply given by $2\bar{\nabla}_{[\lambda}\tilde{f}^\lambda_{\nu]\mu}$, which gives the perturbed Einstein tensor as

$$\delta H_{\mu\nu} = 2\bar{\nabla}_{[\lambda}\tilde{f}^\lambda_{\nu]\mu} - \tilde{\eta}^{\alpha\beta}\bar{\nabla}_{[\lambda}\tilde{f}^\lambda_{\alpha]\beta}\tilde{\eta}_{\mu\nu} + \frac{K}{r_c^2}[\bar{q}^{\alpha\beta}\tilde{h}_{\alpha\beta}\tilde{\eta}_{\mu\nu} - 3\tilde{h}_{\mu\nu}] \tag{123}$$

and, after expanding the connection tensors,

$$\delta H_{\mu\nu} = \frac{1}{2}[2\tilde{\eta}^{\alpha\beta}\bar{\nabla}_\beta\bar{\nabla}_{(\mu}\tilde{h}_{\nu)\alpha} - \bar{\nabla}_\mu\bar{\nabla}_\nu\tilde{h}^\alpha{}_\alpha - \bar{\nabla}^2\tilde{h}_{\mu\nu} - (\bar{\nabla}_\alpha\bar{\nabla}_\beta\tilde{h}^{\alpha\beta} - \bar{\nabla}^2\tilde{h}^\alpha{}_\alpha)\tilde{\eta}_{\mu\nu}] + \frac{K}{r_c^2}[\bar{q}^{\alpha\beta}\tilde{h}_{\alpha\beta}\tilde{\eta}_{\mu\nu} - 3\tilde{h}_{\mu\nu}]. \tag{124}$$

The perturbed Einstein tensor in the Einstein-Hilbert frame is then obtained via a conformal transformation as

$$\delta\tilde{G}_{\mu\nu} = \delta H_{\mu\nu} + 2\tilde{f}^\alpha_{\mu\nu}\bar{\nabla}_\alpha\ln b - \tilde{\eta}_{\mu\nu}[2\tilde{h}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\ln b + 2\tilde{\eta}^{\alpha\beta}\tilde{f}^\lambda_{\alpha\beta}\bar{\nabla}_\lambda\ln b + \tilde{h}^{\alpha\beta}t_\alpha t_\beta(t^\rho\bar{\nabla}_\rho\ln b)^2] + \tilde{h}_{\mu\nu}[2\bar{\nabla}^2\ln b - e^{4\bar{\phi}}(t^\beta\bar{\nabla}_\beta\ln b)^2]. \tag{125}$$

Changing connection to $\hat{\nabla}_\mu$ (see Appendix C) and combining terms gives

$$\delta\tilde{G}_{\mu\nu}=\frac{K}{r_c^2}[\bar{q}^{\alpha\beta}\tilde{h}_{\alpha\beta}\tilde{\eta}_{\mu\nu}-3\tilde{h}_{\mu\nu}]-\frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}_\nu\tilde{h}^\alpha{}_\alpha-\frac{1}{2}\tilde{\eta}^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\tilde{h}_{\mu\nu}+\frac{1}{2}[\tilde{\eta}^{\rho\lambda}\hat{\nabla}_\rho\hat{\nabla}_\lambda\tilde{h}^\alpha{}_\alpha-\tilde{\eta}^{\alpha\rho}\tilde{\eta}^{\beta\lambda}\hat{\nabla}_\alpha\hat{\nabla}_\beta\tilde{h}_{\rho\lambda}]\tilde{\eta}_{\mu\nu}$$
$$+\tilde{\eta}^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_{(\mu}\tilde{h}_{\nu)\beta}+2e^{4\bar{\phi}}[\hat{\nabla}^2\bar{\phi}-6(t^\rho\hat{\nabla}_\rho\bar{\phi})^2]t^\alpha t^\beta\tilde{h}_{\alpha\beta}t_\mu t_\nu-e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\bar{\phi})t^\lambda\hat{\nabla}_\lambda(\bar{q}^\alpha{}_\beta\tilde{h}^\beta{}_\alpha)\tilde{\eta}_{\mu\nu}$$
$$+(t^\rho\hat{\nabla}_\rho\bar{\phi})\{[2\tilde{\eta}^{\alpha\beta}t^\lambda\hat{\nabla}_\alpha\tilde{h}_{\beta\lambda}-t^\beta\hat{\nabla}_\beta\tilde{h}^\alpha{}_\alpha]t_\mu t_\nu+e^{4\bar{\phi}}[t^\lambda\hat{\nabla}_\lambda\tilde{h}_{\mu\nu}-2t^\alpha\hat{\nabla}_{(\mu}\tilde{h}_{\nu)\alpha}+2t^\alpha t^\beta t_{(\mu}\hat{\nabla}_{\nu)}\tilde{h}_{\alpha\beta}]\}$$
$$-2e^{4\bar{\phi}}\{(t^\rho\hat{\nabla}_\rho\bar{\phi})[t^\lambda\hat{\nabla}_\lambda(t_\alpha t^\beta\tilde{h}^\alpha{}_\beta)-t^\lambda\tilde{\eta}^{\alpha\beta}\hat{\nabla}_\alpha\tilde{h}_{\lambda\beta}]-t_\alpha t^\beta\tilde{h}^\alpha{}_\beta\hat{\nabla}^2\bar{\phi}\}\tilde{\eta}_{\mu\nu}$$
$$+e^{4\bar{\phi}}(t^\lambda\hat{\nabla}_\lambda\ln b)[t^\beta\hat{\nabla}_\beta\tilde{h}_{\mu\nu}-2t^\beta\hat{\nabla}_{(\mu}\tilde{h}_{\nu)\beta}-4(t^\rho\hat{\nabla}_\rho\bar{\phi})t^\alpha t^\beta\tilde{h}_{\alpha\beta}t_\mu t_\nu]$$
$$-e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\ln b)[t^\lambda\hat{\nabla}_\lambda\tilde{h}^\alpha{}_\alpha-2t^\rho\tilde{\eta}^{\alpha\beta}\hat{\nabla}_\alpha\tilde{h}_{\beta\rho}+8(t^\rho\hat{\nabla}_\rho\bar{\phi})t_\alpha t^\beta\tilde{h}^\alpha{}_\beta]\tilde{\eta}_{\mu\nu}$$
$$+e^{4\bar{\phi}}[2\hat{\nabla}^2\ln b-4(t^\lambda\hat{\nabla}_\lambda\ln b)(t^\rho\hat{\nabla}_\rho\bar{\phi})-(t^\lambda\hat{\nabla}_\lambda\ln b)^2][\tilde{h}_{\mu\nu}+t_\alpha t^\beta\tilde{h}^\alpha{}_\beta\tilde{\eta}_{\mu\nu}]. \tag{126}$$

Now let us perform the contractions like in all the above cases. Contracting (126) with $t^\mu t^\nu$ gives

$$t^\mu t^\nu\delta\tilde{G}_{\mu\nu}=-\frac{K}{r_c^2}[e^{-4\bar{\phi}}\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu}+3t^\mu t^\nu\tilde{h}_{\mu\nu}]+\frac{1}{2}e^{-4\bar{\phi}}[\bar{q}^{\alpha\mu}\bar{q}^{\beta\nu}\hat{\nabla}_\alpha\hat{\nabla}_\beta\tilde{h}_{\mu\nu}-\Delta(\bar{q}^{\alpha\beta}\tilde{h}_{\alpha\beta})]$$
$$+(t^\rho\hat{\nabla}_\rho\ln b)[t^\lambda\hat{\nabla}_\lambda(\bar{q}^{\alpha\beta}\tilde{h}_{\alpha\beta})-2t^\rho\bar{q}^{\alpha\beta}\hat{\nabla}_\alpha\tilde{h}_{\beta\rho}]. \tag{127}$$

Contracting (126) with $t^\mu\bar{q}^\nu{}_\alpha$ gives

$$t^\mu\bar{q}^\nu{}_\alpha\delta\tilde{G}_{\mu\nu}=-\frac{3K}{r_c^2}t^\mu\bar{q}^\nu{}_\alpha\tilde{h}_{\mu\nu}+\frac{1}{2}t^\mu\bar{q}^\nu{}_\alpha\bar{q}^{\rho\beta}\hat{\nabla}_\rho\hat{\nabla}_\mu\tilde{h}_{\nu\beta}-\frac{1}{2}t^\mu\bar{q}^\nu{}_\alpha\hat{\nabla}_\mu\hat{\nabla}_\nu(\bar{q}^{\rho\beta}\tilde{h}_{\rho\beta})+\frac{1}{2}q^\nu{}_\alpha\hat{\nabla}_\rho\hat{\nabla}_\nu(\bar{q}^{\rho\beta}t^\mu\tilde{h}_{\mu\beta})$$
$$-\frac{1}{2}t^\mu\bar{q}^\nu{}_\alpha\Delta\tilde{h}_{\mu\nu}-(t^\lambda\hat{\nabla}_\lambda\ln b)\bar{q}^\nu{}_\alpha\hat{\nabla}_\nu(t_\mu t^\beta\tilde{h}^\mu{}_\beta)+[2\hat{\nabla}^2\ln b-4(t^\lambda\hat{\nabla}_\lambda\ln b)(t^\rho\hat{\nabla}_\rho\bar{\phi})$$
$$-(t^\lambda\hat{\nabla}_\lambda\ln b)^2]t_\mu\bar{q}^\nu{}_\alpha\tilde{h}^\mu{}_\nu \tag{128}$$

and, likewise, contracting (126) with $\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta$ yields

$$\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta\delta\tilde{G}_{\mu\nu}=\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta\frac{K}{r_c^2}[\bar{q}^{\rho\lambda}\tilde{h}_{\rho\lambda}\bar{q}_{\mu\nu}-3\tilde{h}_{\mu\nu}]+\bar{q}^\mu{}_{(\alpha}\bar{q}^\nu_{\beta)}\bar{q}^{\rho\lambda}\hat{\nabla}_\rho\hat{\nabla}_\mu\tilde{h}_{\nu\lambda}-e^{4\bar{\phi}}\bar{q}^\mu{}_{(\alpha}\bar{q}^\nu_{\beta)}t^\rho t^\lambda\hat{\nabla}_\rho\hat{\nabla}_\mu\tilde{h}_{\nu\lambda}$$
$$-\frac{1}{2}\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta\hat{\nabla}_\mu\hat{\nabla}_\nu\tilde{h}^\rho{}_\rho-\frac{1}{2}\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta\Delta\tilde{h}_{\mu\nu}+\frac{1}{2}e^{4\bar{\phi}}\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta t^\rho t^\lambda\hat{\nabla}_\rho\hat{\nabla}_\lambda\tilde{h}_{\mu\nu}$$
$$+\frac{1}{2}[\Delta\tilde{h}^\mu{}_\mu-e^{4\bar{\phi}}t^\rho t^\lambda\hat{\nabla}_\rho\hat{\nabla}_\lambda(\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu})-\bar{q}^{\mu\rho}\bar{q}^{\nu\lambda}\hat{\nabla}_\mu\hat{\nabla}_\nu\tilde{h}_{\rho\lambda}+2e^{4\bar{\phi}}t^\nu\hat{\nabla}_\nu\hat{\nabla}_\mu(\bar{q}^{\mu\rho}t^\lambda\tilde{h}_{\rho\lambda})]\bar{q}_{\alpha\beta}$$
$$+e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\bar{\phi})[2t^\lambda\bar{q}^{\mu\nu}\hat{\nabla}_\mu\tilde{h}_{\lambda\nu}-t^\lambda\hat{\nabla}_\lambda(\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu})]\bar{q}_{\alpha\beta}$$
$$+e^{4\bar{\phi}}[t^\rho\hat{\nabla}_\rho\bar{\phi}+t^\lambda\hat{\nabla}_\lambda\ln b][\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta t^\rho\hat{\nabla}_\rho\tilde{h}_{\mu\nu}-2\bar{q}^\mu{}_{(\alpha}\bar{q}^\nu_{\beta)}t^\rho\hat{\nabla}_\mu\tilde{h}_{\nu\rho}]$$
$$-e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\ln b)[t^\lambda\hat{\nabla}_\lambda\tilde{h}^\mu{}_\mu-2t^\rho\bar{q}^{\mu\nu}\hat{\nabla}_\mu\tilde{h}_{\nu\rho}+2t^\rho\hat{\nabla}_\rho(t_\mu t^\nu\tilde{h}^\mu{}_\nu)]\bar{q}_{\alpha\beta}$$
$$+e^{4\bar{\phi}}[2\hat{\nabla}^2\ln b-4(t^\lambda\hat{\nabla}_\lambda\ln b)(t^\rho\hat{\nabla}_\rho\bar{\phi})-(t^\lambda\hat{\nabla}_\lambda\ln b)^2][\bar{q}^\mu{}_\alpha\bar{q}^\nu{}_\beta\tilde{h}_{\mu\nu}+t_\mu t^\nu\tilde{h}^\mu{}_\nu\bar{q}_{\alpha\beta}]. \tag{129}$$

Taking the trace of the above equation, by contracting with $\bar{q}^{\alpha\beta}$, yields

$$\bar{q}^{\mu\nu}\delta\tilde{G}_{\mu\nu}=-\tfrac{1}{2}\bar{q}^{\mu\nu}\bar{q}^{\rho\lambda}\hat{\nabla}_\rho\hat{\nabla}_\mu\tilde{h}_{\nu\lambda}+2e^{4\bar{\phi}}t^\nu\hat{\nabla}_\nu\hat{\nabla}_\mu(\bar{q}^{\mu\rho}t^\lambda\tilde{h}_{\rho\lambda})+\tfrac{1}{2}\Delta(\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu})-\Delta(t_\mu t^\nu\tilde{h}^\mu{}_\nu)$$
$$-e^{4\bar{\phi}}t^\rho t^\lambda\hat{\nabla}_\rho\hat{\nabla}_\lambda(\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu})+2e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\bar{\phi})[2t^\lambda\bar{q}^{\mu\nu}\hat{\nabla}_\mu\tilde{h}_{\lambda\nu}-t^\lambda\hat{\nabla}_\lambda(\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu})]$$
$$-e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\ln b)[2t^\lambda\hat{\nabla}_\lambda(\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu})+3t^\rho\hat{\nabla}_\rho(t_\mu t^\nu\tilde{h}^\mu{}_\nu)-4t^\rho\bar{q}^{\mu\nu}\hat{\nabla}_\mu\tilde{h}_{\nu\rho}]$$
$$+e^{4\bar{\phi}}[2\hat{\nabla}^2\ln b-4(t^\lambda\hat{\nabla}_\lambda\ln b)(t^\rho\hat{\nabla}_\rho\bar{\phi})-(t^\lambda\hat{\nabla}_\lambda\ln b)^2][\bar{q}^{\mu\nu}\tilde{h}_{\mu\nu}+3t_\mu t^\nu\tilde{h}^\mu{}_\nu] \tag{130}$$

which gives the traceless part as

$$\left[\bar{q}^{\mu}{}_{\alpha}\bar{q}^{\nu}{}_{\beta} - \frac{1}{3}\bar{q}^{\mu\nu}\bar{q}_{\alpha\beta}\right]\delta\tilde{G}_{\mu\nu} = \left[\bar{q}^{\mu}{}_{\alpha}\bar{q}^{\nu}{}_{\beta} - \frac{1}{3}\bar{q}^{\mu\nu}\bar{q}_{\alpha\beta}\right]\left\{-\frac{3K}{r_c^2}\tilde{h}_{\mu\nu} + \bar{q}^{\rho\lambda}\hat{\nabla}_\rho\hat{\nabla}_\mu\tilde{h}_{\nu\lambda} - e^{4\bar{\phi}}t^\rho t^\lambda\hat{\nabla}_\rho\hat{\nabla}_\mu\tilde{h}_{\nu\lambda} - \frac{1}{2}\Delta\tilde{h}_{\mu\nu} - \frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}_\nu(\bar{q}^{\rho\lambda}\tilde{h}_{\rho\lambda}) + \frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}_\nu(t_\rho t^\lambda\tilde{h}^\rho{}_\lambda) + \frac{1}{2}e^{4\bar{\phi}}t^\rho t^\lambda\hat{\nabla}_\rho\hat{\nabla}_\lambda\tilde{h}_{\mu\nu} + e^{4\bar{\phi}}[t^\rho\hat{\nabla}_\rho\bar{\phi} + t^\rho\hat{\nabla}_\rho \ln b][t^\lambda\hat{\nabla}_\lambda\tilde{h}_{\mu\nu} - 2t^\lambda\hat{\nabla}_\mu\tilde{h}_{\nu\lambda}] + e^{4\bar{\phi}}[2\hat{\nabla}^2\ln b - 4(t^\lambda\hat{\nabla}_\lambda \ln b)(t^\rho\hat{\nabla}_\rho\bar{\phi}) - (t^\lambda\hat{\nabla}_\lambda \ln b)^2]\tilde{h}_{\mu\nu}\right\}. \quad (131)$$

The above contractions are then combined with their counterparts coming from the right-hand side of the generalized Einstein equations.

V. IRREDUCIBLE DECOMPOSITION OF METRIC PERTURBATIONS

A. Harmonic mode decomposition

1. Einstein metric decomposition

Let us write the Einstein metric perturbation into irreducible parts. This yields

$$\tilde{h}_{\mu\nu} = 2e^{-4\bar{\phi}}\tilde{\Xi}t_\mu t_\nu - 2t_{(\mu}\bar{q}^\alpha_{\nu)}\hat{\nabla}_\alpha\tilde{\zeta} - 2t_{(\mu}\tilde{r}_{\nu)} + \tfrac{1}{3}\tilde{\chi}\bar{q}_{\mu\nu} + \left(\bar{q}^\alpha{}_\mu\bar{q}^\beta{}_\nu - \tfrac{1}{3}\bar{q}_{\mu\nu}\bar{q}^{\alpha\beta}\right)\hat{\nabla}_\alpha\hat{\nabla}_\beta\tilde{\nu} + 2\hat{\nabla}_\alpha\tilde{f}_{(\mu}\bar{q}^\alpha_{\nu)} + \tilde{\chi}_{\mu\nu}, \quad (132)$$

where $t^\mu\tilde{r}_\mu = t^\mu\tilde{f}_\mu = t^\mu\tilde{\chi}_{\mu\nu} = 0$.

The variables above are classified as follows: Scalar modes ($\tilde{\Xi}$, $\tilde{\zeta}$, $\tilde{\chi}$, and $\tilde{\nu}$), vector modes ($\tilde{r}_\mu$ and $\tilde{f}_\mu$) obeying $\bar{q}^{\mu\nu}\hat{\nabla}_\mu\tilde{r}_\nu = 0$ and $\bar{q}^{\mu\nu}\hat{\nabla}_\mu\tilde{f}_\nu = 0$, and tensor modes $\tilde{\chi}_{\mu\nu}$ obeying $\bar{q}^{\mu\nu}\tilde{\chi}_{\mu\nu} = 0$ and $\bar{q}^{\lambda\mu}\hat{\nabla}_\lambda\tilde{\chi}_{\mu\nu} = 0$.

2. Vector field decomposition

The vector field perturbation is decomposed as

$$\alpha_\mu = -\tilde{\Xi}t_\mu + \bar{q}^\nu{}_\mu\hat{\nabla}_\nu\alpha + \beta_\mu \quad (133)$$

with $t^\mu\beta_\mu = 0$. It contains a scalar mode α, given by $\Delta\alpha = \bar{q}^{\mu\nu}\hat{\nabla}_\mu\alpha_\nu$, and two vector modes β_μ obeying $\bar{q}^{\mu\nu}\hat{\nabla}_\mu\beta_\nu = 0$.

The "electric field" is also decomposed as

$$E_\mu = \bar{q}^\nu{}_\mu\hat{\nabla}_\nu E + \epsilon_\mu \quad (134)$$

with E being a scalar mode given by $\Delta E = \bar{q}^{\mu\nu}\hat{\nabla}_\mu E_\nu$ and ϵ_μ two vector modes obeying, as usual, $\bar{q}^{\mu\nu}\hat{\nabla}_\mu\epsilon_\nu = 0$.

3. Matter metric decomposition

The matter-frame metric is decomposed similarly to the Einstein-frame metric as

$$h_{\mu\nu} = 2\Xi t_\mu t_\nu - 2t_{(\mu}\bar{q}^\alpha{}_{\nu)}\hat{\nabla}_\alpha\zeta - 2t_{(\mu}r_{\nu)} + \tfrac{1}{3}\chi\bar{q}_{\mu\nu} + (\bar{q}^\alpha{}_\mu\bar{q}^\beta{}_\nu - \tfrac{1}{3}\bar{q}_{\mu\nu}\bar{q}^{\alpha\beta})\hat{\nabla}_\alpha\hat{\nabla}_\beta\nu + 2\hat{\nabla}_\alpha f_{(\mu}\bar{q}^\alpha{}_{\nu)} + \chi_{\mu\nu}, \quad (135)$$

where $t^\mu r_\mu = t^\mu f_\mu = t^\mu\chi_{\mu\nu} = 0$.

The variables above are classified in the same way as in the Einstein frame as follows: Scalar modes (Ξ, ζ, χ, and ν), vector modes (r_μ and f_μ) obeying $\bar{q}^{\mu\nu}\hat{\nabla}_\mu r_\nu = 0$ and $\bar{q}^{\mu\nu}\hat{\nabla}_\mu f_\nu = 0$, and tensor modes $\chi_{\mu\nu}$ obeying $\bar{q}^{\mu\nu}\chi_{\mu\nu} = 0$ and $\bar{q}^{\lambda\mu}\hat{\nabla}_\lambda\chi_{\mu\nu} = 0$.

4. Relations between different frame variables

The relations between Einstein and matter-frame modes can be read off from (132) and (135) with the help of (91). They are as follows:

$$\tilde{\Xi} = \Xi + \varphi, \quad (136)$$

$$\tilde{\zeta} = \zeta - (1 - e^{-4\bar{\phi}})\alpha, \quad (137)$$

$$\tilde{\chi} = \chi + 6\varphi, \quad (138)$$

$$\tilde{\nu} = \nu, \quad (139)$$

$$\tilde{r}_\mu = r_\mu - (1 - e^{-4\bar{\phi}})\beta_\mu, \quad (140)$$

$$\tilde{f}_\mu = f_\mu, \quad (141)$$

$$\tilde{\chi}_{\mu\nu} = \chi_{\mu\nu}. \quad (142)$$

5. Fluid velocity field decomposition

The vector field perturbation is decomposed as

$$\theta_\mu = -\Xi t_\mu + \bar{q}^\nu{}_\mu\hat{\nabla}_\nu\theta + \upsilon_\mu, \quad (143)$$

with $t^\mu\upsilon_\mu = 0$.

It contains a scalar mode θ, given by $\Delta\theta = \bar{q}^{\mu\nu}\hat{\nabla}_\mu\theta_\nu$, and two vector modes υ_μ obeying $\bar{q}^{\mu\nu}\hat{\nabla}_\mu\upsilon_\nu = 0$.

6. Fluid shear decomposition

The shear perturbation is written as

$$\hat{\Sigma}_{\mu\nu} = (\bar{\rho} + \bar{P})\bar{q}^{\alpha}{}_{\mu}\bar{q}^{\beta}{}_{\nu}[(\hat{\nabla}_{\alpha}\hat{\nabla}_{\beta} - \tfrac{1}{3}\bar{q}_{\alpha\beta}\Delta)\Sigma + 2\sigma_{(\alpha,\beta)} + \sigma_{\alpha\beta}], \tag{144}$$

with $t^{\mu}\sigma_{\mu} = t^{\mu}\sigma_{\mu\nu} = 0$. The variable Σ is a scalar mode, σ_{μ} a vector mode obeying $\bar{q}^{\mu\nu}\sigma_{\mu,\nu} = 0$, and $\sigma_{\mu\nu}$ a tensor mode obeying, as usual, $\bar{q}^{\mu\nu}\sigma_{\mu\nu} = 0$ and $\bar{q}^{\mu\nu}\sigma_{\alpha\mu,\nu} = 0$.

B. Gauge nonfixed equations for the scalar modes

All scalar modes can be decomposed in terms of a complete set of eigenmodes of the Laplace-Beltrami operator. For example, a variable A can be written as $A(x^{\hat{a}}) = \int d^3kY(x^{\hat{a}}, k_{\hat{b}})\tilde{A}(k_{\hat{b}})$, where the eigenmodes $Y(x^{\hat{a}}, k_{\hat{b}})$ obey $(\Delta + k^2)Y = 0$. In the special case of a flat hypersurface with trivial topology, the eigenmodes are simply given by $Y = e^{ik_{\hat{a}}x^{\hat{a}}}$ and the integral transform above is a Fourier transform. The wave number k takes values depending on the geometry and topology of the spatial hypersurface. In the case of trivial topology, k takes values $k = \sqrt{k_*^2 - (K/r_c^2)}$, where k_* is continuous, obeying $k_* \geq 0$ for a flat or negatively curved spatial hypersurface, and $k_* = \frac{N}{r_c}$ where N is an integer obeying $N \geq 3$ for a positively curved spatial hypersurface. Let us also choose the same coordinate system defined in Sec. III.

1. Fluid equations

The density contrast equation for scalar modes is

$$\dot{\delta} = -3\frac{\dot{a}}{a}(C_s^2 - w)\delta + (1 + w)\left(-k^2\theta - \frac{1}{2}\dot{\chi} + k^2\zeta\right), \tag{145}$$

while the momentum divergence equation is

$$\dot{\theta} = -\Xi - \frac{\dot{a}}{a}(1 - 3w)\theta + \frac{C_s^2}{1 + w}\delta - \frac{\dot{w}}{1 + w}\theta - \frac{2}{3}\left(k^2 - \frac{3K}{r_c^2}\right)\Sigma. \tag{146}$$

2. Scalar field equation

The two equations equivalent to the scalar field equation are

$$\dot{\gamma} = -3\frac{\dot{b}}{b}\gamma + \frac{\bar{\mu}}{a}e^{-3\bar{\phi}}k^2(\varphi + \dot{\bar{\phi}}\alpha) + \frac{e^{\bar{\phi}}}{a}\bar{\mu}\,\dot{\bar{\phi}}[\dot{\tilde{\chi}} - 2k^2\tilde{\zeta}] + 8\pi Gae^{-3\bar{\phi}}\bar{\rho}[(1 + 3C_s^2)\delta - (1 + 3w)(\tilde{\Xi} + 2\varphi)] \tag{147}$$

and

$$\dot{\varphi} = -\frac{1}{2U}ae^{-\bar{\phi}}\gamma - \dot{\bar{\phi}}\,\tilde{\Xi}. \tag{148}$$

3. Vector field equation

The scalar modes of the perturbed vector field evolve according to the two first-order equations

$$K_B\left(\dot{E} + \frac{\dot{b}}{b}E\right) = -\bar{\mu}\,\dot{\bar{\phi}}(\varphi - \dot{\bar{\phi}}\alpha) + 8\pi Ga^2(1 - e^{-4\bar{\phi}})(\bar{\rho} + \bar{P})(\theta - \alpha) \tag{149}$$

and

$$\dot{\alpha} = E - \tilde{\Xi} + \left(\dot{\bar{\phi}} - \frac{\dot{a}}{a}\right)\alpha. \tag{150}$$

4. Generalized Einstein equations

The scalar modes of the perturbed generalized Einstein equations yield the Hamiltonian constraint equation

$$\frac{1}{3}\left(k^2 - \frac{3K}{r_c^2}\right)(\tilde{\chi} + k^2\nu) + e^{4\bar{\phi}}\frac{\dot{b}}{b}\left[\dot{\tilde{\chi}} - 2k^2\tilde{\zeta} + 6\frac{\dot{b}}{b}\tilde{\Xi}\right] + ae^{3\varphi}\dot{\bar{\phi}}\gamma - K_Bk^2E = 8\pi Ga^2\bar{\rho}[\delta - 2\varphi], \tag{151}$$

the momentum constraint equation

$$-\frac{1}{3}(\dot{\tilde{\chi}} + k^2\dot{\nu}) + \frac{K}{r_c^2}(2\tilde{\zeta} + \dot{\nu}) - 2\frac{\dot{b}}{b}\tilde{\Xi} = 8\pi Ga^2e^{-4\bar{\phi}}(\bar{\rho} + \bar{P})\theta + 2\bar{\mu}\,\dot{\bar{\phi}}\,\varphi, \tag{152}$$

and the two propagation equations

$$-\ddot{\tilde{\chi}} + 2k^2(\dot{\tilde{\zeta}} + e^{-4\bar{\phi}}\tilde{\Xi}) - \frac{1}{3}e^{-4\bar{\phi}}\left(k^2 - \frac{3K}{r_c^2}\right)(\tilde{\chi} + k^2\nu) - 2\frac{\dot{b}}{b}[\dot{\tilde{\chi}} + 3\dot{\tilde{\Xi}} - 2k^2\tilde{\zeta}] - 2\dot{\bar{\phi}}[\dot{\tilde{\chi}} - 2k^2\tilde{\zeta}] + 3\frac{\mu}{U}ae^{-\bar{\phi}}\dot{\bar{\phi}}\gamma + 6\left[-2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} - 4\dot{\bar{\phi}}\frac{\dot{b}}{b}\right]\tilde{\Xi} = 24\pi Ga^2e^{-4\bar{\phi}}\bar{\rho}(C_s^2\delta - 2w\varphi) \tag{153}$$

and

$$\ddot{\tilde{\nu}} + e^{-4\bar{\phi}}\left[2\tilde{\Xi} - \frac{1}{3}\tilde{\chi} - \frac{1}{3}k^2\tilde{\nu}\right] + 2\dot{\tilde{\zeta}} + 2\left[\frac{\dot{b}}{b} + \dot{\bar{\phi}}\right][\dot{\tilde{\nu}} + 2\tilde{\zeta}] = 16\pi Ga^2e^{-4\bar{\phi}}(\bar{\rho} + \bar{P})\Sigma. \tag{154}$$

C. Gauge nonfixed equations for the vector modes

Let ℓ_{μ}, m_{μ}, n_{μ} be an orthonormal triad of dual vector fields, normalized with respect to $\eta_{\mu\nu}$, which give

$$\bar{q}_{\mu\nu} = \ell_{\mu}\ell_{\nu} + m_{\mu}m_{\nu} + n_{\mu}n_{\nu}, \tag{155}$$

TABLE I. Table of generic symbols. Note: A bar above a symbol means the adaptation to the FLRW symmetries of the same unbarred symbol.

Symbol	Short description and notes	Page defined
$\tilde{g}_{\mu\nu}$, $\tilde{g}^{\mu\nu}$, $\tilde{\nabla}_\mu$	Generic metric and its inverse in the Einstein frame and associated compatible connection	3
$g_{\mu\nu}$, $g^{\mu\nu}$, ∇_μ	Generic metric and its inverse in the matter frame and associated compatible connection	3
$D^\alpha{}_{\mu\nu}$	Connection tensor for transforming ∇_μ to $\tilde{\nabla}_\mu$	5
$\tilde{\gamma}_{\mu\nu}$, ${}^{(\tilde{\gamma})}\tilde{\nabla}_\mu$	FLRW metric in the Einstein frame and associated compatible connection	5
$\gamma_{\mu\nu}$, ${}^{(\gamma)}\nabla_\mu$	FLRW metric in the matter frame and associated compatible connection	6
a	Scale factor in the matter frame	6
b	Scale factor in the Einstein frame; related to a as $b = ae^{\bar{\phi}}$	6
$\tilde{\eta}_{\mu\nu}$, $\bar{\nabla}_\mu$	Conformal static metric in the Einstein frame and associated compatible connection	6
$\eta_{\mu\nu}$, $\hat{\nabla}_\mu$	Conformal static metric in the matter frame and associated compatible connection	6
$\tilde{C}^\alpha{}_{\mu\nu}$	Connection tensor for transforming ${}^{(\tilde{\gamma})}\tilde{\nabla}_\mu$ to $\bar{\nabla}_\mu$	6
$C^\alpha{}_{\mu\nu}$	Connection tensor for transforming ${}^{(\gamma)}\nabla_\mu$ to $\hat{\nabla}_\mu$	6
$E^\alpha{}_{\mu\nu}$	Connection tensor for transforming $\bar{\nabla}_\mu$ to $\hat{\nabla}_\mu$	6
$s^\mu{}_\nu$	Projector along A^μ	6
$q^\mu{}_\nu$	Projector perpendicular to A^μ	6
A_μ	TeVeS dual vector field, unit-timelike w.r.t. $\tilde{g}_{\mu\nu}$, $A^\mu = \tilde{g}^{\mu\nu}A_\nu$	3
t_μ	Geodesic and unit-timelike vector field w.r.t. $\eta_{\mu\nu}$, $t^\mu = \eta^{\mu\nu}t_\nu$; related to A_μ as $A_\mu = ae^{-\bar{\phi}}t_\mu$	6
$F_{\mu\nu}$	TeVeS field strength tensor, $F^\mu{}_\nu = \tilde{g}^{\mu\alpha}F_{\alpha\nu}$	4
ϕ	TeVeS scalar field	4
μ	TeVeS auxiliary nondynamical scalar field	4
$V(\mu)$	TeVeS free function of the scalar field μ	4
$F(\mu)$	Bekenstein's original TeVeS free function; related to V as $V(\mu) = \frac{4\pi G^2}{\ell^2}\sigma_B^4 F(G\sigma_B^2)$	4
$U(\mu)$	Function of μ which enters the scalar field equations of motion	7
σ_B	Bekenstein's auxiliary nondynamical scalar field; related to μ as $\mu = 8\pi G\sigma_B^2$	4
Γ^μ	Vector field related to the gradient of the TeVeS scalar field ϕ	4
$\tilde{g}$	Determinant of the Einstein-Hilbert frame metric $\tilde{g}_{\mu\nu}$	3
g	Determinant of the matter-frame metric $g_{\mu\nu}$	4
$\tilde{R}$	Scalar curvature of the Einstein-Hilbert frame metric $\tilde{g}_{\mu\nu}$	3
$\tilde{G}_{\mu\nu}$	Einstein tensor of the Einstein-Hilbert frame metric $\tilde{g}_{\mu\nu}$	4
$\bar{G}_{\mu\nu}$	Einstein tensor of the Einstein-Hilbert frame metric $\tilde{\gamma}_{\mu\nu}$	7
$Y_{\mu\nu}$	Contribution to the Einstein equations not coming from the matter stress-energy tensor	4
$T_{\mu\nu}$	Generic matter stress-energy tensor	4
$S_{\mu\nu}$	Generic matter "stress-energy" tensor in the Einstein-Hilbert frame	4
j_μ	Generic matter current; source of the vector field equations	4
J	Generic matter scalar source; source of the scalar field equation	4
ρ	Fluid density	4
P	Fluid pressure	4
u^μ	Fluid velocity, unit-timelike w.r.t. $g_{\mu\nu}$, $u_\mu = g_{\mu\nu}u^\nu$	4
$\Sigma_{\mu\nu}$	Fluid shear; see also the definition of $\hat{\Sigma}_{\mu\nu} = \frac{1}{a^2}\Sigma_{\mu\nu}$ on page 10	4
w	Fluid equation of state parameter, $w = P/\rho$	7
G	Bare gravitational constant	3
$G_{\rm eff}$	Effective gravitational coupling strength for FLRW dynamics	8
K_B	Bekenstein's constant giving the coupling of the TeVeS vector field to gravity	4
ℓ	Bekenstein's constant giving the overall scale of V.	4
λ	Lagrange multiplier enforcing the unit-timelike constraint on A_μ	4
r_c	Radius of curvature of the spatial hypersurfaces of the FLRW spacetime	6
K	Integer curvature of the spatial hypersurfaces of the FLRW spacetime taking values in $\{-1, 0, 1\}$	7
H	Physical Hubble parameter	8
Δ	Laplace-Beltrami operator defined as $\Delta = \bar{q}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu$	11

TABLE II. Table of perturbation symbols. Note: A δ in front of a symbol also denotes a perturbation of the corresponding symbol about FLRW background, except in a few special cases: variation of the action, generic metric perturbation $\delta g_{\mu\nu}$, and the tensor $\delta H_{\mu\nu}$.

Symbol	Short description and notes	Page defined
φ	Perturbation of the TeVeS scalar field ϕ	9
γ^μ	Perturbation of the vector field Γ^μ	10
γ	Perturbation of the auxiliary scalar field Γ	10
$\tilde{h}_{\mu\nu}$	Perturbation of the Einstein-Hilbert frame metric	9
$\tilde{f}^\alpha{}_{\mu\nu}$	Connection tensor for the transformation $\tilde{\nabla}_\mu \to \bar{\nabla}_\mu$	9
$h_{\mu\nu}$	Perturbation of the matter-frame metric	9
$f^\alpha{}_{\mu\nu}$	Connection tensor for the transformation $\nabla_\mu \to \hat{\nabla}_\mu$	9
$\delta H_{\mu\nu}$	Perturbed Einstein tensor of $\tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}$	12
α_μ	Perturbation of the TeVeS vector field A_μ	9
E_μ	"Electric field" part of $F_{\mu\nu}$	9
$B_{\mu\nu}$	"Magnetic field" part of $F_{\mu\nu}$	9
θ_μ	Fluid velocity perturbation	10
δ	Fluid density contrast $\delta = \delta\rho/\bar{\rho}$	10
C_s^2	Fluid sound speed $C_s^2 = \delta P/\delta\rho$	10
$\tilde{h}$, $\tilde{\zeta}$, $\tilde{\chi}$, $\tilde{\nu}$	Generic scalar modes of Einstein-frame metric perturbations	14
$\tilde{f}$, $\tilde{r}$	Generic vector modes of Einstein-frame metric perturbations	14
$\tilde{\chi}_{\mu\nu}$	Tensor modes of Einstein-frame metric perturbations	14
h, ζ, χ, ν	Generic scalar modes of matter-frame metric perturbations	14
f, r	Generic vector modes of matter-frame metric perturbations	14
$\chi_{\mu\nu}$	Tensor modes of matter-frame metric perturbations	14
$H^{(T)}$	Tensor mode of the metric	18
α, E	Generic scalar modes of the vector field perturbations; E is gauge invariant	14
β, ϵ	Generic vector modes of the vector field perturbations; both are gauge invariant	14
σ_μ	Generic vector mode of the fluid shear	15
$\sigma_{\mu\nu}$	Generic tensor mode of the fluid shear	15
θ	Generic scalar mode of the fluid velocity perturbation	15
v	Generic vector mode of the fluid velocity perturbation	15
Σ	Scalar mode of the fluid shear	15
$\sigma^{(v)}$	Vector mode of the fluid shear	18
$\sigma^{(T)}$	Tensor mode of the fluid shear	18
k	Wave number	15
ℓ^μ, m^μ, n^μ	Orthonormal triad of vector fields	16
ξ^μ, $\hat{\xi}^\mu$	Infinitesimal vector fields used in gauge transformations	19
ξ, ψ	Scalar modes of ξ^μ	19
ω, ω^μ	Vector modes of ξ^μ	19
$\tilde{\Psi}$, $\tilde{\Phi}$	Metric potentials in the Einstein frame for conformal Newtonian gauge	20
Ψ, Φ	Metric potentials in the matter frame for conformal Newtonian gauge	20
h, η	Metric potentials for conformal synchronous gauge	21

and together with t_μ they form an orthonormal tetrad for the metric $\eta_{\mu\nu}$.

Without loss of generality, let ℓ_μ be the direction of propagation of plane waves. Thus, all vector modes are orthogonal to ℓ_μ, for example, $\ell^\mu \beta_\mu = 0$. Each vector mode X can then be decomposed into its two polarizations:

$$X_\mu = X^+ m_\mu + X^- n_\mu. \tag{156}$$

As it turns out, there is no mixing between the two polarizations. Moreover, they obey identical equations. The "+" and "−" labels can therefore be dropped without any confusion.

Vector modes can also be decomposed in terms of a complete set of eigenmodes of the Laplace-Beltrami operator, just like scalar modes. The spectrum of the wave number k is modified though, to reflect the spin-one nature of the vector modes. In this case (again for trivial topology), k takes the values $k = \sqrt{k_*^2 - 2(K/r_c^2)}$, where k_* is continuous, obeying $k_* \geq 0$ for a flat or negatively curved spatial hypersurface, and $k_* = \frac{N}{r_c}$, where N is an integer obeying $N \geq 3$ for a positively curved spatial hypersurface.

Let us now find the equations for vector modes.

1. Fluid equation

The vector mode fluid equation becomes

$$\dot{v} = -\left[(1-3w)\frac{\dot{a}}{a} + \frac{\dot{w}}{1+w}\right]v - \left(k^2 - \frac{2K}{r_c^2}\right)\sigma^{(v)}, \tag{157}$$

where $\sigma^{(v)}$ is the vector mode of the fluid shear, contained in σ_μ.

2. Vector field equation

The two first-order equations for the vector field are

$$\dot{\beta} = \epsilon + \left(\dot{\bar{\phi}} - \frac{\dot{a}}{a}\right)\beta \tag{158}$$

and

$$K_B\left[\dot{\epsilon} + \frac{\dot{b}}{b}\epsilon + \left(k^2 + \frac{2K}{r_c^2}\right)e^{-4\bar{\phi}}\beta\right] = \dot{\bar{\phi}}^2\beta + 8\pi G a^2(1-e^{-4\bar{\phi}})(\bar{\rho}+\bar{P})(v-\beta). \tag{159}$$

3. Generalized Einstein equations

The vector mode momentum constraint is

$$\left(k^2 - \frac{2K}{r_c^2}\right)(\dot{\tilde{f}} + \tilde{r}) = -16\pi G a^2 e^{-4\bar{\phi}}(\bar{\rho}+\bar{P})v \tag{160}$$

and the propagation equation is

$$\ddot{\tilde{f}} + \dot{\tilde{r}} + 2\left(\frac{\dot{b}}{b} + \dot{\bar{\phi}}\right)(\dot{\tilde{f}} + \tilde{r}) = 16\pi G a^2 e^{-4\bar{\phi}}(\bar{\rho}+\bar{P})\sigma^{(v)}. \tag{161}$$

D. Equations for the tensor modes

Using the orthonormal basis defined above, one can do a similar decomposition for the tensor modes into two polarizations. The tensor mode perturbation $\chi_{\mu\nu}$ decomposes into a basis which is written as symmetrized combinations of m_μ and n_ν. There are three possibilities, namely, $m_\mu m_\nu$, $n_\mu n_\nu$, and $m_\mu n_\nu + m_\nu n_\mu$. However, the traceless condition on the tensor modes implies that the coefficient of the first two must have opposite sign, hence there are only two independent polarizations given by

$$\chi_{\mu\nu} = H^+(m_\mu n_\nu + m_\nu n_\mu) + H^\times(m_\mu m_\nu - n_\mu n_\nu). \tag{162}$$

Since there is no mixing of polarizations again, both will be denoted as $H^{(T)}$.

Tensor modes are again decomposed in terms of a complete set of eigenmodes of the Laplace-Beltrami operator. The spectrum of the wave number k is again modified, to reflect the spin-two nature of the tensor modes. In this case (again for trivial topology), k takes the values $k = \sqrt{k_*^2 - 3(K/r_c^2)}$, where k_* is continuous, obeying $k_* \geq 0$ for a flat or negatively curved spatial hypersurface, and $k_* = \frac{N}{r_c}$, where N is an integer obeying $N \geq 3$ for a positively curved spatial hypersurface.

The tensor modes then obey the equation

$$\ddot{H}^{(T)} + 2\left(\frac{\dot{b}}{b} + \dot{\bar{\phi}}\right)\dot{H}^{(T)} + e^{-4\bar{\phi}}\left(k^2 + \frac{2K}{r_c^2}\right)H^{(T)} = 16\pi G a^2 e^{-4\bar{\phi}}(\bar{\rho}+\bar{P})\sigma^{(T)}, \tag{163}$$

where $\sigma^{(T)}$ is the tensor mode of the fluid shear, contained in $\sigma_{\mu\nu}$.

VI. SUMMARY

I have taken a covariant approach to formulate the linear perturbation theory about a spatially homogeneous and isotropic spacetime. The covariant approach is particularly useful in theories with two metrics, where there are two different metric-compatible connections, one for each metric.

The field equations were perturbed covariantly without adhering to a particular gauge or perturbation mode. This allows one to check explicitly that the equations are indeed invariant under infinitesimal gauge transformations. Mode decomposition was performed covariantly, and the equations for each perturbation mode were found, again without assuming a particular gauge. Special gauges for scalar modes are given in Appendix B. While I have not considered the perturbed Boltzmann equation for thermalized fluids, this will remain unchanged when expressed in matter-frame variables.

This completes the linear perturbation theory for Bekenstein's TeVeS theory about a FLRW cosmological background. The equations presented here can be used to study the formation of linear structure and the cosmic microwave background in this theory, as was initiated in [20].

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APPENDIX A: ANY UNIT-TIMELIKE VECTOR FIELD IS GEODESIC IN A FLRW UNIVERSE

In this section I prove that any unit-timelike vector field in a FLRW universe obeys the geodesic equation.

Let $g_{\mu\nu}$ be the Robertson-Walker metric with scale factor a. Consider now a unit-timelike vector field t^μ tangent to a geodesic congruence of curves. As a property of Robertson-Walker metrics, t^μ is always orthogonal to a hypersurface of homogeneity and isotropy (see, for example, [43]). (Of course, at least one such vector field exists, e.g. $t_\mu = \nabla_\mu t$ for some scalar function $t \in C^\infty M$). Let x^μ, y^μ, and z^μ be three unit-spacelike vector fields, which, along with t^μ, complete an orthonormal basis on TM. As a property of FLRW, they can be related to three linearly independent Killing vectors of M, as $x^\mu = \frac{1}{a}\xi^\mu_{(1)}$, $y^\mu = \frac{1}{a}\xi^\mu_{(2)}$, and $z^\mu = \frac{1}{a}\xi^\mu_{(3)}$. With the above considerations we have that

$$t^\mu\nabla_\mu t_\nu = t^\mu\nabla_\nu t_\mu = x^\mu\nabla_\nu x_\mu = 0, \tag{A1}$$

$$x^\mu\nabla_\mu x_\nu = (\mathcal{L}_t \ln a)t_\nu, \tag{A2}$$

where, in the last relation, I have used the fact that $\mathcal{L}_x f = 0$ for any function $f \in C^\infty M$.

Now the vector field

$$A^\mu = (1 + c^2)^{1/2} t^\mu + c x^\mu \tag{A3}$$

is also unit-timelike by construction, for any choice of $c \in C^\infty M$. The isotropy of M implies that there is no loss of generality in (A3).

Now consider $A^\mu\nabla_\mu A_\nu$. Using (A3) one gets

$$\begin{aligned} A^\mu\nabla_\mu A_\nu = {}& c[ct_\nu + (1 + c^2)^{1/2}x_\nu](\mathcal{L}_t \ln c) \\ & + c(1 + c^2)^{1/2}(t^\mu\nabla_\mu x_\nu + x^\mu\nabla_\mu t_\nu) \\ & + c^2(\mathcal{L}_t \ln a)t_\nu. \end{aligned}$$

Now consider the term $t^\mu\nabla_\mu x_\nu + x^\mu\nabla_\mu t_\nu$ which can be expanded as

$$\begin{aligned} t^\mu\nabla_\mu x_\nu + x^\mu\nabla_\mu t_\nu = {}& (\mathcal{L}_t \ln a)x_\nu + (y^\alpha t^\beta\nabla_\beta x_\alpha \\ & + y^\alpha x^\beta\nabla_\beta t_\alpha)y_\nu + (z^\alpha t^\beta\nabla_\beta x_\alpha \\ & + z^\alpha x^\beta\nabla_\beta t_\alpha)z_\nu, \end{aligned}$$

where (A1) and (A2) have been used. However, the coefficient of y_μ in the above relation is zero, since

$$\begin{aligned} y^\alpha t^\beta\nabla_\beta x_\alpha + y^\alpha x^\beta\nabla_\beta t_\alpha &= y^\alpha t^\beta\nabla_\beta x_\alpha + y^\alpha\mathcal{L}_x t_\alpha \\ &\quad - y^\alpha t^\beta\nabla_\alpha x_\beta \\ &= 2y^\alpha t^\beta\nabla_{[\beta}x_{\alpha]} \\ &= 2y^\alpha t^\beta\partial_\beta\xi^{(1)}_\alpha = 0. \end{aligned} \tag{A4}$$

The same holds for the coefficient of z_μ for the same reason and therefore

$$A^\mu\nabla_\mu A_\nu = c[ct_\nu + (1 + c^2)^{1/2}x_\nu]\mathcal{L}_t \ln(ac). \tag{A5}$$

Therefore, the choice $c = \frac{c_0}{a}$ for any constant c_0 means that the unit-timelike vector field A^μ given by

$$A^\mu = \left[1 + \frac{c_0^2}{a^2}\right]^{1/2} t^\mu + \frac{c_0}{a}x^\mu \tag{A6}$$

is geodesic. However, any unit-timelike vector field can be related to t^μ by (A6) which completes the proof.

APPENDIX B: GAUGE CHOICES

1. Gauge transformations

Consider a vector field ξ^μ generating a local one-parameter family of local diffeomorphisms (gauge transformations). Then, under a gauge transformation, any tensor $\mathbf{T}$ transforms as

$$\mathbf{T} \rightarrow \mathbf{T} + \mathcal{L}_\xi\mathbf{T}, \tag{B1}$$

where $\mathcal{L}_\xi\mathbf{T}$ is the Lie derivative of $\mathbf{T}$ along ξ^μ.

Let us define a new vector field $\hat{\xi}^\mu$ by $\xi^\mu = \frac{1}{a}\hat{\xi}^\mu$ and $\xi_\mu = g_{\mu\nu}\xi^\nu = a\hat{\xi}_\mu$. Now perform a split as

$$\hat{\xi}_\mu = -\xi t_\mu + \bar{q}^\nu{}_\mu\hat{\nabla}_\nu\psi + \omega_\mu, \tag{B2}$$

where $t^\mu\omega_\mu = 0$. The above vector field thus consists of two scalar modes ξ and ψ given by $\xi = t^\mu\hat{\xi}_\mu$ and $\Delta\psi = \bar{q}^{\mu\nu}\hat{\nabla}_\mu\hat{\xi}_\nu$, and two vector modes ω_μ which obey $\bar{q}^{\mu\nu}\hat{\nabla}_\mu\omega_\nu = 0$.

Now one can find the gauge transformations for all the perturbed variables. The scalar field perturbation transforms as

$$\varphi' = \varphi - \frac{1}{a}(t^\mu\hat{\nabla}_\mu\bar{\phi})\xi, \tag{B3}$$

while the auxiliary scalar field perturbation γ transforms as

$$\begin{aligned} \gamma' = {}& \gamma + 2\frac{e^{\bar{\phi}}}{a^2}U\xi[\hat{\nabla}^2\bar{\phi} - (t^\rho\hat{\nabla}_\rho\bar{\phi})^2 \\ & + (t^\rho\hat{\nabla}_\rho\bar{\phi})(t^\nu\hat{\nabla}_\nu \ln a)] \end{aligned} \tag{B4}$$

$$\begin{aligned} = {}& \gamma + \left[6\bar{\mu}\frac{e^{\bar{\phi}}}{a^2}(t^\mu\hat{\nabla}_\mu \ln b)(t^\nu\hat{\nabla}_\nu\bar{\phi}) \right. \\ & \left. + 8\pi G e^{-3\bar{\phi}}(\bar{\rho} + 3\bar{P})\right]\xi. \end{aligned} \tag{B5}$$

The vector field perturbation transforms as

$$\alpha'_\mu = \alpha_\mu + \frac{1}{a}[\hat{\nabla}_\mu\xi + (t^\nu\hat{\nabla}_\nu\bar{\phi})\xi t_\mu] \tag{B6}$$

which gives

$$\alpha' = \alpha + \frac{1}{a}\xi, \tag{B7}$$

whereas β_μ is gauge invariant as expected.

The vector field tensor $F_{\mu\nu}$ vanishes for the background, meaning that E, ϵ_μ, and $B_{\mu\nu}$ are all gauge invariant.

The matter metric perturbation transforms as

$$h'_{\mu\nu} = h_{\mu\nu} + \frac{2}{a}[\hat{\nabla}_{(\mu}\hat{\xi}_{\nu)} + (t^\rho\hat{\nabla}_\rho \ln a)t_{(\mu}\hat{\xi}_{\nu)} - (t^\rho\hat{\nabla}_\rho \ln a)\xi\eta_{\mu\nu}] \quad \text{(B8)}$$

which gives

$$\Xi' = \Xi + \frac{1}{a}t^\mu\hat{\nabla}_\mu\xi, \quad \text{(B9)}$$

$$\zeta' = \zeta + \frac{1}{a}[t^\mu\hat{\nabla}_\mu\psi - (t^\mu\hat{\nabla}_\mu \ln a)\psi + \xi], \quad \text{(B10)}$$

$$\chi' = \chi - \frac{2}{a}[k^2\psi + 3(t^\mu\hat{\nabla}_\mu \ln a)\xi], \quad \text{(B11)}$$

$$\nu' = \nu + \frac{2}{a}\psi, \quad \text{(B12)}$$

$$h' = h + \frac{1}{a}[t^\mu\hat{\nabla}_\mu\omega - (t^\mu\hat{\nabla}_\mu \ln a)\omega], \quad \text{(B13)}$$

$$f' = f + \frac{\omega}{a}, \quad \text{(B14)}$$

whereas the Einstein-frame metric perturbations transform as

$$\tilde{\Xi}' = \tilde{\Xi} + \frac{1}{a}[t^\mu\hat{\nabla}_\mu\xi - (t^\mu\hat{\nabla}_\mu\bar{\phi})\xi], \quad \text{(B15)}$$

$$\tilde{\zeta}' = \tilde{\zeta} + \frac{1}{a}[t^\mu\hat{\nabla}_\mu\psi - (t^\mu\hat{\nabla}_\mu \ln a)\psi + e^{-4\bar{\phi}}\xi], \quad \text{(B16)}$$

$$\tilde{\chi}' = \tilde{\chi} - \frac{2}{a}[k^2\psi + 3(t^\mu\hat{\nabla}_\mu \ln b)\xi], \quad \text{(B17)}$$

$$\tilde{\nu}' = \tilde{\nu} + \frac{2}{a}\psi, \quad \text{(B18)}$$

$$\tilde{h}' = \tilde{h} + \frac{1}{a}[t^\mu\hat{\nabla}_\mu\omega - (t^\mu\hat{\nabla}_\mu \ln a)\omega], \quad \text{(B19)}$$

$$\tilde{f}' = \tilde{f} + \frac{\omega}{a}. \quad \text{(B20)}$$

The Lie derivative of the fluid velocity is $\mathcal{L}_\xi u_\mu = \hat{\nabla}_\mu\xi$ and therefore the fluid velocity transforms as

$$\theta'_\mu = \theta_\mu + \frac{1}{a}\hat{\nabla}_\mu\xi. \quad \text{(B21)}$$

Now both the energy density and pressure are scalars given by $\rho = u_\mu u_\nu T^{\mu\nu}$ and $P = \frac{1}{3}q_{\mu\nu}T^{\mu\nu}$, and so

$$\delta' = \delta + \frac{3}{a}(1+w)(t^\rho\hat{\nabla}_\rho \ln a)\xi, \quad \text{(B22)}$$

$$\theta' = \theta + \frac{1}{a}\xi, \quad \text{(B23)}$$

$$\frac{1}{\bar{\rho}}\delta P' = \frac{1}{\bar{\rho}}\delta P + \frac{\xi}{a}[3w(1+w)(t^\rho\hat{\nabla}_\rho \ln a) - t^\rho\hat{\nabla}_\rho w], \quad \text{(B24)}$$

$$\Sigma' = \Sigma, \quad \text{(B25)}$$

$$v' = v. \quad \text{(B26)}$$

Using the gauge transformations above, a lengthy calculation shows that the gauge nonfixed equations derived in the previous section are all gauge invariant. Given the complexity of the gauge nonfixed equations, such a calculation can offer a powerful test of their correctness.

2. Conformal Newtonian gauge

The conformal Newtonian gauge is defined by

$$\Xi = -\Psi, \quad \text{(B27)}$$

$$\chi = -6\Phi, \quad \text{(B28)}$$

$$\zeta = 0, \quad \text{(B29)}$$

$$\nu = 0. \quad \text{(B30)}$$

From the relation $\tilde{\Xi} = \Xi + \varphi$, we also set $\tilde{\Xi} = -\tilde{\Psi}$ and from $\tilde{\chi} = \chi + 6\varphi$ we set $\tilde{\chi} = -6\tilde{\Phi}$. Therefore, the Einstein-Hilbert frame metric perturbations are given by

$$\tilde{\Psi} = \Psi - \varphi, \quad \text{(B31)}$$

$$\tilde{\Phi} = \Phi - \varphi, \quad \text{(B32)}$$

$$\tilde{\zeta} = -(1 - e^{-4\bar{\phi}})\alpha, \quad \text{(B33)}$$

$$\tilde{\nu} = 0. \quad \text{(B34)}$$

a. Fluid equations

The density contrast equation for scalar modes in the conformal Newtonian gauge evolves as

$$\dot{\delta} = -(1+w)(k^2\theta - 3\dot{\Phi}) - 3\frac{\dot{a}}{a}(C_s^2 - w)\delta, \quad \text{(B35)}$$

whereas the momentum divergence evolves as

$$\dot{\theta} = -\frac{\dot{a}}{a}(1-3w)\theta + \frac{C_s^2}{1+w}\delta - \frac{\dot{w}}{1+w}\theta - \frac{2}{3}\left(k^2 - \frac{3K}{r_c^2}\right)\Sigma + \Psi. \quad \text{(B36)}$$

b. Scalar field equation

The two first-order equations coming from the perturbed scalar field equation are

$$\dot\gamma = -3\frac{\dot b}{b}\gamma + \frac{\bar\mu}{a}e^{-3\bar\phi}k^2(\varphi + \dot{\bar\phi}\alpha) + \frac{e^{\bar\phi}}{a}\bar\mu\,\dot{\bar\phi}[-6\dot{\tilde\Phi} - 2k^2\tilde\zeta] + 8\pi G a e^{-3\bar\phi}[\delta\rho + 3\delta P + (\bar\rho + 3\bar P)(\tilde\Psi - 2\varphi)] \tag{B37}$$

and

$$\dot\varphi = -\frac{1}{2U}ae^{-\bar\phi}\gamma + \dot{\bar\phi}\,\tilde\Psi. \tag{B38}$$

c. Vector field equation

The scalar mode of the perturbed vector field equation is

$$K_B\left(\dot E + \frac{\dot b}{b}E\right) = -\bar\mu\,\dot{\bar\phi}(\varphi - \dot{\bar\phi}\alpha) + 8\pi G a^2(1 - e^{-4\bar\phi})(\bar\rho + \bar P)(\theta - \alpha) \tag{B39}$$

and

$$\dot\alpha = E + \tilde\Psi + \left(\dot{\bar\phi} - \frac{\dot a}{a}\right)\alpha. \tag{B40}$$

d. Generalized Einstein equations

The scalar modes of the perturbed generalized Einstein equations yield for the Hamiltonian constraint

$$-2\left(k^2 - \frac{3K}{r_c^2}\right)\tilde\Phi - 2e^{4\bar\phi}\frac{\dot b}{b}\left[3\dot{\tilde\Phi} + k^2\tilde\zeta + 3\frac{\dot b}{b}\tilde\Psi\right] + ae^{3\bar\phi}\dot{\bar\phi}\gamma - K_B k^2 E = 8\pi G a^2\bar\rho[\delta - 2\varphi], \tag{B41}$$

the momentum constraint equation

$$\dot{\tilde\Phi} + \frac{K}{r_c^2}\tilde\zeta + \frac{\dot b}{b}\tilde\Psi - \bar\mu\,\dot{\bar\phi}\,\varphi = 4\pi G a^2 e^{-4\bar\phi}(\bar\rho + \bar P)\theta, \tag{B42}$$

and the two propagation equations

$$6\ddot{\tilde\Phi} + 2k^2(\dot{\tilde\zeta} - e^{-4\bar\phi}\tilde\Psi) + 2e^{-4\bar\phi}\left(k^2 - \frac{3K}{r_c^2}\right)\tilde\Phi + 2\frac{\dot b}{b}[6\dot{\tilde\Phi} + 3\dot{\tilde\Psi} + 2k^2\tilde\zeta] + 4\dot{\bar\phi}[3\dot{\tilde\Phi} + k^2\tilde\zeta] + 3\frac{\mu}{U}ae^{-\bar\phi}\dot{\bar\phi}\gamma - 6\left[-2\frac{\ddot b}{b} + \frac{\dot b^2}{b^2} - 4\dot{\bar\phi}\frac{\dot b}{b}\right]\tilde\Psi = 24\pi G a^2 e^{-4\bar\phi}\bar\rho(C_s^2\delta - 2w\varphi) \tag{B43}$$

and

$$\tilde\Phi - \tilde\Psi + e^{4\bar\phi}\left[\dot{\tilde\zeta} + 2\left(\frac{\dot b}{b} + \dot{\bar\phi}\right)\tilde\zeta\right] = 8\pi G a^2(\bar\rho + \bar P)\Sigma. \tag{B44}$$

3. Conformal synchronous gauge

The conformal synchronous gauge is defined by $\Xi = 0$ and $\zeta = 0$, which fixes $\tilde\zeta = -(1 - e^{-4\bar\phi})\alpha$ and $\tilde\Xi = \varphi$. Following the standard notation, let us also set $\chi = h$, which gives $\tilde\chi = h + 6\varphi$ and $\nu = -\frac{1}{k^2}(h + 6\eta)$.

a. Fluid equations

The density contrast equation for scalar modes evolves as

$$\dot\delta = -3\frac{\dot a}{a}(C_s^2 - w)\delta - (1 + w)\left(k^2\theta + \frac{1}{2}\dot h\right), \tag{B45}$$

while the momentum divergence evolves as

$$\dot\theta = -\frac{\dot a}{a}(1 - 3w)\theta + \frac{C_s^2}{1 + w}\delta - \frac{\dot w}{1 + w}\theta - \frac{2}{3}\left(k^2 - \frac{3K}{r_c^2}\right)\Sigma. \tag{B46}$$

b. Scalar field equation

The two equations equivalent to the scalar field equation are

$$\dot\gamma = -3\frac{\dot b}{b}\gamma + \frac{\bar\mu}{a}e^{-3\bar\phi}k^2(\varphi + \dot{\bar\phi}\alpha) + \frac{e^{\bar\phi}}{a}\bar\mu\,\dot{\bar\phi}[\dot h + 6\dot\varphi - 2k^2\tilde\zeta] + 8\pi G a e^{-3\bar\phi}[\delta\rho + 3\delta P - 3(\bar\rho + 3\bar P)\varphi] \tag{B47}$$

and

$$\dot\varphi = -\frac{1}{2U}ae^{-\bar\phi}\gamma - \dot{\bar\phi}\varphi. \tag{B48}$$

c. Vector Bekenstein equation

The scalar mode of the perturbed vector Bekenstein equation is given by the two first-order equations

$$K_B\left(\dot E + \frac{\dot b}{b}E\right) = -\bar\mu\,\dot{\bar\phi}(\varphi - \dot{\bar\phi}\alpha) + 8\pi G a^2(1 - e^{-4\bar\phi})(\bar\rho + \bar P)(\theta - \alpha) \tag{B49}$$

and

$$\dot\alpha = E - \varphi + \left(\dot{\bar\phi} - \frac{\dot a}{a}\right)\alpha. \tag{B50}$$

d. Generalized Einstein equations

The scalar modes of the perturbed generalized Einstein equations yield for the Hamiltonian constraint

$$2\left(k^2 - \frac{3K}{r_c^2}\right)(\varphi - \eta) + e^{4\bar\phi}\frac{\dot b}{b}\left[\dot h - 2k^2\tilde\zeta + 6\frac{\dot a}{a}\varphi\right] + ae^{3\varphi}\left(\dot{\bar\phi} - \frac{3}{U}\frac{\dot b}{b}\right)\gamma - K_B k^2 E = 8\pi G a^2\bar\rho[\delta - 2\varphi], \quad \text{(B51)}$$

the momentum constraint equation

$$2k^2\dot\eta + \frac{K}{r_c^2}[2k^2\dot{\tilde\zeta} - \dot h - 6\dot\eta] - 2k^2\left(\frac{\dot a}{a} + \bar\mu\,\dot{\bar\phi}\right)\varphi + \frac{k^2}{U}ae^{-\bar\phi}\gamma = 8\pi G a^2 e^{-4\bar\phi}(\bar\rho + \bar P)k^2\theta, \quad \text{(B52)}$$

and the two propagation equations

$$-\ddot h - 6\ddot\varphi + 2k^2\dot{\tilde\zeta} + \frac{6K}{r_c^2}e^{-4\bar\phi}\varphi + 2e^{-4\bar\phi}\left(k^2 - \frac{3K}{r_c^2}\right)\eta - 2\frac{\dot b}{b}[\dot h + 9\dot\varphi - 2k^2\tilde\zeta] - 2\dot{\bar\phi}[\dot h + 6\dot\varphi - 2k^2\tilde\zeta] + 3\frac{\mu}{U}ae^{-\bar\phi}\dot{\bar\phi}\gamma + 6\left[-2\frac{\ddot b}{b} + \frac{\dot b^2}{b^2} - 4\dot{\bar\phi}\frac{\dot b}{b}\right]\varphi = 24\pi G a^2 e^{-4\bar\phi}\bar\rho(C_s^2\delta - 2w\varphi) \quad \text{(B53)}$$

and

$$\ddot h + 6\ddot\eta - 2e^{-4\bar\phi}k^2\eta - 2k^2\dot{\tilde\zeta} + 2\left[\frac{\dot b}{b} + \dot{\bar\phi}\right][\dot h + 6\dot\eta - 2k^2\tilde\zeta] = -16\pi G a^2 e^{-4\bar\phi}(\bar\rho + \bar P)k^2\Sigma. \quad \text{(B54)}$$

4. α gauge

In the α gauge, one sets the vector field perturbation to zero, $\alpha = 0$, which fixes $\tilde\zeta = \zeta$. Since setting $\alpha = 0$ essentially fixes the gauge variable ξ, one is no longer allowed to set $\Xi = 0$. Therefore, this gauge cannot be put in synchronous form. As a further gauge fixing condition, let $\nu = 0$, which can still be done, as the gauge variable ψ was not fixed at that point.

a. Fluid equations

The density contrast equation for scalar modes evolves as

$$\dot\delta = -3\frac{\dot a}{a}(C_s^2 - w)\delta + (1 + w)\left(-k^2\theta - \frac{1}{2}\dot\chi + k^2\zeta\right), \quad \text{(B55)}$$

while the momentum divergence evolves as

$$\dot\theta = -\Xi - \frac{\dot a}{a}(1 - 3w)\theta + \frac{C_s^2}{1 + w}\delta - \frac{\dot w}{1 + w}\theta - \frac{2}{3}\left(k^2 - \frac{3K}{r_c^2}\right)\Sigma. \quad \text{(B56)}$$

b. Scalar field equation

The two equations equivalent to the scalar field equation are

$$\dot\gamma = -3\frac{\dot b}{b}\gamma + \frac{\bar\mu}{a}e^{-3\bar\phi}k^2\varphi + \frac{e^{\bar\phi}}{a}\bar\mu\,\dot{\bar\phi}[\dot{\tilde\chi} - 2k^2\zeta] + 8\pi G a e^{-3\bar\phi}\bar\rho[(1 + 3C_s^2)\delta - (1 + 3w)(E + 2\varphi)] \quad \text{(B57)}$$

and

$$\dot\varphi = -\frac{1}{2U}ae^{-\bar\phi}\gamma - \dot{\bar\phi}E. \quad \text{(B58)}$$

c. Vector field equation

The scalar mode of the perturbed vector field equation is given by the first-order equation

$$K_B\left(\dot E + \frac{\dot b}{b}E\right) = -\bar\mu\,\dot{\bar\phi}\,\varphi + 8\pi G a^2(1 - e^{-4\bar\phi})(\bar\rho + \bar P)\theta. \quad \text{(B59)}$$

d. Generalized Einstein equations

The scalar modes of the perturbed generalized Einstein equations yield for the Hamiltonian constraint

$$\frac{1}{3}\left(k^2 - \frac{3K}{r_c^2}\right)\tilde\chi + e^{4\bar\phi}\frac{\dot b}{b}\left[\dot{\tilde\chi} - 2k^2\zeta + 6\frac{\dot b}{b}E\right] + ae^{3\varphi}\dot{\bar\phi}\gamma - K_B k^2 E = 8\pi G a^2\bar\rho[\delta - 2\varphi], \quad \text{(B60)}$$

the momentum constraint equation

$$-\frac{1}{3}\dot{\tilde\chi} + \frac{2K}{r_c^2}\zeta - 2\frac{\dot b}{b}E = 8\pi G a^2 e^{-4\bar\phi}(\bar\rho + \bar P)\theta + 2\bar\mu\,\dot{\bar\phi}\,\varphi, \quad \text{(B61)}$$

and the two propagation equations

$$-\ddot{\tilde\chi} + 2k^2(\dot\zeta + e^{-4\bar\phi}E) - \frac{1}{3}e^{-4\bar\phi}\left(k^2 - \frac{3K}{r_c^2}\right)\tilde\chi - 2\frac{\dot b}{b}[\dot{\tilde\chi} + 3\dot E - 2k^2\zeta] - 2\dot{\bar\phi}[\dot{\tilde\chi} - 2k^2\zeta] + 3\frac{\mu}{U}ae^{-\bar\phi}\dot{\bar\phi}\gamma + 6\left[-2\frac{\ddot b}{b} + \frac{\dot b^2}{b^2} - 4\dot{\bar\phi}\frac{\dot b}{b}\right]E = 24\pi G a^2 e^{-4\bar\phi}\bar\rho(C_s^2\delta - 2w\varphi) \quad \text{(B62)}$$

and

$$e^{-4\bar{\phi}}\left[2E - \frac{1}{3}\tilde{\chi}\right] + 2\dot{\zeta} + 4\left[\frac{\dot{b}}{b} + \dot{\bar{\phi}}\right]\zeta = 16\pi G a^2 e^{-4\bar{\phi}}(\bar{\rho} + \bar{P})\Sigma. \quad \text{(B63)}$$

APPENDIX C: SPECIFICS OF THE PERTURBATIONS OF THE EINSTEIN TENSOR

In this section I write out explicitly all the perturbed terms of the Einstein tensor to help the reader follow the calculations.

Changing connection to $\hat{\nabla}_\mu$ gives

$$\bar{\nabla}_\beta \bar{\nabla}_\mu \tilde{h}^\beta{}_\nu = \tilde{\eta}^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\mu \tilde{h}_{\beta\nu} + 2e^{4\bar{\phi}}\hat{\nabla}^2\bar{\phi}[t^\alpha t^\beta \tilde{h}_{\alpha\beta}t_\mu t_\nu - t^\lambda t_\mu \tilde{h}_{\lambda\nu}] + 2(t^\rho\hat{\nabla}_\rho\bar{\phi})\{\tilde{\eta}^{\alpha\beta}t^\lambda\hat{\nabla}_\alpha\tilde{h}_{\beta\lambda}t_\mu t_\nu + e^{4\bar{\phi}}[2t^\alpha t^\beta\hat{\nabla}_\alpha\tilde{h}_{\beta\nu}t_\mu - t^\alpha\hat{\nabla}_\mu\tilde{h}_{\alpha\nu} + t^\alpha t^\beta\hat{\nabla}_\mu\tilde{h}_{\alpha\beta}t_\nu]\} + 8e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\bar{\phi})^2[t^\lambda t_\mu\tilde{h}_{\lambda\nu} - 2t^\alpha t^\beta\tilde{h}_{\alpha\beta}t_\mu t_\nu], \quad \text{(C1)}$$

$$\bar{\nabla}_\mu\bar{\nabla}_\nu\tilde{h}^\alpha{}_\alpha = \hat{\nabla}_\mu\hat{\nabla}_\nu\tilde{h}^\alpha{}_\alpha + 2(t^\rho\hat{\nabla}_\rho\bar{\phi})t_\mu t_\nu t^\beta\hat{\nabla}_\beta\tilde{h}^\alpha{}_\alpha, \quad \text{(C2)}$$

$$\bar{\nabla}^2\tilde{h}_{\mu\nu} = \tilde{\eta}^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\tilde{h}_{\mu\nu} - 2e^{4\bar{\phi}}\{2\hat{\nabla}^2\bar{\phi}t_{(\mu}t^\lambda\tilde{h}_{\nu)\lambda} + (t^\rho\hat{\nabla}_\rho\bar{\phi})[t^\lambda\hat{\nabla}_\lambda\tilde{h}_{\mu\nu} - 4t^\alpha t^\beta\hat{\nabla}_\alpha\tilde{h}_{\beta(\mu}t_{\nu)}] + 4(t^\rho\hat{\nabla}_\rho\bar{\phi})^2[t^\alpha t^\beta\tilde{h}_{\alpha\beta}t_\mu t_\nu - 2t^\beta t_{(\mu}\tilde{h}_{\nu)\beta}]\}, \quad \text{(C3)}$$

$$\bar{\nabla}_\alpha\bar{\nabla}_\beta\tilde{h}^{\alpha\beta} = \tilde{\eta}^{\alpha\mu}\tilde{\eta}^{\beta\nu}\hat{\nabla}_\alpha\hat{\nabla}_\beta\tilde{h}_{\mu\nu} + 2e^{8\bar{\phi}}\{(t^\rho\hat{\nabla}_\rho\bar{\phi}) \times [3t^\lambda\hat{\nabla}_\lambda(t^\mu t^\nu\tilde{h}_{\mu\nu}) - 2e^{-4\bar{\phi}}t^\lambda\tilde{\eta}^{\mu\nu}\hat{\nabla}_\mu\tilde{h}_{\lambda\nu}] - 2t^\mu t^\nu\tilde{h}_{\mu\nu}[\hat{\nabla}^2\bar{\phi} - 6t^\rho\hat{\nabla}_\rho\bar{\phi})^2]\}, \quad \text{(C4)}$$

$$\bar{\nabla}^2\tilde{h}^\alpha{}_\alpha = \tilde{\eta}^{\rho\lambda}\hat{\nabla}_\rho\hat{\nabla}_\lambda\tilde{h}^\alpha{}_\alpha - 2(t^\rho\hat{\nabla}_\rho\bar{\phi})e^{4\bar{\phi}}t^\lambda\hat{\nabla}_\lambda\tilde{h}^\alpha{}_\alpha, \quad \text{(C5)}$$

$$2\tilde{f}^\alpha_{\mu\nu}\bar{\nabla}_\alpha \ln b = e^{4\bar{\phi}}(t^\lambda\hat{\nabla}_\lambda \ln b)[t^\beta\hat{\nabla}_\beta\tilde{h}_{\mu\nu} - 2t^\beta\hat{\nabla}_{(\mu}\tilde{h}_{\nu)\beta} - 4(t^\rho\hat{\nabla}_\rho\bar{\phi})t^\alpha t^\beta\tilde{h}_{\alpha\beta}t_\mu t_\nu], \quad \text{(C6)}$$

$$2\tilde{h}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta \ln b = 2e^{8\bar{\phi}}t^\alpha t^\beta\tilde{h}_{\alpha\beta}[-\hat{\nabla}^2 \ln b + 2(t^\lambda\hat{\nabla}_\lambda \ln b)(t^\rho\hat{\nabla}_\rho\bar{\phi})], \quad \text{(C7)}$$

$$2\tilde{\eta}^{\alpha\beta}\tilde{f}^\lambda_{\alpha\beta}\bar{\nabla}_\lambda \ln b = e^{4\bar{\phi}}(t^\gamma\hat{\nabla}_\gamma \ln b)[t^\rho\hat{\nabla}_\rho\tilde{h}^\alpha{}_\alpha - 2t^\beta\hat{\nabla}_\alpha\tilde{h}^\alpha{}_\beta] \quad \text{(C8)}$$

$$= e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho \ln b)[t^\lambda\hat{\nabla}_\lambda\tilde{h}^\alpha{}_\alpha - 2t^\rho\tilde{\eta}^{\alpha\beta}\hat{\nabla}_\alpha\tilde{h}_{\beta\rho} + 8e^{4\bar{\phi}}(t^\rho\hat{\nabla}_\rho\bar{\phi})t^\alpha t^\beta\tilde{h}_{\alpha\beta}], \quad \text{(C9)}$$

$$\bar{\nabla}^2 \ln b = e^{4\bar{\phi}}[\hat{\nabla}^2 \ln b - 2(t^\lambda\hat{\nabla}_\lambda \ln b)(t^\rho\hat{\nabla}_\rho\bar{\phi})]. \quad \text{(C10)}$$

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