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Gauge-invariant formulation of second-order cosmological perturbations

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Gauge-invariant treatments of the second-order cosmological perturbation in a four dimensional homogeneous isotropic universe filled with the perfect fluid are completely formulated without any gauge fixing. We derive all components of the Einstein equations in the case where the first order vector and tensor modes are negligible. These equations imply that the tensor and the vector mode of the second-order metric perturbations may be generated by the scalar-scalar mode coupling of the linear order perturbations as the result of the nonlinear effects of the Einstein equations.

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To clarify the relation between scenarios of the early universe and observational data such as the cosmic microwave background (CMB) anisotropies, the general relativistic cosmological *linear* perturbation theory has been developed to a high degree of sophistication during the last 25 years [1]. Recently, the first order approximation of the early universe from a homogeneous isotropic one is revealed by the observation of the CMB by Wilkinson Microwave Anisotropy Probe (WMAP) [2] and is suggested that fluctuations in the early universe are adiabatic and Gaussian at least in the first order approximation. One of the next theoretical tasks is to clarify the accuracy of these results, for example, through the non-Gaussianity. To accomplish this, the *second-order* cosmological perturbation theory is necessary. So, the perturbation theory beyond the linear order has been investigated by many authors [3–6] and is a topical subject, in particular, to study the non-Gaussianity generated during the inflation [7] and that will be observed in CMB data [8].

In this paper, we show the gauge-invariant formulation of the general relativistic second-order cosmological perturbations on Friedmann-Robertson-Walker (FRW) universe filled with the perfect fluid. This short paper is prepared to show the aspect of the special importance of our companion paper [9], briefly. The formulation in this paper is one of the applications of the gauge-invariant formulation of the second-order perturbation theory on a generic background spacetime developed in Refs. [9,10]. We treat all perturbative variables in the gauge-invariant manner. We also derive all components of the second-order Einstein equations of cosmological perturbations in terms of these gauge-invariant variables without any gauge fixing.

First, we explain the "*gauge*" in general relativistic perturbations [4,5,11]. To explain this, we have to explain what we are doing in perturbation theories, at first. In perturbation theories, we treat two spacetimes. One is the physical spacetime $\mathcal{M}$ which we will describe by perturbations. In cosmological perturbations, $\mathcal{M}$ is an universe with small inhomogeneities. Another is the background spacetime $\mathcal{M}_0$ which is prepared for perturbative analyses. In cosmological perturbations, $\mathcal{M}_0$ is the FRW universe with the metric

$$g_{ab} = a^2(\eta)(-(d\eta)_a(d\eta)_b + \gamma_{ij}(dx^i)_a(dx^j)_b), \tag{1}$$

where γ_{ij} is the metric on a maximally symmetric 3-space with curvature constant K. We note that $\mathcal{M}$ and $\mathcal{M}_0$ are different manifolds.

We also note that, in perturbation theories, we always write equations in the form

$$Q(``p") = Q_0(p) + \delta Q(p). \tag{2}$$

Equation (2) gives the relation between variables on $\mathcal{M}$ and $\mathcal{M}_0$. Actually, $Q(``p")$ in the right hand side (rhs) of Eq. (2) is a variable on $\mathcal{M}$, while $Q_0(p)$ and $\delta Q(p)$ in the left hand side (lhs) of Eq. (2) are those on $\mathcal{M}_0$. Further, the point "$p$" in $Q(``p")$ is on $\mathcal{M}$, while the point p in $Q_0(p)$ and $\delta Q(p)$ is on $\mathcal{M}_0$. Since Eq. (2) is a field equation, we implicitly identify these two points "p" and p through Eq. (2) by a map $\mathcal{M}_0 \to \mathcal{M}$: $p \in \mathcal{M}_0 \mapsto ``p" \in \mathcal{M}$. This identification is the "*gauge choice*" in perturbation theories [11].

Moreover, the above gauge choice between $\mathcal{M}_0$ and $\mathcal{M}$ is not unique in theories with general covariance. Rather, Eq. (2) involves the degree of freedom corresponding to the choice of the map $\mathcal{X}$: $\mathcal{M}_0 \mapsto \mathcal{M}$. This degree of freedom is the "*gauge degree of freedom*" in general relativistic perturbation theory. By virture of the general covariance, there is no preferred coordinate system and we have no guiding principle to choose the identification map $\mathcal{X}$. So, we may always change the gauge choice $\mathcal{X}$.

To develop this understanding of the "gauge", we introduce a parameter λ for the perturbation and the $4+1$-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$ so that $\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}$ and $\mathcal{M} = \mathcal{M}_\lambda = \mathcal{N}|_{\mathbb{R}=\lambda}$. On this extended manifold $\mathcal{N}$, the gauge choice is regarded as a diffeomorphism $\mathcal{X}_\lambda$: $\mathcal{N} \to \mathcal{N}$ such that $\mathcal{X}_\lambda$: $\mathcal{M}_0 \to \mathcal{M}_\lambda$. Further, a gauge choice $\mathcal{X}_\lambda$ is introduced as an exponential map with the generator ${}^{\mathcal{X}}\eta^a$ which is chosen so that its integral curve in $\mathcal{N}$ is transverse to each $\mathcal{M}_\lambda$ everywhere on $\mathcal{N}$

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[4]. Points lying on the same integral curve are regarded as the "*same point*" by the gauge choice $\mathcal{X}_\lambda$.

Now, we define the first and the second-order perturbation of the physical variable Q on $\mathcal{M}_\lambda$ by the pulled-back variable $\mathcal{X}^*_\lambda Q$ on $\mathcal{M}_0$, which is induced by the gauge choice $\mathcal{X}_\lambda$ and expanded as

$$\mathcal{X}^*_\lambda Q_\lambda = Q_0 + \lambda \mathcal{L}_{\mathcal{X}\eta} Q \Big|_{\mathcal{M}_0} + \frac{1}{2}\lambda^2 \mathcal{L}^2_{\mathcal{X}\eta} Q \Big|_{\mathcal{M}_0} + O(\lambda^3). \tag{3}$$

$Q_0 = Q|_{\mathcal{M}_0}$ is the background value of Q and all terms in Eq. (3) are evaluated on $\mathcal{M}_0$. Since Eq. (3) is just the perturbative expansion of $\mathcal{X}^*_\lambda Q_\lambda$, we may regard that the first and the second-order perturbations of Q are given by ${}^{(1)}_{\mathcal{X}}Q = \mathcal{L}_{\mathcal{X}\eta} Q|_{\mathcal{M}_0}$, and ${}^{(2)}_{\mathcal{X}}Q = \mathcal{L}^2_{\mathcal{X}\eta} Q|_{\mathcal{M}_0}$, respectively.

Suppose that we have two gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ with the generators ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$, respectively. When these generators have the different tangential components to each $\mathcal{M}_\lambda$, these $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are regarded as "*different gauge choices*". Further, the "*gauge transformation*" is regarded as the change of the gauge choice $\mathcal{X}_\lambda \to \mathcal{Y}_\lambda$, which is given by the diffeomorphism

$$\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda \colon \mathcal{M}_0 \to \mathcal{M}_0. \tag{4}$$

The diffeomorphism Φ_λ does change the point identification. Further, Φ_λ induces a pull-back from the representation ${}^{\mathcal{X}}Q_\lambda$ in the gauge choice $\mathcal{X}_\lambda$ to the representation ${}^{\mathcal{Y}}Q_\lambda$ in the gauge choice $\mathcal{Y}_\lambda$ as

$${}^{\mathcal{Y}}Q_\lambda = \mathcal{Y}^*_\lambda Q|_{\mathcal{M}_0} = (\mathcal{X}^{-1}_\lambda \mathcal{Y}_\lambda)^*(\mathcal{X}^*_\lambda Q)|_{\mathcal{M}_0} = \Phi^*_\lambda {}^{\mathcal{X}}Q_\lambda. \tag{5}$$

As discussed in Ref. [5], ${}^{\mathcal{Y}}Q_\lambda = \Phi^*_\lambda {}^{\mathcal{X}}Q_\lambda$ is expanded as

$${}^{\mathcal{Y}}Q_\lambda = {}^{\mathcal{X}}Q + \lambda \mathcal{L}_{\xi_{(1)}} {}^{\mathcal{X}}Q + \frac{\lambda^2}{2}(\mathcal{L}_{\xi_{(2)}} + \mathcal{L}^2_{\xi_{(1)}}){}^{\mathcal{X}}Q + O(\lambda^3), \tag{6}$$

where $\xi^a_{(1)}$ and $\xi^a_{(2)}$ are the generators of Φ_λ.

Comparing Eq. (6) with $\mathcal{Y}^*_\lambda \circ (\mathcal{X}^{-1}_\lambda)^* {}^{\mathcal{X}}Q$ in terms of the generators ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$, we find $\xi^a_{(1)} = {}^{\mathcal{Y}}\eta^a - {}^{\mathcal{X}}\eta^a$ and $\xi^a_{(2)} = [{}^{\mathcal{Y}}\eta, {}^{\mathcal{X}}\eta]^a$. Further, Eqs. (3) and (6) yield

$${}^{(1)}_{\mathcal{Y}}Q - {}^{(1)}_{\mathcal{X}}Q = \mathcal{L}_{\xi_{(1)}} Q_0, \tag{7}$$

$${}^{(2)}_{\mathcal{Y}}Q - {}^{(2)}_{\mathcal{X}}Q = 2\mathcal{L}_{\xi_{(1)}} {}^{(1)}_{\mathcal{X}}Q + \{\mathcal{L}_{\xi_{(2)}} + \mathcal{L}^2_{\xi_{(1)}}\} Q_0. \tag{8}$$

These are the gauge transformation rules for the first and the second-order perturbations, respectively.

Inspecting gauge transformation rules (7) and (8), we define the gauge-invariant variables of each order. First, we consider the metric perturbations. The metric on $\mathcal{M}_\lambda$ is expanded through the gauge choice $\mathcal{X}_\lambda$ as

$$\mathcal{X}^*_\lambda \bar{g}_{ab} = g_{ab} + \lambda {}_{\mathcal{X}}h_{ab} + \frac{\lambda^2}{2} {}_{\mathcal{X}}l_{ab} + O^3(\lambda). \tag{9}$$

Since the $\eta = \text{const}$ hypersurfaces in the metric (1) are maximally symmetric, the components $\{h_{\eta\eta}, h_{i\eta}, h_{ij}\}$ of the first order metric perturbation h_{ab} is classified into the three sets of variables $\{h_{\eta\eta}, h_{(VL)}, h_{(L)}, h_{TL}\}$, $\{h_{(V)i}, h_{(TV)i}\}$, and $h_{(TT)ij}$, which are defined by

$$\begin{aligned} h_{\eta i} &=: D_i h_{(VL)} + h_{(V)i}, \qquad D^i h_{(V)i} = 0, \\ h_{ij} &=: a^2 h_{(L)}\gamma_{ij} + a^2 h_{(T)ij}, \qquad \gamma^{ij} h_{(T)ij} = 0, \\ h_{(T)ij} &=: \left(D_i D_j - \frac{1}{3}\gamma_{ij}\Delta\right) h_{(TL)} + 2D_{(i}h_{(TV)j)} + h_{(TT)ij}, \\ & D^i h_{(TV)i} = 0, \qquad D^i h_{(TT)ij} = 0. \end{aligned} \tag{10}$$

The uniqueness of the decomposition (10) is guaranteed by the existence of the Green functions of operators $\Delta := D^i D_i$, $\Delta + 2K$, and $\Delta + 3K$, where D_i is the covariant derivative associated with the metric γ_{ij}. In terms of the variables given in (10), we define a vector field X_a by

$$X_a := X_\eta (d\eta)_a + X_i (dx^i)_a, \tag{11}$$

$$X_\eta := h_{(VL)} - \tfrac{1}{2}a^2 \partial_\tau h_{(TL)}, \tag{12}$$

$$X_i := a^2 (h_{(TV)i} + \tfrac{1}{2} D_i h_{(TL)}), \tag{13}$$

where X_a is transformed as ${}_{\mathcal{Y}}X_a - {}_{\mathcal{X}}X_a = \xi_{(1)a}$ under the gauge transformation (7). We also define the variables

$$-2a^2 \overset{(1)}{\Phi} := h_{\eta\eta} - 2(\partial_\eta - \mathcal{H})X_\eta, \tag{14}$$

$$-2a^2 \overset{(1)}{\Psi} := a^2 (h_{(L)} - \tfrac{1}{3}\Delta h_{(TL)}) + 2\mathcal{H} X_\eta, \tag{15}$$

$$a^2 \overset{(1)}{\nu}_i := h_{(V)i} - a^2 \partial_\eta h_{(TV)i}, \tag{16}$$

$$\overset{(1)}{\chi}_{ij} := h_{(TT)ij}, \tag{17}$$

where $\mathcal{H} = \partial_\eta a / a$. Further, we denote

$$\begin{aligned} \mathcal{H}_{ab} = {} & -2a^2 \overset{(1)}{\Phi} (d\eta)_a (d\eta)_b + 2a^2 \overset{(1)}{\nu}_i (d\eta)_{(a}(dx^i)_{b)} \\ & + a^2(-2\overset{(1)}{\Psi}\gamma_{ij} + \overset{(1)}{\chi}_{ij})(dx^i)_a (dx^j)_b, \end{aligned} \tag{18}$$

where $D^i \overset{(1)}{\nu}_i = \overset{(1)}{\chi}_{[ij]} = \overset{(1)}{\chi}{}^i{}_i = D^i \overset{(1)}{\chi}_{ij} = 0$. The tensor field $\mathcal{H}_{ab}$ is gauge-invariant under the gauge transformation (7). In the cosmological perturbations [1], $\{\overset{(1)}{\Phi}, \overset{(1)}{\Psi}\}$, $\overset{(1)}{\nu}_i$, $\overset{(1)}{\chi}_{ij}$ are called the scalar, vector, and tensor modes, respectively. In terms of the variables $\mathcal{H}_{ab}$ and X_a, the original first order metric perturbation h_{ab} is given by

$$h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab}. \tag{19}$$

Since the scalar mode dominates in the early universe, we assume that $\overset{(1)}{\nu}_i = \overset{(1)}{\chi}_{ij} = 0$ in this paper.

Now, we consider the decomposition of the second-order metric perturbation l_{ab} into the gauge-invariant and variant parts. Through the above variables X_a and h_{ab}, we first consider the variable $\hat{L}_{ab}$ defined by

$$\hat{L}_{ab} := l_{ab} - 2\mathcal{L}_X h_{ab} + \mathcal{L}_X^2 g_{ab} \tag{20}$$

Under the gauge transformation rules (7) and (8), the variable $\hat{L}_{ab}$ is transformed as

$$_{\mathcal{Y}}\hat{L}_{ab} - {}_{\mathcal{X}}\hat{L}_{ab} = \mathcal{L}_\sigma g_{ab}, \qquad \sigma^a := \xi^a_{(2)} + [\xi_{(1)}, X]^a. \tag{21}$$

We note that this gauge transformation (21) has the same form as that of h_{ab}. Then, as shown above, there exist a vector field Y_a and a tensor field $\mathcal{L}_{ab}$ such that the second-order metric perturbation l_{ab} is given by

$$l_{ab} =: \mathcal{L}_{ab} + 2\mathcal{L}_X h_{ab} + (\mathcal{L}_Y - \mathcal{L}_X^2) g_{ab}. \tag{22}$$

The variables $\mathcal{L}_{ab}$ and Y^a are the gauge-invariant and variant parts of l_{ab}, respectively, and the components of $\mathcal{L}_{ab}$ are given by

$$\mathcal{L}_{ab} = -2a^2\overset{(2)}{\Phi}(d\eta)_a(d\eta)_b + 2a^2\overset{(2)}{\nu}_i(d\eta)_{(a}(dx^i)_{b)} + a^2(-2\overset{(2)}{\Psi}\gamma_{ij} + \overset{(2)}{\chi}_{ij})(dx^i)_a(dx^j)_b, \tag{23}$$

where $D^i\overset{(2)}{\nu}_i = \overset{(2)}{\chi}_{[ij]} = \overset{(2)}{\chi}{}^i{}_i = D^i\overset{(2)}{\chi}_{ij} = 0$. The vector field Y_a is transformed as $_{\mathcal{Y}}Y_a - {}_{\mathcal{X}}Y_a = \sigma_a$ under the gauge transformations (7) and (8).

Further, by using the above variables X_a and Y_a, we can find the gauge-invariant variables for the perturbations of an arbitrary field as

$${}^{(1)}\mathcal{Q} :== {}^{(1)}Q - \mathcal{L}_X Q_0, \tag{24}$$

$${}^{(2)}\mathcal{Q} := {}^{(2)}Q - 2\mathcal{L}_X {}^{(1)}Q - \{\mathcal{L}_Y - \mathcal{L}_X^2\}Q_0. \tag{25}$$

Through the gauge transformation rules (7) and (8), we can easily check that these variables are gauge-invariant up to the first and the second order, respectively.

As the matter contents, in this paper, we consider the perfect fluid whose energy-momentum tensor is given by

$$\bar{T}_a{}^b = (\bar{\epsilon} + \bar{p})\bar{u}_a\bar{u}^b + \bar{p}\delta_a{}^b. \tag{26}$$

We expand these fluid components $\bar{\epsilon}$, $\bar{p}$, and $\bar{u}_a$ as Eq. (3):

$$\bar{\epsilon} = \epsilon + \lambda\overset{(1)}{\epsilon} + \tfrac{1}{2}\lambda^2\overset{(2)}{\epsilon} + O(\lambda^3), \tag{27}$$

$$\bar{p} = p + \lambda\overset{(1)}{p} + \tfrac{1}{2}\lambda^2\overset{(2)}{p} + O(\lambda^3), \tag{28}$$

$$\bar{u}_a = u_a + \lambda\overset{(1)}{u}_a + \tfrac{1}{2}\lambda^2\overset{(2)}{u}_a + O(\lambda^3). \tag{29}$$

Following the definitions (24) and (25), we easily obtain the corresponding gauge-invariant variables for these perturbations of the fluid components:

$$\begin{aligned}
\overset{(1)}{\mathcal{E}} &:= \overset{(1)}{\epsilon} - \mathcal{L}_X\epsilon,\\
\overset{(1)}{\mathcal{P}} &:= \overset{(1)}{p} - \mathcal{L}_X p,\\
\overset{(1)}{\mathcal{U}}_a &:= \overset{(1)}{(u_a)} - \mathcal{L}_X u_a,\\
\overset{(2)}{\mathcal{E}} &:= \overset{(2)}{\epsilon} - 2\mathcal{L}_X\overset{(1)}{\epsilon} - \{\mathcal{L}_Y - \mathcal{L}_X^2\}\epsilon,\\
\overset{(2)}{\mathcal{P}} &:= \overset{(2)}{p} - 2\mathcal{L}_X\overset{(1)}{p} - \{\mathcal{L}_Y - \mathcal{L}_X^2\}p,\\
\overset{(2)}{\mathcal{U}}_a &:= \overset{(2)}{(u_a)} - 2\mathcal{L}_X\overset{(1)}{u}_a - \{\mathcal{L}_Y - \mathcal{L}_X^2\}u_a.
\end{aligned} \tag{30}$$

Through $\bar{g}^{ab}\bar{u}_a\bar{u}_b = g^{ab}u_au_b = -1$ and neglecting the rotational part in $\overset{(1)}{\mathcal{U}}_a$, we see that

$$\overset{(1)}{\mathcal{U}}_a = -a\overset{(1)}{\Phi}(d\eta)_a + aD_i\overset{(1)}{v}(dx^i)_a. \tag{31}$$

We also expand the Einstein tensor as

$$\bar{G}_a{}^b = G_a{}^b + \lambda{}^{(1)}G_a{}^b + \tfrac{1}{2}\lambda^{2(2)}G_a{}^b + O(\lambda^3). \tag{32}$$

Eqs. (19) and (22) give each order perturbation of the Einstein tensor which is decomposed as

$$\overset{(1)}{G}{}_a{}^b = {}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] + \mathcal{L}_X G_a{}^b, \tag{33}$$

$${}^{(2)}G_a{}^b = {}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] + 2\mathcal{L}_X{}^{(1)}G_a{}^b + \{\mathcal{L}_Y - \mathcal{L}_X^2\}G_a{}^b \tag{34}$$

as expected from Eqs. (24) and (25). Here, ${}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}]$ and ${}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}]$ are gauge-invariant parts of the first and the second order perturbations of the Einstein tensor, respectively. On the other hand, the energy momentum tensor (26) is also expanded as

$$\bar{T}_a{}^b = T_a{}^b + \lambda{}^{(1)}T_a{}^b + \tfrac{1}{2}\lambda^{2(2)}T_a{}^b + O(\lambda^3) \tag{35}$$

and ${}^{(1)}T_a{}^b$ and ${}^{(2)}T_a{}^b$ are also given in the form

$${}^{(1)}T_a{}^b = {}^{(1)}\mathcal{T}_a{}^b + \mathcal{L}_X T_a{}^b, \tag{36}$$

$${}^{(2)}T_a{}^b = {}^{(2)}\mathcal{T}_a{}^b + 2\mathcal{L}_X{}^{(1)}T_a{}^b + \{\mathcal{L}_Y - \mathcal{L}_X^2\}T_a{}^b \tag{37}$$

through the definitions (30) and (31) of the gauge-invariant variables of the fluid components. Here, ${}^{(1)}\mathcal{T}_a{}^b$ and ${}^{(2)}\mathcal{T}_a{}^b$ are the gauge-invariant part of the first and the second-order perturbation of the energy-momentum tensor, respectively. Then, the first and the second-order perturbations of the Einstein equation are necessarily given in term of gauge invariant variables:

$${}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] = 8\pi G{}^{(1)}\mathcal{T}_a{}^b, \tag{38}$$

$${}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] = 8\pi G{}^{(2)}\mathcal{T}_a{}^b. \tag{39}$$

The traceless scalar part of the spatial component of Eq. (38) yields $\overset{(1)}{\Psi} = \overset{(1)}{\Phi}$, and the other components of Eq. (38) gives well-known equations [1]. Neglecting the first order vector and tensor modes and using $\overset{(1)}{\Psi} = \overset{(1)}{\Phi}$, the components of $\overset{(2)}{\mathcal{U}}_a$ are given by

$$\overset{(2)}{\mathcal{U}}_a = a((\overset{(1)}{\Phi})^2 - D_i \overset{(1)}{v} D^i \overset{(1)}{v} - \overset{(2)}{\Phi})(d\eta)_a + a(D_i \overset{(2)}{v} + \overset{(2)}{\mathcal{V}})(dx^i)_a, \quad (40)$$

where $D^i \overset{(2)}{\mathcal{V}}_i = 0$.

All components of Eq. (39) are summarized as follows: As the scalar parts of Eq. (39), we have

$$4\pi G a^2 \overset{(2)}{\mathcal{E}} = (-3\mathcal{H}\partial_\eta + \Delta + 3K - 3\mathcal{H}^2)\overset{(2)}{\Phi} - \Gamma_0 + \frac{3}{2}\left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right) - \frac{9}{2}\mathcal{H}\partial_\eta(\Delta + 3K)^{-1}\left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right), \quad (41)$$

$$8\pi G a^2(\epsilon + p)D_i\overset{(2)}{v} = -2\partial_\eta D_i\overset{(2)}{\Phi} - 2\mathcal{H}D_i\overset{(2)}{\Phi} + D_i\Delta^{-1}D^k\Gamma_k - 3\partial_\eta D_i(\Delta + 3K)^{-1}\left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right), \quad (42)$$

$$4\pi G a^2 \overset{(2)}{\mathcal{P}} = (\partial_\eta^2 + 3\mathcal{H}\partial_\eta - K + 2\partial_\eta\mathcal{H} + \mathcal{H}^2)\overset{(2)}{\Phi} - \frac{1}{2}\Delta^{-1}D^iD_j\Gamma_i{}^j + \frac{3}{2}(\partial_\eta^2 + 2\mathcal{H}\partial_\eta) \times (\Delta + 3K)^{-1}\left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right), \quad (43)$$

$$\overset{(2)}{\Psi} - \overset{(2)}{\Phi} = \frac{3}{2}(\Delta + 3K)^{-1}\left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right). \quad (44)$$

where

$$\Gamma_0 := 8\pi G a^2(\epsilon + p)D^i\overset{(1)}{v}D_i\overset{(1)}{v} - 3D_k\overset{(1)}{\Phi}D^k\overset{(1)}{\Phi} - 3(\partial_\eta\overset{(1)}{\Phi})^2 - 8\overset{(1)}{\Phi}\Delta\overset{(1)}{\Phi} - 12(K + \mathcal{H}^2)(\overset{(1)}{\Phi})^2, \quad (45)$$

$$\Gamma_i := -16\pi G a^2(\overset{(1)}{\mathcal{E}} + \overset{(1)}{\mathcal{P}})D_i\overset{(1)}{v} + 12\mathcal{H}\overset{(1)}{\Phi}D_i\overset{(1)}{\Phi} - 4\overset{(1)}{\Phi}\partial_\eta D_i\overset{(1)}{\Phi} - 4\partial_\eta\overset{(1)}{\Phi}D_i\overset{(1)}{\Phi}, \quad (46)$$

$$\Gamma_i{}^j := 16\pi G a^2(\epsilon + p)D_i\overset{(1)}{v}D^j\overset{(1)}{v} - 4D_i\overset{(1)}{\Phi}D^j\overset{(1)}{\Phi} - 8\overset{(1)}{\Phi}D_iD^j\overset{(1)}{\Phi} + 2(3D_k\overset{(1)}{\Phi}D^k\overset{(1)}{\Phi} + 4\overset{(1)}{\Phi}\Delta\overset{(1)}{\Phi} + (\partial_\eta\overset{(1)}{\Phi})^2 + 4(2\partial_\eta\mathcal{H} + K + \mathcal{H}^2)(\overset{(1)}{\Phi})^2 + 8\mathcal{H}\overset{(1)}{\Phi}\partial_\eta\overset{(1)}{\Phi})\gamma_i{}^j. \quad (47)$$

As the vector parts of Eq. (39), we have

$$8\pi G a^2(\epsilon + p)\overset{(2)}{\mathcal{V}}_i = \frac{1}{2}(\Delta + 2K)\overset{(2)}{\nu}_i + (\Gamma_i - D_i\Delta^{-1}D^k\Gamma_k), \quad (48)$$

$$\partial_\eta(a^2\overset{(2)}{\nu}_i) = 2a^2(\Delta + 2K)^{-1}\{D_i\Delta^{-1}D^kD_l\Gamma_k{}^l - D_k\Gamma_i{}^k\}. \quad (49)$$

As the tensor parts of Eq. (39), we have the evolution equation of ${}^{(2)}\chi_{ij}$

$$(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \Delta)^{(2)}\chi_{ij} = 2\Gamma_{ij} - \frac{2}{3}\gamma_{ij}\Gamma_k{}^k - 3\left(D_iD_j - \frac{1}{3}\gamma_{ij}\Delta\right)(\Delta + 3K)^{-1} \times \left(\Delta^{-1}D^kD_l\Gamma_k{}^l - \frac{1}{3}\Gamma_k{}^k\right) + 4(D_{(i}(\Delta + 2K)^{-1}D_{j)}\Delta^{-1}D^lD_k\Gamma_l{}^k - D_{(i}(\Delta + 2K)^{-1}D^k\Gamma_{j)k}), \quad (50)$$

Eqs. (49) and (50) imply that the second-order vector and tensor modes may be generated due to the scalar-scalar mode coupling of the first order perturbation.

Further, the Eqs. (41) and (44) are reduced to the single equation for $\overset{(2)}{\Phi}$

$$(\partial_\eta^2 + 3\mathcal{H}(1 + c_s^2)\partial_\eta - c_s^2\Delta + 2\partial_\eta\mathcal{H} + (1 + 3c_s^2)(\mathcal{H}^2 - K))\overset{(2)}{\Phi} = 4\pi G a^2\left\{\tau\overset{(2)}{\mathcal{S}} + \frac{\partial c_s^2}{\partial\epsilon}(\overset{(1)}{\mathcal{E}})^2 + 2\frac{\partial c_s^2}{\partial S}\overset{(1)}{\mathcal{E}}\overset{(1)}{\mathcal{S}} + \frac{\partial\tau}{\partial S}(\overset{(1)}{\mathcal{S}})^2\right\} - c_s^2\Gamma_0 + \frac{1}{6}\Gamma_k{}^k + \frac{3}{2}\left(c_s^2 + \frac{1}{3}\right)\left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right) - \frac{3}{2}(\partial_\eta^2 + (2 + 3c_s^2)\mathcal{H}\partial_\eta)(\Delta + 3K)^{-1} \times \left(\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right). \quad (51)$$

Here, we have used the second-order perturbation of the equation of state for the fluid components

$$\overset{(2)}{\mathcal{P}} = c_s^2\overset{(2)}{\mathcal{E}} + \tau\overset{(2)}{\mathcal{S}} + \frac{\partial c_s^2}{\partial\epsilon}\overset{(1)}{\mathcal{E}}{}^2 + 2\frac{\partial c_s^2}{\partial S}\overset{(1)}{\mathcal{E}}\overset{(1)}{\mathcal{S}} + \frac{\partial\tau}{\partial S}\overset{(1)}{\mathcal{S}}{}^2, \quad (52)$$

where $\overset{(1)}{\mathcal{S}}$ and $\overset{(2)}{\mathcal{S}}$ are the gauge-invariant entropy perturbation of the first and second-order, respectively, we denoted that $c_s^2 := \partial\bar{p}/\partial\bar{\epsilon}$ and $\tau := \partial\bar{p}/\partial\bar{S}$. The Eq. (51) will be useful to discus the second-order effect in the CMB anisotropy, for example, the non-Gaussianity generated by the nonlinear effects [8].

We gave all components of the second-order perturbation of the Einstein equation in terms of gauge-invariant variables. This is the main result of this paper. The similar equations are also obtained in the case where the matter content of the universe is a single scalar field [9]. We also note that we have derived the second-order perturbation equations without using explicit solutions to the first order perturbations. Therefore, the Eqs. (41)–(50) are applicable to any expansion law of the background universe. In particular, in the case where $a(\eta) = 1$, these equations are those for the second post-Minkowski perturbations. Besides the explicit forms of the second-order Einstein equations, the gauge-invariant expressions (38) and (39) of the first and the second-order Einstein equation are true if the decomposition formula (19) is true. Hence, the gauge-invariant formulation of the second-order perturbation shown here is also applicable not only to cosmological perturbations but also any other situations of the general relativistic perturbations.

In the case of cosmological perturbations, Eqs. (41)–(50) make possible to discuss the nonlinear effects in the evolution of the universe. In many works [7], the second-order effects have been investigated through the conserved quantities which corresponds to the Bardeen parameter in the linear theory. It will be interesting to clarify these conservation laws by using the gauge-invariant formulation in this paper.

Further, the rotational part of the fluid velocity in Eqs. (48) of the vector mode is also important in the early universe because this part of the fluid velocity is related to the generation of the magnetic field in the early universe [12]. The generation of the tensor mode by Eq. (50) is also interesting, since this is one of the generation process of gravitational waves. We have already known that the fluctuations of the scalar mode exist in the early universe from the anisotropy of the CMB. Hence, the generation of the vector mode and tensor mode due to the second-order perturbation will give the lower limit of these modes in the early universe.

Thus, the formulation shown here have very wide applications and we leave these our future works. These issues are interesting not only from the theoretical point of view but also from the observational point of view. Because of its very wide applications, the gauge-invariant formulation shown here will play a key role in general relativistic perturbations.

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