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# $f(R)$ gravity without a cosmological constant

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In this work we consider the possibility of describing the current evolution of the universe, without the introduction of any cosmological constant or dark energy (DE), by modifying the Einstein-Hilbert (EH) action. In the context of the $f(R)$ gravities within the metric formalism, we show that it is possible to find an action without cosmological constant which exactly reproduces the behavior of the EH action with cosmological constant. In addition the $f(R)$ action is analytical at the origin having Minkowski and Schwarzschild solutions as vacuum solutions. The found $f(R)$ action is highly nontrivial and must be written in terms of hypergeometric functions but, in spite of looking somewhat artificial, it shows that the cosmological constant, or more generally the DE, is not a logical necessity.

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One of the most important recent scientific discoveries is the accelerated expansion of the universe. Different data from type Ia supernovae [1] observation, large structure information and delicate measurements of the cosmic microwave background (CMB) anisotropies (particularly those from the Wilkinson Microwave Anisotropy Probe (WMAP) [2]) have concluded that our universe is expanding at an increasing rate. This fact sets the very urgent problem of finding the cause for this speed-up.

Usual explanations belong to one of the following three classes: First one reconciles this acceleration with General Relativity (GR) by invoking a strange cosmic fluid, DE, with a state equation $p = \omega\rho$ where $\omega$ is very close to $-1$, i.e. the fluid has a large negative pressure. For the particular case $\omega = -1$ this fluid behaves just as a cosmological constant $\Lambda$. Within this approach recent data obtained by WMAP correspond to the cosmological parameters [2]: $\Omega_M h^2 = 0.1493^{+0.007}_{-0.013}$, $\Omega_\Lambda = 0.72 \pm 0.04$ and $H_0(t = t_0) = 100\ h\,\mathrm{kms}^{-1}\,\mathrm{Mpc}^{-1}$ with $h = 0.71^{+0.03}_{-0.03}$ and $t_0 \equiv t_{\mathrm{today}}$. The main problem of this kind of description is that the fitted $\Lambda$ value seems to be about 55 orders of magnitude smaller than expected (the cosmological constant problem). The second type of explanations consider a dynamical DE by introducing a new scalar field. Finally the third one is trying to explain the cosmic acceleration as a consequence of new gravitational physics [3]. EH action modifications have been widely considered in the literature [4], firstly to describe inflation, and more recently to describe the current cosmic speed-up, or even both cosmological eras simultaneously.

The simplest way of modifying EH action is by adding some function $f(R)$ with the required properties (see [5] for a recent review on $f(R)$ gravities). For example in [6] it was introduced a gravitational model where $f(R) = -\mu^4/R$, being the total gravitational action proportional to $R - \mu^4/R$. This proposal has very interesting cosmological properties and triggered a lot of work on $f(R)$ gravities applied to cosmology. However this kind of actions with negative powers of the curvature has the very serious drawback of not having vacuum solutions with vanishing curvature. For instance in the mentioned model the vacuum constant curvature solution is $R = \pm\sqrt{3}\mu^2$. Thus, even if one succeeds in reproducing cosmic acceleration, paradigmatic GR vacuum solutions assumed to play a major role in any fundamental theory of gravity, such as Minkowski or Schwarzschild, are excluded. Other $f(R)$ functions recently considered in the literature face similar problems and moreover could be in conflict with Solar System experiments [7] while some other models could agree with Supenovae data [8].

In this work we address the issue of finding a $f(R)$ gravity able to reproduce the current cosmic speed-up without any cosmological constant but having $R = 0$ as vacuum solution. From a more formal point of view we are seeking for a $f(R)$ gravity having the same Friedmann-Robertson-Walker (FRW) solution as the standard EH action with cosmological constant for nonrelativistic matter ($p = 0$), but being analytical at $R = 0$. Clearly the $f(R)$ expansion at $R = 0$ must start at the $R^2$ term to avoid having cosmological constant or to redefine the Newton constant.

In order to consider such as the standard EH cosmological solution with DE as $f(R)$ gravity cosmological solution without DE in the same setting, we start from a general action $S = S_G + S_M + S_{DE}$ where $S_G$ is the gravitational action given by:

$$S_G = \frac{1}{2\kappa}\int d^4x\sqrt{|g|}(R + f(R)) \tag{1}$$

with $\kappa \equiv 8\pi G$ and $f(R)$ being any arbitrary function of the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$, where the Ricci tensor is given by $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ and the curvature tensor is defined with the convention $R^\alpha_{\beta\mu\nu} \sim \partial_\beta\Gamma^\alpha_{\mu\nu}$ where $\Gamma^\alpha_{\mu\nu}$ are the symbols for the Levi-Civita connection since we are using the metric formalism (see [9] for exhaustive research on

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nonmetric formalisms and references therein). $S_M$ and $S_{DE}$ are the actions describing matter (also including dark matter) and DE (which, in particular, includes any possible cosmological constant $\Lambda$) respectively. For these two actions the corresponding energy-momentum tensor are given by $T_X^{\mu\nu} = 2|g|^{-1/2}\delta S_X/\delta g_{\mu\nu}$ with X = M (matter) or X = DE. For the sake of simplicity DE will be assumed to follow $p_{DE} = -\rho_{DE}$ as state equation (i.e. it is just a cosmological constant) where $\rho_{DE} \equiv \Lambda/\kappa$. Thus in our notation $(T_{DE})^\mu_\nu = -\rho_{DE}\delta^\mu_\nu$. Assuming that matter (including dark matter) can be described as a perfect fluid, the corresponding energy-momentum tensor is $(T_M)^\mu_\nu = -\mathrm{diag}(\rho_M, -p_M, -p_M, -p_M)$.

In the metric formalism field tensorial equations are found by performing variations of the above action (1) with respect to the metric. Thus the equations are:

$$(1 + f'(R))R_{\mu\nu} - \frac{1}{2}(R + f(R))g_{\mu\nu} + \mathcal{D}_{\mu\nu}f'(R) = \kappa T_{\mu\nu} \tag{2}$$

where $'$ represents the derivative with respect to $R$, $\mathcal{D}_{\mu\nu} \equiv D_\mu D_\nu - g_{\mu\nu}\Box$, $\Box \equiv D_\alpha D^\alpha$ and $D$ is the usual covariant derivative. By computing covariant derivative of (2) we find the equations of motion $D_\mu T^\mu_\nu = 0$ independently from $f(R)$ and $\Lambda$. In the following we will be interested in the cosmological solutions of the above equations with flat spatial sections. Thus we will consider the line element

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2 d\Omega_2^2) \tag{3}$$

From this metric the matter equations of motion in the general case read:

$$\dot{\rho}_M + 3(1 + \omega_M)\rho_M \tfrac{\dot{a}}{a} = 0, \; \partial_k \rho_M = 0 \tag{4}$$

where $k$ runs through $r$, $\theta$ and $\phi$, the dot represents the time derivative and we have assumed the matter state equation to be $p_M = \omega_M \rho_M$. Equations in (4) can be integrated to give:

$$\rho_M(t) = \rho_M(t_0)\left(\frac{a(t_0)}{a(t)}\right)^{3(1+\omega_M)} \tag{5}$$

where $t_0$ is the present time. In a flat universe it is possible to write the scalar curvature in terms of scale parameter $a \equiv a(t)$ and its derivatives as:

$$R = 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] \tag{6}$$

The most recent cosmological data quoted in the introduction are compatible with a cosmological model based on a flat FRW metric like (3) together with standard Einstein field equations with cosmological constant $\Lambda \neq 0$ and nonrelativistic (dust) matter (including dark matter), i.e. $p_M = 0$. In this case we can use the Eq. (2) with $f(R) \equiv 0$ and $T^\mu_\nu = -\mathrm{diag}(\rho_M + \rho_{DE}, -\rho_{DE}, -\rho_{DE}, -\rho_{DE})$. Thus the field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa(T_M)_{\mu\nu} \tag{7}$$

and the matter equation of motion $D_\mu(T_M)^\mu_\nu = 0$.

The $\mu, \nu = t$ component of (7) can be written as follows

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{\kappa}{3}\rho_{M0} + \frac{\Lambda}{3} \tag{8}$$

where the 0 subindex means that we are using standard EH cosmology equations with cosmological constant. This notation will be relevant later on when we will compare standard cosmology with the results coming from the $f(R)$ action for gravity that we have found in this work. The above equation can be solved exactly to find:

$$a_0(t) = \left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left[\frac{3}{2}H_0(t_0)(t - t_0)\sqrt{\Omega_\Lambda} + \mathrm{arcsinh}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_M}}\right)\right] \tag{9}$$

where we have used the notation: $H_0(t) \equiv \dot{a}_0(t)/a_0(t)$, $\Omega_\Lambda \equiv \Lambda/3H_0^2(t_0)$ and $\Omega_M \equiv \kappa\rho_{M0}(t_0)/3H_0^2(t_0)$ with the condition $a_0(t_0) = 1$. On the other hand, by taking the trace of (2) in this case, ie. $f(R) \equiv 0$ and dust, we find:

$$R_0(t) - 4\Lambda = \kappa\rho_{M0}(t) \tag{10}$$

Now we consider again Eq. (2) but in the case where we have an arbitrary function $f(R)$ in the action and dust matter but not DE contributing to total energy-momentum tensor, so that $T^\mu_\nu = -\mathrm{diag}(\rho_M, 0, 0, 0)$. Thus in this case the $\mu, \nu = t$ component of (2), becomes

$$3(1 + f'(R))\frac{\ddot{a}}{a} - \frac{1}{2}(R + f(R)) - 3\frac{\dot{a}}{a}\dot{R}f''(R) = -\kappa\rho_M \tag{11}$$

where we have used that for flat FRW universes $R^t_t = 3\ddot{a}/a$ and $\mathcal{D}^t_t f'(R) = -3\dot{a}\dot{R}f''(R)/a$, In (11) we have eliminated the subindex 0 in the different quantities to avoid any confusion with the previous case. At this stage it is clear that Eq. (11) solutions will depend on $f(R)$ and lead to different evolutions of the universe for the same initial conditions. However, our approach to the problem will be to find a function $f(R)$ so that the solution $a(t)$ of (11) will be exactly the same as the solution in (9) that we got by using standard cosmology and which fits the present cosmological data. In other words we want to find a $f(R)$ such that $a(t) = a_0(t)$ for the same initial (or better present, i.e. $t = t_0$, conditions). If it were possible to find this function $f(R)$ then it would be possible to avoid the necessity for introducing any cosmological constant just by modifying the gravitational sector of the action. In the following we will show that such a function happens to exist and we will give its precise form. In order to do that we first notice that having $a(t) \equiv a_0(t)$ in this period of universe life clearly implies $R(t) \equiv R_0(t)$ and then we can substitute $R$ by $R_0$ in

(11). On the other hand we will write the matter density as the former matter density plus a new contribution, ie. $\rho_M(t) = \rho_{M0}(t) + \Delta\rho(t)$.

Assuming that matter for arbitrary $f(R)$ is still nonrelativistic (i.e. dust) in this cosmological era we have

$$\Delta\rho(t) = \Delta\rho(t_0)\left(\frac{a_0(t_0)}{a_0(t)}\right)^3 \tag{12}$$

where according to (5) particularized for $a = a_0$, $\omega_M = 0$ and (10) we can write (12) as follows

$$\Delta\rho(t) = -\eta\frac{R_0 - 4\Lambda}{\kappa} \tag{13}$$

where we have introduced the parameter $\eta \equiv -\Delta\rho(t_0)/\rho_{M0}(t_0)$ so that matter density is written as $\rho_M(t;\eta) = (1-\eta)\rho_{M0}(t)$. Finally the last term on the l.h.s. of (11) can be written in terms of the scalar curvature by differentiating (10) and using (4). Thus we get

$$3(R_0 - 3\Lambda)(R_0 - 4\Lambda)f''(R_0) + \left(-\frac{1}{2}R_0 + 3\Lambda\right)f'(R_0) - \frac{1}{2}f(R_0) - \Lambda - \eta(R_0 - 4\Lambda) = 0 \tag{16}$$

This last equation can be considered as a differential equation for the function $f(R)$ (in the following we will omit the subindex 0 in $R$ since no confusion is possible). (16) is a second order linear equation so two initial conditions are needed to solve it: $f(0)$ and $f'(0)$ for instance. The natural choice for these initial conditions will be the following: Firstly we do not want to have any cosmological constant in our action, so that $f(0) = 0$. Secondly we want to recover the standard EH action for low scalar curvatures without redefine the Newton constant, i.e. $f'(0) = 0$. Moreover we want $f(R)$ to be an analytical function at the origin so that $R = 0$ should be a solution for the field equations in vacuum. This is an extremely important requirement since it allows Minkowski and Schwarzschild to be vacuum solutions.

With these initial conditions (16) can be solved by using standard methods. A particular solution is:

$$f_p(R) = -\eta R + 2\Lambda(\eta - 1) \tag{17}$$

The homogeneous equation associated with (16) is a Gauss equation solved in terms of hypergeometric functions $2F_1$. The general solution of the homogeneous equation can be written as:

$$f_h(R) = \Lambda(K_+ f_+(R) + K_- f_-(R)) \tag{18}$$

where

$$f_\pm(R) = \alpha_2^{-a_\pm}F_1(a_\pm, 1 + a_\pm - c; 1 + a_\pm - a_\mp; -\alpha^{-1}) \tag{19}$$

with $\alpha = 3 - R/\Lambda$, $a_\pm = -(7 \pm \sqrt{73})/12$ and $c = -1/2$. The $\eta$ dependent constants $K_\pm$ must be determined from the initial conditions given above. Numerically we find: $K_+ = 0.6436(-0.9058\eta + 0.0596)$ and $K_- = 0.6436(-0.2423\eta + 3.4465)$.

The hypergeometric functions given in (19) are generally defined in the whole complex plane. However we want to have a real gravitational action. In principle this is very easy to achieve since the coefficients in Eq. (16) and the constants $K_+$ and $K_-$ are all of them real. Then it is obvious that the real part of (19) is a proper solution of homogeneous equation associated with (16). Thus the function we are seeking can be written as:

$$f(R) \equiv f_p(R) + \text{Re}[f_h(R)] \tag{21}$$

Nevertheless situation is a bit more complicated. The homogeneous equation has three regular singular points at $R_1 = 3\Lambda$, $R_2 = 4\Lambda$ and $R_3 = \infty$. This results in the solution $f_h(R)$ having two branch points $R_1$ and $R_2$. More concretely there are two cuts along the real axis: one from minus infinity to $R_1$ and another from $R_2$ to infinity. Thus one must be quite careful when interpreting (21). From minus infinity to $R_1$ there is only one Riemann sheet of $f_h(R)$ where $f(0)$ and $f'(0)$ vanish and therefore this is the one that we have to use to define $f(R)$. From $R_1$ to $R_2$ the real part of $f_h(R)$ is well defined. Finally from $R_2$ to infinity there is only one Riemann sheet producing a smooth behavior of $f(R)$. To reach this sheet one must understand $R$ in the above equation as $R + i\epsilon$.

At the present moment we do not know if this analytical structure has any fundamental meaning or it is just an artefact of our construction. Much more important is the fact that the function $R + f_p(R) + f_h(R)$, which is the analytical extension of our Lagrangian, is analytical at $R = 0$, having at this point the local behavior $R + \mathcal{O}(R^2)$. Therefore our generalized gravitational Lagrangian $R + f(R)$ does guarantee that $R = 0$ is a vacuum solution while it reproduces the current evolution of the universe without any cosmological constant. Once the $f(R)$ function has been obtained it is possible to check out our result by solving (11) in terms of $a(t)$ for the $f(R)$ given in (21). The numerical solution $a(t)$ shows a nice agreement with $a_0(t)$ given in (9).Thus we can be sure that our gravitational action proportional to $R + f(R)$ provides the same cosmic evolution as EH action $R - 2\Lambda$ in a dust matter universe. Therefore, our model will verify, in the same range of precision, all the experimental tests that the standard cosmological model verifies in the present era.

Notice also that in principle this can achieved for any value of $\eta$, i.e. for any desired amount of matter. Nevertheless some restrictions should be imposed over the parameter $\eta$. For instance it is obvious that in a dust matter dominated universe $\rho_M(t;\eta) \geq 0$ implies $\eta \leq 1$.

Much more stringent bounds can be set on $\eta$ by demanding our model to work properly back in time up to Big Bang Nucleosynthesis (BBN) era. Observations indicate that the cosmological standard model fits correctly primordial light elements abundances during BBN with a 10% of relative error for $H_0(t)$. Therefore by the time of BBN, departure of our model from the standard cosmology must not be too large and (11) should give similar behavior to the one given by the standard Friedmann Eq. (8) where now $\rho_{M0}(t) = \rho_0^{\rm dust}(t) + \rho_0^{\rm radiation}(t)$.

At BBN era cosmological constant is negligible compared with dust and radiation densities. The scalar curvature is of order $10^{-39}$ eV$^2$ (with $\hbar = c = 1$ for these calculations) and by that time dust and radiation densities are of the order of $10^{16}$ eV$^4$ and $10^{21}$ eV$^4$ respectively. Since $R \simeq R_0$ we can rewrite (11) as a modified Friedmann equation as follows

$$H^2(t) = H_0^2(t)\left\{\frac{10^5 R - \eta R + \frac{1}{2}(Rf'(R) - f(R))}{10^5 R[1 + f'(R) - 3f''(R)(1-\eta)R]}\right\} \tag{22}$$

As it was commented above, it is required that $H^2(t) = H_0^2(t)(1 \pm 0.2)$ for curvatures of order $R_{\rm BBN}$. This implies that the second factor on the r.h.s. of(22) should be between 0.8 and 1.2 by that period. Thus in order to match our $f(R)$ gravity model with the standard cosmology at the BBN times we need to tune $\eta$ to a value about 0.065. Therefore the matter content of our model is not too different from the one in the standard cosmology and the difference is in fact smaller than experimental precision in [2].

To conclude we have succeeded in finding a $f(R)$ gravity which exactly reproduces the same evolution of the universe, from BNN up to the present time, as standard cosmology, but without the introduction of any form of DE or cosmological constant. In particular our model reproduces - by construction - a DE dominated universe at low curvatures like other models previously considered do by using rather different $f(R)$ functions (see for instance [10]). The gravitational lagrangian $R + f(R)$ is analytical at the origin and consequently $R = 0$ is a vacuum solution of the field equations. Therefore Minkowski, Schwarzschild and other important $R = 0$ GR solutions with no DE are also solutions for this $f(R)$ gravity. The price we have to pay for all those good properties is that our lagrangian, considered as a function of $R$, has a very complicated analytical structure with cuts along the real axis from infinity to $R = 3\Lambda$ and from $R = 4\Lambda$ to infinity. Obviously the only reasonable interpretation of our action is as some kind of effective action. For example several types of nontrivial $f(R)$ gravities have been derived from compactifications of 4 + n dimensional gravity in the context of String/M-theory [11]. In classical physics one typically starts from some action principle, obtains the corresponding field equations and finally solves them for some initial or boundary conditions. In this work we have proceeded in the opposite way: we started from solutions obtained in the standard cosmological model and then we have searched for an action that, possessing certain properties, gives rise to field equations having the same solutions. Classical actions are of course real but effective quantum actions usually have a complex structure coming from loops and related to unitarity. The presence of an imaginary part in the action, evaluated on some classical configuration, indicates quantum lost of stability by particle emission of this configuration [12]. Therefore it is tempting to think that our action could have some interpretation in terms of an effective quantum action. However, our action determination procedure does not allow to make such a kind of statement.

The complicated structure of this action may be an indication that the cosmological constant problem is even much harder to solve than we have previously thought. Obviously much more insight and research are needed in order to get further progress in this issue.

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