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Interpretations and implications of negative binomial distributions of multiparticle productions

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The number of particles produced in high energy experiments is approximated by a negative binomial distribution. Deriving a representation of the distribution from a stochastic equation, conditions for the process to satisfy the distribution are clarified. Based on them, it is proposed that multiparticle production consists of spontaneous and induced production. The rate of the induced production is proportional to the number of existing particles. The ratio of the two production rates remains constant during the process. The "NBD space" is also defined where the number of particles produced in its subspaces follows negative binomial distributions with different parameters.

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I. INTRODUCTION

In high energy experiments, accelerated particles are collided to reproduce the conditions that occurred in the early universe. Many particles are created in both circumstances. In the experiments, particles from selected events are analyzed to understand features of elementary particle processes. However, because multiparticle production process as a whole cannot be handled by simple perturbative calculations, a complete understanding of the phenomena has not been achieved.

It is known that multiplicity distributions of particles produced in the experiments are well described by the negative binomial distributions (NBD) [1–5]. Experiments show that it holds as a general aspect of multiparticle production processes, regardless of the types of colliding particles, such as $\bar{p}p$, e^+e^-, πp, pN, or NN, for a wide range of the energies, $\sqrt{s}$.

Negative binomial distributions appear as a convolution of k Bose-Einstein (or geometric) distributions[6,7]. To explain the distributions, models and interpretations of the multiparticle production phenomena have been proposed, such as branching of QCD partons [8], a Gamma mixing distribution to the Poisson [9], and productions of clans [10].

Information theory derives NBD as the most probable distribution from limited knowledge [11,12]. Also, it is providing new aspects of multiparticle production especially based on Tsallis entropy [13–15].

To comprehend the general features of multiparticle production processes, new interpretations of negative binomial distributions are being attempted in this report. In the following, a "particle" means a QCD parton as in the branching models.

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II. MATHEMATICAL FORMULAS AND CONDITIONS OF NBD

A stochastic equation of NBD with parameter k is expressed as

$$\frac{dP_n}{dt} = \lambda(t)\{-(k+n)P_n + (k+n-1)P_{n-1}\}, \qquad (1)$$

for integer $n \geq 1$. For $n = 0$, the second term of the right hand side is omitted. $P_n(t)$ represents a probability that an event contains n particles which have been produced after collision. t is most simply regarded as time, but it is interpreted differently as a variable of parton branching in Refs. [8,16–18].

The $\lambda(t)$ denotes a particle production rate at t. Unlike the references, the production rate here is assumed t-dependent. At t_0, the initial t of the production, $\lambda(t)$ becomes nonzero and multiparticle production begins. $\lambda(t)$ returns to zero and multiparticle production ends before the observation. In addition, different from the references, the representation (1) of the stochastic equation corresponds to an initial condition $P_n(t_0) = \delta_{n0}$. And the factor $(k+n)\lambda(t)$, multiplied on n-particle probability, will be used to separate two types of production afterwards.

From Eq. (1), $P_n(t)$ is represented with $P_{n-1}(t)$,

$$P_n(t) = \int_{t_0}^{t} dt_n \exp\left\{-\int_{t_n}^{t} dt'_n (k+n)\lambda(t'_n)\right\} \times (k+n-1)\lambda(t_n)P_{n-1}(t_n).$$

By successive substitutions, $P_n(t)$ is expressed as follows:

$$P_n(t) = \int_{t_0}^{t} dt_n \exp\left\{-\int_{t_n}^{t} dt'_n (k+n)\lambda(t'_n)\right\}(k+n-1)\lambda(t_n)$$
$$\int_{t_0}^{t_n} dt_{n-1} \exp\left\{-\int_{t_{n-1}}^{t_n} dt'_{n-1}(k+n-1)\lambda(t'_{n-1})\right\}(k+n-2)\lambda(t_{n-1})$$
$$\ldots$$
$$\int_{t_0}^{t_2} dt_1 \exp\left\{-\int_{t_1}^{t_2} dt'_1 (k+1)\lambda(t'_1)\right\}k\lambda(t_1)\exp\left(-\int_{t_0}^{t_1} dt'_0 k\lambda(t'_0)\right), \tag{2}$$

which leads to a NBD formula

$$P_n(t) = \frac{\Gamma(k+n)}{\Gamma(1+n)\Gamma(k)} \exp\left(-k\int_{t_0}^{t} dt'\lambda(t')\right) \times \left\{1 - \exp\left(-\int_{t_0}^{t} dt'\lambda(t')\right)\right\}^n. \tag{3}$$

In the case of $k = 1$, the negative binomial distribution becomes Bose-Einstein distributions. The negative binomial distribution expressed by (3) will be specified by NBD$(k, \int_{t_0}^{t} dt'\lambda(t'))$ in this report.

Equation (2) (and also (1)) is most easily illustrated in a simple particle branching picture, temporarily assuming k as an initial number of particles at t_0. After t_0, particles are produced one by one, the i-th particle at t_i, where $i = 1, 2, \ldots, n$ and $t_0 < t_1 < \ldots < t_n < t$. The probability that the i-th particle is produced between t_i and $t_i + dt_i$ is considered as $(k + i - 1)\lambda(t_i)dt_i$. It is supposed that each of $(k + i - 1)$ particles which exist just before t_i branches with the same rate of $\lambda(t_i)$. The factor $\exp\{-\int_{t_i}^{t_{i+1}} dt'_i(k + i)\lambda(t'_i)\}$ means a probability that $(k + i)$ particles remain without branching from t_i to t_{i+1}.

In order to satisfy a negative binomial distribution, the following conditions on $\lambda(t)$ and k are required for the multiparticle production.

(i) $\lambda(t)$ is common to all existing particles at any t in each event or collision.

(ii) $\int_{t_0}^{T} dt\lambda(t)$ is the same for all events, where T denotes the t at the observations. (It is possible that $\lambda(t)$ differs event by event.)

(iii) k remains constant at any t in each event.

(iv) k is the same for all events.

These consequences of negative binomial distributions provide general features of multiparticle production, at least as approximations. Or these requirements provide a basis to discuss the limits of application of negative binomial distributions.

III. INTERPRETATIONS: SPONTANEOUS AND INDUCED PRODUCTION

Experiments [1] revealed that k is not an integer and decreases with $\sqrt{s}$. These contradict the simple assumption that k is a number of initial particles or partons. (For example, number of partons in (anti)protons is expected to increase with $\sqrt{s}$ as observed in jet production.) In addition, if k is the initial number of particles, there should exist a fixed number of partons at initial time t_0 in every event. To avoid this feature, slightly different distributions from negative binomial distribution were proposed [16,17,19] by assuming simple distributions of initial particle numbers. Instead of modifying the NBD, other interpretations for k will be considered in this report: constant ratios of different production rates.

First, k is possibly interpreted as a ratio of particle production probabilities from initial particles and from produced particles. It is assumed that the production or the branching rate from the initial particles as a whole is written as $k\lambda(t)$ at any t, while the production rate from each particle produced after t_0 is expressed as the $\lambda(t)$. By this interpretation, the behavior of k with $\sqrt{s}$ becomes more understandable. At lower $\sqrt{s}$, particles produced by branching rarely initiate additional production because their energies are too low. Then, the productions by initial particles are dominant, and k becomes large. Valence quarks in colliding hadrons or q and $\bar{q}$ from e^+e^- annihilations may be regarded as the initial particles here.

In this context, the initial particles and the produced particles are treated differently with the different production rates. Furthermore, the initial particles are not observed or counted as final particles unlike the produced particles, although the former continue to exist and repeat branching with the latter until production finishes with $\lambda(t) = 0$. In Eq. (1), n represents the number of particles produced after t_0 and observed at t which results in a negative binomial distribution. Contributions from the initial particles are included in k, but not in the number of particles, n, to be detected.

However, both types of particles should only be QCD partons in terms of QCD theory. Instead of employing different types of particles, negative binomial distributions are reinterpreted introducing two types of particle production: "spontaneous" and "induced" production.

Spontaneous production occurs independently of the other particles. On the other hand, a presence of particles causes an additional particle to be produced in the case of induced production. The rate of the induced production is proportional to the number of particles. The ratio between the spontaneous production rate and the induced production rate (per one existing particle) is assumed constant, and the value is identified as k of the negative binomial distribution. As seen in the Eq. (1) the transition rate at t from the n particle state to the $n + 1$ state is expressed $(k + n)\lambda(t)$, where $\lambda(t)$ means the induced production rate for one particle.

The words spontaneous and induced are borrowed from theories of the photon emission from atoms. The two categories of emission were first proposed by A. Einstein. Quantum mechanics later provided the basis, proving that the rates of the two emissions are always the same ($k = 1$) [20].

It has already been indicated in a different manner that negative binomial distributions of multiparticle productions can emerge as a result of "partial stimulated (induced) emission" [10]. It was attributed to Bose-Einstein interference of identical Bosons, and the model is treated separately from branching processes. In their models, pions are assumed as the identical Bosons. And it is believed that the theory is rejected by the NA22 experiments [2].

On the contrary, particles in this report corresponds to QCD partons. The t_0 can be shifted backwards in time even till the colliding particles are completely separated. It means that the spontaneous and the induced production here are supposed to occur during the parton branching, before hadronization of the pions. So, the Boson that could cause the Bose-Einstein interference is regarded as gluon or quark-antiquark pair.

In branching pictures, spontaneous production represents a production from vacuum. The vacuum may relate to the flows of initial or colliding particles. The induced production corresponds to a branching from existing particles where the branching rate from each particle is supposed equal. In these branching pictures, the NBD may appear even without any kind of interference if the assumed relations are at least approximately satisfied between the production from vacuum and from present particles.

However, as mentioned previously, (iii) and (iv), the negative binomial parameter k for the multiparticle productions remains constant at any t in any events. Meanwhile, the ratio of the spontaneous and the induced emission is always 1 for photons of any frequencies and polarizations that are emitted from any kind of atoms at any temperatures. This comparison may lead to formulation of a comparable theory for multiparticle production, where the ratio of the spontaneous and the induced production is unchanged but not unity. With the generation operator, $\mathbf{a}^\dagger$, which satisfies usual commutation relation for Bosons, acting on an n particle state, $|n\rangle$,

$$\mathbf{a}^\dagger|n\rangle = \sqrt{n + k}|n + 1\rangle,$$

is expected in those theories.

IV. STATISTICAL FEATURES AND COMPARISONS WITH OTHER MODELS

The amount of the spontaneous production in an event follows a Poisson distribution of average $k \int_{t_0}^{t} dt' \lambda(t')$. The number of induced particles originating from one spontaneously produced particle obeys the Bose-Einstein distribution, NBD$(1, \int_{t_s}^{t} dt' \lambda(t'))$,

$$P_n(t; t_s) = \exp\left(-\int_{t_s}^{t} dt' \lambda(t')\right)\left\{1 - \exp\left(-\int_{t_s}^{t} dt' \lambda(t')\right)\right\}^n \tag{4}$$

where t_s denotes the t of the spontaneous production. In addition, (4) holds for particles originating from an induced particle at t_s, instead of spontaneously produced one.

Because k is the average number of spontaneously produced particles, it becomes unnecessary to require noninteger and fixed value of particle numbers for each event.

As seen in Refs. [21,22], a convolution of Poisson and Bose-Einstein distribution leads to a distribution slightly different from a negative binomial distribution. The difference comes from the fact that the Bose-Einstein distribution (4) depends on t_s, and differs for each spontaneous production. The Bose-Einstein distribution for each cluster in the references is assumed identical.

Similarly, we can proceed to a comparison with the popular clan models [10]. The clan model assumes that the clans, or clusters, are produced according to a Poisson distribution, and all the clan decay, producing particles that follow an identical logarithmic distribution. Then the final multiplicities obey negative binomial distributions. It is easily shown that by integrating zero-truncated form of (4) with t_s after multiplying by the production probability density $\lambda(t_s)dt_s/\int_{t_0}^{t} dt' \lambda(t')$, the logarithmic distribution is obtained. An explanation for the logarithmic distribution to appear in the clan model is derived from the context of this report. And the clan is identified as a cluster of particles originating from a single spontaneous production.

Statistical features of the model provided here is similar with the clan model. But the physical entity such as clan is not assumed. Actually, a cluster initiated by an spontaneous production at t, and that by a induced production at the same t, are not supposed to be different, except the latter resides in another cluster. Also, only Bose-Einstein distribution is used, which is derived from a minimal requirement in information theory [11].

V. NBD SPACE AND RAPIDITY

The interpretation of negative binomial distributions are extended using a generating function:

$$G(z) = \left(\frac{\exp(-\int_{t_0}^{t} dt' \lambda(t'))}{1 - \{1 - \exp(-\int_{t_0}^{t} dt' \lambda(t'))\} z} \right)^k. \quad (5)$$

Suppose that N_0 independent regions are created at t_0. In the i-th region, the ratio of the spontaneous and the induced production rates is assumed k_i, so that the number of particles produced in the i-th region, n_i, follow a negative binomial distribution with k_i. k_i can differ for regions and N_0 can be different for events, but the sum $k = \sum_{i=1}^{N_0} k_i$ is required to be constant for all events. Also, the integral of the induced production rate $\int_{t_0}^{t} dt' \lambda(t')$ must be the same for all regions. Then the total multiplicity $n = \sum_{i=1}^{N_0} n_i$ results in negative binomial distribution with k because the generating function of the sum becomes a product of generating functions of all regions.

By advancing the above arguments with the generating function (5), "NBD space" will be defined in the following. Experiments show that the parameter k varies when the size of rapidity region to observe particles is changed [1–3]. It requires a mathematical framework of a space of particle production, where the number of particles produced in its subspace follows NBD with different parameters.

Setting the number of the independent regions N_0 to infinity, $N_0 \to \infty$, and introducing a continuous coordinate r, the NBD parameter k is expressed as $k = \int dr \kappa(r)$ with a density function $\kappa(r)$.

In the $r - t$ plane, the spontaneous production rate at a point (r, t) is expressed $\kappa(r)\lambda(t) dr dt$. Each $r = const$ line on the plane is regarded as an independent region of particle productions. It means that if a particle is produced spontaneously at a point (r, t_s), induced productions or particle branchings follow on the same line of $r = \text{const}$, according to (4). And all particles from the successive productions should be observed with the same r. Total number of particles in the whole space obeys the $\text{NBD}(k, \int_{t_0}^{t} dt' \lambda(t'))$. The number of particles, produced in any rectangular region of $[r_1, r_2] \times [t_0, t]$ in the $r - t$ plane, or observed in any range of $[r_1, r_2]$ at any t, follows the $\text{NBD}(\int_{r_1}^{r_2} dr \kappa(r), \int_{t_0}^{t} dt' \lambda(t'))$. By changing the detection area $[r_1, r_2]$, the NBD parameter k varies.

In the Ref. [10], the behavior of their variables, a and b, estimated by the results of the UA5 experiments, are shown with respect to the size of the rapidity windows. As the size is enlarged, a increases and b becomes flat. Here, a and b are replaced by $k(1 - \exp(-\int_{t_0}^{t} dt' \lambda(t')))$ and $1 - \exp(-\int_{t_0}^{t} dt' \lambda(t'))$. Then the behavior obtained by the experiment implies that $k(= \int_{r_1}^{r_2} dr \kappa(r))$ increases with the rapidity window size and the other NBD parameter $\int_{t_0}^{t} dt' \lambda(t')$ is independent of rapidity. It means that rapidity possesses common features with the variable r for the NBD space.

However, particles branched from the same origin are not detected with the same rapidity in real situations. So rapidity does not have the ideal feature of r. Instead, particles from a common origin, or cluster, exhibits a spread in rapidity. This is the reason why the variable b in [10] gradually decreases as the rapidity region become small enough. Variable b relates to the number of particles in a cluster, and it decreases when a part of them is lost outside of the region.

In spite of the differences, the variation of the parameter k with the size of rapidity regions implies the necessity of a coordinate, r, on which a density function of spontaneous production, $\kappa(r)$, is defined. The coordinate r could be just a rapidity of a spontaneous production. In that case, the number of observed particles in a limited rapidity window does not obey negative binomial distributions in a strict sense, because particles move in or out at the window edges. But the multiplicity can be approximated by NBD when the window size is large enough compared to the spread of a cluster. By narrowing the size, the NBD fit is expected to be gradually deteriorated, because the effect from the edges of the window can not be ignored.

Let us assume that clustering of observed particles stemmed from the same spontaneous production is enabled in experiments. Then, by counting all particles in a cluster only if the center of the cluster resides in the window, the NBD approximation for limited rapidity window would be improved.

In addition, suppose that particles initiated by a common origin, regardless of spontaneous or induced production, form a cluster in some phase space, and suppose that the spread size of the cluster correlates with t of the original production. Then, the details of the model in this report, such as $\lambda(t)$ and $\kappa(r)$, as well as differences from the clan model, can be investigated from experimental data.

VI. CONCLUSIONS

In this report, representations (2) and (3) for the negative binomial distributions are derived by integrating the stochastic Eq. (1). By interpreting the formulas, conditions on multiparticle production processes, (i)-(iv), are listed. Then it is proposed that multiparticle production processes are composed of spontaneous and induced production. The parameter k of negative binomial distributions is interpreted as a constant ratio of the two production rates.

Comparisons with the other models, especially with the clan model, are performed. The clan results in a cluster of particles originating from a spontaneous production. But no differece is expected from a cluster initiated by an induced production, if t of the productions is the same.

The generating function (5) of negative binomial distributions is used to introduce a continuous coordinate r on which a density function of k is defined. Then, the "NBD space" is constructed where particles produced in its subspaces follow negative binomial distributions of various parameters. Also, the relation between r and rapidity is discussed.

The understanding of negative binomial distributions here would contribute to the interpretations of NBD phenomena in a variety of fields.

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