Chiral lattice gauge theories from warped domain walls and Ginsparg-Wilson fermions

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We propose a construction of a 2-dimensional lattice chiral gauge theory. The construction may be viewed as a particular limit of an infinite warped 3-dimensional theory. We also present a ''single-site'' construction using Ginsparg-Wilson fermions which may avoid, in both 2 and 4 dimensions, the problems of waveguide-Yukawa models.

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I. INTRODUCTION AND SUMMARY

Understanding the strong-coupling behavior of chiral gauge theories is an outstanding problem of great interest, both on its own and for its possible relevance to phenomenology: the Standard Model of elementary particle physics is a chiral gauge theory and additional strong chiral gauge dynamics at the (multi-) TeV scale may be responsible for breaking the electroweak symmetry and fermion mass generation.

The only clues of the strong-coupling behavior of nonsupersymmetric chiral gauge theories come from 't Hooft anomaly matching and most attractive channel arguments. Large-N expansions, including the recently considered gravity duals in the anti-de Sitter (AdS)/conformal field theories (CFT) (QCD) framework, do not apply to chiral gauge theories. Thus, the space-time lattice regularization remains, to this day, the only way to advance our limited knowledge of chiral gauge dynamics.

The lattice, however, fails to reproduce the physics of a chiral gauge theory due to the presence of extra, unwanted fermion ''doubler'' modes. The difficulty of this problem is encoded in a no-go theorem [1]. Recent reviews of the different approaches to lattice chiral gauge theories are [2,3]; see also Ref. [4] for a new approach.

It has long been known that the extra fermions *can* be removed from the spectrum by sacrificing the gauge symmetry, at least perturbatively [5,6]. Recently, a proposal was made to do that in a new way [7,8]. The fermion masses must be chosen in a nontrivial way to break the appropriate global symmetries in order to reproduce the anomalies of the target theory, while still maintaining the appropriate light fermion modes. The gauge symmetry can then be restored through a limiting process inspired by a 5-dimensional model in AdS space. Unfortunately for this approach, called ''warped domain-wall fermions,'' the associated Goldstone mode is strongly coupled leaving the model's status somewhat uncertain.

The warped domain-wall fermion model bears some similarity to the ''waveguide model'' [9,10], which is known not to give a chiral theory [11,12]. There are, however, significant differences between the models—in particular, in the warped case the source of the gauge boson and fermion mass are decoupled and so further investigation of this model is necessary.

In this paper, we first consider a construction analogous to that of [7] in 3-dimensional AdS space in an attempt to construct a 2-dimensional chiral gauge theory. We describe a limit which results in a 2-dimensional chiral gauge theory without a strongly coupled Goldstone mode.

We then propose a related, simplified ''one-site'' model which consists of only a 2-dimensional lattice theory where massless fermions are introduced using the Ginsparg-Wilson (GW) mechanism [13,14] for imposing a modified chiral symmetry. The symmetries and anomalous Ward identities in this model are as expected in the target theory. Furthermore, a preliminary strong-coupling analysis (a more detailed study of this model is in progress), which is also expected to hold in the 4-dimensional version of the construction, indicates that no new unwanted light fermions appear and the fermion spectrum of the unbroken gauge theory remains chiral in an appropriately taken limit.

In addition to providing some insight on the workings of the 4-dimensional warped domain-wall fermion construction, the models presented here are interesting for their own sake as they are the simplest examples of chiral gauge theories. It is also clear that 2-dimensional models are the most amenable to numerical tests and to this end alone it is desirable to have an appropriate formulation; in the process, we will see that the warped 2-dimensional case presents a number of subtleties compared to the 4-dimensional warped domain-wall fermions.

This paper is organized as follows. We begin in Section II, where we describe previous work on lattice chiral gauge theories within the ''waveguide model.'' We

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review the earlier arguments showing that the waveguide model gives rise to a vectorlike spectrum of massless fermions, both at small and large Yukawa coupling.

In Section III, we explain how the proposal of [7] addresses the difficulty with obtaining a chiral spectrum using a warped AdS background. Motivated by the strong fermion/Goldstone mode coupling found in the 4-dimensional implementation of the proposal, we construct and study the much simpler 2-dimensional version in detail. Within perturbation theory and in the deconstructed description, we show that there are no bulk Goldstone/fermion strong interactions in this model and that the desired spectrum and separation of scales can be achieved while at weak coupling (two appendices describe various important technical details). Our results of Section III indicate that the ‘‘warped domain-wall’’ framework for lattice chiral gauge theories is still of interest and deserves further study, including a full lattice implementation.

In Section IV, we present another proposal: the ‘‘single-site model.’’ It is related to the ‘‘warped domain-wall’’ in that it is also motivated by considering the waveguide model and its failure to give a chiral fermion spectrum, this time in the strong-Yukawa-coupling regime. Our main observation here is that using GW fermions helps avoid the left/right mixing that leads to a vectorlike fermion spectrum. To also obtain a massless gauge boson, we have to make use of the strong-Yukawa symmetric phase of the Yukawa-Higgs theory. We give a plausibility argument as to why we believe this phase can be realized in our construction without fine tuning. Further analytical and numerical work on the ‘‘one-site’’ model supporting our proposal will appear in [15].

In Section V, we conclude with a summary of the proposals, a list of outstanding issues, and an outlook for future work.

II. DOMAIN-WALL FERMIONS IN 2 DIMENSIONS

We review here some features of fermions in 2 dimensions and discuss some relevant previous work on chiral gauge theories. By understanding the shortcomings of previous attempts we will see how our construction differs in important ways.

A 2-dimensional Dirac fermion has two complex components:

$$\Psi \equiv \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \tag{1}$$

We will call the ψ_- field left-handed and the ψ_+ field right-handed. We work with light-cone coordinates:

$$x^{\pm} \equiv t \pm x \longrightarrow 2\partial_{\pm} = \partial_t \pm \partial_x, \tag{2}$$

so that the Lagrangian for a charged, massive, Dirac fermion is:

$$\mathcal{L} = 2i\bar{\psi}_-(\partial_+ - iA_+)\psi_- + 2i\bar{\psi}_+(\partial_- - iA_-)\psi_+ + m_D\bar{\psi}_+\psi_- + m_D^*\bar{\psi}_-\psi_+. \tag{3}$$

The bar on these one-component, complex fields indicates complex conjugation, while the subscript D on the mass indicates that it is a Dirac type mass term: it does not break a gauge symmetry. We will later introduce masses of the Majorana type which would break gauge symmetry: $m_M\psi_+\psi_- +$ h.c.. By Lorentz invariance there are no other types of mass terms: any mass must couple left- and right-handed fermions. This restriction of mass terms will be important later.

The domain-wall fermions [9] arise from consideration of a theory with a third dimension labeled with the coordinate z. With the appropriate mass terms there will be a light left-handed mode localized at one end of the extra dimension and a light right-handed mode localized at the other end. We will consider the theory with the z direction on a lattice, keeping the other two directions as a continuum. Upon discretization the kinetic term in the z direction contributes to mass and mixing terms for the fermions on adjacent sites:

$$\bar{\psi}\partial_z\psi \longrightarrow \frac{\bar{\psi}_i(\psi_i - \psi_{i-1})}{\delta z}. \tag{4}$$

This is sometimes referred to as the deconstructed model [16]. Alternatively we may think of this as a lattice in all three directions where the lattice spacing in the two x^{μ} directions is much smaller than the lattice spacing in the z direction. Indeed, we will never take the z lattice spacing to zero.

If we place two domain-wall lattices back-to-back with one being charged and one being neutral we have the ‘‘waveguide’’ approach. The Lagrangian is given by:

$$\begin{aligned}&\sum_{i=1}^{k}[2i\bar{\psi}_{i-}\partial_+\psi_{i-} + 2i\bar{\psi}_{i+}\partial_-\psi_{i+} + (m_i\bar{\psi}_{i-}\psi_{i+} + \text{h.c.})] \\ &\quad + \sum_{i=2}^{k}(m_i'\bar{\psi}_{i+}\psi_{i-1,-} + \text{h.c.}) + \sum_{i=k+1}^{N}[2i\bar{\psi}_{i-}(\partial_+ - iA_+)\psi_{i-} \\ &\quad + 2i\bar{\psi}_{i+}(\partial_- - iA_-)\psi_{i+} + (m_i\bar{\psi}_{i-}\psi_{i+} + \text{h.c.})] \\ &\quad + \sum_{i=k+2}^{N}(m_i'\bar{\psi}_{i+}\psi_{i-1,-} + \text{h.c.})\end{aligned} \tag{5}$$

and schematically, this Lagrangian is represented in Fig. 1. The masses m_i and m_i' depend upon the details of the discretization and are not important at this point. Note that the gauge mode is independent of the sites: we are treating it as a 2-dimensional degree of freedom. In order to couple the charged and uncharged fermions in a gauge invariant manner we introduce a charged scalar, ϕ, and the coupling:

$$y\bar{\psi}_{k+1,+}\phi\psi_{k,-} + \text{h.c.}. \tag{6}$$

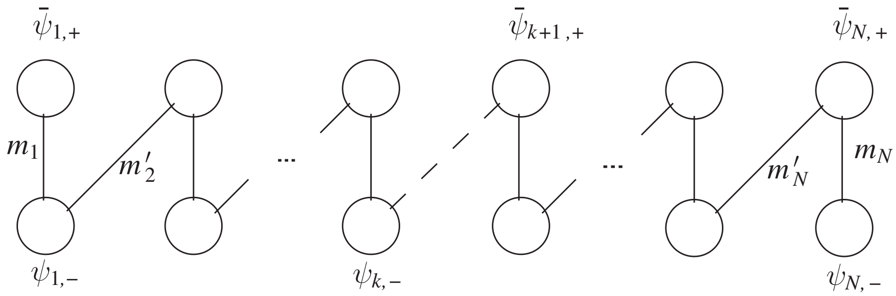

FIG. 1. The waveguide approach to a chiral gauge theory. Circles represent Weyl fermions. Solid lines represent mass terms and a charged scalar couples the charged fermion, $\bar\psi_{k+1,+}$, to the neutral fermion, $\psi_{k,-}$.

The phases of the analogous model in both 2 and 4 dimensions were analyzed for both weak and strong-Yukawa couplings y in [11,12], with the conclusion that the theory is nonchiral in every case. The simplest possibility is for $y = 0$. In this case one can easily see that the model falls apart into two disconnected theories. One is the fully gauged waveguide part and the other is the ungauged part of the domain wall. Each of these two parts themselves form a domain-wall model, and each of these will either have zero modes localized at both ends or at neither end. Thus the boundary of the waveguide will act as a domain-wall boundary itself. Nothing qualitatively different happens for small nonzero Yukawa, as long as the field ϕ does not acquire a vacuum expectation value (VEV).

However, if the scalar does obtain a VEV, then the light fermion mode localized at the waveguide boundary could be eliminated using the opposite chirality fermion localized on the other side of the waveguide boundary via the mass term $y\psi_{k+1,+}\langle\phi\rangle\psi_{k,-}$. The problem with this approach is rooted in the fact that the gauge field does not fluctuate in the extra dimension. The fermion mass obtained this way will be of the order $m_f \sim y\langle\phi\rangle$. However, in this Higgs' mechanism, the gauge boson will also pick up a mass of order $m_0 \sim g\langle\phi\rangle$. To get to an unbroken chiral theory one would like $m_0 \ll m_f$, however their mass ratio is given by g/y. Since in 4 dimensions the Yukawa is an IR free coupling, at low energies its value will be determined by g, and it seems that no hierarchy between the masses is possible in the weak-coupling region. In the next section, we will explain how the "warped domain-wall" proposal avoids this problem by separating the scales of gauge boson and fermion masses.

In the opposite limit of strong Yukawa coupling, the phase structure of the $g = 0$ lattice-Higgs-Yukawa model with a fixed-length Higgs field was also analyzed in [12] via a strong-coupling expansion in y and it was again found that the spectrum of the model was vectorlike. This is most easily seen at leading order in $1/y$ by first rescaling the fermion fields at the boundary of the waveguide, see Eq. (6), by $1/\sqrt{y}$, thus making their kinetic terms vanish at $y \to \infty$. In our deconstructed picture of Fig. 1 this results in removing the two circles adjacent to the dashed line; thus, after the rescaling, the remaining charged $\Psi_{k+1,-}$ loses its Wilson term in the 2-dimensional noncompact lattice directions not shown—recall that the Wilson term couples $\Psi_{k+1,-}$ to the now absent $\Psi_{k+1,+}$. Naturally, the loss of the Wilson term results in the appearance of a plethora of charged and neutral massless states near the waveguide boundary, localized at the new boundaries of the split waveguide, leading once more to a vectorlike spectrum [12]. Thus, it was concluded that also in the strong-Yukawa limit it is not possible to get a chiral gauge theory from domain-wall fermions. In Section IV, we will explain how using the GW mechanism of imposing a modified chiral symmetry on the lattice avoids the mixing of light and mirror modes in the Yukawa coupling that led to the appearance of doublers [12].

In [7] it was argued that the situation is different when one allows the gauge field to fluctuate in the extra dimension and when one is considering a nontrivial background metric along the extra dimension. In this case the scaling of the gauge boson mass could be different from that of the fermion mass in the presence of a symmetry-breaking VEV on one of the domain-wall boundaries. This led to a possibility of recovering a chiral gauge theory in the limit when the warping (the background curvature of the extra dimension) is increased to infinity. We will repeat much of that argument below in the context of a 2-dimensional domain-wall theory.

III. A WARPED 3-DIMENSIONAL THEORY

A. Gauge fields

The key feature of this construction, as in the 4-dimensional case [7], is the separation of scales which is made possible by an appropriate introduction of curvature. It is known that in a theory with a compact extra dimension and gauge symmetry breaking at one boundary, the mass of the gauge boson is not set by the VEV of the Higgs field alone [17]. The lightest gauge mode bends in the extra dimension and the associated gradient terms also contribute to the mass. In the limit that the Higgs VEV goes to infinity, the gauge field is repelled from the location of symmetry breaking and the mass of the gauge boson is independent of the VEV. In flat space this mass is parametrically the same as the masses of the Kaluza-Klein (KK) modes, but in a warped background there is the possibility of a separation of the lightest mode from the KK modes. We chose our warping and boundary conditions so as to achieve that separation.

In order to study the strong coupling of a 2-dimensional chiral gauge theory, we require a hierarchy not just between the KK modes and the mass of the lightest gauge boson, but we must have the strong-coupling scale of the theory lie between these two scales:

$$m_{\mathrm{KK}} \gg \Lambda_{\chi\mathrm{GT}} \gg m_{A_0}. \tag{7}$$

The scale m_{KK} is the scale at which all of the 3-dimensional physics enters. Much below this scale we are left with a 2-dimensional theory, as demonstrated by some simple checks of the mass spectrum and in Appendix B. The mass of the lightest gauge mode, m_{A_0}, sets the scale of the gauge symmetry breaking, and much above this scale the theory has an unbroken gauge symmetry. The low energy 2-dimensional gauge theory has a gauge coupling, g_2, with units of mass, and therefore the gauge coupling itself sets the scale of strong coupling for this gauge theory. At tree level in the 2-dimensional theory, we have $\Lambda_{\chi\mathrm{GT}} = g_2$.

We first describe our construction in terms of a continuum theory living in a slice of 3-dimensional AdS space, AdS_3. The metric is given by:

$$ds^2 = \left(\frac{R}{z}\right)^2 (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \tag{8}$$

where x^μ are the two flat directions and z is the extra, warped direction. The space is bounded in the z direction. One end, $z = R$, is called the UV brane. The other end, $z = R'$, is called the IR brane, where $R' \gg R$. We will break the gauge symmetry with a Higgs mechanism on the UV brane. Note that this is different from [7] and many phenomenological models [18], where the gauge symmetry is broken at the IR end. The reason for the difference is that the scaling of the mass of the lightest gauge boson depends crucially on the number of dimensions as can be seen below.

The gauge field action is:

$$\int d^3x \sqrt{g} \bigg[-\frac{1}{4g_3^2} F_{MN} F^{MN} + \delta(z-R)\bigg(\frac{1}{2} D_\mu \phi^* D^\mu \phi - V(\phi)\bigg)\bigg], \tag{9}$$

where ϕ is a UV-brane localized Higgs field, which we will take below to have a fixed VEV. Then, the bulk equation of motion for the KK modes of the transverse components of the gauge field is:

$$\frac{R}{z} \partial_z \left(\frac{z}{R} \partial_z f_n(z)\right) = -m_n^2 f_n(z), \tag{10}$$

where m_n is the 2-dimensional mass squared of the n-th KK mode. The solutions for the KK modes are Bessel functions:

$$f_n(z) = A_n J_0(m_n z) + B_n Y_0(m_n z). \tag{11}$$

The boundary conditions also come from requiring that the boundary terms in the variation of the action vanish; without boundary terms in the action, the allowed boundary conditions are Dirichlet or Neumann. By choosing the Higgs VEV large enough, we have effectively Dirichlet boundary conditions at the UV end (see [18]). We choose Neumann boundary conditions at the IR end because we want the gauge group to be unbroken there. This leads to a mass spectrum well approximated by:

$$m_{A_n} \sim \frac{n\pi}{R'} \to m_{\mathrm{KK}} \equiv \frac{\pi}{R'}, \tag{12}$$

except for the lightest mode which has a mass:

$$m_{A_0}^2 = \frac{2}{R'^2} \frac{1}{\ln(R'/R)} \left(1 + \mathcal{O}\left(\frac{1}{\ln(R'/R)}\right)\right). \tag{13}$$

We can see right away that the physics of the 3-dimensional theory, set by the KK scale m_{KK} can be separated from the physics of the gauge symmetry breaking for large $\ln(R'/R)$.

The powers of the ratios of R/z in the KK Eq. (10) depend on the dimensionality of the space: had this been a slice of AdSM_5, then both R/z factors would have been inverted. By breaking the gauge symmetry on the UV brane we find the same scaling for the mass of the gauge boson as in 5-dimensions with IR-brane breaking [19].

For the deconstruction description we choose a small dimensionless lattice spacing parameter, a, and let the physical lattice spacing scale across the space: $\delta z_i \sim z_i$, ensuring that the interval (8) $\delta z_i / z_i$ between two neigboring lattice points is i-independent; Eq. (14) below implicitly defines the exact expression for δz_i. It is helpful to define a local warp factor which encodes how much the metric warps from one lattice site to the next:

$$w = \frac{1}{1+a} \to z_i = w^{1-i} R. \tag{14}$$

If there are N slices, then $R' \equiv z_N = w^{1-N} R$.

We may consider the deconstructed Lagrangian coming from Eq. (9) in the $A_z = 0$ gauge. Alternatively this may be viewed as the Lagrangian for the N, coupled, 2-dimensional gauge theories in unitary gauge. The UV boundary Higgs is left in:

$$-\frac{1}{4} \sum_{i=1}^{N} \frac{a z_i}{R g_3^2} [F_{\mu\nu}^i]^2 + \frac{1}{2} D_\mu^1 \phi^* D^{1\mu} \phi - V(\phi) + \frac{1}{2} \sum_{i=1}^{N-1} \frac{a z_i^2}{R g_3^2} \frac{(A_\mu^{i+1} - A_\mu^i)^2}{(a z_i)^2} + \ldots \tag{15}$$

The dots represent interaction terms in non-Abelian theories, the D^1 is a covariant derivative under the first gauge group, and we have suppressed coordinates in the x^μ direction.

With this discretization, the 2-dimensional gauge theory on each slice, i, has a gauge coupling given by:

$$\frac{1}{g_i^2} = \frac{a z_i^2}{R g_3^2}. \tag{16}$$

If we put $\langle\phi\rangle = 0$ so that the gauge symmetry was unbroken by the UV boundary Higgs field then there would be a massless gauge mode comprised of equal parts of the gauge modes from each site. The corresponding low energy gauge coupling is then approximately given (at tree level) by:

$$\frac{1}{g_2^2} = \sum_{i=1}^{N} \frac{1}{g_i^2} = \frac{R'^2}{2 R g_3^2}. \tag{17}$$

In the presence of the UV-brane Higgs mechanism, the low energy gauge coupling comes from considering the overlap of the wave functions for the gauge boson and fermions. However, as the gauge symmetry is restored the above approximation becomes exact.

To find the physical mass spectrum in the gauge sector we need to rescale each gauge boson in order to have the canonical kinetic normalization: $-1/4$. Doing so gives a gauge mass matrix,

$$(aR)^2 M_{\text{Gauge}} = \begin{pmatrix} 1+v_0^2 & -w & 0 & 0 & & & \\ -w & 2w^2 & -w^3 & 0 & \cdots & & \\ 0 & -w^3 & 2w^4 & -w^5 & & & \\ \vdots & & & \ddots & & & 0 \\ & & 0 & -w^{2N-5} & w^{2N-4} & -w^{2N-3} \\ & & 0 & 0 & -w^{2N-3} & w^{2N-2} \end{pmatrix}, \tag{18}$$

where v_0 is the VEV of the UV boundary Higgs. In practice it is sufficient to take $v_0 = 1$ in order to reproduce the mass given by Eq. (13).

Furthermore, the discretization (14) implies $\ln(R'/R) \approx Na$. Thus, our hierarchy of mass scales, Eq. (7), can be written, using (12), (13), (16), and (17), as:

$$1 \gg a z_i^2 g_i^2 \gg \frac{1}{Na}. \tag{19}$$

We see then that the site gauge couplings must be small in comparison to the local energy scale $1/z_i$. By choosing to hold a fixed, we satisfy this hierarchy requirement by letting the gauge couplings scale as:

$$g_i^2 \sim \frac{1}{z_i^2 \sqrt{N}} \tag{20}$$

and taking the large N limit. With these gauge couplings smaller than other mass scales at site i, we do not expect significant corrections to this tree-level relation.

Finally, we note that the 3-dimensional case is different than AdS_5; the fact that we can take N large and keep the individual gauge couplings (20) in AdS_3 small should not come as a surprise—the 3-dimensional bulk theory is superrenormalizable, in contrast to the 5-dimensional case where taking large N is ultimately responsible for entering the strong-coupling domain [7].

B. Fermions

The presence of an equal number of left- and right-handed fermions on a lattice means that we must find a way to remove the fermion of one handedness by making it heavy. In the 4-dimensional construction of Ref. [7] this could be done indirectly through Majorana mass terms which only give a mass to one Weyl fermion in a Dirac pair. However, in 2 dimensions Lorentz invariance requires that all mass terms connect a left- to a right-handed fermion. While it might be possible to remove one fermion in a Lorentz violating manner, we will choose a different approach: exchanging an unwanted, light, charged fermion for a light neutral fermion.

We will increase the number of fermions, but make the new fermions neutral under the gauge symmetry. In the absence of the gauge symmetry breaking at the UV boundary, our low energy spectrum would contain light left- and right-handed charged Weyl modes, l_- and l_+ respectively. It would also have two neutral Weyl modes, n_- and n_+. We then use the gauge breaking Higgs mechanism from the previous subsection to generate an effective mass term in the low energy theory between the unwanted charged Weyl fermion and one of the neutral fermions:

$$y\langle\phi\rangle \bar{l}_+ n_-. \tag{21}$$

This mass leaves l_- as the only charged fermion in the low energy spectrum.

All four of these modes, l_-, l_+, n_-, n_+, may be realized as domain-wall fermions. In the charged sector, we will maintain consistency with the warped AdS_3 background even though this may not always be necessary. In the neutral sector we will, for simplicity, use flat space domain-wall fermions as in Section II. However, it is important that the fermions in this model are able to reproduce the anomaly for a single, light, left-handed fermion. Since the fermions are coming from the domain wall, there are an equal and finite number of left- and right-handed fermions in the measure of the path integral. The phase of the measure is therefore well defined, so the anomaly will not arise in the usual continuum manner.

The anomaly arises, instead, from symmetry-breaking Majorana masses in the neutral sector and through the couplings (of the form of Eq. (21)) of neutral and charged modes. We will discuss anomalies and symmetries in more detail in Section III C.

We now describe how to obtain the light (before including effects of the UV-brane Higgs) spectrum of charged, l_-, l_+, and neutral, n_-, n_+, modes. In order to leave a full neutral Weyl spinor $n_\pm$ light we must introduce, in our construction, two neutral Dirac spinors, $n^1_\pm$ and $n^2_\pm$, so that there will still be a light neutral Weyl spinor after all of the necessary Majorana and Dirac mass terms are included. In total we will use $3N$ 2-dimensional Dirac spinors for the remainder of this subsection.

We will again use a deconstruction description for the fermions. Alternatively, these fermions may be thought of as ordinary Wilson fermions living on a 2-dimensional lattice in the small lattice spacing limit. We will use the following fermion basis for expressing the mass matrix:

$$\vec{\Psi}^T_+ = (\eta^1_{1+}, \eta^1_{2+}, \ldots \eta^1_{N+}, \bar{\eta}^1_{1+}, \bar{\eta}^1_{2+}, \ldots \bar{\eta}^1_{N+}, \eta^2_{1+}, \eta^2_{2+}, \ldots \eta^2_{N+}, \bar{\eta}^2_{1+}, \bar{\eta}^2_{2+}, \ldots \bar{\eta}^2_{N+}, \psi_{1+}, \psi_{2+}, \ldots \psi_{N+}, \bar{\psi}_{1+}, \bar{\psi}_{2+}, \ldots \bar{\psi}_{N+}), \tag{22}$$

and likewise for $\vec{\Psi}_-$. In this $6N$ vector of Weyl fermions, the η's are all neutral, while the ψ_i's are charged under the i-th gauge group. Again, the bar means complex conjugation. We need to include both the barred and unbarred Weyl fermions in this vector so that we may include the Majorana mass terms. The mass matrix,

$$i\vec{\Psi}^T_+ M \vec{\Psi}_-, \tag{23}$$

is almost block diagonal. In addition to the above mass matrix, the complete Lagrangian also involves the usual kinetic terms (Wilson, if a 2-dimensional lattice description is used) for the neutral $\eta^{1,2}_{i\pm}$, $i = 1, \ldots, N$ as well as kinetic terms for the charged fermions $\psi^i_\pm$ in the warped AdS_3 background (see Appendix A for details).

The mass matrix (23) can be thought of as representing the mass and hopping terms for Dirac fermions in a slice of flat 3-dimensional space—the first $4N \times 4N$ elements—with $\eta^1_{1\pm}$ and $\eta^2_{1\pm}$ being localized at the left end while the $\eta^1_{N\pm}$ and $\eta^2_{N\pm}$ "live" at the right end. The right end of the flat space ends at the UV brane of the AdS_3 slice, where $\psi_{1\pm}$ lives. Finally, the ψ_N lives at the IR-brane end of AdS_3 (this picture is further visualized by the plots representing the locations of the various modes in Figs. 2 and 4). The gauge fields are 3-dimensional and propagate in the AdS_3 part of the lattice.

We will start by describing the mass and hopping terms (23) in its $2N \times 2N$ diagonal blocks,

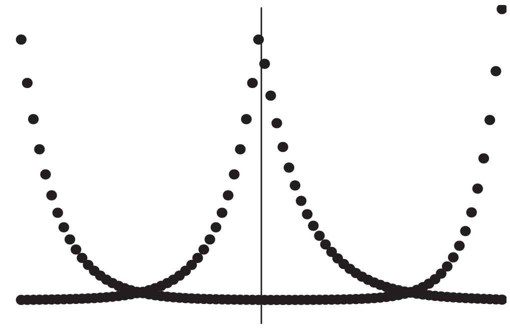

FIG. 2. The wave functions of the exponentially light modes before adding masses which couple the neutral and charged sectors. The right half is charged with the l_- mode localized on the far right. The n_+ mode is localized on the far left. The two modes in the middle, n_- and l_+ will pick up a mass though the Yukawa coupling with the Higgs on the wall.

$$M = \text{diag}(M_{\eta 1}, M_{\eta 2}, M_\psi), \tag{24}$$

and add terms coupling the η's to the ψ's at the end of this subsection. Each of these mass matrices breaks up into $N \times N$ blocks which represent either Dirac- or Majorana-type masses. For example, for each $\eta^{1,2}$ and ψ, we have a $2N \times 2N$ mass matrix of the form:

$$M_\eta = \begin{pmatrix} M_{\eta\text{Majorana}} & M_{\eta\text{Dirac}} \\ M_{\eta\text{Dirac}} & M_{\eta\text{Majorana}} \end{pmatrix}. \tag{25}$$

If we chose $M_{\eta\text{Majorana}} = 0$ and

$$aRM_{\eta D} = \begin{pmatrix} 1 & 0 & & & \\ -(1+\epsilon) & 1 & 0 & & \\ 0 & -(1+\epsilon) & \ddots & 0 & \\ & & & 1 & 0 \\ & & & -(1+\epsilon) & 1 \end{pmatrix}, \tag{26}$$

then we would have an exponentially light, left-handed Weyl mode localized closer to the N-th slice and an exponentially light right-handed Weyl mode peaked at site 1. We will call these modes n^1_- and n^1_+, respectively.

If we now add an equal Majorana and Dirac mass term for the light neutral modes, then a Majorana-Weyl spinor will remain massless. In the low energy theory this mass term has the appearance:

$$\frac{m}{a}(n^1_+ n^1_- + \bar{n}^1_+ n^1_- + \text{h.c.}), \tag{27}$$

so that the imaginary part of n^1_+, stays massless and m is a number of order one. (For now, the imaginary part of n^1_- is also massless; note, however, that it is localized near the UV brane of the warped part of space and will further get a mass by coupling to the UV-brane localized charged fermions (32) below.) This additional mass (27) between the light neutral modes is described by an entry in the top right corner of both the Dirac and Majorana part of the mass matrix, modifying Eq. (25) accordingly:

$$aRM_{\eta 1M} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ & \ddots & 0 & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix},$$

$$aRM_{\eta 1D} = \begin{pmatrix} 1 & 0 & & & 1 \\ -(1+\epsilon) & 1 & 0 & & \\ 0 & -(1+\epsilon) & \ddots & 0 & \\ & & & 1 & 0 \\ & & & -(1+\epsilon) & 1 \end{pmatrix}. \tag{28}$$

For the η^2 masses we can completely repeat this structure with only a change in the sign of the Majorana type mass relative to the Dirac type mass in the top right corner of the matrix (28), or equivalently, by adding a mass term to the low energy theory of the form (29):

$$\frac{m}{a}(n_+^2 n_-^2 - \bar{n}_+^2 n_-^2 + \text{h.c.}). \tag{29}$$

This leaves the real part of n_+^2 as an exponentially light mode. The n_-^2 mode is also exponentially light. However, it is localized near the UV brane of the warped part of space and will further get a mass by coupling to the UV-brane localized charged fermions, see Eq. (32). Combining the modes of n^1 and n^2, then, we have a massless Weyl mode localized near site-1 (the leftmost end in Fig. 2), which in the limit $N \to \infty$ exhibits a chiral symmetry:

$$n_+ \equiv \Im(n_+^1) + i\Re(n_+^2) \to e^{i\alpha} n_+. \tag{30}$$

For the charged fermions we make use of the AdS_3 background. The continuum action of a Dirac fermion in a slice of AdS_3, its zero mode solutions, and the relevant boundary conditions are given in Appendix A, where also the discretized version is presented. As shown there, the Majorana masses for the charged fermions are zero and the Dirac type masses are given by:

$$-2aRwM_{D\psi} = \begin{pmatrix} \alpha & 0 & \dots & & \\ -\sqrt{w}w & \alpha w & 0 & \dots & \\ 0 & \sqrt{w}w^2 & \alpha w^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \alpha w^{N-2} & 0 \\ 0 & 0 & \dots & -\sqrt{w}w^{N-1} & \alpha w^{N-1} \end{pmatrix}, \tag{31}$$

where $\alpha \sim 1$ is defined in Appendix A and $w = 1/(1+a)$ is the local warp factor. As shown in the appendix, by appropriately choosing α, we have a light left-handed mode, l_-, peaked at site N (the IR brane). In addition, the unwanted right-handed companion, l_+, is peaked at site 1 (the UV brane).

Finally, to obtain the desired chiral spectrum, we need to couple the neutral to the charged modes through the UV boundary Higgs. Before doing that, however, let us take stock of the spectrum. As shown in Fig. 2, there are two light Weyl modes: l_+ and l_-, localized near the UV brane (middle of figure) and IR brane (right-hand side) of the warped part of space. There are also four light Majorana-Weyl modes: $\Im(n_-^1)$, $\Im(n_+^1)$, $\Re(n_-^2)$, and $\Re(n_+^2)$. The $\Im(n_+^1)$, $\Re(n_+^2)$—the n_+ of (30))—are localized at the leftmost end while the $\Im(n_-^1)$, $\Re(n_-^2)$ are close to the UV brane. By showing the wave functions, the figure tells us where each mode is localized. Because the + and − modes are spatially separated we may give a mass to the l_+ mode without significant impact on the l_- mode. This separation is the reason for using the domain-wall fermions in the first place. In the large N limit the l_- mode becomes massless. In addition, in that limit, the massless neutral modes decouple because their wave function has zero overlap with the charged fermions and gauge modes.

The masses which couple neutral and charged modes via the UV-brane Higgs show up as off-diagonal entries in the mass matrix from Eq. (24). Instead of writing a large matrix, their contribution can be written simply in terms of the Lagrangian as:

$$i\phi(y_1\bar{\psi}_{0+}\eta_{N-}^1 + y_2\bar{\psi}_{0+}\eta_{N-}^2) + \text{h.c.}, \tag{32}$$

where $\langle\phi\rangle y_{1,2}$ are of order one. These masses are sufficient to lift the mass of the l_+ mode to the KK scale and to transmit the breaking of the chiral flavor symmetry to reproduce the anomaly as we will discuss below.

The qualitative expectations for the mass spectrum—based on known domain-wall fermion spectra and on fermion spectra in AdS backgrounds, see Appendix A—can be further substantiated by numerically solving for the eigenvalues and eigenvectors of the mass matrices for various values of N. Figure 3 shows the mass spectrum resulting from the diagonalization of the mass matrices in Eqs. (18), (24), and (32). This figure includes the results of all mass terms discussed in this section. The mass ratios of the first KK modes to the lightest gauge mode, m_{A1}^2/m_{A0}^2 and m_{f1}^2/m_{A0}^2, are plotted as a function of N. The KK modes are getting heavier than the light gauge mode as expected. The log of the ratio of the light fermion to the light gauge mode, $2\log(m_{f0}/m_{A0})$, is also plotted and shows that one light charged fermion remains.

C. Anomalies

The condition for a 2-dimensional $U(1)$ gauge theory to be free of gauge anomalies is that the sum of the squares of the charges of the left- and right-handed modes must cancel:

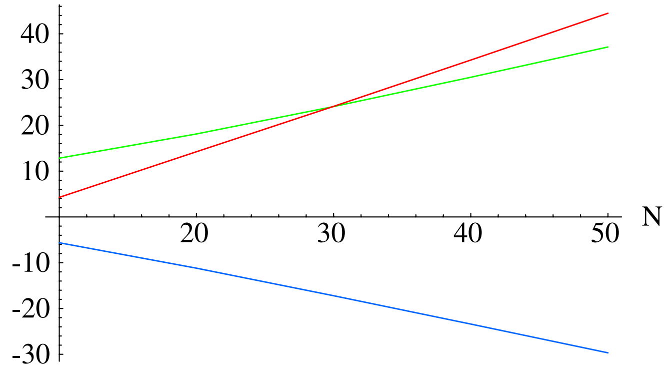

FIG. 3 (color online). (Color online) Mass ratios of the light modes as a function of lattice size, N. The growing lines give $(\frac{m_{KK}}{m_{A0}})^2$, the ratio the KK modes' mass to that of the light gauge boson. The green (highest intercept) line is for the first gauge KK mode, while the red (middle intercept) line is for the first fermion KK mode. The falling (negative intercept) line gives the mass of the lightest fermion mode, m_{f0}, by showing $2\ln\frac{m_{f0}}{m_{A0}}$. Clearly, there is an exponentially light fermion in the spectrum.

$$\sum_i q_{i,\mathrm{left}}^2 - \sum_j q_{j,\mathrm{right}}^2 = 0. \tag{33}$$

An example of such a theory which we will use here is the "345" theory where there are left-handed fermions of charge 3 and 4 as well as right-handed fermions of charge 5:

$$3_-, 4_-, 5_+. \tag{34}$$

Before adding gauge breaking mass terms, our warped domain-wall construction necessarily contains the mirror fermions as well, 3_+, 4_+, 5_-. As mentioned above, the Lorentz structure requires that left- and right-handed modes be lifted in pairs and so at least one neutral mode is needed in order to provide enough mass terms to remove the unwanted charged modes.

If only Dirac masses are contained in the theory, then one global fermion number $U(1)$ symmetry will remain. A current of charged fermions may end up in neutral modes, with the gauge charge absorbed by the Higgs field. However, the target continuum theory violates fermion number and the 't Hooft operator has the (schematic) form:

$$(3_-)^3 \partial_+ (4_-)^4 (\bar{5}_+)^5, \tag{35}$$

where 4_- denotes a Weyl fermion field representing a left-handed Weyl fermion of $U(1)$ charge 4, etc. Therefore Majorana masses are needed in order to introduce a violation of fermion number into the lattice (this has already been noted in [20]).

To describe them, recall that, as discussed in the previous subsection, we introduced two neutral Dirac modes $\eta^{1,2}$, which led, before adding any of the mass terms (27), (29), and (32), to four Weyl fermions—the left-handed ones ($n_-^{1,2}$) localized near the UV brane and the right-handed ones ($n_+^{1,2}$), localized at the far left in the "flat slice bulk." The wave functions of the lightest modes of $\eta^{1,2}$ as well as those of the 345 Dirac fields are shown in Fig. 4.

We now add the equal or opposite strength Majorana and Dirac masses, (27) and (29), to the neutral modes, so that one massless neutral Dirac mode remains—the n_+ at the far left end and n_- near the UV brane. Finally, we add the mass terms (32), coupling the unwanted charged mirror modes and the n_- mode. In addition, we add Majorana mass terms of the form $\eta^1_{N-}\psi^3_{1+}$ (and similar for all unwanted charged modes (ψ^3_{1+}, ψ^4_{1+}, ψ^5_{1-}, including appropriate powers of the Higgs field) violating the fermion number symmetry. Note that there are an equal number of left- and right-handed fields to which we are giving a mass.

With all of the masses discussed above, the only remaining exact symmetry in the theory is the global part of the gauge symmetry, the 345 symmetry. However, the 't Hooft operator preserves also another global symmetry, 133, where the 4_- and 5_+ transform with 3 times the phase of the 3_-. We speculate that either this 133 symmetry will emerge in the IR, or else we have found a theory which preserves no symmetry beyond the gauged 345, which can happen if 133-violating operators (e.g., 4-fermi operators in 2 dimensions) remain relevant in the IR. Later on, we will present a "one-site" model with GW fermions, where both the 345 and 133 symmetries are exact symmetries of the partition function while the fermion number has the correct anomaly.

D. Scalar couplings

While the gauge couplings appear to be perturbative, we verify here that the longitudinal mode of the gauge boson is not strongly coupled to the fermion wave functions as it is in the AdS$_5$ case. We begin by writing in the gauge terms and then calculating the relevant Yukawa coupling.

We will do this calculation in the 3-dimensional continuum language in AdS where an analytic expression for the gauge boson wave function may be found. We take the

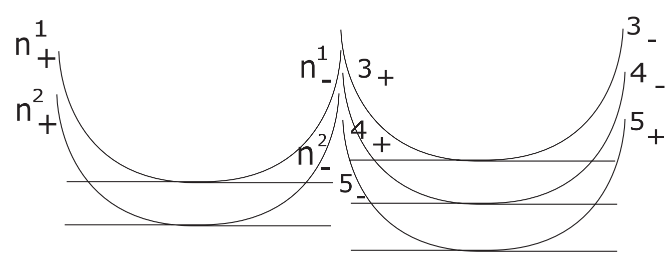

FIG. 4. A schematic representation of the location of all (exponentially light) modes which are needed for the 345 theory to have the correct anomaly properties. The right half is gauged; the left is neutral. After introducing the UV-brane mass terms from Eqs. (27), (29), and (32), the only remaining light modes will be the 3_-, 4_-, and 5_+ from the right-hand side as well as one Weyl combination of the n^1_+ and n^2_+ from the left-hand side.

lightest longitudinal gauge mode to be:

$$A_\mu = f_0(z)\partial_\mu \varphi(x). \tag{36}$$

By using the previously found wave function for arbitrary mass, Eq. (11), and considering the kinetic terms for the gauge field we can find the proper normalization:

$$f_0(z) = \sqrt{R}\ln(z/R). \tag{37}$$

To see how this longitudinal mode couples to the fermions we look at the gauge terms in the fermion kinetic term:

$$\int d^2x dz\left(\frac{R}{z}\right)^2(-2i)(-ig_3)(\bar{\psi}_- A_+ \psi_- + \bar{\psi}_+ A_- \psi_+), \tag{38}$$

where we are working in the $A_3 = 0$ gauge. In terms of the longitudinal mode after an integration by parts we have

$$2g_3 \int d^2x dz\left(\frac{R}{z}\right)^2 f_0(z)\varphi[\partial_+(\bar{\psi}_-\psi_-) + \partial_-(\bar{\psi}_+\psi_+)]. \tag{39}$$

After expanding these derivatives we have four terms involving derivatives on the fermions. By making use of the equations of motion we can turn these four terms into expressions involving the fermion masses, the ∂_z, and the gauge modes. The terms involving the bulk mass and gauge modes cancel, leaving us with

$$g_3 \int d^2x dz\left(\frac{R}{z}\right)^2 f_0(z)\varphi\bigg[-\partial_z\bar{\psi}_+\psi_- - \bar{\psi}_-\partial_z\psi_+ - \partial_z\bar{\psi}_-\psi_+ - \bar{\psi}_+\partial_z\psi_- + \frac{2}{z}(\bar{\psi}_+\psi_- + \bar{\psi}_-\psi_+)\bigg]. \tag{40}$$

We can now perform an integration by parts in z on two of these terms (for example the first two). Most terms cancel (including the $2/z$ terms) leaving us with:

$$g_3 \int d^2x dz\left(\frac{R}{z}\right)^2 \partial_z f_0(z)\varphi(x)(\bar{\psi}_+\psi_- + \bar{\psi}_-\psi_+). \tag{41}$$

The fermion kinetic term has the same leading factor of $(R/z)^2$ and so we should rescale the fields to get canonical kinetic terms. This leaves us with a Yukawa coupling between the longitudinal mode, φ, and the fermions:

$$y(z) = g_3\partial_z f_0 = \frac{g_3\sqrt{R}}{z}. \tag{42}$$

From the scaling requirements in Section III A we found that we must have $g_3^2 R \ll 1$. This means that our Yukawa coupling is smaller than the local scale $1/z$ which sets the fermion masses at that location. In addition, this Yukawa coupling has units of mass as we expect for a coupling between a 2-dimensional scalar and two 2-dimensional fermions.

To see that this Yukawa really is perturbative we can estimate the size of loop corrections to the fermion mass and kinetic terms. Consider one-loop contributions to the fermion two-point function which involve one scalar and one fermion in the loop. The external fermion legs are in the site basis (at sites i and j), but the propagator in the loop must be in the mass eigenstate basis. This gives a contribution:

$$y_i y_j \sum_k \alpha_{ik}\alpha_{jk}\int d^2p\,\frac{\gamma^\mu p_\mu - m_\psi^{(k)}}{p^2 - m_\psi^{(k)2} + i\epsilon} \times \frac{1}{(q-p)^2 - m_\varphi^2 + i\epsilon}, \tag{43}$$

where α relates the site and KK bases:

$$\Psi_i = \sum_k \alpha_{ik}\Psi^{(k)}. \tag{44}$$

After combining the denominators and shifting the momentum we see that on dimensional grounds the result scales as

$$\sum_k \alpha_{ik}\alpha_{jk}\frac{y_i y_j}{m_\psi^{(k)2}}(\gamma^\mu q_\mu - m_\psi^{(k)}). \tag{45}$$

First, let us heuristically argue that this is small and then compute this matrix using our numerical solutions for the wave functions. The KK modes tend to be localized in that part of the space which corresponds to their mass: $z \sim 1/m$. If the modes were exactly localized then the matrix α would be diagonal. Furthermore α is unitary and the masses scale like $m^{(k)} \sim 1/(az_k)$ for most of the KK modes. In that case we have

$$Rg_3^2 a^2 \delta_{ij}(\gamma^\mu q_\mu - m_\psi^{(i)}). \tag{46}$$

This is clearly a small correction to the action at each site.

In fact the matrix relating site and KK bases is not diagonal, but we can find a numerical solution for α_{ij} and the KK masses. We may then calculate this sum over KK modes in Eq. (45). (This sum includes the charged mode, l_+, which became heavy along with the neutral mode, n_-). Doing this shows that the results are in fact a small number (of order a^2) times our leading small factor of Rg_3^2. Note that the exchange of the fermion zero modes has been ignored because it is IR divergent. However, that IR divergence is present in the target theory as well, so it is to be expected (see Appendix B for a calculation of the domain-wall beta function in the deconstucted version of the theory).

IV. A ONE-SITE CONSTRUCTION

In this section, we present a one-site model using GW fermions which has precisely the light field spectrum discussed at length above. It also exhibits *exactly* the right set of symmetries and anomalies to be a candidate for a lattice formulation of the 345 theory. The advantage of this formulation is that its chiral symmetries—which are only expected to emerge at large N in the warped domain-wall model—are exact symmetries of the lattice theory. Thus one can study their consequences, including the associated exact (anomalous or nonanomalous) lattice Ward identities.

Furthermore, a strong-coupling analysis at the end of this section indicates that the spectrum of this theory is chiral and that this proposal may be a road to constructing the fermion measure for chiral gauge theories with GW fermions starting from a vectorlike theory, where the measure is well defined.

In essence, the idea is to consider a ''one-site limit'' of our construction of Section III, using 2-dimensional GW fermions in order to implement exact lattice chiral symmetries. Schematically, the field content and couplings of the model are represented in Fig. 5. There are, for the 345 $U(1)$ theory, three 2-dimensional Dirac fermions, Ψ_3, Ψ_4, Ψ_5, charged under the $U(1)$ gauge group with charges 3, 4, 5, respectively. There is also a neutral Dirac fermion, Ψ_0.

The fermion fields live on the sites, labeled by $\{x\}$, of a 2-dimensional lattice and their lattice action consists of kinetic terms:

$$S_{\rm kin} = \sum_{q=0,3,4,5} \sum_{x,y} \bar{\Psi}_q(x) D_q(x,y) \Psi_q(y), \qquad (47)$$

where D_q is the GW operator for a fermion of charge q, obeying the GW relation (for a review of the GW relation and exact chiral symmetry on the lattice, see, for example [3] and references therein):

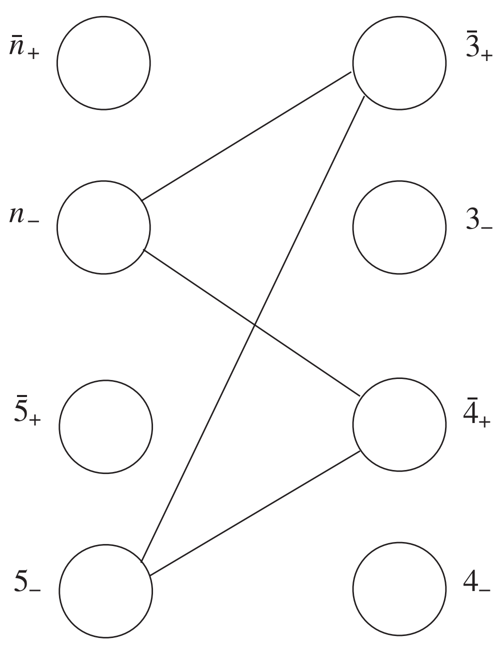

FIG. 5. The 2-dimensional model using GW fermions. The lines represent arbitrary $\mathcal{O}(1/a)$ masses of both Dirac and Majorana type. Because of the chiral symmetry present in the GW formulation, each fermion is exactly massless before an explicit mass term is added. Therefore, four modes remain massless: n_+, 3_-, 5_+, and 4_-.

$$\{D_q, \gamma_5\} = D_q \gamma_5 D_q. \qquad (48)$$

Here γ_5 is the appropriate matrix in 2 dimensions and the lattice spacing has been set (from now on) to unity. The lattice action (47) has a large number of exact global symmetries:

$$\prod_{q=0,3,4,5} U(1)_{q,-} \times U(1)_{q,+}, \qquad (49)$$

where $U(1)_{q,\pm}$ acts only on the Dirac fermion field of charge q as follows:

$$\Psi_q \to e^{i\alpha_{q,\pm} P_\pm} \Psi_q \qquad \bar{\Psi}_q \to \bar{\Psi}_q e^{-i\alpha_{q,\pm} \hat{P}_\mp}, \qquad (50)$$

where $P_\pm = (1 \pm \gamma_5)/2$ and $\hat{P}_\pm = (1 \pm \hat{\gamma}_5)/2$ with $\hat{\gamma}_5 \equiv (1-D)\gamma_5$ ($\hat{\gamma}_5^2 = 1$ follows from the GW relation (48); also note that Ψ_q and $\bar{\Psi}_q$ transform differently, which is perfectly natural in Euclidean space). The projector used for every Ψ_q involves the appropriate GW operator D_q.

That the symmetries in Eq. (50) are all exact follows from $\hat{P}_\mp D = D P_\pm$—yet another consequence of the GW relation (48). Furthermore, the measure of integration is not invariant under any individual $U(1)_{q,+}$ or $U(1)_{q,-}$. Instead, under a $U(1)_{q,\pm}$ transformation (50) with parameter $\alpha_{q,\pm}$, the measure changes:

$$\begin{aligned} U(1)_{q,\pm}\colon \prod_{r=0,3,4,5} d\bar{\Psi}_r d\Psi_r \to & \prod_{r=0,3,4,5} d\bar{\Psi}_r d\Psi_r \\ & \times [1 - i\alpha_{q,\pm} \mathrm{Tr}(P_\pm - \hat{P}_\mp)] \\ = & \prod_{r=0,3,4,5} d\bar{\Psi}_r d\Psi_r \\ & \times \left[1 \pm i\alpha_{q,\pm} \mathrm{Tr}\left(\gamma_5 - \frac{1}{2} D_q \gamma_5\right)\right]. \end{aligned} \qquad (51)$$

Eq. (51) implies that for vectorlike symmetries $U(1)_{qV}$ ($\alpha_{q,+} = \alpha_{q,-}$), there is no Jacobian and thus they are true symmetries of the theory. On the other hand [21,22], since $\mathrm{Tr}(\gamma_5 - \frac{1}{2} D_q \gamma_5) = n_+^0 - n_-^0$ (the difference between the number of left- and right-handed zero modes of D_q), the continuum violation of charge for anomalous symmetries is reproduced by the nonzero Jacobian.

To construct our candidate 345 chiral lattice theory, we introduce a unitary Higgs field, $\phi(x)$, living on the lattice sites (we assume that the issues with building a UV completion, or the necessity thereof, see [23], for the unitary Higgs field are independent of the problem of chirality on the lattice). We will use $\phi(x)$ to write all possible Dirac and Majorana mass terms that violate all symmetries (49) of the kinetic term (47) except:

$$U(1)_{3,-} \times U(1)_{4,-} \times U(1)_{5,+} \times U(1)_{0,+}. \qquad (52)$$

The explicit form of the mass matrix is described in what follows, Eq. (55), and is also schematically indicated in Fig. 5. The lattice path integral measure is not invariant under all four $U(1)$ symmetries (52) of action. It only respects three linear combinations: the $U(1)_{345}$ and the $U(1)_{133}$ chiral symmetries—linear combinations of $U(1)_{3,-} \times U(1)_{4,-} \times U(1)_{5,+}$ with coefficients 345 and 133, respectively—and the $U(1)_{0,+}$, which acts only on the $n_+ \equiv P_+ \Psi_0$ neutral field, whose dynamics are expected to decouple from the physics of the charged sector.

The 345 and 133 $U(1)$'s are exact global symmetries of the partition function. On the other hand, the third linear combination of the first three $U(1)$'s in Eq. (52)—the fermion number symmetry of the light fields, which can be taken to be the "111" symmetry—has an anomaly exactly reproduced by the Jacobian, Eq. (51), of the corresponding transformation of the measure; see [21,22], and references in [3]. Thus, the lattice theory obeys exact Ward identities, including the anomalous ones. For example, using (51) one finds that the 111 transform of an operator $\mathcal{O}$ obeys the exact lattice Ward identity:

$$\langle \delta_{\alpha_{111}} \mathcal{O} \rangle = i \frac{\alpha}{2} \langle \mathcal{O} \,\mathrm{Tr}[\gamma_5 (D_3 + D_4 - D_5)] \rangle. \quad (53)$$

The continuum limit expansion $\mathrm{Tr}\gamma_5 D_q \sim \int d^2x \epsilon^{\mu\nu} F_{\mu\nu}$ [22] implies that the anomalous Ward identity (53) has a continuum limit exactly as expected.

To ensure that the dynamics of this theory reproduce that of the desired unbroken chiral gauge theory, we next focus our attention on the coupling of the Higgs field to the fermions, as well as on its kinetic term (i.e., the mass term for the gauge field). In particular, we will study the possible existence of the strong-Yukawa-coupling symmetric phase (recall again the strong-coupling analysis of [12] which showed that in the waveguide model the spectrum in this phase was vectorlike). Remarkably, as we find below, to leading order in the strong-Yukawa-coupling expansion and small gauge coupling—precisely the regime where the waveguide idea broke down—there appear no new massless modes and the spectrum of the unbroken gauge theory is now chiral.

We begin by writing the most general mass matrix which breaks the symmetries of Eq. (49) to the four chiral $U(1)$'s of (52). To this end, we relate the Dirac fields Ψ_q to their chiral components: $\Psi_{q,\pm} \equiv P_\pm \Psi_q$, $\bar{\Psi}_{q,\pm} \equiv \bar{\Psi}_q \hat{P}_\mp$; note that the definition of the $\bar{\Psi}_\pm$ chiral modes is now both momentum and gauge-background dependent. We then write down the most general Dirac and Majorana couplings—giving mass of order the inverse cutoff—to the fields:

$$X_+ = (\Psi^T_{3,+} \bar{\Psi}_{3,+} \Psi^T_{4,+} \bar{\Psi}_{4,+}) \qquad Y_- = \begin{pmatrix} \Psi_{5,-} \\ \bar{\Psi}^T_{5,-} \\ \Psi_{0,-} \\ \bar{\Psi}^T_{0,-} \end{pmatrix}, \quad (54)$$

where T denotes transposition (we treat unbarred Dirac spinors as columns and barred ones as rows) of the form:

$$S_{\mathrm{mass}} = \lambda \sum_x X_+(x) M Y_-(x). \quad (55)$$

The structure of S_{mass} is evident from Fig. 5, where both Dirac and Majorana masses are to be included for the connected fields. We note that if Majorana masses are omitted, there will be extra unbroken chiral symmetries and unlifted zero modes in an instanton background, resulting in failure to reproduce the 't Hooft vertex (35); moreover, consistent with the symmetry argument, a careful analysis shows that with Dirac masses only, the mass matrix (55) has a zero eigenvalue at the end of the Brillouin zone; details will be given in a future publication.

Instead of writing explicitly the entire matrix M, we give an example of a Dirac mass term: $\bar{\Psi}_{0,-}(\phi^*)^3 \Psi_{3,+} + \bar{\Psi}_{3,+} \phi^3 \Psi_{0,-}$, and of a Majorana mass of the form: $\bar{\Psi}_{5,-} \gamma_2 \phi^8 (\bar{\Psi}_{3,+})^T - \Psi^T_{3,+} \gamma_2 (\phi^*)^8 \Psi_{5,-}$. Here γ_2 is the (hermitean) 2d gamma matrix that appears when Majorana masses are written using Dirac spinors, while ϕ is the unitary Higgs field. Thus, the explicit form of M in (55) contains appropriate powers of ϕ and γ_2-insertions. The general mass matrix (55) violates all $U(1)$ symmetries from (49) and preserves the desired $U(1)_{3,-} \times U(1)_{4,-} \times U(1)_{5,+} \times U(1)_{0,+}$ symmetry (52).

The total action of our lattice model is, finally:

$$S = S_{\mathrm{Wilson}} + S_{\mathrm{kin}} + S_{\mathrm{mass}} + \frac{\kappa}{2} \sum_x \sum_{\hat{\mu}} [2 - (\phi(x)^* U(x, \hat{\mu}) \phi(x + \hat{\mu}) + \mathrm{h.c.})], \quad (56)$$

where S_{kin} is defined in (47), S_{mass}—in (55), the last term is the kinetic term for the charge-1 unitary lattice-Higgs field ϕ, and S_{Wilson} is the usual plaquette action for the link variables $U(x, \hat{\mu})$ (appropriately modified to restrict the gauge field path integral to admissible gauge field backgrounds, see [3]).

In the broken phase, when $\langle \phi \rangle \neq 0$, we already analyzed the fermion spectrum and found that there are four light modes: the charged $\Psi_{3,-}$, $\Psi_{4,-}$, $\Psi_{5,+}$, and the neutral $\Psi_{0,+}$. The dynamical issue that needs to be addressed is the existence of an unbroken phase where $\langle \phi \rangle = 0$, such that the gauge boson is "massless." The gauge symmetry can thus be thought of as "emerging" in the IR [24]. The essential idea behind the "Foerster-Nielsen-Ninomiya (FNN) mechanism" [24] is that integrating out the fluctuations of the unitary Higgs field, whose correlation length in the symmetric phase is a few lattice spacings, results in renormalization of the gauge coupling plus a tower of higher-dimensional gauge invariant local operators which are irrelevant for the long-distance physics of the gauge theory. The fact that lattice-Higgs models exhibit such behavior is well known; for the 2-dimensional case, see also [23,25] for a general analysis in various dimensions. Previous discussions of the use of this mechanism to the lattice definition of chiral gauge theories are given in Refs. [26,27].

In our case, an important requirement further to the "restoration" of the gauge symmetry at distances larger than a few lattice spacings should be that there are no new massless fermions in addition to the desired massless chiral spectrum.

To study the continuum limit in the asymptotically free theory it is sufficient to begin at leading order in the $g \to 0$ expansion; in fact, apart from a few comments, here we will confine our analysis to this limit. This freezes the gauge degrees of freedom to $U = 1$. The resulting theory is a unitary Higgs-Yukawa model whose phase structure can be studied in various limits. We are interested in the symmetric phase of the lattice $O(2)$ model and will take $\kappa < \kappa_c$ (simple random-walk intuition leads to the estimate $\kappa_c^{-1} \geq 2d$ in d dimensions on a hypercubic lattice [24]) while also taking the $\lambda \to \infty$ limit. Recall that this was precisely the limit where the strong-coupling analysis of the waveguide showed that new massless fermions were appearing at the waveguide boundary, see our discussion at the end of Section II and Ref. [12].

Since $d\Psi = d\Psi_+ d\Psi_-$, the lattice partition function factorizes, in a trivial gauge background, into a product $Z = Z_{\rm light} \times Z_{\rm mirror}$:

$$\begin{aligned} Z_{\rm light} &= \int \prod_x d\Psi_{3,-} d\Psi_{4,-} d\Psi_{5,+} d\Psi_{0,+} e^{-S_{\rm kin}(\Psi^{light})} \\ Z_{\rm mirror} &= \int \prod_x d\Psi_{3,+} d\Psi_{4,+} d\Psi_{5,-} d\Psi_{0,-} d\phi \\ &\quad \times e^{-S^{\rm mirror}_{\rm kin}(\Psi^{\rm mirror}) - S_\kappa(\phi) - S_{\rm mass}(\Psi^{\rm mirror})}. \end{aligned} \tag{57}$$

For conciseness, we omitted the conjugate fields in the measure and denoted collectively by Ψ_{light} the fields $\Psi_{3,-}$, $\Psi_{4,-}$, $\Psi_{5,+}$, $\Psi_{0,+}$, and by $\Psi_{\rm mirror}$ the heavy charged mirrors $\Psi_{3,+}$, $\Psi_{4,+}$, $\Psi_{5,-}$, and the neutral $\Psi_{0,-}$. The mass term is given by Eq. (55) and the kinetic term for ϕ by (56).

The most important point is the splitting of the kinetic terms (47) into light and mirror modes in (57). This follows from the identity (note that it also holds in an arbitrary gauge background):

$$\bar{\Psi}_q D_q \Psi_q = \bar{\Psi}_{q,+} D_q \Psi_{q,+} + \bar{\Psi}_{q,-} D_q \Psi_{q,-}, \tag{58}$$

where the cross terms vanish due to the GW relation (48). Thus the mirror and light partition functions factorize at $g = 0$; recall our discussion of Section II showing that the lack of factorization in the kinetic terms was the cause of failure of the waveguide. Of course, for $g \neq 0$ the factorization of the measure depends on the gauge field (see discussion in the following paragraphs), but we are only interested in the spectrum of the Yukawa-Higgs model at this point. We stress that the GW relation was crucial in order for (58) to hold; we know of no other way to achieve (58) and hence the factorization (57) on the lattice.

Finally, let us study $Z_{\rm mirror}$ and its effect on the light modes, in the $\lambda \to \infty$ and $\kappa < \kappa_c$ limit. Of particular concern is the possible appearance of extra massless states and the associated vanishing of the mirror determinant. To this end, we redefine the mirror fermion fields in (57) ($\Psi_{3,+}$, $\Psi_{4,+}$, $\Psi_{5,-}$, and the singlet $\Psi_{0,-}$) by $1/\sqrt{\lambda}$. This multiplies their kinetic terms by $1/\lambda$. Thus, as $\lambda \to \infty$, the mirror fields kinetic terms vanish, and the mirror action consists solely of a mass term given by (55) with $\lambda = 1$. We can now perform the integral over the mirror fermions in $Z_{\rm mirror}$, leading to a factor of det$M$—by construction manifestly nonzero and ϕ-independent. Hence, to this order of the strong-coupling expansion, there are no new massless states.

Admittedly, the argument of the previous paragraph is oversimplified. The true story is more complicated, due to the fact that the $\bar{\Psi}_\pm$ chiral components are somewhat smeared due to the nonlocality of the chiral projectors, and will be explained in [15]. Nevertheless, the results there indicate that the scalar dynamics are not significantly affected by the fermions' quantum fluctuations, with the conclusion that the "FNN mechanism" continues to apply, and that there are no light mirror modes.

Ideally, turning on a small gauge coupling will not cause a dramatic rearrangement of the spectrum. While the $g \neq 0$ case clearly deserves further detailed study, we expect that the effect of the mirror fermions on the gauge field and the light chiral fermions is parametrically suppressed by $1/\lambda$ (the one notable exception should occur if the massless fermion spectrum is anomalous, when the gauge coupling is turned on, a mass for the gauge boson is generated [28,29], with the details controlled by the ultraviolet physics). To argue for this, we note that the factorization of the partition function into mirror and light, Eq. (57), due to (58), occurs also in fixed nontrivial gauge backgrounds (details, including the factorization of the fermion measure, are under investigation and will be given elsewhere). The integral over the mirror fermions can now be performed as in the $U = 1$ case above, by noting that the $U \neq 1$ gauge field background interacts with the mirror fermions only through their kinetic terms. Thus, one expects that all effects of the mirror fermions on the gauge field effective action are local and of order $1/\lambda$. Taking into account the kinetic terms of ϕ and the mirror fermions in a strong-coupling expansion leads to corrections to κ: $\kappa \to \kappa + \mathcal{O}(1/\lambda)$, as well as to other $\mathcal{O}(1/\lambda)$ terms, like $\sum_x (\phi(x))^3 (\phi(x+\mu)^*)^3$, etc., including higher powers of ϕ. A detailed study of the phase diagram away from the $1/\lambda$ expansion is beyond our scope here; we stress again that this stage of our analysis—the $g = 0$ analysis of the Yukawa-Higgs model—was precisely where the waveguide model failed [12] to reproduce the chiral gauge theory spectrum.

We should also note that nothing (except for the need, coming from 2-dimensional Lorentz invariance, to introduce the spectator neutral fermions) about the proposal considered in this section is intrinsically 2-dimensional. In fact, all the steps and relevant properties, including the factorization (58) of the GW fermion kinetic terms and the existence of a "high-temperature" disordered phase of the compact Higgs variables, hold in a 4-dimensional theory as well, particularly in the Abelian case considered here. A more detailed study of a similar construction of non-Abelian chiral theories will be given elsewhere.

Finally, we reiterate why we think that this "one-site" proposal is of interest. It is a) a full lattice proposal (not deconstructed—all dimensions are latticized) of a local action and measure for a chiral gauge theory, b) the realization of both the anomalous and anomaly-free global symmetries is exactly as in the target continuum theory, and, c) we gave plausibility arguments why the FNN mechanism may work and the breaking of gauge symmetry be irrelevant in the infrared.

While we have not proven that the proposal results in a chiral lattice gauge theory, we believe that the three points above warrant its presentation and further study. Clearly, the study of the $g = 0$ dynamics currently underway [15] has to be followed by a detailed study of the $g \neq 0$ case, both in perturbation theory and nonperturbatively, and by a convincing demonstration that an unbroken lattice gauge theory with a chiral spectrum of fermions has been constructed.

V. SUMMARY, RELATION BETWEEN THE TWO MODELS, AND OUTLOOK

Let us first summarize the main results of this paper.

(1) We began by a study of the earlier proposal of the warped domain-wall model [7]. Motivated by the strong-coupling issues encountered by this proposal in 4 dimensions [7], we turned to the simpler 2-dimensional case, where the target 2-dimensional chiral theory is the IR limit of a 3-dimensional theory in a slice of AdS_3. We studied in detail the spectrum and perturbative expansion of the deconstructed version of the theory (it is expected that this analysis is adequate also for small enough lattice spacing in the 2 dimensions of the target theory). We showed through a perturbative analysis that in the $N \to \infty$ limit the IR theory has massless gauge bosons and a chiral spectrum of massless fermions in the weak-coupling regime. We found no strong coupling of the Goldstone mode to the fermions, in contrast to the 4-dimensional case. Thus, while our 2-dimensional study has nothing to say about the viability of warped domain walls in the physically interesting case of 4 dimensions, it indicates that this proposal is still of interest and worthy of further study. We believe that it is a useful first step towards the full lattice study of this proposal (which still awaits implementation).

(2) Next, we proposed a purely 2-dimensional lattice theory, a "one-site model." It uses the GW mechanism of exact realization of chiral global symmetries at finite lattice spacing. The model has modified, momentum and gauge-background dependent, chiral symmetries, which reduce to the usual continuum chiral symmetries for the low-lying modes. The exact chiral symmetry also ensures that the Ward identities at finite lattice spacing are the ones of the continuum theory. We argued, in a preliminary strong-Yukawa-coupling analysis, for the existence of an unbroken phase with a chiral spectrum of fermions (at $g = 0$), in contrast to the analogous phase of the waveguide model. Admittedly, a more detailed analysis is needed; in this regard, we note that the forthcoming results of [15] offer a strong indication that the plausibility arguments given in Section IV indeed hold.

The common theme of the two proposals is that the chiral spectrum is obtained after a particular limit of a vectorlike theory is taken, which decouples the mirrors while keeping the gauge boson massless. Thus, both proposals are similar to the "waveguide" models.

The first proposal uses warping and localization to address the weak-coupling problems of the waveguide models, discussed in Section II and Ref. [11]) where the Higgs-Yukawa sector of the theory is in the broken phase. On the other hand, the "one-site" proposal is inherently a strong-coupling one—it was motivated by the observation that GW fermions avoid the left-right mixing that led to a vectorlike spectrum in the strong-Yukawa limit of the waveguide model [12]. The mechanism of the "one-site" proposal depends on the existence of the strong-coupling symmetric phase of the Higgs-Yukawa theory and on the validity of the "FNN mechanism."

For finite values of N and a we expect that these theories are not equivalent—the global symmetries are different since the 1-site model respects the 133 symmetry, while the warped domain-wall model does not. In addition, the GW fermion model has a massless gauge mode, while the domain-wall model has just a light gauge mode. Thus, if they are the same it could only be in the intermediate energy regime.

Which of the two proposed lattice theories is more amenable to study in practical simulations is a question that we have not touched upon. We note that, once the two noncompact directions are latticized via Wilson fermions, the action of our proposed "warped domain wall" will be manifestly reflection positive. On the other hand, the question of reflection positivity (Hermiticity in real time) of the single-site model deserves further study; to this end, a Hamiltonian formulation might be desirable. Here, we only point out that nonpositivity may be irrelevant in the continuum limit—examples of nonreflection positive lattice actions appear commonly in lattice constructions of supersymmetric target theories (for a recent review, see [30]). These lattice actions preserve some exact nilpotent supersymmetries; in fact, demanding invariance under these is the ultimate reason for nonpositivity. Despite nonpositivity, however, it has been argued or shown [31,32] that in the continuum limit the models possess a positive self-adjoint transfer matrix.

Another issue left for future work is the positivity of the fermion determinant. We have nothing to say about it here and only note that both the overlap formulation of Ref. [33] and the construction via GW fermions of Ref. [14] have a complex measure problem: the Euclidean effective action for 4-dimensional fermions in nonreal representations is generally expected to be complex [34].

Finally, we note that our proposal suggests a way to define the fermion measure for the construction of [14] by obtaining the unbroken chiral gauge theory—our light partition function $Z_{\rm light}$ of (57))—from a particular limit of a vectorlike theory. The fermion measure in our vectorlike models is well defined and hard questions of how its phase or dimensionality change as one varies the gauge background do not arise. To this end, it would be desirable to understand in more detail the behavior of our "single-site" model in topologically nontrivial backgrounds.

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APPENDIX A: FERMIONS IN CONTINUUM AND DISCRETIZED SLICE OF ADS$_3$

For completeness, in this appendix we present the formulas relevant for the description of fermions in a slice of AdS$_3$ in the continuum and on the lattice (deconstructed version). These expressions are relevant for obtaining the mass matrix (31) of the charged fermions.

The bulk action for a 3-dimensional Dirac fermion in a slice of AdS$_3$, in terms of their 2-dimensional Weyl components, is:

$$S_\Psi^{\rm bulk} = \int d^2x \int_R^{R'} dz \left(\frac{R}{z}\right)^2 \Big[2i\bar{\psi}_-\partial_+\psi_- + 2i\bar{\psi}_+\partial_-\psi_+ - \frac{i}{2}(\bar{\psi}_+\partial_z\psi_- - \partial_z\bar{\psi}_-\psi_+ + \bar{\psi}_-\partial_z\psi_+ - \partial_z\bar{\psi}_+\psi_-) + i\frac{mR}{z}(\bar{\psi}_-\psi_+ - \bar{\psi}_+\psi_-)\Big], \tag{A1}$$

where m is a real mass parameter (odd under 3d parity) and R is the AdS$_3$ curvature radius (the contribution of the spin connection is "hidden" and can be recovered upon integrating by parts some of the z-derivatives). The equations of motion are trivially solved for the fermion zero modes ($\partial_+ = \partial_- = 0$) and yield two solutions of opposite chirality:

$$\psi_-^{(0)} = c_1 z^{1-mR}, \qquad \psi_+^{(0)} = c_2 z^{1+mR}. \tag{A2}$$

Clearly, the zero modes (A2) can be localized anywhere with the right choice of mR. To see that, let us substitute the ψ_- zero mode (and set $\psi_+ = 0$) into the action (A1):

$$\psi_-^{(0)} = \chi_-(x)\left(\frac{z}{R}\right)^{1-mR}, \tag{A3}$$

where $\chi_-(x)$ is now an x^0, x^1 dependent function—the wave function of the zero mode; note that $\chi(x)$ has the same mass dimension (one) as ψ. Clearly, only the $\bar{\psi}_-\partial_+\psi_-$ term in (A1) contributes, giving the following 2-dimensional action for $\chi_-(x)$:

$$S_{c_1} = R\int d^2x 2i\bar{\chi}_-(x)\partial_+\chi_-(x)\int_1^{R'/R} dy y^{-2mR} = \frac{R(R'^{1-2mR} - R^{1-2mR})}{1-2mR}\int d^2x 2i\bar{\chi}_-(x)\partial_+\chi_-(x). \tag{A4}$$

We interpret (A4) by taking various limits: if $mR > 1/2$, we can certainly take $R' \to \infty$, i.e., the IR brane to infinity, and still have a finite action 2-dimensional mode. This implies that the χ_- zero mode is localized near the UV brane if $mR > 1/2$. If $mR < 1/2$ the localization is nearer the IR brane (this is clear from (A2): for, say, negative mR, we have $\psi_-^{(0)}$ growing at large-z, indicating IR localization).

Clearly, the story is the opposite for the $\psi_+^{(0)}$ zeromode—the two cases simply differ by the sign of mR—so when $mR < -1/2$ (regime where $\psi_-^{(0)}$ was IR-localized) we have UV localization of $\psi_+^{(0)}$. Conversely, when $mR > -1/2$, $\psi_+^{(0)}$ is IR-localized.

Next, we discretize the following continuum action, obtained from (A1) upon integration by parts and dropping of the boundary terms—this is needed, as in [7], in order to obtain Wilson terms and hence no doublers in the discretized bulk:

$$S_\Psi^{\rm bulk} = \int d^2x \int_R^{R'} dz \left(\frac{R}{z}\right)^2 \Big[2i\bar{\psi}_-\partial_+\psi_- + 2i\bar{\psi}_+\partial_-\psi_+ - i(\bar{\psi}_+\partial_z\psi_- - \partial_z\bar{\psi}_-\psi_+) + i\frac{mR-1}{z}(\bar{\psi}_-\psi_+ - \bar{\psi}_+\psi_-)\Big]. \tag{A5}$$

Now we have N 2d Dirac fermions $(\psi_-^k, \psi_+^k)^T$, $k = 1, \ldots N$, each charged under the corresponding gauge group, $\psi_\pm^k \to g_k\psi_\pm^k$; $\partial_\pm \to D_\pm = \partial_\pm + iA_\pm^k$. Gauge invariant "hopping" terms between the different groups can be written using the unitary bifundamental links U_k, $k = 1, \ldots N-1$.

The bulk lattice fermion Lagrangian we thus (note that ∂_z in (A5) is replaced by the symmetric lattice derivative) obtain is:

$$L_\Psi^{\rm bulk} = i\sum_{k=1}^{N} aRw^k[2\bar\psi_-^k D_+\psi_-^k + 2\bar\psi_+^k D_-\psi_+^k] - \sum_{k=1}^{N-1} w^{2k+1}(\bar\psi_+^{k+1}U_k\psi_-^k - \bar\psi_-^k U_k^\dagger \psi_+^{k+1}) - \sum_{k=1}^{N} w^{2k-1}(1-awmR)(\bar\psi_-^k\psi_+^k - \bar\psi_+^k\psi_-^k). \tag{A6}$$

Similar to the gauge field case, we define new fermion fields:

$$\psi_\pm^k \to \psi_\pm^k (2w^k aR)^{-(1/2)}, \tag{A7}$$

and the complex conjugate for the $\bar\psi$ fields. The new fields now have proper canonical dimension $1/2$. We also define:

$$\alpha \equiv 1 - awmR, \tag{A8}$$

in terms of which the new bulk Lagrangian is:

$$L_\Psi^{\rm bulk} = i\sum_{k=1}^{N} \bar\psi_-^k D_+\psi_-^k + \bar\psi_+^k D_-\psi_+^k - \sum_{k=1}^{N-1} \frac{w^{k+(1/2)}}{2awR}(\bar\psi_+^{k+1}U_k\psi_-^k - \bar\psi_-^k U_k^\dagger \psi_+^{k+1}) - \sum_{k=1}^{N} w^{k-1}\frac{\alpha}{2awR}(\bar\psi_-^k\psi_+^k - \bar\psi_+^k\psi_-^k). \tag{A9}$$

The mass matrix of Eq. (31) can be then easily read off Eq. (A9). It is also possible to use $L_\Psi^{\rm bulk}$ to find analytically the zero modes in the discretized version and show that they are well approximated by the continuum expressions (A2) at large N.

APPENDIX B: DOMAIN-WALL β-FUNCTION

Our warped domain-wall construction arising from AdS_3 has the spectrum of fermions that we expect for a chiral gauge theory. As explained in Section III B the tower of KK modes becomes heavy and the light neutral modes decouple in the $N \to \infty$ limit. Since this theory is perturbative in the large N limit, we expect naïve decoupling arguments to hold for individual modes.

One might wonder, though, if the large number of modes could have a nontrivial contribution even in the IR. However, the masses of most of the KK modes for our deconstructed AdS_3 are given approximately by:

$$m_n = \mathcal{O}\left(\frac{w^{-n}}{aR'}\right). \tag{B1}$$

Only the lightest and heaviest handful of modes deviate from this expression (this spectrum of modes is different from the 3-dimensional continuum where the spacing is linear in n). These masses are rising exponentially, and so we expect that the contribution to an IR propagator from all N modes is not much larger than the contribution from just one mode at the KK scale: $1/R'$.

To verify this expectation, we outline here a 2-dimensional continuum calculation of the β-function for our theory in the IR using all of the fermions in the entire mass matrix. We may write our fermion Lagrangian as

$$\frac{i}{2}(\vec\Psi_-^\dagger, \vec\Psi_+^\dagger)\begin{pmatrix} D_+ & -\tilde M^T \\ \tilde M & D_- \end{pmatrix}\begin{pmatrix}\vec\Psi_- \\ \vec\Psi_+\end{pmatrix}, \tag{B2}$$

where $\tilde M$ is related to the full fermion mass matrix given in Eq. (23). The only difference being that some rows were interchanged since our action here is written with $\Psi^\dagger$ rather than Ψ^T. The appropriate covariant derivative is $D_\pm = \partial_\pm - i\hat g \hat f A_{\pm,0}(x)$, where $A_{\pm,0}(x)$ is the 2-dimensional wave function of the lightest gauge mode, $\hat g$ is the charge matrix, diagonal with either the site charges g_i for charged modes or zero for neutral modes, and the matrix $\hat f$ has the wave function of the gauge boson zero mode down its diagonal entries: $\hat f_{i,j} = f_i\delta_{i,j}$, so that these factors together reproduce the gauge coupling of the lightest mode. We are only interested in the lightest gauge mode since we want the one-loop beta function at low energies. The corresponding wave function is given in Eq. (37), or it can be found from the gauge boson mass matrix.

The momentum space fermion propagator can be written as:

$$G(p) = -2i\begin{pmatrix} p_+ & -\tilde M^T \\ \tilde M & p_- \end{pmatrix}^{-1}. \tag{B3}$$

The one-loop correction to the A_+A_+ correlator is:

$$\int d^2p\,{\rm Tr}\left[\begin{pmatrix}\frac{\hat g\hat f}{2} & 0\\ 0 & 0\end{pmatrix}G(p)\begin{pmatrix}\frac{\hat g\hat f}{2} & 0\\ 0 & 0\end{pmatrix}G(q-p)\right], \tag{B4}$$

and similarly for the A_-A_- or A_+A_- correlators. These are all well defined matrices and may be manipulated numerically. More specifically, we can plot the momentum dependence of the integrand for the one-loop correction in the IR and verify that it is what we expect for a chiral theory. Only the light left-handed modes: 3_- and 4_-, should contribute to the A_+A_+ correlator and only the light right-handed mode: 5_+ should contribute to A_-A_-. The A_+A_- correlator only gets contributions from the massive modes and so it should be suppressed by the KK scale (whatever regulator is used to define our formal continuum 2-dimensional perturbative expansion also contributes to the A_+A_-, with a coefficient that can be determined solely by demanding gauge invariance in the anomaly-free theory, hence we need not specify it; see, e.g., the calculation of the 2-dimensional anomaly in [35]).

Since the 345 theory is anomaly-free, the contribution from the left- and right-handed modes to their respective A_+A_+ and A_-A_- correlators will be equal; it is therefore sufficient to consider the anomalous theory of subsection III B, where there was only one light charged field, l_-. By numerically calculating and plotting the momentum dependence of the integrand of Eq. (B4) we see that there is a pole at both $p_+ = 0$ and $q_+ - p_+ = 0$. The coefficient of this pole is in fact approximately g_2^2, the charge of the light fermions under the restored gauge symmetry. The related expressions for the $A_- A_-$ and $A_- A_+$ correlators do not show any momentum poles. We therefore conclude that this theory gives the appropriate chiral β-function in the IR.

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