Axial anomaly of QED in a strong magnetic field and noncommutative anomaly

N. Sadooghi[1,2,*] and A. Jafari Salim[1,†]

[1] *Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran-Iran*
[2] *Institute for Studies in Theoretical Physics and Mathematics (IPM), School of Physics, P.O. Box 19395-5531, Tehran-Iran*
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The Adler-Bell-Jackiw (ABJ) anomaly of a 3 + 1 dimensional QED is calculated in the presence of a strong magnetic field. It is shown that in the regime with the lowest Landau level (LLL) dominance a dimensional reduction from $D = 4$ to $D = 2$ dimensions occurs in the longitudinal sector of the low energy effective field theory. In the chiral limit, the resulting anomaly is therefore comparable with the axial anomaly of a two-dimensional massless Schwinger model. It is further shown that the $U_A(1)$ anomaly of QED in a strong magnetic field is closely related to the *nonplanar* axial anomaly of a conventional noncommutative $U(1)$ gauge theory.

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I. INTRODUCTION

A. Motivation

In the early days of current algebra, before the development of QCD, it was realized both in model field theories such as linear sigma model of the baryons and in QED that the flavor-singlet axial current's conservation is broken by quantum fluctuations, the famous Adler-Bell-Jackiw (ABJ) triangle anomaly [1], $\langle\partial_\mu j_5^\mu\rangle = -\frac{g^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}$ (for recent reviews of $U(1)$ axial anomaly, see [2]). Later on, it was shown that in QCD the axial-flavor symmetry, although it is a perfect classical symmetry of massless quarks, is broken by quantum effects. This symmetry and its corresponding anomaly are sensitive to the strong coupling, topological excitations in QCD, and also play an important role in the properties of the theory's vacuum. The far reaching consequences of the discovery of the quantum anomalies, in general, include quantitative predictions of physical amplitudes from anomaly in global symmetries such as in two photons decay of pions, and restriction of consistent gauge theory models of particle physics from cancellation of anomalies in local symmetries such as in electroweak theory. In studying various field theories, it is therefore important to calculate the anomalies of their various global and local symmetries.

In this paper, we will derive the axial anomaly of 3 + 1 dimensional QED in the presence of a strong magnetic field, where an approximation to the regime of lowest Landau level (LLL) dominance is justified. Our result includes two main observations, which will be worked out in this paper:

The first observation is that our resulting $U_A(1)$ anomaly in the presence of a strong magnetic background field is comparable, as expected, with the axial anomaly of an ordinary 1 + 1 dimensional massless Schwinger model [3]. This is indeed related to the dimensional reduction $D \to D - 2$ in the dynamics of fermion pairing in a magnetic field, which causes a generation of a fermion dynamical mass even at the weakest attractive interaction between fermion in the regime of LLL dominance [4].

The second, and probably the more interesting observation is the connection of the $U_A(1)$ anomaly of QED in the LLL regime with the axial anomaly of a conventional noncommutative $U(1)$ gauge theory (see [5] for a review of noncommutative field theory (NCFT), and [6] for a recent review of noncommutative anomalies). The connection between the dynamics in relativistic field theories in a strong homogeneous magnetic field and that in NCFT has been recently studied in [7]. In particular, it is shown that the effective action of QED in the LLL approximation is closely connected to the dynamics of a *modified* noncommutative QED, in which the UV/IR mixing [8] is absent—a phenomenon which is also observed in the Nambu-Jona-Lasinio (NJL) model [7] and in the scalar $O(N)$ model [9] in the presence of a strong magnetic field. The UV/IR mixing in the ordinary noncommutative field theory manifests itself in the singularity of field theory amplitudes in two limits of small noncommutativity parameter Θ and large UV cutoff M of the theory. As it is argued in [7], the reason for the absence of UV/IR mixing in the modified noncommutative field theories is the appearance of a dynamical form factor $\sim\exp(-\mathbf{q}_\perp^2/4|eB|)$ for the photon fields in the regime of LLL dominance.

Because of this connection between the modified noncommutative field theory and the ordinary one, it is important to calculate the Adler-Bell-Jackiw (ABJ) anomaly of modified noncommutative QED and to compare it with the anomaly of an ordinary noncommutative $U(1)$ gauge theory. As we will show later, the $U_A(1)$ anomaly of QED in the presence of a strong magnetic field is in particular comparable with the *nonplanar* anomaly of a conventional noncommutative QED (see the second part of this section for a review of anomalies in noncommutative QED and more detailed comparison).

The organization of this paper is as follows: As next, we will summarize our results by presenting some necessary

*Electronic address: sadooghi@sharif.edu
†Electronic address: jafari_amir@mehr.sharif.edu

technical details on the relation between the $U_A(1)$ anomaly of a 3 + 1 dimensional QED in the presence of a strong magnetic field and the anomaly of a 1 + 1 dimensional massless Schwinger model on the one hand and the nonplanar anomaly of a conventional noncommutative QED on the other hand. Then in Sec. II, after giving a brief review on the effective action of QED in a strong magnetic field, we will derive the anomaly of QED in the LLL approximation and eventually compare it with the nonplanar anomaly of the ordinary noncommutative $U(1)$ gauge theory. In Sec. III, we will calculate the two-point vertex function of the photon and determine the spectrum of a 3 + 1 dimensional QED in the regime of LLL dominance. Sec. IV is devoted to conclusions.

B. Technical details

1. QED in a strong magnetic field and Massless Schwinger model

The well established magnetic catalysis of dynamical chiral symmetry breaking is a universal phenomenon in 3 + 1 dimensional QED in a strong constant magnetic field and leads to a dimensional reduction $D \to D-2$ in a magnetic field. This is why the $U_A(1)$ anomaly of 3 + 1 dimensional QED is comparable with the axial anomaly of a 1 + 1 dimensional massless Schwinger model

$$\langle \partial_\mu j_5^\mu(x) \rangle = \frac{g}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}(x). \tag{1.1}$$

Recognizing this as the two-dimensional version of the triangle anomaly, the divergence of the axial vector current j_5^μ is linear rather than quadratic in field strength tensor $F^{\mu\nu}$. In the ordinary 1 + 1 dimensional QED, it is easy to derive (1.1) from the vacuum polarization tensor $\Pi_{\mu\nu}(q)$. It is enough to use the relation between the vector current j^μ and the axial vector current j_5^μ using the properties of Dirac γ-matrices in two-dimensions, $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$, to get

$$\langle j_5^\mu(q) \rangle = -\epsilon^{\mu\nu}\langle j_\nu(q)\rangle = \epsilon^{\mu\nu}\frac{g}{\pi}\left(A_\nu(q) - \frac{q^\nu q^\rho}{q^2}A_\rho(q)\right). \tag{1.2}$$

Here, $j_\nu(q)$ is defined by the vacuum polarization tensor $\Pi_{\mu\nu}(q)$

$$\langle j_\nu(q)\rangle = \int dx e^{iqx}\langle j_\nu(x)\rangle = -\frac{1}{g}\Pi_{\nu\rho}(q)A^\rho(q). \tag{1.3}$$

Using the methods familiar from four dimensions, the one-loop contribution of $\Pi_{\mu\nu}(q)$ in two dimensions can be computed and reads

$$\Pi_{\mu\nu}(q) = (g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2), \qquad \text{with} \quad \Pi(q^2) = \frac{g^2}{\pi}\frac{1}{q^2}. \tag{1.4}$$

On substituting back (1.4) into (1.3), the expression on the second line of (1.2) is then found. Multiplying this expression with q_μ yields the desired axial anomaly in two-dimensional momentum space

$$q^\mu\langle j_5^\mu(q)\rangle = \frac{g}{\pi}\epsilon^{\mu\nu}q_\mu A_\nu(q). \tag{1.5}$$

Transforming back into the coordinate space, the anomaly of an ordinary massless Schwinger model is given by (1.1). In Sec. II, we will use this method to determine the anomaly of QED in the presence of a strong magnetic field in the momentum space. Assuming that the constant magnetic field is directed in x_3 direction, we find

$$q_\mu\langle \mathcal{J}_5^\mu(q)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{2\pi^2} \times e^{-(\mathbf{q}_\perp^2/2|eB|)}\epsilon^{12ab}q_aA_b(\mathbf{q}_\parallel, \mathbf{q}_\perp), \quad a,b = 0,3,$$

where the symbols $\perp$ and $\parallel$ are related to the (1,2) and (0,3) components, respectively. To determine the anomaly in the coordinate space, we will compactify two transverse coordinates $\mathbf{x}_\perp$ around a circle with radius R to study, in particular, the role played by $\mathbf{q}_\perp = \mathbf{0}$. Taking the decompactification limit $R \to \infty$, it turns out that the zero transverse mode does not contribute to the unintegrated form of the anomaly

$$\langle \partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{2\pi^2} \times e^{(\nabla_\perp^2/2|eB|)}\epsilon^{12ab}\bar{F}_{ab}(\mathbf{x}_\parallel, \mathbf{x}_\perp), \tag{1.6}$$

where $\bar{F}_{ab} = \partial_a\bar{A}_b - \partial_a\bar{A}_b$, and the nonzero transverse modes are defined by $\bar{A}_a = A_a - A_a^{(0)}$. Here, the zero mode of the gauge field $A_a^{(0)}$ is constant along the transverse directions $\mathbf{x}_\perp$ and is defined by

$$A_a^{(0)}(\mathbf{x}_\parallel, \mathbf{q}_\perp = 0) \equiv \int_{-R}^{+R} d^2x_\perp A_a^{(0)}(\mathbf{x}_\parallel, \mathbf{x}_\perp).$$

In Sec. III, we will calculate the 1PI effective action for two photons in the LLL approximation, and determine the vacuum polarization tensor of 3 + 1 dimensional QED in a strong magnetic field. In particular, we will show that the spectrum of this theory consists of a massive photon of mass $M_\gamma^2 \sim e^2|eB|$. This is again in analogy to what happens in the ordinary Schwinger model whose free neutral boson picks up a mass $m_\gamma = \frac{g}{\sqrt{\pi}}$.[1] Indeed, the emergence of a finite mass arising from the 1PI effective action of two photons in LLL approximation confirms the previous results from [4,10]. There, the photon mass of QED in a magnetic field was calculated from the photon propagator $\mathcal{D}_{\mu\nu}$ of QED in one-loop approximation with fermions from the LLL

[1] According to [3], the photon mass m_γ in the Schwinger model is one-loop exact.

$$i\mathcal{D}_{\mu\nu}(q) = \frac{g^{\perp}_{\mu\nu}}{q^2} + \frac{q^{\|}_{\mu}q^{\|}_{\nu}}{q^2\mathbf{q}^2_{\|}} + \frac{(g^{\|}_{\mu\nu} - q^{\|}_{\mu}q^{\|}_{\nu}/\mathbf{q}^2_{\|})}{q^2 + \mathbf{q}^2_{\|}\Pi(\mathbf{q}^2_{\perp}, \mathbf{q}^2_{\|})} - \xi\frac{q_{\mu}q_{\nu}}{(q^2)^2}, \tag{1.7}$$

with ξ an arbitrary gauge parameter. Here, $\Pi(\mathbf{q}^2_{\perp}, \mathbf{q}^2_{\|})$ is given by $\Pi(\mathbf{q}^2_{\perp}, \mathbf{q}^2_{\|}) = e^{-(\mathbf{q}^2_{\perp}/2|eB|)}\Pi(\mathbf{q}^2_{\|})$, and $\Pi(\mathbf{q}^2_{\|})$ is calculated in [4,10] explicitly. As it turns out, since the LLL fermions couple only to the longitudinal (0,3) components of the photon fields, no polarization effects are present in the transverse (1,2) component of $\mathcal{D}_{\mu\nu}(q)$. Therefore as the full propagator, one can take the Feynman-like noncovariant propagator

$$\mathcal{D}_{\mu\nu}(q) = i\frac{g^{\|}_{\mu\nu}}{q^2 + \mathbf{q}^2_{\|}\Pi(\mathbf{q}^2_{\|}, \mathbf{q}^2_{\perp})}.$$

It was shown in [4] that the kinematic region mostly responsible for generating the fermion mass is that with the dynamical mass m_{dyn} satisfying $m^2_{\text{dyn}} \ll |\mathbf{q}^2_{\|}| \ll |eB|$ and $|\mathbf{q}^2_{\perp}| \ll |eB|$. In that region, which is indeed the relevant regimes for the LLL approximation, the fermions can be treated as massless [11] and the polarization operator can be calculated in one-loop approximation. Here, one uses the asymptotic behavior of $\Pi(\mathbf{q}^2_{\|})$ [4], i.e.,

$$\Pi(\mathbf{q}^2_{\|}) \simeq +\frac{N_f\alpha_b|eB|}{3\pi m^2_{\text{dyn}}} \quad \text{for } m^2_{\text{dyn}} \gg |\mathbf{q}^2_{\|}|, \tag{1.8}$$

$$\Pi(\mathbf{q}^2_{\|}) \simeq -\frac{2N_f\alpha_b|eB|}{\pi\mathbf{q}^2_{\|}} \quad \text{for } m^2_{\text{dyn}} \ll |\mathbf{q}^2_{\|}|, \tag{1.9}$$

with N_f the number of flavors and $\alpha_b \equiv \frac{e_b^2}{4\pi}$ the running coupling. Hence (1.9) implies that

$$\frac{1}{q^2 + \mathbf{q}^2_{\|}\Pi(\mathbf{q}^2_{\|}, \mathbf{q}^2_{\perp})} \simeq \frac{1}{q^2 - M^2_{\gamma}}, \quad \text{with} \quad M^2_{\gamma} = \frac{2N_f\alpha_b|eB|}{\pi}.$$

The appearance of a finite photon mass is indeed a reminiscent of the Higgs effect in 1 + 1 dimensional Schwinger model [3]. Note that although the IR dynamics of QED in the presence of a magnetic field is very different from that in the Schwinger model, the tensor and spinor structure of this dynamics is exactly the same as in the Schwinger model [4].

2. Anomalies of noncommutative QED and $U_A(1)$ anomaly in a strong magnetic field

The main observation in this paper is related to the connection of the $U_A(1)$ anomaly of QED in a strong magnetic field and the *nonplanar* anomaly of the ordinary noncomutative $U(1)$ gauge theory. At this stage, before describing the similarities between these two anomalies, it will be instructive to summarize some of the previous results of the anomalies in noncommutative QED [6,12–14]:[2]

As is well-known, noncommutative field theory (NCFT) is characterized by a $\star$-deformation of the ordinary commutative field theory. The noncommutative Moyal $\star$-product is defined by

$$(f \star g)(x) \equiv f(x+\xi)\exp\left(\frac{i\Theta^{\mu\nu}}{2}\frac{\partial}{\partial\xi^{\mu}}\frac{\partial}{\partial\zeta^{\nu}}\right)g(x+\zeta)|_{\xi=\zeta=0}. \tag{1.10}$$

Because of the specific structure of the $\star$-product, noncommutative field theories can be regarded as nonlocal field theories involving higher order derivatives between the interacting fields. Perturbatively, the theory consists therefore of planar and nonplanar diagrams. The latter are usually the source of the appearance of the above mentioned UV/IR mixing phenomenon, which is shown to modify the anomalies in noncommutative field theory too [13,14]. The main observation in [12,13] was that the noncommutative $U(1)$ gauge theory consists of two different *global* axial vector currents; the covariant $J^{\mu}_5 = \psi_{\beta} \star \bar{\psi}_{\alpha}(\gamma^{\mu}\gamma^5)^{\alpha\beta}$, and the invariant axial vector current $j^{\mu}_5 = \bar{\psi}_{\alpha} \star \psi_{\beta}(\gamma^{\mu}\gamma^5)^{\alpha\beta}$. Naively, one would expect that these two currents have the same anomaly. But, as it is shown in [6], due to the properties of $\star$-product, only the *integrated* form of two anomalies are the same

$$\int d^2x_{\perp}\langle D_{\mu}J^{\mu}_5(x)\rangle = \int d^2x_{\perp}\langle\partial_{\mu}j^{\mu}_5(x)\rangle, \tag{1.11}$$

with $\mathbf{x}_{\perp}$ denoting the noncommutative directions. Their *unintegrated* forms are indeed different; while the anomaly corresponding to the covariant current, arising from the planar diagrams of the theory, is the expected $\star$-deformation of the ABJ anomaly [12]

$$\langle D_{\mu}J^{\mu}_5(x)\rangle = -\frac{e^2}{16\pi^2}F_{\mu\nu}(x) \star \tilde{F}^{\mu\nu}(x), \tag{1.12}$$

the anomaly in $\langle\partial_{\mu}j^{\mu}_5\rangle$, receives contribution from nonplanar diagrams and is therefore affected by the noncommutative UV/IR mixing. The nonplanar (invariant) anomaly of $\langle\partial_{\mu}j^{\mu}_5\rangle$ is calculated in [13,14] using various regularization methods. Quoting, in particular, the result from [14], where the nonplanar anomaly is calculated using the well-known Fujikawa's path integral method, the divergence of the invariant axial vector current is

[2]Other aspects of the anomalies of NCFT have been studied in [15]. For a more complete list of references see also [6].

$$\langle\partial_\mu j_5^\mu(x)\rangle = \lim_{M\to\infty} -\frac{e^2}{16\pi^2}\int\frac{d^4k}{(2\pi)^4}\times\int\frac{d^4p}{(2\pi)^4}e^{-(M^2\tilde{q}^2/4)}e^{-ikx}F_{\mu\nu}(k)\times\frac{\sin(k\times p)}{k\times p}\tilde{F}^{\mu\nu}(p)e^{-ipx}+\cdots, \quad (1.13)$$

where M is the UV regulator. Here, we have used the notations $q\equiv k+p$, $\tilde{q}^\mu\equiv\Theta^{\mu\nu}q_\nu$ and $k\times p\equiv k_\mu\tilde{p}^\mu/2$. To show the celebrated UV/IR mixing in the case of nonplanar anomaly (1.13), we have considered two limits $\tilde{q}^2\gg\frac{1}{M^2}$ and $\tilde{q}^2\ll\frac{1}{M^2}$, separately. As it turns out the limit $\tilde{q}^2\gg\frac{1}{M^2}$ is equivalent with taking first the limit $M\to\infty$ and then $|\tilde{q}|\to 0$. In this case, even before taking $|\tilde{q}|\to 0$, the anomaly vanishes. In the opposite case, i.e. by taking first $|\tilde{q}|\to 0$ and then $M\to\infty$, a finite anomaly arises. Note that this limit can be understood as the limit $\tilde{q}^2\ll\frac{1}{M^2}$. In this case the exponent $\exp(-\frac{M^2\tilde{q}^2}{4})\to 1$ and we are left with a finite nonplanar anomaly,

$$\langle\partial_\mu j_5^\mu\rangle = -\frac{e^2}{16\pi^2}F_{\mu\nu}\star'\tilde{F}^{\mu\nu}+\cdots, \quad (1.14)$$

where the generalized $\star'$-product is defined by

$$(f\star' g)(x)\equiv f(x+\xi)\frac{\sin(\frac{\Theta_{\mu\nu}}{2}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\zeta})}{\frac{\Theta_{\mu\nu}}{2}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\zeta}}g(x+\zeta)|_{\xi=\zeta=0}. \quad (1.15)$$

The ellipsis in (1.13) and (1.14) indicate the contribution of the expansion of a noncommutative Wilson line, which is to be attached to $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ in order to restore the $\star$-gauge invariance of the result [14].

In [6] we argue that the above results remain only correct when we assume that the noncommutative $U(1)$ gauge theory is an effective field theory which consists of a natural, large but finite cutoff M. However, considering the noncommutative field theory as a fundamental field theory and taking the limit of $M\to\infty$ first, the nonplanar anomaly vanishes except for the one point in the momentum space, $|\tilde{q}|=0$; As it can easily be checked in (1.13), in this case, the phase factor $\exp(-M^2\tilde{q}^2/4)=1$, and this leads to a finite nonplanar anomaly. This is in accordance with the arguments in [16], where it is emphasized that a nonvanishing nonplanar anomaly is indeed necessary to guarantee that the covariant and invariant currents have the same *integrated* axial anomaly [see the argument leading to (1.11)].

To compute the nonplanar anomaly in such a fundamental theory, without any natural cutoff, it is necessary to perform in addition to the familiar UV regularization, an appropriate IR regularization. In [6], the IR regulator is introduced by compactifying each space coordinates to a circle with radius R. Assuming that the noncomutativity is between the spacial coordinates $\mathbf{x}_\perp=(x_1,x_2)$, and denoting the other two directions by $\mathbf{x}_\parallel=(x_0,x_3)$, the unintegrated form of the nonplanar (invariant) anomaly is given by

$$\langle\partial_\mu j_5^\mu(x)\rangle = -\frac{e^2}{16\pi^2}\frac{1}{(2R)^2}\times\int_{-R}^{+R}d^2y_\perp F^{\alpha\beta}(\mathbf{x}_\parallel,\mathbf{y}_\perp)\tilde{F}_{\alpha\beta}(\mathbf{x}_\parallel,\mathbf{y}_\perp). \quad (1.16)$$

Here, taking the decompactification (IR) limit $R\to\infty$ the anomaly "density" vanishes due to $1/R^2$ dependence on the r.h.s. of (1.16). To obtain the desired finite result, we should integrate both sides over the noncommutative directions $\mathbf{x}_\perp$—this removes the R dependence on the r.h.s. of (1.16)—and then, take the limit $R\to\infty$. This situation is as if a finite charge is evenly distributed over the space giving zero density but still being totally nonzero [6]. The integrated form of the nonplanar (invariant) anomaly becomes

$$\int_{-R}^{+R}d^2x_\perp\langle\partial_\mu j_5^\mu(x)\rangle = -\frac{e^2}{16\pi^2}\frac{1}{(2R)^2}\int_{-R}^{+R}d^2x_\perp\int_{-R}^{+R}d^2y_\perp F^{\alpha\beta}(\mathbf{x}_\parallel,\mathbf{y}_\perp)\tilde{F}_{\alpha\beta}(\mathbf{x}_\parallel,\mathbf{y}_\perp), \quad (1.17)$$

which survives the limit $R\to\infty$, i.e.

$$\int_{-\infty}^{+\infty}d^2x_\perp\langle\partial_\mu j_5^\mu(x)\rangle = -\frac{e^2}{16\pi^2}\int_{-\infty}^{+\infty}d^2y_\perp F^{\alpha\beta}(\mathbf{x}_\parallel,\mathbf{y}_\perp)\tilde{F}_{\alpha\beta}(\mathbf{x}_\parallel,\mathbf{y}_\perp). \quad (1.18)$$

Thus, the expression for the nonplanar anomaly turns out to be independent of noncommutative coordinates $\mathbf{x}_\perp$. Although this can be interpreted as a dimensional reduction in the space-time coordinates, but the dimensional reduction seems to be not complete here. This is because the nonplanar anomaly (1.18) depends, as in any ordinary $3+1$ dimensional field theory, quadratically on the field strength tensor. This is in contrast to our result (1.6) on the anomaly of QED in the presence of a strong magnetic field, which depends, as in a two-dimensional theory, linearly on the field strength tensor, at least in the one-loop level. Further comparison shows that while the *unintegrated* form of the nonplanar anomaly (1.16) receives contribution *only* from zero noncommutative mode of the Fourier transformed of $\mathcal{F} \equiv F\tilde{F}$, the *unintegrated* $U_A(1)$ anomaly of QED in the LLL approximation receives additional contribution from nonzero transverse modes. To show this, we have to compactify the transverse coordinates along a circle with radius R. In the decompactification limit $R \to \infty$, the zero transverse mode reappears in the *integrated* axial anomaly of QED in a strong magnetic field, i.e. we have

$$\int_{-\infty}^{+\infty} d^2x_\perp \langle \partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\mathrm{sign(eB)}}{4\pi^2} \times \int_{-\infty}^{+\infty} d^2x_\perp \epsilon^{12ab} F_{ab}(\mathbf{x}_\parallel, \mathbf{x}_\perp), \tag{1.19}$$

with the field strength tensor F_{ab} consisting of the nonzero and zero transverse modes, $F_{ab} = \bar{F}_{ab} + F_{ab}^{(0)}$. The mechanism for the reappearance of the zero mode in the integrated form of the QED anomaly in the LLL approximation is similar to what happens in the noncommutative case [see how (1.18) arises from (1.16)].

II. $U(1)$ AXIAL ANOMALY IN A STRONG MAGNETIC FIELD IN THE LLL APPROXIMATION

In the first part of this section, we will briefly review some results from [11] on the effective action of QED in a strong magnetic field. This will help us to set up our notations. We then use the LLL fermion propagator to determine the $U_A(1)$ anomaly of QED in a strong magnetic field.

To put the QED dynamics in a magnetic field under control, we will consider, as in [11], the case with a large number of fermion flavors N_f. As is well-known the magnetic field is a strong catalyst for dynamical chiral symmetry breaking and even the weakest possible attraction between the fermions is enough for dynamical mass generation. It is shown in [4] that the dynamical mass behaves as $m_{\mathrm{dyn}} \simeq \sqrt{|eB|}\exp(-N_f)$ for a large running coupling $\tilde{\alpha}_b \equiv N_f\alpha_b$. Thus, in the limit of large N_f, the dynamical mass satisfies $m_{\mathrm{dyn}} \ll \sqrt{|eB|}$. This assumption guarantees, in particular, that no dynamical symmetry breaking occurs, and as a consequence no light (pseudo) Nambu-Goldstone bosons are produced. The low energy effective theory will then consist only of photons and is given only in terms of these fields. As for the current fermion mass m, it is chosen to satisfy the condition $m \ll \sqrt{|eB|}$, which implies that the magnetic field is very strong, and this is in fact a guarantee that the LLL approximation is reliable.

The effective action for photons is given by integrating out the fermions and reads

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)}, \qquad \Gamma^{(0)} = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}, \qquad \Gamma^{(1)} = -iN_f \mathrm{Tr}\ln(i\not{D} - m), \tag{2.1}$$

with $D_\mu \equiv \partial_\mu - ieA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the vector field $A_\mu = A_\mu^{\mathrm{cl.}} + \tilde{A}_\mu$, where the classical part $A_\mu^{\mathrm{cl.}} \equiv \langle 0|A_\mu|0\rangle$ and $\tilde{A}_\mu$ is the fluctuating part. To proceed, it is useful to choose the symmetric gauge

$$A_\mu^{\mathrm{cl.}} = \frac{B}{2}(0, x_2, -x_1, 0).$$

This leads to a magnetic field directed in x_3 dimensions. From now on, the longitudinal (0,3) directions are denoted by $\mathbf{x}_\parallel = (x_0, x_3)$, and the transverse directions (1,2) by $\mathbf{x}_\perp = (x_1, x_2)$. Using the Schwinger proper time formalism [17], it is possible to derive the fermion propagator in this gauge. It is given by

$$\mathcal{S}_F(x, y) = \exp\left(\frac{ie}{2}(x-y)^\mu A_\mu^{\mathrm{ext.}}(x+y)\right)S(x-y) = e^{(ieB/2)\epsilon^{ab}x_a y_b}S(x-y), \qquad a, b = 1, 2, \tag{2.2}$$

where the first factor containing the external $A_\mu^{\mathrm{ext.}}$ is the Schwinger line integral [17]. Further, the Fourier transform of the translationally invariant part $S(x-y)$ reads

$$\tilde{S}(k) = i\int_0^\infty ds e^{-ism^2}\exp\left(is\left[k_\parallel^2 - \frac{k_\perp^2}{eBs\cot(eBs)}\right]\right) \times \{(m + \gamma^\parallel\cdot\mathbf{k}_\parallel)(1 - \gamma^1\gamma^2\tan(eBS)) - \gamma^\perp\cdot\mathbf{k}_\perp(1 + \tan^2(eBs))\}, \tag{2.3}$$

where $\mathbf{k}_\parallel = (k_0, k_3)$ and $\gamma_\parallel = (\gamma_0, \gamma_3)$ and $\mathbf{k}_\perp = (k_1, k_2)$ and $\gamma_\perp = (\gamma_1, \gamma_2)$. After performing the integral over s, $\tilde{S}(k)$ can be decomposed as follows

$$\tilde{S}(k) = ie^{-(k_\perp^2/|eB|)}\sum_{n=0}^\infty(-1)^n\frac{D_n(eB, k)}{k_\parallel^2 - m^2 - 2|eB|n}, \tag{2.4}$$

with $D_n(eB, k)$ expressed through the generalized Laguerre polynomials L_m^α

$$D_n(eB, k) = (\gamma^\parallel\cdot\mathbf{k}_\parallel + m)\{2\mathcal{O}[L_n(2\rho) - L_{n-1}(2\rho)] + 4\gamma^\perp\cdot k_\perp L_{n-1}^{(1)}(2\rho)\}. \tag{2.5}$$

Here, we have introduced $\rho \equiv \frac{k_\perp^2}{|eB|}$ and

$$\mathcal{O} \equiv \frac{1}{2}(1 - i\gamma^1\gamma^2\,\mathrm{sign}(eB)). \tag{2.6}$$

The lowest Landau level (LLL) is determined by $n = 0$. Thus, the full fermion propagator (2.2) in the LLL approximation can be decomposed into two independent transverse and longitudinal parts [7]

$$\mathcal{S}_F(x, y) = S_\parallel(\mathbf{x}_\parallel - \mathbf{y}_\parallel)P(\mathbf{x}_\perp, \mathbf{y}_\perp), \tag{2.7a}$$

with the longitudinal part

$$S_\parallel(\mathbf{x}_\parallel - \mathbf{y}_\parallel) = \int\frac{d^2k_\parallel}{(2\pi)^2}e^{i\mathbf{k}_\parallel\cdot(\mathbf{x}-\mathbf{y})^\parallel}\frac{i\mathcal{O}}{\gamma^\parallel\cdot\mathbf{k}_\parallel - m}, \tag{2.7b}$$

and the transverse part

$$P(\mathbf{x}_\perp, \mathbf{y}_\perp) = \frac{|eB|}{2\pi} \exp\left(\frac{ieB}{2}\epsilon^{ab}x^a y^b - \frac{|eB|}{4}(\mathbf{x}_\perp - \mathbf{y}_\perp)^2\right), \quad a, b = 1, 2. \tag{2.7c}$$

Note that the longitudinal part (2.7b) is nothing else but the fermion propagator in two dimensions. In particular, the matrix $\mathcal{O}$, defined in (2.6), is the projector on the fermion (antifermion) states with the spin polarized along (opposite to) the magnetic field [7]. Further, the Schwinger line integral included in the transverse part (2.7c) is responsible for the noncommutative feature of the effective action of QED in the LLL approximation

$$\Gamma_{\text{LLL}} = \Gamma^{(0)} + \Gamma^{(1)}_{\text{LLL}},$$
$$\Gamma^{(1)}_{\text{LLL}} = -\frac{iN_f|eB|}{2\pi}\int d^2x_\perp \,\text{Tr}_\|(\mathcal{O}\ln(i\gamma^\|(\partial_\| - ie\mathcal{A}_\|)) - m)_\star, \tag{2.8}$$

where $\Gamma^{(0)}$ is defined in (2.1). Here, the $\star$ is the Moyal $\star$-product defined in (1.10). The appearance of this product on the r.h.s. of the above equation shows that the effective QED in the LLL dominant regime is indeed an effective noncommutative field theory. In (2.8) the longitudinal *smeared* gauge fields $\mathcal{A}_\| = (\mathcal{A}_0, \mathcal{A}_3)$ is defined as

$$\mathcal{A}_\|(x) \equiv e^{(\nabla_\perp^2/4|eB|)} A_\|(x). \tag{2.9}$$

Note that here, since the one-loop contribution to the effective action includes only the longitudinal $\mathcal{A}_\|$ field, the spin structure of the LLL dynamics is (1 + 1) dimensional [4,11], i.e. the LLL fermions couple only to longitudinal components of the photon field. Note further that the Gaussian-like form factor $e^{\nabla_\perp^2/4|eB|}$ in the definition of the smeared field arises essentially from the Schwinger line integral in the transverse part of the fermion propagator (2.7a)–(2.7c), and is responsible for the noncommutative property of the effective action in the regime of LLL dominance and the cancellation of the UV/IR mixing [8] of the modified noncommutative field theory [7,9,11,18].

Using these results, it is easy to calculate the n-point vertex function of longitudinal photon in the LLL (from now on we will omit the subscription $\|$ in the longitudinal gauge field $\mathcal{A}_\|$)

$$\Gamma^{(n)}_{\text{LLL}} = i\frac{(ie)^n N_f|eB|}{2\pi n}\int d^2x_\perp d^2x_1^\| \cdots d^2x_n^\| \,\text{tr}(S_\|(\mathbf{x}_1^\| - \mathbf{x}_2^\|) \times \hat{\mathcal{A}}(\mathbf{x}^\perp, \mathbf{x}_2^\|)\cdots S_\|(\mathbf{x}_n^\| - \mathbf{x}_1^\|)\hat{\mathcal{A}}(\mathbf{x}^\perp, \mathbf{x}_1^\|))_\star,$$

where $S_\|(\mathbf{x}_\| - \mathbf{y}_\|)$ is defined in (2.7b) and $\hat{\mathcal{A}} \equiv \gamma_\| \cdot \mathcal{A}^\|$. At this stage, we have all the necessary tools to determine the ABJ anomaly in this modified noncommutative $U(1)$ gauge theory.

Let us now consider the axial vector current associated with the $U_A(1)$ symmetry of the *original* QED Lagrangian

$$\mathcal{J}_5^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x). \tag{2.10}$$

To determine $\langle\partial_\mu\mathcal{J}_5^\mu(x)\rangle$ we will calculate first $\langle\mathcal{J}_5^\mu(q)\rangle$ in the momentum space, and then, build $q_\mu\langle\mathcal{J}_5^\mu\rangle$ in analogy to what is performed in (1.5) to determine the anomaly in the ordinary two-dimensional Schwinger model from (1.3). To do so, we will use the LLL fermion propagator (2.7a)–(2.7c). Note that here, in contrast to the ordinary 3 + 1 dimensional QED gauge theory where a triangle diagram of one axial and two vector current was responsible for the emergence of the anomaly, the two-point function of longitudinal photons which gives rise to the anomaly of our modified noncommutative $U(1)$. This is in analogy to the 1 + 1 dimensional QED and can be again regarded as a consequence of the dimensional reduction in the presence of a strong magnetic background field.

In the momentum space the vacuum expectation value of the axial vector current $\mathcal{J}_5^\mu(x)$ is given by

$$\langle\mathcal{J}_5^\mu(q)\rangle = \int d^4x e^{-iqx}\langle\mathcal{J}_5^\mu(x)\rangle. \tag{2.11}$$

In the first order of perturbation theory, we have

$$\langle\mathcal{J}_5^\mu(q)\rangle = -e\int d^4x d^4y e^{-iqx}\langle\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\bar{\psi}(y)\not{A}(y) \times \psi(y)\rangle, \tag{2.12}$$

that, after contacting the fermionic fields, leads to

$$\langle\mathcal{J}_5^\mu(q)\rangle = -e\int d^4x d^4y e^{-iqx}\,\text{tr}(\gamma^\mu\gamma^5 S_\|(\mathbf{x}_\| - \mathbf{y}_\|) \times P(\mathbf{x}_\perp, \mathbf{y}_\perp)\not{A}(y)S_\|(\mathbf{y}_\| - \mathbf{x}_\|)P(\mathbf{y}_\perp, \mathbf{x}_\perp)). \tag{2.13}$$

Substituting $S_\|(\mathbf{x}_\| - \mathbf{y}_\|)$ from (2.7b), we get

$$\langle\mathcal{J}_5^\mu(q)\rangle = -e\int d^4x d^4y\int\frac{d^4p}{(2\pi)^4}A_\nu(p)e^{ipy}\int\frac{d^2k_\|}{(2\pi)^2}\frac{d^2\ell_\|}{(2\pi)^2} \times e^{-iqx}e^{i\mathbf{k}_\|\cdot(\mathbf{x}-\mathbf{y})^\|}e^{i\ell_\|\cdot(\mathbf{y}-\mathbf{x})^\|} \times \text{tr}\left(\gamma^\mu\gamma^5 P(\mathbf{x}_\perp, \mathbf{y}_\perp)\frac{i}{\gamma^\| \cdot \mathbf{k}_\| - m} \times \mathcal{O}\gamma^\nu P(\mathbf{y}_\perp, \mathbf{x}_\perp)\frac{i}{\gamma^\| \cdot \ell_\| - m}\mathcal{O}\right). \tag{2.14}$$

Now using

$$\int d^2x_\perp d^2y_\perp e^{-i\mathbf{q}_\perp\cdot\mathbf{x}^\perp}P(\mathbf{x}_\perp, \mathbf{y}_\perp)e^{i\mathbf{p}_\perp\cdot\mathbf{y}^\perp}P(\mathbf{y}_\perp, \mathbf{x}_\perp) = 2\pi|eB|\delta^2(\mathbf{p}_\perp - \mathbf{q}_\perp)e^{-(\mathbf{q}_\perp^2/2|eB|)}, \tag{2.15}$$

and performing the integration over $\mathbf{x}_\|$ and $\mathbf{y}_\|$ coordinates, we arrive first at

$$\langle \mathcal{J}_5^\mu(q)\rangle = \frac{e|eB|}{2\pi} e^{-(\mathbf{q}_\perp^2/2|eB|)} A_\nu(q) \int \frac{d^2k_\parallel}{(2\pi)^2} \times \frac{\mathrm{tr}(\gamma^\mu\gamma^5(\not{k}_\parallel + m)\mathcal{O}\gamma^\nu(\not{k}_\parallel - \not{q}_\parallel + m)\mathcal{O})}{(\mathbf{k}_\parallel^2 - m^2)((\mathbf{k}_\parallel - \mathbf{q}_\parallel)^2 - m^2)}. \tag{2.16}$$

To calculate this integral, let us first concentrate on the expression in the nominator. Using the property of the matrix $\mathcal{O}$ as the projector in the longitudinal direction, $\mathcal{O}\gamma\mathcal{O} = \mathcal{O}\gamma_\parallel$, we get

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\gamma^5(\not{k}_\parallel + m)\mathcal{O}\gamma^\nu(\not{k}_\parallel - \not{q}_\parallel + m)\mathcal{O})A_\nu(q) &= \mathrm{tr}(\gamma^\mu\gamma^5(\not{k}_\parallel + m)\gamma_\parallel^\nu(\not{k}_\parallel - \not{q}_\parallel + m)\mathcal{O})A_\nu^\parallel(q)\\ &= \frac{1}{2}\,\mathrm{tr}(\gamma^\mu\gamma^5(\not{k}_\parallel + m)\gamma_\parallel^\nu(\not{k}_\parallel - \not{q}_\parallel + m))A_\nu^\parallel(q)\\ &\quad - \frac{i}{2}\,\mathrm{sign}(eB)\,\mathrm{tr}(\gamma^\mu\gamma^5(\not{k}_\parallel + m)\gamma_\parallel^\nu(\not{k}_\parallel - \not{q}_\parallel + m)\gamma^1\gamma^2)A_\nu^\parallel(q).\end{aligned}$$

Here, we have used the definition of $\mathcal{O}$ in (2.6). To calculate the traces of Dirac γ-matrices, we use the relations $\mathrm{tr}(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\alpha\beta\rho\sigma}$, and

$$\mathrm{tr}\,(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\tau) = 4ig_{\eta\xi}(\epsilon^{\alpha\beta\rho\eta}s^{\sigma\lambda\tau\xi} - \epsilon^{\sigma\lambda\tau\eta}s^{\alpha\beta\rho\xi}), \tag{2.17}$$

with $s^{\alpha\beta\rho\sigma} \equiv g^{\alpha\beta}g^{\rho\sigma} - g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}$. After some straightforward calculation, (2.16) can be written as

$$\langle \mathcal{J}_5^\mu(q)\rangle = -\frac{e|eB|\mathrm{sign}(eB)}{\pi} e^{-(\mathbf{q}_\perp^2/2|eB|)} A_b(q) \int \frac{d^2k_\parallel}{(2\pi)^2} \times \frac{\{\epsilon^{12\mu b}(m^2 - \mathbf{k}_\parallel\cdot(\mathbf{k}-\mathbf{q})^\parallel) + (\epsilon^{12\mu c}g^{ab} + g^{bc}\epsilon^{12\mu a})\mathbf{k}_a^\parallel(\mathbf{k}-\mathbf{q})_c^\parallel\}}{(\mathbf{k}_\parallel^2 - m^2)((\mathbf{k}_\parallel - \mathbf{q}_\parallel)^2 - m^2)}, \tag{2.18}$$

where we have omitted the symbol $\parallel$ on the gauge field A_μ and the metric $g_{\mu\nu}$. Instead, we have used the indices a, b, $c = 0, 3$, to denote the projection into the longitudinal directions $\mathbf{x}_\parallel = (x_0, x_3)$. Note that due to the antisymmetry of the $\epsilon^{\alpha\beta\gamma\rho}$ tensor, μ in $\epsilon^{12\mu c}$ on the r.h.s. must be chosen to be $\mu = 0, 3$. This is indeed the first signature for the dimensional reduction from $D = 4$ to $D = 2$ dimensions in the result of the anomaly. The rest of the calculation is a straightforward computation of the two-dimensional Feynman integral over $\mathbf{k}_\parallel$. Introducing first the Feynman parameter $0 \le \alpha \le 1$ and then performing a finite shift of integration $\mathbf{k}_\parallel \to \mathbf{k}_\parallel + \alpha\mathbf{q}_\parallel$, we arrive first at

$$\begin{aligned}\langle \mathcal{J}_5^\mu(q)\rangle = &-\frac{e|eB|\mathrm{sign}(eB)}{\pi} e^{-(\mathbf{q}_\perp^2/2|eB|)}\epsilon^{12\mu a}A^b(q)\int_0^1 d\alpha\left[\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{2\mathbf{k}_a^\parallel\mathbf{k}_b^\parallel}{(\mathbf{k}_\parallel^2-\Delta)^2} - \int\frac{d^2k_\parallel}{(2\pi)^2}\frac{g_{ab}}{(\mathbf{k}_\parallel^2-\Delta)}\right]\\ &-\frac{2e|eB|\mathrm{sign}(eB)}{\pi}e^{-(\mathbf{q}_\perp^2/2|eB|)}\epsilon^{12\mu a}A^b(q)(\mathbf{q}_\parallel^2 g_{ab} - \mathbf{q}_a^\parallel\mathbf{q}_b^\parallel)\int_0^1 d\alpha\alpha(1-\alpha)\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{1}{(\mathbf{k}_\parallel^2-\Delta)^2},\end{aligned} \tag{2.19}$$

where $\Delta \equiv m^2 - \alpha(1-\alpha)\mathbf{q}_\parallel^2$. As for the first two integrals, it can be shown that although they are both infinite in the UV limit $|\mathbf{k}_\parallel| \to \infty$, they cancel each other, and we are left with

$$\langle \mathcal{J}_5^\mu(q)\rangle = -\frac{ie|eB|\mathrm{sign}(eB)}{2\pi^2} e^{-(\mathbf{q}_\perp^2/2|eB|)}\epsilon^{12\mu a}A^b(q) \times (\mathbf{q}_\parallel^2 g_{ab} - \mathbf{q}_a^\parallel\mathbf{q}_b^\parallel)\int_0^1 d\alpha\alpha(1-\alpha)\frac{1}{\Delta}. \tag{2.20}$$

In the chiral limit, $m \to 0$, using the definition of Δ, we get[3]

$$\langle \mathcal{J}_5^\mu(q)\rangle = +\frac{ie|eB|\mathrm{sign}(eB)}{2\pi^2} e^{-(\mathbf{q}_\perp^2/2|eB|)}\epsilon^{12\mu a}A^b(q) \times \left(g_{ab} - \frac{\mathbf{q}_a^\parallel\mathbf{q}_b^\parallel}{\mathbf{q}_\parallel^2}\right). \tag{2.21}$$

The ABJ anomaly of QED with N_f flavors in a strong magnetic field in the LLL approximation is then found by multiplying (2.21) with q_μ and using the antisymmetry property of ϵ^{12ab} tensor

$$q_\mu\langle \mathcal{J}_5^\mu(q)\rangle = +\frac{ieN_f|eB|\mathrm{sign}(eB)}{2\pi^2} \times e^{-(\mathbf{q}_\perp^2/2|eB|)}\epsilon^{12ab}q_a^\parallel A_b(\mathbf{q}_\parallel, \mathbf{q}_\perp). \tag{2.22}$$

Before transforming this result back into the coordinate space, let us compare it with (1.13), the nonplanar anomaly of the invariant current $j_5^\mu \equiv \psi^\beta \star \bar\psi^\alpha(\gamma^\mu\gamma^5)_{\alpha\beta}$ of the

[3]The chiral limit is taken just to isolate the anomaly in the divergence of the axial vector current arising from the quantum effects. According to the arguments in [11], the fermions can be treated as massless in the regime of LLL dominance.

ordinary noncommutative QED. Assuming that the noncommutativity is defined only between two coordinates x_1 and x_2, (1.13) can be written as

$$q_\mu \langle j_5^\mu(q)\rangle = \lim_{M\to\infty} -\frac{ie^2}{16\pi^2} e^{-((M\theta)^2\mathbf{q}_\perp^2)/4} \times \int \frac{d^4p}{(2\pi)^4} F_{\mu\nu}(q-p)\frac{\sin(q\times p)}{q\times p} \times \tilde{F}^{\mu\nu}(p) + \cdots, \tag{2.23}$$

with $q \equiv k + p$. Here, we have transformed (1.13) into the momentum space and replaced the phase factor $e^{-M^2\tilde{q}^2/4}$ with $\tilde{q}_\mu \equiv \Theta_{\mu\nu}q^\nu$ by $e^{-(M\theta)^2\mathbf{q}_\perp^2/4}$, where θ is defined by $\Theta_{ij} \equiv \theta\epsilon_{ij}$, with $i, j = 1, 2$—this gives us the possibility to compare (2.23) with (2.22). As we have argued in Sec. I, the phase factor $e^{-(M\theta)^2\mathbf{q}_\perp^2/4}$ is indeed the origin of the appearance of UV/IR mixing; Assuming that the ordinary noncommutative $U(1)$ gauge theory is a fundamental theory and taking the limit $M \to \infty$, the nonplanar anomaly vanishes everywhere in the momentum space except for the point $\mathbf{q}_\perp = \mathbf{0}$. In this case the phase factor

$$e^{-(M\theta)^2\mathbf{q}_\perp^2/4} = 1, \quad \text{for } \mathbf{q}_\perp = \mathbf{0},$$

and the nonplanar anomaly turns out to be given by (1.16), where a compactification around a circle with the radius R is performed. Only in this way it could be shown in [6] that although the *unintegrated* form of the nonplanar anomaly vanishes in the decompactification limit $R \to \infty$, the *integrated* form of the nonplanar anomaly is finite and is given by (1.18). Note that (1.16) can also be interpreted as if the unintegrated nonplanar anomaly receives a finite contribution *only* from the zero mode of the Fourier transformed of $\mathcal{F} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu}$ in the noncommutative coordinates $\mathbf{x}_\perp$, i.e. from

$$\langle \partial_\mu j_5^\mu(x)\rangle \sim \tilde{\mathcal{F}}(\mathbf{x}_\parallel, \mathbf{q}_\perp = \mathbf{0}) \equiv \frac{1}{(2R)^2}\int_{-R}^{+R} d^2x_\perp \mathcal{F}(\mathbf{x}_\parallel, \mathbf{x}_\perp). \tag{2.24}$$

This contribution vanishes in the $R \to \infty$ limit.

The above analysis of the nonplanar anomaly of the ordinary noncommutative QED shows the special role played by $\mathbf{q}_\perp = \mathbf{0}$ in the expression (2.22) for the anomaly of QED in a strong magnetic field in the LLL approximation. Note that in this case the magnetic field provides a natural cutoff for the modified noncommutative QED in the LLL approximation, and can be compared with the product of the UV cutoff M and the IR cutoff θ of the ordinary noncommutative field theory; $\sqrt{|eB|} \sim (M\theta)^{-1}$, where here, in contrast to the ordinary noncommutative case, *both* M and θ are kept finite, so that $\mathbf{q}_\perp^2 \ll |eB|$ is correct and thus the reliability of the LLL approximation is guaranteed.

To transform (2.22) back into the coordinate space and to be specially careful about the role played by $\mathbf{q}_\perp = \mathbf{0}$, we compactify, as in the case of the ordinary noncommutativity, two transverse coordinates $\mathbf{x}_\perp$ around a circle with radius R. Multiplying both sides of (2.22) with e^{iqx} and integrating (summing) over the continuous (discrete) momenta $q_\parallel$ ($q_\perp$), we have first

$$\langle \partial_\mu \mathcal{J}_5^\mu(x)\rangle = -\frac{eN_f|eB|\text{sign}(eB)}{2\pi^2}\int \frac{d^2q_\parallel}{(2\pi)^2}\frac{1}{(2R)^2}\sum_{\mathbf{q}_\perp} e^{i(\mathbf{q}_\parallel\cdot\mathbf{x}^\parallel + \mathbf{q}_\perp\cdot\mathbf{x}^\perp)} e^{-(\mathbf{q}_\perp^2/2|eB|)}\epsilon^{12ab} q_a A_b(\mathbf{q}_\parallel, \mathbf{q}_\perp), \tag{2.25}$$

where we have used the relation

$$\int \frac{d^2q_\perp}{(2\pi)^2} \to \frac{1}{(2R)^2}\sum_{\mathbf{q}_\perp}, \quad \text{with } \mathbf{q}_\perp \equiv \frac{\pi \mathbf{n}_q}{R}. \tag{2.26}$$

To proceed, we separate the sum over discrete transverse momenta $\mathbf{q}_\perp$ into the nonzero $\mathbf{q}_\perp \neq \mathbf{0}$ and the zero mode $\mathbf{q}_\perp = \mathbf{0}$, so that (2.25) can be written as

$$\langle \partial_\mu \mathcal{J}_5^\mu(x)\rangle = -\frac{eN_f|eB|\text{sign}(eB)}{2\pi^2}\epsilon^{12ab}\bigg(\int \frac{d^2q_\parallel}{(2\pi)^2}\frac{1}{(2R)^2}\sum_{\mathbf{q}_\perp\neq\mathbf{0}} e^{i(\mathbf{q}_\parallel\cdot\mathbf{x}^\parallel + \mathbf{q}_\perp\cdot\mathbf{x}^\perp)} e^{-(\mathbf{q}_\perp^2/2|eB|)} q_a\bar{A}_b(\mathbf{q}_\parallel, \mathbf{q}_\perp \neq \mathbf{0}) + \int \frac{d^2q_\parallel}{(2\pi)^2}\frac{1}{(2R)^2} e^{i\mathbf{q}_\parallel\cdot\mathbf{x}^\parallel} q_a A_b^{(0)}(\mathbf{q}_\parallel, \mathbf{q}_\perp = \mathbf{0})\bigg). \tag{2.27}$$

Substituting the Fourier transformed of the nonzero transverse modes $\bar{A}$,

$$\bar{A}_b(\mathbf{q}_\parallel, \mathbf{q}_\perp \neq \mathbf{0}) = \int_{-\infty}^{+\infty} d^2y_\parallel \int_{-R}^{+R} d^2y_\perp \bar{A}_b(\mathbf{y}_\parallel, \mathbf{y}_\perp) e^{-i(\mathbf{q}_\parallel\cdot\mathbf{y}^\parallel + \mathbf{q}_\perp\cdot\mathbf{y}^\perp)}, \tag{2.28}$$

and the zero transverse modes $A^{(0)}$,

$$A_b^{(0)}(\mathbf{q}_\parallel, \mathbf{q}_\perp = \mathbf{0}) = \int_{-\infty}^{+\infty} d^2y_\parallel \int_{-R}^{+R} d^2y_\perp A_b^{(0)}(\mathbf{y}_\parallel, \mathbf{y}_\perp) e^{-i\mathbf{q}_\parallel\cdot\mathbf{y}^\parallel}, \tag{2.29}$$

into the r.h.s. of (2.27) and using the identities

$$\int \frac{d^2q_\parallel}{(2\pi)^2} e^{i\mathbf{q}_\parallel\cdot(\mathbf{x}-\mathbf{y})^\parallel} = \delta^2(\mathbf{x}_\parallel - \mathbf{y}_\parallel), \quad \text{and} \quad \frac{1}{(2R)^2}\sum_{\mathbf{q}_\perp\neq 0} e^{i\mathbf{q}_\perp\cdot(\mathbf{x}-\mathbf{y})^\perp} = \delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp),$$

we arrive after integrating over $\mathbf{y}_\parallel$ and $\mathbf{y}_\perp$ at

$$\langle\partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{2\pi^2}\bigg(e^{(\nabla_\perp^2/2|eB|)}\epsilon^{12ab}\partial_a\bar{A}_b(\mathbf{x}_\parallel,\mathbf{x}_\perp) + \frac{1}{(2R)^2}\int_{-R}^{+R} d^2y_\perp \epsilon^{12ab}\partial_a A_b^{(0)}(\mathbf{x}_\parallel,\mathbf{y}_\perp)\bigg).$$

Using now the notations

$$\bar{F}_{ab} \equiv \partial_a\bar{A}_b - \partial_b\bar{A}_a, \quad \text{and} \quad F_{ab}^{(0)} \equiv \partial_a A_b^{(0)} - \partial_b A_a^{(0)},$$

the $U_A(1)$ anomaly of QED in a strong magnetic field for finite compactification length R reads

$$\langle\partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{4\pi^2}\bigg(e^{\nabla_\perp^2/2|eB|}\epsilon^{12ab}\bar{F}_{ab}(\mathbf{x}_\parallel,\mathbf{x}_\perp) + \frac{1}{(2R)^2}\int_{-R}^{+R} d^2y_\perp \epsilon^{12ab}F_{ab}^{(0)}(\mathbf{x}_\parallel,\mathbf{y}_\perp)\bigg). \tag{2.30}$$

Apart from the fact that the anomaly of $3+1$ dimensional QED in the strong magnetic field is, in contrast to the ordinary noncommutative QED, linear in $F_{\mu\nu}$, at least at this one-loop level, the above situation is the same as in (1.16), i.e. by taking the decompactification limit $R\to\infty$, the contribution from the zero mode vanishes and we are left with the R independent first term in (2.30) from the contribution of the nonzero transverse modes to the anomaly

$$\langle\partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{4\pi^2} e^{(\nabla_\perp^2/2|eB|)}\epsilon^{12ab}\bar{F}_{ab}(\mathbf{x}_\parallel,\mathbf{x}_\perp). \tag{2.31}$$

A finite nonvanishing contribution of the zero transverse mode to the axial anomaly of QED in a strong magnetic field arises only when we integrate, as in the case of nonplanar anomaly [see (1.17) and (1.18)], over $\mathbf{x}_\perp$ on both sides of (2.30).[4] In this way, the R-dependence in the second term of (2.30) cancels and the integrated form of the anomaly of QED in a strong magnetic field becomes

$$\int_{-R}^{+R} d^2x_\perp\langle\partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{4\pi^2}\bigg(\int_{-R}^{+R} d^2x_\perp e^{(\nabla_\perp^2/2|eB|)}\epsilon^{12ab}\bar{F}_{ab}(\mathbf{x}_\parallel,\mathbf{x}_\perp) + \frac{1}{(2R)^2}\int_{-R}^{+R} d^2x_\perp\int_{-R}^{+R} d^2y_\perp\epsilon^{12ab}F_{ab}^{(0)}(\mathbf{x}_\parallel,\mathbf{y}_\perp)\bigg),$$

which survives even in $R\to\infty$ limit, i.e.,

$$\int_{-\infty}^{+\infty} d^2x_\perp\langle\partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{4\pi^2}\bigg(\int_{-\infty}^{+\infty} d^2x_\perp e^{(\nabla_\perp^2/2|eB|)}\epsilon^{12ab}\bar{F}_{ab}(\mathbf{x}_\parallel,\mathbf{x}_\perp) + \int_{-\infty}^{+\infty} d^2y_\perp\epsilon^{12ab}F_{ab}^{(0)}(\mathbf{x}_\parallel,\mathbf{y}_\perp)\bigg).$$

Expanding now the phase factor $e^{\nabla_\perp^2/2|eB|}$ and neglecting the surface term arising from the term including the transverse derivatives $\nabla_\perp$, we get

$$\int_{-\infty}^{+\infty} d^2x_\perp\langle\partial_\mu \mathcal{J}_5^\mu(x)\rangle = \frac{ieN_f|eB|\text{sign}(eB)}{4\pi^2} \times \int_{-\infty}^{+\infty} d^2x_\perp\epsilon^{12ab}F_{ab}(\mathbf{x}_\parallel,\mathbf{x}_\perp), \tag{2.32}$$

with $F_{ab}=\bar{F}_{ab}+F_{ab}^{(0)}$ including the nonzero $\bar{F}_{ab}$ *and* the zero transverse modes $F_{ab}^{(0)}$. Hence, in contrast to the nonplanar anomaly of the ordinary noncommutative QED, where the integrated anomaly in the limit $R\to\infty$ receives contribution *only* from the zero modes of the fields in noncommutative directions, here, the integrated form of the axial anomaly of QED in the LLL approximation includes both nonzero and zero transverse modes.

[4]In the case of nonplanar anomaly, $\mathbf{x}_\perp=(x_1,x_2)$ are the noncommutative directions.

III. TWO-POINT VERTEX FUNCTION OF PHOTONS IN THE LLL APPROXIMATION AND THE PHOTON MASS

In this section, the two-point effective vertex function of photons in the LLL approximation will be determined. In particular, the special role played by the zero transverse mode of the photon field will be studied in detail. We start with the expression of the two-point vertex function at one-loop level

$$\Gamma^{(2)}_{\rm LLL} = -(ie)^2 N_f \int d^4x d^4y \, \mathrm{tr}(\mathcal{S}_F(x,y) \not{A}(y) \times \mathcal{S}_F(y,x)\not{A}(y-x)). \quad (3.1)$$

Substituting $\mathcal{S}_F(x,y)$ from (2.7a)–(2.7c), using the relation (2.15), and integrating over $\mathbf{x}_\parallel$ and $\mathbf{y}_\parallel$, we arrive after a straightforward calculation at

$$\Gamma^{(2)}_{\rm LLL} = -\frac{e^2N_f|eB|}{2\pi}\int\frac{d^4q}{(2\pi)^4}\frac{d^2k_\parallel}{(2\pi)^2}e^{-(\mathbf{q}_\perp^2/2|eB|)}\mathrm{tr}\bigg(\frac{(\not{\mathbf{k}}_\parallel+m)}{\mathbf{k}_\parallel^2-m^2}\mathcal{O}\gamma^a\frac{(\not{\mathbf{k}}_\parallel+\not{\mathbf{q}}_\parallel+m)}{((\mathbf{k}_\parallel+\mathbf{q}_\parallel)^2-m^2)}\gamma^\nu\bigg)A_a(q)A_\nu(-q),$$

with $a = 0, 3$ and $\nu = 0,1,2,3$. Here, as in the previous section, we have used the property of the $\mathcal{O}$ matrix (2.6), $\mathcal{O}\gamma^\mu\mathcal{O} = \mathcal{O}\gamma^m$, with $\mu = 0,\cdots,3$ and $m = 0, 3$. Using further the definition of $\mathcal{O}$, and following the standard procedure to evaluate the two-dimensional Feynman integrals, i.e. introducing the Feynman parameter α and performing a shift of integration variable $\mathbf{k}_\parallel \to \mathbf{k}_\parallel - \alpha\mathbf{q}_\parallel$, we arrive first at

$$\Gamma^{(2)}_{\rm LLL} = -\frac{e^2N_f|eB|}{\pi}\int\frac{d^4q}{(2\pi)^4}e^{-(\mathbf{q}_\perp^2/2|eB|)}A^a(q)A^b(-q)\int_0^1 d\alpha\bigg[\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{2\mathbf{k}_a^\parallel\mathbf{k}_b^\parallel}{(\mathbf{k}_\parallel^2-\Delta)^2}-\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{g_{ab}}{(\mathbf{k}_\parallel^2-\Delta)}\bigg] - \frac{2e^2N_f|eB|}{\pi}\int\frac{d^4q}{(2\pi)^4}e^{-(\mathbf{q}_\perp^2/2|eB|)}A^a(q)A^b(-q)(\mathbf{q}_\parallel^2g_{ab}-\mathbf{q}_a^\parallel\mathbf{q}_b^\parallel)\int_0^1 d\alpha\,\alpha(1-\alpha)\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{1}{(\mathbf{k}_\parallel^2-\Delta)^2}, \quad (3.2)$$

where $\Delta \equiv m^2 - \alpha(1-\alpha)\mathbf{q}_\parallel^2$. Here, as in the ordinary $1 + 1$ dimensional Schwinger model, the two-point vertex function (3.2) is closely related to $\langle j_5^\mu(q)\rangle$ from (2.19). As in that case, the first two integrals cancel each other and we are left with

$$\Gamma^{(2)}_{\rm LLL} = -\frac{ie^2N_f|eB|}{2\pi^2}\int\frac{d^4q}{(2\pi)^4}e^{-(\mathbf{q}_\perp^2/2|eB|)}A^a(q)A^b(-q)(\mathbf{q}_\parallel^2g_{ab}-\mathbf{q}_a^\parallel\mathbf{q}_b^\parallel)\int_0^1 d\alpha\,\alpha(1-\alpha)\frac{1}{\Delta}. \quad (3.3)$$

Taking again the chiral limit $m \to 0$, we get

$$\Gamma^{(2)}_{\rm LLL} = -\frac{ie^2N_f|eB|}{2\pi^2}\int\frac{d^4q}{(2\pi)^4}e^{-(\mathbf{q}_\perp^2/2|eB|)}A^a(q)\bigg(g_{ab}-\frac{\mathbf{q}_a^\parallel\mathbf{q}_b^\parallel}{\mathbf{q}_\parallel^2}\bigg)A^b(-q). \quad (3.4)$$

To extract the one-loop vacuum polarization tensor $\Pi_{\mu\nu}$ of the modified noncommutative QED from this result, we use, as was suggested in [7,11], the definition of the smeared fields in the momentum space

$$\mathcal{A}_\mu(q) \equiv e^{-(\mathbf{q}_\perp^2/4|eB|)}A_\mu(q). \quad (3.5)$$

Using now the general relation between the two-point vertex function and the vacuum polarization tensor of the smeared photon fields

$$\Gamma^{(2)}_{\rm LLL} = \int\frac{d^4q}{(2\pi)^4}\mathcal{A}^a(q)(-i\Pi_{ab}(\mathbf{q}_\parallel))\mathcal{A}^b(-q), \quad (3.6)$$

and comparing with (3.4), the projection of the vacuum polarization tensor $\Pi_{\mu\nu}$ into the longitudinal directions, Π_{ab} with $a, b = 0, 3$, reads

$$\Pi_{ab}(\mathbf{q}_\parallel) = (g_{ab}\mathbf{q}_\parallel^2 - \mathbf{q}_a^\parallel\mathbf{q}_b^\parallel)\Pi(\mathbf{q}_\parallel), \quad \text{with } \Pi(\mathbf{q}_\parallel) = \frac{M_\gamma^2}{\mathbf{q}_\parallel^2}, \quad (3.7)$$

and the photon mass

$$M_\gamma^2 = \frac{e^2N_f|eB|}{2\pi^2}. \quad (3.8)$$

According to the structure of $\Pi_{ab}(\mathbf{q}_\parallel)$ in (3.7), there are no polarization effects in the transverse directions, and the strong screening effect appears only in the longitudinal components of the photon propagator $\sim(g^{ab}\mathbf{q}_\parallel^2 - \mathbf{q}_\parallel^a\mathbf{q}_\parallel^b)$.

In an alternative treatment, it is possible to work with the ordinary gauge fields A_μ instead of the smeared fields $\mathcal{A}_\mu$. After redefining (3.6) in terms of A_μ, the vacuum polarization tensor is given by

$$\tilde{\Pi}_{ab}(\mathbf{q}_\parallel,\mathbf{q}_\perp) = (g_{ab}\mathbf{q}_\parallel^2 - \mathbf{q}_a^\parallel\mathbf{q}_b^\parallel)\tilde{\Pi}(\mathbf{q}_\parallel,\mathbf{q}_\perp), \quad \text{with } \tilde{\Pi}(\mathbf{q}_\parallel,\mathbf{q}_\perp) = \frac{M_\gamma^2(\mathbf{q}_\perp)}{\mathbf{q}_\parallel^2}, \quad (3.9)$$

and the "effective mass"

$$M_\gamma^2(\mathbf{q}_\perp) = \frac{e^2N_f|eB|}{2\pi^2}e^{-(\mathbf{q}_\perp^2/2|eB|)},$$

depending on the transverse coordinates. Keeping in mind that the LLL approximation is only valid when $\mathbf{q}_\perp^2 \ll |eB|$, the phase factor $e^{-\mathbf{q}_\perp^2/2|eB|}$ can be neglected, and we arrive at the same momentum independent mass (3.8). This confirms the result computed in [4,10] using similar arguments.

Now, in analogy to the evaluation of the anomaly in the previous chapter, we will check the special role played by the momentum $\mathbf{q}_\perp = \mathbf{0}$. To do this we compactify two transverse coordinates around a circle with radius R. The two-point vertex function (3.4) is therefore given by

$$\Gamma^{(2)}_{\text{LLL}} = -\frac{ie^2 N_f |eB|}{2\pi^2} \int_{-\infty}^{+\infty} \frac{d^2 q_\parallel}{(2\pi)^2} \frac{1}{(2R)^2} \sum_{\mathbf{q}_\perp} e^{-(\mathbf{q}_\perp^2/2|eB|)} A^a(q) \left(g_{ab} - \frac{\mathbf{q}_a^\parallel \mathbf{q}_b^\parallel}{\mathbf{q}_\parallel^2} \right) A^b(-q), \tag{3.10}$$

where we have used (2.26). Separating the zero and the nonzero modes, we have first

$$\Gamma^{(2)}_{\text{LLL}} = -\frac{ie^2 N_f |eB|}{2\pi^2} \int_{-\infty}^{+\infty} \frac{d^2 q_\parallel}{(2\pi)^2} \frac{1}{(2R)^2} \sum_{\mathbf{q}_\perp \neq \mathbf{0}} e^{-(\mathbf{q}_\perp^2/2|eB|)} \bar{A}^a(\mathbf{q}_\parallel, \mathbf{q}_\perp) \left(g_{ab} - \frac{\mathbf{q}_a^\parallel \mathbf{q}_b^\parallel}{\mathbf{q}_\parallel^2} \right) \bar{A}^b(-\mathbf{q}_\parallel, -\mathbf{q}_\perp)$$
$$- \frac{ie^2 N_f |eB|}{2\pi^2} \int_{-\infty}^{+\infty} \frac{d^2 q_\parallel}{(2\pi)^2} \frac{1}{(2R)^2} A^{a,(0)}(\mathbf{q}_\parallel, \mathbf{q}_\perp = \mathbf{0}) \left(g_{ab} - \frac{\mathbf{q}_a^\parallel \mathbf{q}_b^\parallel}{\mathbf{q}_\parallel^2} \right) A^{b,(0)}(-\mathbf{q}_\parallel, -\mathbf{q}_\perp = \mathbf{0}),$$

where $\bar{A}$ denotes the nonzero and $A^{(0)}$ the zero transverse modes of QED in a strong magnetic field. Using the Fourier transformations (2.28) and (2.29) for the gauge fields, we get

$$\Gamma^{(2)}_{\text{LLL}} = -\frac{ie^2 N_f |eB|}{2\pi^2} \int_{-\infty}^{+\infty} d^2 y_\parallel \int_{-R}^{+R} d^2 y_\perp e^{(\nabla_\perp^2/2|eB|)} \bar{A}^a(\mathbf{y}_\parallel, \mathbf{y}_\perp) \left(g_{ab} - \frac{\partial_a \partial_b}{\nabla_\parallel^2} \right) \bar{A}^b(\mathbf{y}_\parallel, \mathbf{y}_\perp)$$
$$- \frac{ie^2 N_f |eB|}{2\pi^2} \int_{-\infty}^{+\infty} d^2 y_\parallel \frac{1}{(2R)^2} \int_{-R}^{+R} d^2 y_\perp A^{a,(0)}(\mathbf{y}_\parallel, \mathbf{y}_\perp) \left(g_{ab} - \frac{\partial_a \partial_b}{\nabla_\parallel^2} \right) \int_{-R}^{+R} d^2 z_\perp A^{b,(0)}(\mathbf{y}_\parallel, \mathbf{z}_\perp).$$

Taking the limit $R \to \infty$, the second term vanishes and we are left with

$$\Gamma^{(2)}_{\text{LLL}} = -\frac{ie^2 N_f |eB|}{2\pi^2} \int_{-\infty}^{+\infty} d^2 y_\parallel d^2 y_\perp e^{(\nabla_\perp^2/2|eB|)} \bar{A}^a(\mathbf{y}_\parallel, \mathbf{y}_\perp) \left(g_{ab} - \frac{\partial_a \partial_b}{\nabla_\parallel^2} \right) \bar{A}^b(\mathbf{y}_\parallel, \mathbf{y}_\perp). \tag{3.11}$$

Hence the zero transverse mode does not contribute to the vacuum polarization tensor of the theory.

IV. CONCLUSIONS

In this paper, we have calculated the $U_A(1)$ anomaly of a $3 + 1$ dimensional QED in a strong and homogeneous magnetic field in the lowest Landau level (LLL) approximation. Because of the well established dimensional reduction $D \to D - 2$ in the dynamics of fermion pairing in a magnetic field, this anomaly is comparable with the axial anomaly of a $1 + 1$ dimensional Schwinger model. On the other hand, it is comparable with the axial anomaly of the ordinary noncommutative field theory. The motivation behind this comparison was the recently explored connection between the dynamics of relativistic field theories in a strong magnetic field in the LLL dominant regime and that in conventional noncommutative field theories. Different aspects of the axial anomaly of noncommutative $U(1)$ gauge theory are studied widely in the literature. Among other results, it is well-known that noncommutative QED consists of *two* anomalous axial vector current, whose anomalies receive contribution from planar and nonplanar diagrams of the theory, separately. Nonplanar (invariant) axial anomaly is affected by UV/IR mixing, a phenomenon which is absent in the dynamics of a $3 + 1$ dimensional QED in a strong magnetic field.

Apart from the fact that the axial anomaly of QED in the regime of LLL dominance depends, in contrast to the ordinary $3 + 1$ dimensional QED, linearly on the field strength tensor $F_{\mu\nu}$, it is comparable with the *nonplanar* anomaly of noncommutative $U(1)$ gauge theory. To show this, we have compactified the transverse directions to the external magnetic field along a circle with radius R. A procedure which was also performed in the case of nonplanar anomaly of noncommutative QED to explore the UV/IR mixing phenomenon. We have shown that in the limit $R \to \infty$, the *unintegrated* form of the axial anomaly of QED in the LLL approximation receives contribution *only* from the nonzero modes of the field strength tensor in the transverse directions. In the case of nonplanar anomaly, however, the zero mode of the Fourier transformed of $\mathcal{F} = F\tilde{F}$ in the noncommutative coordinates contributes to the *unintegrated* form of the nonplanar anomaly only for finite R. In the limit $R \to \infty$, the zero mode contribution and thus the unintegrated form of the nonplanar anomaly vanish. We have further shown that the contribution from the zero transverse mode of the field strength tensor, i.e. the mode which is constant in the transverse direction to the external magnetic field, reappears in the *integrated* version of the axial anomaly of the QED in the LLL approximation. The mechanism of this reappearance is quite similar with the mechanism in which the *integrated* form of the nonplanar anomaly was shown to be finite in $R \to \infty$ limit. The main reason for all these effects, is the fact that QED in a strong magnetic field consists of a natural UV cutoff $M \sim \sqrt{eB}$ which is kept large but finite, whereas noncommutative field theories are treated as fundamental theories with infinitely large UV cutoff.

Further, motivated by the connection between the axial anomaly and the vacuum polarization tensor in the ordinary two-dimensional Schwinger model, we have calculated the two-point vertex function of QED in the LLL approximation. We have shown that the theory consists of a massive photon of mass $M_\gamma \sim e^2|eB|$. This is in analogy to the case of ordinary two-dimensional Schwinger model whose massive photon picks up a mass $m_\gamma^2 = g^2/\pi$. This can be again interpreted as a signature for the above mentioned dimensional reduction from $D = 4$ to $D = 2$ dimensions.

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