PHYSICAL REVIEW D **74,** 103007 (2006)

Cosmological coincidence problem in interacting dark energy models

H. Mohseni Sadjadi* and M. Alimohammadi†

Department of Physics, University of Tehran, North Karegar Avenue, Tehran, Iran

(Received 16 October 2006; published 29 November 2006)

The interacting dark energy model with interaction term $Q = \lambda_m H\rho_m + \lambda_d H\rho_d$ is considered. By studying the model near the transition time, in which the system crosses the $\omega = -1$ phantom divide line, the conditions needed to overcome the coincidence problem is investigated. The phantom model, as a candidate for dark energy, is considered, and for two specific examples, the quadratic and exponential phantom potentials, it is shown that it is possible the system crosses the $\omega = -1$ line, meanwhile the coincidence problem is alleviated, the two facts that have root in observations.

DOI: 10.1103/PhysRevD.74.103007 PACS numbers: 95.36.+x, 98.80.Cq, 98.80.Jk

I. INTRODUCTION

Nowadays based on astrophysical data it is believed that the universe is accelerating [1]. The origin of this acceleration is still unknown and different models have been proposed to elucidate this subject. One picture is the assumption that nearly 70% of the universe is composed of a smooth energy component with negative pressure dubbed as dark energy. A simple candidate for dark energy is the cosmological constant [2] which suffers from conceptual problems such as fine-tuning and coincidence problems [3]. Therefore alternative models, e.g., introducing dynamical exotic fields such as scalar fields with suitably chosen potentials, have been introduced [4].

In dark energy models, the ratio of matter to dark energy density, r, is expected to decrease rapidly (proportional to the scale factor) as the universe expands, but observations show that these densities are of the same order today. To solve this problem (known as coincidence problem), one can adopt an evolving dark energy field with suitable nongravitational interaction with matter [5,6]. Various models corresponding to different forms of interaction, leading to a constant or slowly varying (soft coincidence) r at late times, have been proposed [7].

Some present data seems to favor an evolving dark energy, corresponding to an equation of state parameter less than $\omega = -1$ at present epoch (phantom regime) from $\omega > -1$ in the near past (quintessence regime) [8]. So another cosmological coincidence problem may be proposed: why $\omega = -1$ crossing occurred at the present time [9].

In [10], it was shown that $\omega = -1$ crossing in models including matter and phantom scalar field is either impossible or unstable with respect to cosmological perturbations. However, this transition may be possible for scalar-tensor theories [11], multifield models [12], and coupled dark energy models with specific couplings [13,14].

In [15], the transition from quintessence to the phantom phase in the quintom model was considered in the slow roll approximation. By studying the Friedmann equations near the transition time, it was shown that, in the noninteracting quintom model, $r \simeq 0$ at transition time. This lies in the fact that the main part of the dark energy at transition time corresponds to the quintom potential. By considering interaction between cold dark matter and dark energy, the mutual energy exchange between two fluids will be allowed and the coincidence problem may be alleviated.

In this paper we consider the dark energy model composed of a phantom scalar field interacting with cold dark matter. We try to elucidate the connection between the coincidence problem and $\omega = -1$ crossing (second cosmological coincidence problem).

It may be worth noting that the phantom models suffer from the quantum instability problem. Because the phantom fields have negative kinetic energy, it is possible that a phantom particle decays into an arbitrary number of phantoms and ordinary particles, such as gravitons. It can be shown that the decay rates of these interactions are infinite, which indicates that the phantom models are dramatically unstable. But if we think of these models as the low-energy effective theories, with the fundamental fields having positive kinetic energy, then we should use a momentum cutoff Λ in calculating the decay rates. In this way, it can be shown that, for $\Lambda \sim M_{\mathrm{pl}}$, the lifetimes can become larger than the age of the universe when one chooses suitable phantom-gravity interaction potentials, and this removes the quantum instability of these kinds of phantom models [16].

The Scheme of the paper is as follows. After the Introduction, we consider the dark energy model with interaction term $Q = \lambda_m H\rho_m + \lambda_d H\rho_d$ in Sec. II. By restricting ourselves to times $t \ll h_0^{-1}$ around the transition time (h_0 is the Hubble parameter), we study the general properties of interacting dark energy models and the necessary conditions needed to cross the $\omega = -1$ line are obtained. These results are insensitive to the origin of the dark energy. In Sec. III we assume that dark energy is composed of a phantom scalar field interacting with cold dark matter. After a general discussion, we illustrate, via two specific examples, how the necessary conditions for

*Electronic address: mohseni@phymail.ut.ac.ir
†Electronic address: alimohmd@ut.ac.ir

$\omega = -1$ crossing can alleviate the coincidence problem. It is seen that it is possible to tune the parameters such that $r_0 = 3/7$ at transition time.

We use units $\hbar = c = G = 1$ throughout the paper.

II. $\omega = -1$ CROSSING IN INTERACTING DARK ENERGY MODEL

We consider a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) universe containing dark energy and dark matter fluids. In terms of dark energy density ρ_d and matter energy density ρ_m, the Hubble parameter is given by the Friedmann equation

$$H^2 = \frac{8\pi}{3}\rho = \frac{8\pi}{3}(\rho_m + \rho_d), \tag{1}$$

where ρ is the total energy density. By introducing $\Omega_d = \rho_d/\rho$ and $\Omega_m = \rho_m/\rho$, Eq. (1) can be written as $\Omega_d + \Omega_m = 1$, which indicates that the universe is spatially flat. The derivative of the Hubble parameter with respect to the comoving time can be extracted from Einstein equations. The result is

$$\dot{H} = -4\pi(\rho_d + P_d + \rho_m). \tag{2}$$

P_d is the pressure of the dark energy fluid and the dark matter is assumed to be pressureless. The equation of state of the universe is $P = \omega\rho$, where $P = P_d$ is the pressure and ω is the equation of state parameter which can be written as

$$\omega = -1 - \frac{2\dot{H}}{3H^2}. \tag{3}$$

For an accelerated universe we have $\omega < -1/3$. When $-1 < \omega < -1/3$, the universe is in the quintessence phase and when $\omega < -1$, the universe is in the phantom phase. In the following, we assume that the dark matter and dark energy components can interact through the following source term:

$$Q = \lambda_m H\rho_m + \lambda_d H\rho_d, \tag{4}$$

where λ_m and λ_d are two real constants. For special choices such as $\lambda_m = 0$, $\lambda_d = 0$ or $\lambda_d = \lambda_m$, Eq. (4) reduces to the interaction terms which have been considered before [5]. The other forms of interaction terms, not necessarily suitable for our purpose, have been also considered in the literature [17].

Because of the interaction term, we have not the conservation of partial stress-energy tensors of matter and dark energy: $T^{\mu\nu}_{(m)\ ;\nu} = -T^{\mu\nu}_{(d)\ ;\nu} \neq 0$. In fact, the projection of this nonconservation equation along the velocity of the whole (comoving) fluid U_ν (which was taken to be the same as the velocities of each of the fluid components) is [6]

$$U_\nu T^{\mu\nu}_{(m)\ ;\mu} = -U_\nu T^{\mu\nu}_{(d)\ ;\mu} = -Q. \tag{5}$$

Note that the coupling (4) can be written as a scalar as follows:

$$Q = \tfrac{1}{3}U_\mu U_\nu(\lambda_m T^{\mu\nu}_{(m)} + \lambda_d T^{\mu\nu}_{(d)})U^\alpha_{;\alpha}. \tag{6}$$

For the FLRW metric, the equation (5) reduces to

$$\dot{\rho}_d + 3H(\rho_d + P_d) = -Q, \qquad \dot{\rho}_m + 3H\rho_m = Q. \tag{7}$$

Using Eq. (1), Eq. (7) can be written as

$$\begin{aligned} \dot{\rho}_d + (3 + \lambda_d - \lambda_m)H\rho_d + 3HP_d &= -\frac{3}{8\pi}\lambda_m H^3, \\ \dot{\rho}_m + (3 + \lambda_d - \lambda_m)H\rho_m &= \frac{3}{8\pi}\lambda_d H^3. \end{aligned} \tag{8}$$

Using Eq. (8), the evolution equation of the ratio of energy densities of dark matter and dark energy, denoted by $r = \rho_m/\rho_d$, reads

$$\dot{r} = r(r+1)\left(3\omega + \lambda_m + \frac{\lambda_d}{r}\right)H. \tag{9}$$

From

$$\Omega_d = \frac{1}{1+r}, \tag{10}$$

Eq. (9) then results in

$$\omega = -\frac{1}{3H}\frac{\dot{\Omega}_d}{1-\Omega_d} - \frac{\lambda_d\Omega_d}{3(1-\Omega_d)} - \frac{\lambda_m}{3}. \tag{11}$$

In the vicinity of transition time from quintessence to phantom era, $\omega > -1$ goes to $\omega < -1$, so $\dot{H}$ must change sign from $\dot{H} < 0$ to $\dot{H} > 0$. At transition time we have $\dot{H} = 0$ and $\omega = -1$. The dark energy equation of state parameter ω_d is defined through $P_d = \omega_d\rho_d$. Therefore $\omega\rho = \omega_d\rho_d$ or $\Omega_d\omega_d = \omega$. Using

$$\rho_m = \frac{3\omega_d H^2 + 2\dot{H} + 3H^2}{8\pi\omega_d} \qquad \rho_d = -\frac{2\dot{H} + 3H^2}{8\pi\omega_d}, \tag{12}$$

and Eq. (8), one can obtain the following equation for the Hubble expansion:

$$\begin{aligned} &\ddot{H} + (6 + \lambda_d - \lambda_m + 3\omega_d)H\dot{H} \\ &\quad + \frac{3}{2}[(3-\lambda_m)\omega_d + 3 + \lambda_d - \lambda_m]H^3 = \frac{\dot{\omega}_d}{\omega_d}\left(\dot{H} + \frac{3H^2}{2}\right). \end{aligned} \tag{13}$$

For a constant ω_d, we arrive at the result of [18]. At $\dot{H} = 0$ we obtain

$$\ddot{H} = -\tfrac{3}{2}[(3-\lambda_m)\omega_d + 3 + \lambda_d - \lambda_m]H^3. \tag{14}$$

Note that $H > 0$, therefore for a constant ω_d, the sign of $\ddot{H}$ does not change. This shows that in the constant-ω_d approximation, the system can cross the $\omega = -1$ line only once. This is because the transition from quintessence to phantom phase needs positive $\ddot{H}$ (at transition time), while the vice versa needs negative $\ddot{H}$. In the following, we consider ω_d as a function of time.

At transition time, we obtain from Eq. (11)

$$-3\dot{\omega} = (3 - \lambda_m + \lambda_d)\frac{\dot{\Omega}_d}{1-\Omega_d} + \frac{\ddot{\Omega}_d}{H(1-\Omega_d)}, \quad (15)$$

which results in

$$(3 - \lambda_m + \lambda_d)\dot{\Omega}_d + \frac{\ddot{\Omega}_d}{H} \geq 0. \quad (16)$$

Insertion of

$$\dot{\Omega}_d = \frac{8\pi}{3}\frac{\dot{\rho}_d}{H^2}, \qquad \ddot{\Omega}_d = \frac{8\pi}{3H^2}\left(\ddot{\rho}_d - 2\frac{\ddot{H}}{H}\rho_d\right) \quad (17)$$

into Eq. (16), leads to

$$(3 - \lambda_m + \lambda_d)H\dot{\rho}_d + \ddot{\rho}_d - \frac{2\ddot{H}}{H}\rho_d \geq 0, \quad (18)$$

at transition time.

We assume that, in the neighborhood of transition time, the Hubble parameter is a differentiable function of time. The Taylor expansion of H at transition time, which we take as $t = 0$, can be written as [15]

$$H = h_0 + h_1 t^\alpha + O(t^{\alpha+1}), \qquad \alpha \geq 2, \qquad h_1 \neq 0. \quad (19)$$

$h_0 = H(t = 0)$, α is the order of the first nonzero derivative of the Hubble parameter at transition time, and $h_1 = (1/\alpha!)d^\alpha H/dt^\alpha|_{t=0}$. The transition from quintessence to phantom phase occurs if α is an even positive integer and $h_1 > 0$. We also consider the following expansions for Ω_d, ρ_m, and ρ_d at $t = 0$:

$$\begin{aligned}\Omega_d &= u_0 + u_1 t^\beta + O(t^{\beta+1}),\\ \rho_m &= \rho_{m0} + \rho_{m1}t^\gamma + O(t^{\gamma+1}),\\ \rho_d &= \rho_{d0} + \rho_{d1}t^\theta + O(t^{\theta+1}),\end{aligned} \quad (20)$$

respectively. β, γ, and θ are the orders of the first nonzero derivatives of Ω_d, ρ_m, and ρ_d at $t = 0$, respectively. Note that the above expansions are valid until $t \ll h_0^{-1}$, which is completely reasonable since h_0^{-1} is of the order of the age of our universe.

To obtain the relation between the parameters α, β, γ, and θ, we proceed as follows. For $\beta \neq 1$, if we expand both sides of Eq. (11) at $t = 0$, the first resulting term of the right-hand side, with nonvanishing power of t, is $t^{\beta-1}$ while the left-hand side (after t^0) begins with $t^{\alpha-1}$. So if $\beta \neq 1$, we must have $\alpha = \beta$. In this case, $(3/8\pi)H^2\Omega_d = \rho_d$ results in $\theta = \beta(= \alpha)$. For $\beta = 1$, this equation results in $\theta = \beta(= 1)$. Therefore always $\theta = \beta$. From Eq. (12) it is clear that, for a constant ω_d, we must have $\beta = \alpha - 1$ which leads to $\beta = 1$ and $\alpha = 2$.

In the case $\beta \neq 1$, comparing the coefficients of t^0-terms of Eq. (11) gives

$$u_0 = \frac{3 - \lambda_m}{\lambda_d - \lambda_m + 3}, \quad (21)$$

and equating the coefficients of $t^{\alpha-1}$-terms results in

$$h_1 = \frac{h_0 u_1}{2(1 - u_0)}. \quad (22)$$

This relation shows that the transition is possible only if $u_1 > 0$.

In the case $\beta = 1$, the same procedure leads to

$$u_1 = (\lambda_m - \lambda_d - 3)h_0 u_0 + (3 - \lambda_m)h_0, \quad (23)$$

$$(\lambda_d - \lambda_m + 3)h_0 u_k + (k+1)u_{k+1} = 0, \qquad 1 < k \leq \alpha - 2, \quad (24)$$

and

$$h_1 = \frac{(\lambda_d - \lambda_m + 3)h_0^2 u_{\alpha-1} + \alpha h_0 u_\alpha}{2\alpha(1 - u_0)}. \quad (25)$$

The Taylor expansion of r at $t = 0$ is

$$r = r_0 + r_1 t^\beta + O(t^2), \quad (26)$$

where $r_0 = u_0^{-1} - 1$ and $r_1 = -u_1/(u_0^2)$.

III. INTERACTING PHANTOM DARK ENERGY MODEL AND COINCIDENCE PROBLEM

In this section we assume that the origin of the dark energy is a phantom scalar field ϕ. So

$$\rho_d = -\tfrac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P_d = -\tfrac{1}{2}\dot{\phi}^2 - V(\phi), \quad (27)$$

where $V(\phi) > 0$ is the phantom potential. ω_d is given by

$$\omega_d = \frac{-\frac{1}{2}\dot{\phi}^2 - V(\phi)}{-\frac{1}{2}\dot{\phi}^2 + V(\phi)}, \quad (28)$$

therefore $\omega_d < -1$. For $\Omega_d\omega_d < -1$, the universe is in the phantom phase and for $\Omega_d\omega_d > -1$ it is in the quintessence phase.

The field equation of ϕ is

$$\dot{\phi}\left(\ddot{\phi} + 3H\dot{\phi} - \frac{dV}{d\phi}\right) = Q. \quad (29)$$

This can be derived by putting Eq. (27) back into Eq. (7). From Eq. (27) we obtain

$$\dot{\phi}^2 = -(1 + \omega_d)\rho_d \qquad 2V(\phi) = (1 - \omega_d)\rho_d. \quad (30)$$

The second equation of (30) can be written as

$$\dot{\phi} = \frac{dV^{-1}(y)}{dy}\dot{y}, \quad (31)$$

where $y = (1 - \omega_d)\rho_d/2$, and V^{-1} is the inverse function of V. Equation (31) and the first equation of (30) then lead to

$$(1+\omega_d)\rho_d = -\left(\frac{dV^{-1}(y)}{dy}\right)^2 \dot{y}^2, \tag{32}$$

which after some calculation can be rewritten as

$$[\dot{\Omega}_d - \dot{\omega} - 3H(1+\omega)(\Omega_d - \omega)]^2\left(\frac{dV^{-1}(y)}{dy}\right)^2 = -\frac{4}{\rho}(\Omega_d + \omega). \tag{33}$$

This equation together with our previous results may be served to find some necessary conditions for $\omega = -1$ crossing in interacting phantom dark energy models, including the domain to which u_0 belongs. We will try to obtain a relation between the coincidence problem and the behavior of the system at transition time. For example, for the case $\beta = 1$, if one obtains h_1 as a polynomial of u_0, then restricting h_1 to positive values, which is necessary for transition, will restrict the value of u_0 to a subset of (0, 1). By choosing the appropriate parameters, then it becomes possible to prevent r to be 0 or very large. For $\beta \neq 1$ cases, Eq. (21) determines the value of r at transition time, which again can be chosen to be $O(1)$. In this way the occurrence of $\omega = -1$ crossing *and* the alleviation of the coincidence problem can be achieved *simultaneously*.

In the following, we will show these points via some specific examples. In these examples we restrict ourselves to the case $\alpha = 2$.

A. Phantom field with square power law potential

For $V(\phi) = (1/2)m^2\phi^2$, Eq. (33) becomes

$$2m\sqrt{\omega^2 - \Omega_d^2} = \pm[\dot{\Omega}_d - \dot{\omega} - 3H(1+\omega)(\Omega_d - \omega)]. \tag{34}$$

In the following, we adopt that, in the quintessence phase and near the transition time, $\dot{\Omega}_d > 0$ or equivalently $r_1 < 0$ [19]. Therefore

$$2m\sqrt{\omega^2 - \Omega_d^2} = \dot{\Omega}_d - \dot{\omega} - 3H(1+\omega)(\Omega_d - \omega) \tag{35}$$

can be used in the neighborhood of transition time. Taking $\beta = 1$ (the case $\beta \neq 1$ will be discussed later), the expansion of Eq. (35) at $t = 0$ then results in

$$2m\sqrt{1-u_0^2} - \frac{2m(-4h_1 + 3u_0u_1h_0^2)}{3\sqrt{1-u_0^2h_0^2}}t + O(t^2) = u_1 + \frac{4h_1}{3h_0^2} + \left(2u_2 + 4\frac{h_2}{h_0^2} + 4\frac{h_1(1+u_0)}{h_0}\right)t + O(t^2). \tag{36}$$

As a result we arrive at

$$2m\sqrt{1-u_0^2} = u_1 + \frac{4h_1}{3h_0^2}. \tag{37}$$

The necessity of quintessence to phantom phase transition, i.e. $h_1 > 0$, then results in

$$u_1 > 2m\sqrt{1-u_0^2}. \tag{38}$$

Using Eq. (23), we can write the above inequality in terms of u_0:

$$au_0 + b < c\sqrt{1-u_0^2}. \tag{39}$$

We have defined $a = \lambda_m - \lambda_d - 3$, $b = 3 - \lambda_m$, and $c = 2m/h_0$.

To study the solutions of Eq. (39), we consider two situations. The first possibility is

$$au_0 + b \le 0, \tag{40}$$

which leads to $r_1 \ge 0$. This conflicts with the assumption $r_1 < 0$, or equivalently $\dot{\Omega}_d > 0$ at transition time, and therefore is not acceptable. The second possibility is $au_0 + b > 0$ which leads to

$$\mathcal{P}(u_0) := (a^2 + c^2)u_0^2 + 2abu_0 + b^2 - c^2 < 0. \tag{41}$$

If $a^2 - b^2 + c^2 < 0$, $\mathcal{P}$ has no real roots and its sign does not change. But $\mathcal{P}(1) > 0$, therefore Eq. (41) is not satisfied in this case. For $a^2 - b^2 + c^2 > 0$, $\mathcal{P}$ has two roots which we denote by u_{R1} and u_{R2}. Equation (41) is satisfied if the value of Ω_d at transition time is restricted to the intersection of the intervals (u_{R1}, u_{R2}) and (0, 1):

$$u_0 \in (0,1)\bigcap(u_{R1}, u_{R2}). \tag{42}$$

So if $(u_{R1}, u_{R2}) \subset (0,1)$, by choosing the appropriate parameters a, b, and c, we can obtain the desired order of magnitude: $\sim O(1)$ for $r_0 = 1/u_0 - 1$. The Sturm sequences at 0 and 1 are

$$S(0) = \left[b^2 - c^2, 2ab, \frac{c^2(a^2 - b^2 + c^2)}{a^2 + c^2}\right], \tag{43}$$

and

$$S(1) = \left[(a+b)^2, 2(a^2 + c^2 + ab), \frac{c^2(a^2 - b^2 + c^2)}{a^2 + c^2}\right]. \tag{44}$$

Using the Sturm theorem, one can show that for

$$a^2 - b^2 + c^2 > 0, \qquad a^2 + c^2 + ba > 0, \qquad b^2 - c^2 > 0, \qquad ab < 0, \tag{45}$$

the two roots of $\mathcal{P}$ belong to (0, 1). In this way we have

$$\frac{-ab - c\sqrt{a^2 + c^2 - b^2}}{a^2 + c^2} < \Omega_d < \frac{-ab + c\sqrt{a^2 + c^2 - b^2}}{a^2 + c^2} \tag{46}$$

at transition time. As an example, consider the case $\lambda_m = 1$, $\lambda_d = 2$, and $c = 1$. In this case $u_{R1} = 0.258$ and $u_{R2} = 0.682$, therefore $0.46 < r < 2.8$ at transition time. Note that (u_{R1}, u_{R2}) may be more tightened by choosing appropriate a, b, and c, e.g. for $c = 1$, $a = 7$, and $b = -5$, which correspond to $\lambda_m = 8$ and $\lambda_d = -2$, we have $u_{R1} = 0.6$ and $u_{R2} = 0.8$, therefore $0.6 < u_0 < 0.8$ in agreement with the expected value $u_0 = 0.7$ and $r_0 = 3/7$.

In $\beta \neq 1$ cases, we have $\beta = \alpha$. For $\alpha = 2$, following the same method ending to Eq. (36), we expand both sides of Eq. (35). It is obtained, up to the order $O(t)$,

$$m\sqrt{1 - u_0^2} = \frac{2h_1}{3h_0^2} \tag{47}$$

which results in $h_1 > 0$. u_0 and u_1 are given by Eqs. (21) and (22), respectively. Higher orders of t in the Taylor expansion of Eq. (35) determine the other coefficients of the Taylor expansion of H an Ω_d. Since $h_1 > 0$ induces no additional constraint on u_0, the appropriate parameters a, b, and c can lead to the desired values for r_0. For $\alpha > 2$, the aforementioned expansion leads to $m\sqrt{1 - u_0^2} = 0$ which is ruled out by the assumption that $u_0 \neq 1$. Therefore in $\beta \neq 1$, the choice $\alpha = 2$ is the only eligible one.

Equation (35) with $\dot{\Omega}_d < 0$ and Eq. (34) with the minus sign can also be investigated by the same method. In brief, it is shown that, in the interacting phantom model with $V(\phi) = (1/2)m^2\phi^2$ phantom potential and the interaction Q-term of Eq. (4), it is possible to choose the parameters such that *both* the $\omega = -1$ crossing and $r_0 = O(1)$ occur. In the special case which leads to Eq. (47), one can tune the parameters such that r_0 has no choice but the desired value $3/7$.

B. Exponential potential

Consider the following potential:

$$V = v_0 \exp(\lambda\phi), \qquad \lambda > 0, \qquad v_0 > 0. \tag{48}$$

Equation (33) then results in

$$H\tilde{\lambda}(\omega^2 - \Omega_d^2)^{1/2}(\Omega_d - \omega)^{1/2} = \pm[\dot{\Omega}_d - \dot{\omega} - 3H(1 + \omega)(\Omega_d - \omega)], \tag{49}$$

where $\tilde{\lambda} = \sqrt{3/(8\pi)}\lambda$. As the previous example, we consider the upper sign of Eq. (49) which is a result of the assumptions $\dot{\Omega}_d > 0$ and $\dot{\omega} < 0$ in the vicinity of transition time.

By Taylor expansion of both sides of Eq. (49) at transition time, we obtain the following equation for $\alpha = 2$ and $\beta = 1$:

$$\frac{4h_1}{3h_0^3} = -au_0 - b + \tilde{\lambda}(1 - u_0^2)^{1/2}(1 + u_0)^{1/2}. \tag{50}$$

a and b are defined by the same relations as for the first example. $h_1 > 0$ then results in

$$au_0 + b < \tilde{\lambda}(1 - u_0^2)^{1/2}(1 + u_0)^{1/2}. \tag{51}$$

Equation (51) is satisfied in two cases: (i) $au_0 + b < 0$, which makes r_1 negative, or $\dot{\Omega}_d < 0$, and is not acceptable; (ii) $au_0 + b > 0$. In this case we must have

$$u_0^3 + (A^2 + 1)u_0^2 + (2AB - 1)u_0 + B^2 - 1 < 0, \tag{52}$$

where $A = a/(\tilde{\lambda}^2)$ and $B = b/(\tilde{\lambda}^2)$. We also assume $B^2 > 1$. Equation (52) is satisfied only if the polynomial

$$\mathcal{Q}(u_0) := u_0^3 + (A^2 + 1)u_0^2 + (2AB - 1)u_0 + B^2 - 1 \tag{53}$$

has real roots. Following Descartes rule, $B^2 - 1 > 0$ and $2AB - 1 < 0$ are necessary conditions for $\mathcal{Q}(u_0)$ to have two real positive roots. The domain to which u_0 in Eq. (52) belongs is the intersection of $(0, 1)$ and (u_{R1}, u_{R2}), where u_{R1} and u_{R2} are the roots of $\mathcal{Q}(u_0)$. So by appropriate choosing of A and B, one can restrict u_0 to the domain allowed by astrophysical data. To do so, we construct the Sturm sequence corresponding to the cubic polynomial $\mathcal{Q}(u_0)$ at 0 and 1. They are

$$S(0) = \left[B^2 - 1, 2AB - 1, \frac{1}{9}(2AB - 1)(A^2 + 1) + (1 - B^2), \frac{9}{4}\frac{D}{(A^4 + 2A^2 - 6AB + 4)^2}\right], \tag{54}$$

and

$$S(1) = \left[A^2 + 2AB + B^2, 2A^2 + 2AB + 4, \frac{1}{9}(2A^4 + 2A^3B + 3A^2 - 10AB - 9B^2 + 16), \frac{9}{4}\frac{D}{(A^4 + 2A^2 - 6AB + 4)^2}\right]. \tag{55}$$

In the above equations, $D > 0$ is discriminant of the polynomial $\mathcal{Q}(u_0)$. By implying the Sturm theorem, it can be verified that, in order to have two real roots in interval $(0, 1)$, the parameters A and B must satisfy, besides the previous mentioned conditions $B^2 - 1 > 0$ and $2AB - 1 < 0$, the following inequalities:

$$A^2 + AB + 2 > 0, \qquad 2A^4 + 2A^3B + 3A^2 - 10AB - 9B^2 + 16 > 0. \tag{56}$$

For example, for $\tilde{\lambda} = 1$, $\lambda_m = 1$, and $\lambda_d = 2$, we obtain $0.23 < u_0 < 0.73$ which is in agreement with $u_0 \sim 0.7$ obtained from astrophysical data.

Now we consider $\beta \neq 1$. For $\alpha = 2$, the Taylor expansion of both sides of Eq. (49), with upper sign, results in

$$\frac{4}{3}\frac{h_1}{h_0^2} - \tilde{\lambda}\sqrt{1-u_0^2}\sqrt{1+u_0} + \left(4\frac{h_2}{h_0^2} - 2u_2 + 4\frac{h_1(1+u_0)}{h_0} - \frac{2}{3}\frac{\tilde{\lambda}h_1\sqrt{1-u_0^2}}{h_0^2\sqrt{1+u_0}} - \frac{4}{3}\frac{\tilde{\lambda}h_1\sqrt{1+u_0}}{h_0^2\sqrt{1-u_0^2}}\right)t + O(t^2) = 0. \quad (57)$$

Therefore

$$\frac{4h_1}{3h_0^3} = \tilde{\lambda}(1-u_0^2)^{1/2}(1+u_0)^{1/2} \quad (58)$$

which implies $h_1 > 0$. By suitable choosing of λ_m and λ_d, one can obtain the appropriate value for r at transition time. For $\alpha > 2$, the aforementioned expansion leads to $\tilde{\lambda}(1-u_0^2)^{1/2}(1+u_0)^{1/2} = 0$ which is ruled out by the assumption that $u_0 \neq 1$. Therefore $\alpha = 2$ is the only allowed case for $\beta \neq 1$.

IV. CONCLUSION

In this paper, by considering the energy exchange between cold dark matter and dark energy [see Eq. (4)], we study the possibility of simultaneous occurrence of two phenomena, the coincidence problem, and $\omega = -1$ crossing, from $\omega > -1$ to $\omega < -1$. We consider the physical quantities near the transition time $t = 0$, through Eqs. (19) and (20). The transition occurs for positive h_1 and even α, the parameters which have been introduced in Eq. (19).

The equation of state parameter ω is expressed by Eq. (11) and the potential of phantom field, as a candidate of dark energy, enters in Eq. (33). We studied the perturbative solutions of these equations, near $t = 0$, for two specific potentials, i.e. the quadratic and exponential potential. It is shown that always $\theta = \beta$, and for $\beta \neq 1$, $\alpha = \beta$ [see Eq. (20) and its subsequent discussion]. For $\alpha = 2$, as a first acceptable solution for $\omega = -1$ crossing, it is shown that, in both examples, it is possible to choose the parameters such that, besides the satisfaction of dynamical equations, the occurrence of $\omega > -1$ to $\omega < -1$ transition allows the ratio $r_0 = (\rho_m/\rho_d)_{t=0}$ to be around the desired value 3/7. This proves the possibility of solving these two problems in a unique framework.

For $\alpha > 2$ and $\beta = 1$, it can also be shown that it is possible to choose the parameters such that the above-mentioned properties are achieved.

ACKNOWLEDGMENTS

We would like to thank M. Honardoost for useful discussion. This work was partially supported by the "center of excellence in structure of matter" of the Department of Physics.

[1] S. Perlmutter *et al.*, Nature (London) **391**, 51 (1998); D. N. Spergel *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **148**, 175 (2003); A. C. S. Readhead *et al.*, Astrophys. J. **609**, 498 (2004); J. H. Goldstein *et al.*, Astrophys. J. **599**, 773 (2003); M. Tegmark *et al.* (SDSS Collaboration), Phys. Rev. D **69**, 103501 (2004); A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116**, 1009 (1998); S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **517**, 565 (1999).

[2] T. Padmanabhan, Phys. Rep. **380**, 235 (2003); V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000); S. M. Carroll, Living Rev. Relativity **4**, 1 (2001).

[3] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989); I. Zlatev, L.-M. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).

[4] R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998); P. J. E. Peebles and A. Vilenkin, Phys. Rev. D **59**, 063505 (1999); P. J. Steinhardt, L. M. Wang, and I. Zlatev, Phys. Rev. D **59**, 123504 (1999); M. Doran and J. Jaeckel, Phys. Rev. D **66**, 043519 (2002); A. R. Liddle, P. Parson, and J. D. Barrow, Phys. Rev. D **50**, 7222 (1994); R. R. Caldwell, Phys. Lett. B **545**, 23 (2002); R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003); P. T. Silva and O. Bertolami, Astrophys. J. **599**, 829 (2003); Q. Huang and M. Li, J. Cosmol. Astropart. Phys. 08 (2004) 013; J. M. Cline, S. Y. Jeon, and G. D. Moore, Phys. Rev. D **70**, 043543 (2004); V. Faraoni, Classical Quantum Gravity **22**, 3235 (2005); Xin Zhang, Phys. Rev. D **74**, 103505 (2006); Q. Huang and M. Li, J. Cosmol. Astropart. Phys. 03 (2005) 001; M. C. Bento, O. Bertolami, and A. A. Sen, Phys. Rev. D **70**, 083519 (2004); **67**, 063003 (2003).

[5] L. Amendola, Phys. Rev. D **62**, 043511 (2000); W. Zimdahl, D. Pavon, and L. P. Chimento, Phys. Lett. B **521**, 133 (2001); W. Zimdahl, Int. J. Mod. Phys. D **14**, 2319 (2005); D. Pavon and W. Zimdahl, Phys. Lett. B **628**, 206 (2005); G. Mangano, G. Miele, and V. Pettorino, Mod. Phys. Lett. A **18**, 831 (2003); G. Farrar and P. J. E. Peebles, Astrophys. J. **604**, 1 (2004); S. del Campo, R. Herrera, and D. Pavon, Phys. Rev. D **70**, 043540 (2004); R-G. Cai and A. Wang, J. Cosmol. Astropart. Phys. 03 (2005) 002; Micheal S. Berger, and H. Shojaei, Phys. Rev. D **73**, 083528 (2006); Bo Hu and Y. Ling, Phys. Rev. D **73**, 123510 (2006); Hui Li, Z. Guo, and Y. Zhang, Int. J. Mod. Phys. D **15**, 869 (2006); A. P. Billyard and A. A. Coley,

Phys. Rev. D **61**, 083503 (2000); M. Szydlowski, Phys. Lett. B **632**, 1 (2006); M. Szydlowski, T. Stachowiak, and R. Wojtak, Phys. Rev. D **73**, 063516 (2006).
[6] L. P. Chimento, A. S. Jakubi, D. Pavon, and W. Zimdahl, Phys. Rev. D **67**, 083513 (2003).
[7] D. Tocchini-Valentini and L. Amendola, Phys. Rev. D **65**, 063508 (2002); W. Zimdahl and D. Pavon, Gen. Relativ. Gravit. **35**, 413 (2003); D. Pavon, S. Sen, and W. Zimdahl, J. Cosmol. Astropart. Phys. 05 (2004) 009; G. Olivares, F. Atrio-Barandela, and D. Pavon, Phys. Rev. D **71**, 063523 (2005); S. del Campo, R. Herrera, G. Olivares, and D. Pavon, Phys. Rev. D **74**, 023501 (2006); S. del Campo, Ramon Herrera, and Diego Pavon, Phys. Rev. D **71**, 123529 (2005); M. S. Berger and H. Shojaei, Phys. Rev. D **73**, 083528 (2006); X. Zhang, Phys. Lett. B **611**, 1 (2005); Mod. Phys. Lett. A **20**, 2575 (2005).
[8] U. Alam, V. Sahni, T. D. Saini, and A. A. Starobinsky, Mon. Not. R. Astron. Soc. **354**, 275 (2004); U. Alam, V. Sahni, and A. A. Starobinsky, J. Cosmol. Astropart. Phys. 06 (2004) 008; B. Feng, X. L. Wang, and X. M. Zhang, Phys. Lett. B **607**, 35 (2005); D. Huterer and A. Cooray, Phys. Rev. D **71**, 023506 (2005); S. Nesserisa and L. Perivolaropoulos, Phys. Rev. D **72**, 123519 (2005); U. Seljak, A. Slosar, and P. McDonald, J. Cosmol. Astropart. Phys. 10 (2006) 014; S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005); Mingzhe Li, Bo Feng, and Xinmin Zhang, J. Cosmol. Astropart. Phys. 12 (2005) 002; V. K. Onemli and R. P. Woodard, Phys. Rev. D **70**, 107301 (2004); H. Mohseni Sadjadi, Phys. Rev. D **73**, 063525 (2006); S. Nesseris and L. Perivolaropoulos, astro-ph/0610092; Zong-Kuan Guo, Yun-Song Piao, Xinmin Zhang, and Yuan-Zhong Zhang, astro-ph/0608165; B. Guberina, R. Horvat, and H. Nikolic, Phys. Lett. B **636**, 80 (2006); H. Mohseni Sadjadi and M. Honardoost, gr-qc/0609076; X. Zhang and F. Wu, Phys. Rev. D **72**, 043524 (2005); Z. Chang, F. Wu, and X. Zhang, Phys. Lett. B **633**, 14 (2006); V. Sahni and Y. Shtanov, J. Cosmol. Astropart. Phys. 11 (2003) 014.
[9] Hao Wei and Rong-Gen Cai, Phys. Rev. D **73**, 083002 (2006).
[10] A. Vikman, Phys. Rev. D **71**, 023515 (2005).
[11] L. Perivolaropoulos, J. Cosmol. Astropart. Phys. 10 (2005) 001; S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **73**, 103511 (2006).
[12] Z.-K. Guo, Y.-S. Piao, X. Zhang, and Y.-Z. Zhang, Phys. Lett. B **608**, 177 (2005); W. Hu, Phys. Rev. D **71**, 047301 (2005); H. Wei, R. G. Cai, and D. F. Zeng, Classical Quantum Gravity **22**, 3189 (2005); H. Stefancic, Phys. Rev. D **71**, 124036 (2005); R. R. Caldwell and M. Doran, Phys. Rev. D **72**, 043527 (2005); H. Wei and R. G. Cai, Phys. Lett. B **634**, 9 (2006); X. F. Zhang, H. Li, Y. S. Piao, and X. M. Zhang, Mod. Phys. Lett. A **21**, 231 (2006); M. z. Li, B. Feng, and X. m. Zhang, J. Cosmol. Astropart. Phys. 12 (2005) 002.
[13] Zong-Kuan Guo, Rong-Gen Cai, and Yuan-Zhong Zhang, J. Cosmol. Astropart. Phys. 05 (2005) 002.
[14] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005); B. Gumjudpai, T. Naskar, M. Sami, and S. Tsujikawa, J. Cosmol. Astropart. Phys. 06 (2005) 007.
[15] H. Mohseni Sadjadi and M. Alimohammadi, Phys. Rev. D **74**, 043506 (2006); M. Alimohammadi and H. Mohseni Sadjadi, gr-qc/0608016.
[16] S. M. Carroll, M. Hoffman, and M. Trodden, Phys. Rev. D **68**, 023509 (2003); J. M. Cline, S. Jeon, and G. D. Moore, Phys. Rev. D **70**, 043543 (2004).
[17] G. Huey, P. J. Steinhardt, B. A. Ovrut, and D. Waldram. Phys. Lett. B **476**, 379 (2000); L. Amendola and D. Tocchini-Valentini, Phys. Rev. D **66**, 043528 (2002); H. Wei and R. G. Cai, Phys. Rev. D **72**, 123507 (2005).
[18] John D. Barrow and T. Clifton, Phys. Rev. D **73**, 103520 (2006).
[19] B. Wang, Y. Gong, and E. Abdalla, Phys. Lett. B **624**, 141 (2005); H. Kim, H. W. Lee, and Y. S. Myung, Phys. Lett. B **632**, 605 (2006); B. Hu and Y. Ling, Phys. Rev. D **73**, 123510 (2006).