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Issues on 3D noncommutative electromagnetic duality

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We extend the ordinary 3D electromagnetic duality to the noncommutative (NC) space-time through a Seiberg-Witten map to second order in the noncommutativity parameter θ, defining a new scalar field model. There are similarities with the 4D NC duality; these are exploited to clarify properties of both cases. Up to second order in θ, we find that duality interchanges the 2-form θ with its 1-form Hodge dual ${}^\star\theta$ times the gauge coupling constant, i.e., $\theta \to {}^\star\theta g^2$ (similar to the 4D NC electromagnetic duality). We directly prove that this property is false in the third order expansion in both 3D and 4D space-times, unless the slowly varying fields limit is imposed. Outside this limit, starting from the third order expansion, θ cannot be rescaled to attain an S-duality. In addition to possible applications on effective models, the 3D space-time is useful for studying general properties of NC theories. In particular, in this dimension, we deduce an expression that significantly simplifies the Seiberg-Witten mapped Lagrangian to all orders in θ.

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I. INTRODUCTION

The 4D noncommutative (NC) electromagnetic duality, up to the subleading order in θ or in the slowly varying fields limit [1,2], via the Seiberg-Witten map [1], relates two $U(1)$ gauge theories and has a curious property [3–5]: one has θ as its noncommutativity parameter, while the other has ${}^\star\theta g^2$ (where * is the Hodge duality operator and g^2 is the gauge coupling constant). This is more than a simple curiosity; it suggests a consistency problem [3,4]. Employing the standard quantization programme, it is well known that a timelike noncommutativity parameter ($\theta^{\mu\nu}\theta_{\mu\nu} < 0$) leads to unitarity violation [6]. Since θ is spacelike iff ${}^\star\theta$ is timelike, the above results suggest that a modification on the quantization programme of NC theories is necessary [7]; otherwise only lightlike noncommutativity ($\theta^{\mu\nu}\theta_{\mu\nu} = 0$) may be consistent with the $U(1)$ NC theory [4].[1]

Since the role of electromagnetic duality in NC theories is so relevant, in this work we extend it to the 3D space-time and evaluate the necessity of the slowly varying fields limit from a classical field theoretical perspective, in order to find what the fundamental properties of this duality are. Many arguments of Ref. [3] depend on the space-time dimension (e.g., the 4D space-time is the only one in which θ and ${}^\star\theta$ are both 2-forms and the S-dual massless gauge fields are both 1-forms); therefore a natural question is how the NC electromagnetic duality presents itself in other dimensions, and to what extent the properties of the 4D NC electromagnetic duality can be extended to those. From all possibilities, the 3D space-time seems to be a natural option. In this space-time, we establish to second order in θ the dual scalar action (consistently with the rule $\theta \to {}^\star\theta g^2$) and we show that many terms of the Seiberg-Witten mapped action can be considerably simplified. The necessity of the slowly varying fields limit, to preserve the rule $\theta \to {}^\star\theta g^2$ and therefore S-duality,[2] starts from the third order in θ for any space-time dimension (with $D \geq 2$).

This paper is organized as follows: after a review of the ordinary 3D electromagnetic duality, we establish its extension to the NC space-time up to first order in θ, providing the duality map and some physical details. In the fourth section, we extend this duality to second order and, through the Seiberg-Witten differential equation, analyze its behavior to higher orders. Finally, in the last section, we present our conclusions.

II. REVISITING THE 3D ELECTROMAGNETIC DUALITY

To introduce our framework, we briefly review the electromagnetic duality in 3D ordinary space-time. The electromagnetic theory action with a 1-form source J is

$$S_A[A, J] = \int (aF \wedge {}^\star F + eA \wedge {}^\star J), \tag{1}$$

where A is the 1-form potential; the field strength F satisfies, by definition, $F = dA$ and $a = -1/(2g^2)$. To preserve gauge invariance and to satisfy the continuity equation, ${}^\star J$ must be a closed 2-form.

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[1]It should be noted that in this work we are concerned with the issue of duality of NC theories within the field theoretical framework. From the string theory perspective, S-duality of IIB strings in the presence of a magnetic background induces a duality between spatially NC Yang-Mills $\mathcal{N} = 4$ theory with a string model called NCOS (noncommutative open string), as conjectured in Ref. [8]. Although our approach and that of Ref. [8] are quite different, there are similarities in the resulting dualities, like the exchange of θ with ${}^\star\theta g^2$. See our Conclusions for further comments.

[2]In the sense of a global inversion of the coupling constant (string S-duality is not of concern in this approach).

As usual, the dynamics of the electromagnetic fields comes from the equation of motion and the Bianchi identity, namely,

$$d^\star F = -\frac{e}{2a}{}^\star J \quad \text{and} \quad dF = 0. \tag{2}$$

Except for the sign of the first equality, the above equations are valid in any space-time dimension. These equations can be expressed on more "observational grounds" through the electric and magnetic fields given by $E^i = F^{i0}$ and $B = \vec{\nabla} \times \vec{A} = -\epsilon^{ij}\partial_i A_j = -F_{12}$. We adopt the conventions $g = \mathrm{diag}(+,-,-)$, $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, $\vec{E} = (E^1, E^2)$, $\vec{\nabla} = (\partial_1, \partial_2)$, $\epsilon_{12} = \epsilon^{12} = 1$; Greek indices can assume the values 0, 1, or 2, and Latin indices only 1 or 2. Using this notation, Eq. (2) becomes

$$\vec{\nabla} \cdot \vec{E} = \frac{e}{2a}\rho, \tag{3}$$

$$\vec{\nabla} \times B = \dot{\vec{E}} + \frac{e}{2a}\vec{j}, \tag{4}$$

$$\vec{\nabla} \times \vec{E} = -\dot{B}, \tag{5}$$

where $J^\mu = (\rho, \vec{j})$; a dot over a field means a temporal derivative and, by definition, $(\vec{\nabla} \times B)^i = \epsilon^{ij}\partial_j B$. These equations have curious similarities and differences with the usual 4D Maxwell equations. Among the differences, since $\vec{E}$ is a vector and B a pseudoscalar, even in the case without sources, there is no hope of finding a simple duality which simply interchanges electric and magnetic fields. However, Eq. (3) with $\rho = 0$ hints to set $\vec{E} = -\vec{\nabla} \times \phi$, which implies $F^{i0} = -\epsilon^{ij}\partial_j\phi$. Thus, to preserve Lorentz symmetry, we shall set

$$F^{\mu\nu} = \epsilon^{\mu\nu\lambda}\partial_\lambda\phi \qquad (\text{for } J = 0). \tag{6}$$

Consequently, $B = -\dot{\phi}$. Using the map (6), it is straightforward to show that Eqs. (3) and (4), without sources, turn into an identity, while (5) becomes the free scalar field equation, $\partial_\mu\partial^\mu\phi = 0$. Note that there is no violation of the number of degrees of freedom; both descriptions (vectorial and scalar) have 1 degree of freedom.

To conclude this introduction, we shall present the 3D electromagnetic duality with a source J and introduce the master Lagrangian approach [9]. Consider the action

$$S_M[F, \phi] = \int\left[aF \wedge \left({}^\star F + \frac{e}{a}\Lambda\right) - d\phi \wedge F\right], \tag{7}$$

where F is regarded as an independent 2-form and Λ is a 1-form. Equating to zero the variation of the above action with respect to ϕ, we obtain $dF = 0$. This implies, in Minkowski space, that $F = dA$. Replacing F by dA and setting ${}^\star J = d\Lambda$, S_M becomes equivalent to the action in Eq. (1).

On the other hand, the variation of Eq. (7) with respect to F produces

$${}^\star F = \frac{1}{2a}(d\phi - e\Lambda). \tag{8}$$

Inserting Eq. (8) into the master action S_M (and recalling $^{\star\star} = 1$ for any differential form in the 3D space-time), we find

$$\begin{aligned} S_M[F, \phi] &\leftrightarrow -\frac{1}{4a}\int(d\phi - e\Lambda) \wedge {}^\star(d\phi - e\Lambda) \\ &= S_{\phi_\Lambda}[\phi]. \end{aligned} \tag{9}$$

We use the symbol "$\leftrightarrow$" instead of "$=$" to be clear that equivalence of actions (functionals) is to be understood as a correspondence between their equations of motion; that is, if $S_1 \leftrightarrow S_2$, the set of equations of S_1 can be manipulated, using its own equalities, or inserting new redundant ones, to become the set of equations of S_2 (the inverse also proceeds).

The two equations of motion of S_M [$dF = 0$ and Eq. (8)] generate a map between the equations of motion of S_A and S_ϕ, viz.,

$${}^\star dA = \frac{1}{2a}(d\phi - e\Lambda). \tag{10}$$

Applying d on both sides, we find Eq. (2), while the application of $d^\star$ results in $d^\star d\phi = e d^\star\Lambda$, which is the equation of motion of S_ϕ.

III. 3D NC ELECTROMAGNETIC DUALITY TO FIRST ORDER IN θ

The NC version of the $U(1)$ gauge theory, whose gauge group we denote by $U_*(1)$, is given by [10]

$$S_{\hat{A}*} = a\int \hat{F} \wedge_* {}^\star\hat{F}, \tag{11}$$

where a is a constant, $\hat{F} = d\hat{A} - i\hat{A} \wedge_* \hat{A} = \frac{1}{2} \times (\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_*)dx^\mu \wedge dx^\nu$, $[A, B]_* = A * B - B * A$ and

$$(A * B)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu\right)A(x)B(y)|_{y\to x} \tag{12}$$

is the Moyal product. In particular, $[x^\mu, x^\nu]_* = i\theta^{\mu\nu}$. $(\theta^{\mu\nu})$ can be any real and constant antisymmetric matrix.

Since $d\hat{F} \neq 0$, previous duality arguments cannot be directly applied. However, Seiberg and Witten have shown, for infinitesimal gauge transformations, that a $U_*(N)$ gauge theory can be mapped into a $U(N)$ one [1]. As a corollary, also useful for our purposes, this map provides a more direct treatment of the observables [11]. To first order in θ, for the $U(1)$ case, this map reads

$$\hat{A} = [A_\mu - \theta^{\alpha\nu}A_\alpha(\partial_\nu A_\mu - \tfrac{1}{2}\partial_\mu A_\nu)]dx^\mu \tag{13}$$

in which $\hat{A}$ transforms as $\delta_{\hat{\lambda}}\hat{A} = d\hat{\lambda} - 2i\hat{A} \wedge_* \hat{\lambda}$, while A transforms as $\delta_\lambda A = d\lambda$.

Inserting (13) into action (11) one obtains an effective $U(1)$ gauge theory whose action contains explicit θ corrections. The action for this effective theory is denoted by S_{A_θ} and the products between its fields are the ordinary ones. Up to first order in θ,

$$\begin{aligned} S_{A_\theta} &= a \int F \wedge {}^\star F(1 + \langle \theta, F\rangle) \\ &= -a \int (\vec{E}^2 - B^2)(1 - \vec{\theta} \cdot \vec{E} - \theta B) d^3x, \end{aligned} \tag{14}$$

where $F = dA$, $(\vec{\theta})^i = \theta^{i0}$, $\theta = \theta^{12}$, and $\langle , \rangle$ is the scalar product between differential forms,[3] in particular, $\langle F, \theta\rangle = {}^\star({}^\star F \wedge \theta) = \frac{1}{2}\theta^{\mu\nu}F_{\mu\nu}$. In order for S_{A*} to be dimensionless, the constant a must have dimension of length. The term $F^{\mu\nu}F_{\nu\lambda}\theta^{\lambda\kappa}F_{\kappa\mu}d^3x$, which appears in 4D electromagnetism, also occurs in 3D, but it is proportional to ${}^\star F \wedge F\langle F, \theta\rangle$ [12]. The equations of motion are

$$\vec{\nabla} \cdot \vec{D} = 0, \tag{15}$$

$$\vec{\nabla} \times H = \dot{\vec{D}}, \tag{16}$$

$$\vec{\nabla} \times \vec{E} = -\dot{B}. \tag{17}$$

In the above, $\vec{D} = \vec{E}(1 - \vec{\theta} \cdot \vec{E} - \theta B) - \frac{1}{2}\vec{\theta}(\vec{E}^2 - B^2)$ and $H = B(1 - \vec{\theta} \cdot \vec{E} - \theta B) + \frac{1}{2}\theta(\vec{E}^2 - B^2)$ (these definitions are analogous to the one used in Ref. [13]). Equations (5) and (17) are equal because both come from the Bianchi identity. Clearly $\vec{\theta}$ is responsible for a violation of spacial isotropy.

Exploiting the Bianchi identity, we propose the following master action:

$$S_{M_\theta}[F, \phi] = \int [a{}^\star F \wedge F(1 + \langle \theta, F\rangle) - d\phi \wedge F]. \tag{18}$$

We will use the above master action to find the first order duality, and a natural generalization of it will be employed to unveil the duality in higher θ orders.

However, this is not the only possible master action; the following actions also ascertain dualities between the same vector and scalar descriptions of NC 3D electromagnetism:

$$\begin{aligned} S_{M_{\theta,c}}[G, \phi] = \int\Big[& aG \wedge {}^\star G(1 + c\langle \theta, G\rangle) \\ & - \Big(1 + \frac{1}{2}(c-1)\langle \theta, G\rangle\Big) d\phi \wedge G\Big], \end{aligned} \tag{19}$$

$$S_{M'_\theta}[B, A] = \int\Big[-\frac{1}{4a} B \wedge {}^\star B\Big(1 - \frac{1}{2a}\langle \theta, {}^\star B\rangle\Big) - B \wedge dA\Big]. \tag{20}$$

The first one is a generalization of the master action in Eq. (18) by a continuous and arbitrary parameter c, the latter being recovered for $c = 1$. The master $S_{M_{\theta,c}}$ has the interesting feature of balancing the NC contribution between its two terms. Nevertheless, for any c, the models it connects are the same vector and scalar ones that are found by S_{M_θ}. In Eq. (20), A and B are 1-forms. This other equivalent master action appears to be better suited for the inverse of our problem, that is, of finding the vector picture if the scalar one is already known.

Resuming the analysis of (18), from its variation with respect to ϕ, we obtain $dF = 0$, which implies $F = dA$; inserting this result into S_{M*}, S_{A*} is obtained. To settle the other side of duality, the variation in regard to F is evaluated, leading to a nontrivial NC extension of Eq. (8) without a source, namely,

$$\frac{1}{2a} d\phi = {}^\star F(1 + \langle \theta, F\rangle) + \frac{1}{2}\langle F, F\rangle^\star \theta. \tag{21}$$

In the above, the property $F \wedge {}^\star F\langle F, \theta\rangle = \langle F, F\rangle^\star\theta \wedge F$ was employed. Regarding the fields $\vec{D}$ and H, this reads

$$-\frac{1}{2a}\vec{\nabla} \times \phi = \vec{D}, \tag{22}$$

$$-\frac{1}{2a}\dot{\phi} = H. \tag{23}$$

To first order in θ, the inverse of the above relations reads

$${}^\star F = \frac{1}{2a} d\phi\Big(1 - \frac{1}{2a}\langle {}^\star d\phi, \theta\rangle\Big) - \frac{1}{8a^2}\langle d\phi, d\phi\rangle^\star \theta, \tag{24}$$

$$\begin{aligned} \vec{E} = & -\frac{1}{2a}\Big(1 - \frac{1}{2a}\vec{\theta} \cdot \vec{\nabla} \times \phi - \frac{1}{2a}\dot{\phi}\theta\Big)\vec{\nabla} \times \phi \\ & + \frac{1}{8a^2}(\vec{\nabla}\phi \cdot \vec{\nabla}\phi - \dot{\phi}^2)\vec{\theta}, \end{aligned} \tag{25}$$

$$\begin{aligned} B = & -\frac{\dot{\phi}}{2a}\Big(1 - \frac{1}{2a}\vec{\theta} \cdot \vec{\nabla} \times \phi - \frac{1}{2a}\dot{\phi}\theta\Big) \\ & - \frac{1}{8a^2}(\vec{\nabla}\phi \cdot \vec{\nabla}\phi - \dot{\phi}^2)\theta. \end{aligned} \tag{26}$$

The insertion of the F expression into S_{M_θ} leads to a NC extension of the scalar field action, namely,

$$S_{M_\theta} \leftrightarrow -\frac{1}{4a}\int d\phi \wedge {}^\star d\phi\Big(1 - \frac{1}{2a}\langle {}^\star d\phi, \theta\rangle\Big) = S_{\phi_\theta}. \tag{27}$$

[3] In odd dimensional Minkowski space, the internal product of two n-forms A and B is defined by $\langle A, B\rangle = {}^\star({}^\star A \wedge B) = \frac{1}{n!}A_{\mu_1\ldots\mu_n}B^{\mu_1\ldots\mu_n}$.

The correspondence of the equations of motion between vector and scalar models, as expected, is given by $F = dA$ together with Eq. (21) (and its inverse). Indeed, if d is applied on both sides of Eq. (21), with $F = dA$, the equation of motion of S_{A_θ} is obtained, while the application of $d^\star$ in Eq. (24) produces the equation of motion of S_{ϕ_θ}.

It is straightforward to verify that the map (21) correctly relates the Hamiltonians and brackets of both representations.

With the last result, we defined a new scalar field model whose action is, to leading order in the noncommutativity parameter, classically equivalent to the $U(1)$ model of electromagnetic theory in 3D space-time. Although there are cubic terms in the Lagrangian, this duality also holds in the Feynman path integral.[4] An analogous claim was done in Ref. [3] and explicit computation with the path integral for the NC extension of the duality of Maxwell-Chern-Simons and self-dual models was done in Ref. [14], which presents the same resulting duality of Ref. [15], which does not use the partition function approach. This result can be generalized. Schematically, let $\mathcal{L}_1(A)$ and $\mathcal{L}_2(B)$ be two classically equivalent Lagrangians that are related by the master Lagrangian $\mathcal{L}_m(A, B)$ whose partition function is

$$Z = \int \mathcal{D}A\mathcal{D}B \exp\left[-i\int (a_1 A^2 + \theta A^3 + a_2 BA + f(B))d^D x\right]. \tag{28}$$

Introducing two new fields, the integral over A becomes a Gaussian integral, as follows,

$$Z = \int \mathcal{D}A\mathcal{D}B\mathcal{D}C\mathcal{D}D \exp\left[-i\int (a_1 A^2 + \theta ACC + D(C - A) + +a_2 BA + f(B))d^D x\right]. \tag{29}$$

Now integration over A can be readily computed; we should replace A by $\frac{1}{2a_1}(-\theta CC + D - a_2 B)$. Hence, in the above theory, if classical action duality holds for any θ and partition function duality holds for $\theta = 0$, partition function duality also holds for $\theta \neq 0$. The same arguments are valid to the NC scalar/vector duality presented here.

IV. HIGHER θ ORDER DUALITY

To second order in θ, (14) reads[5] [3,16,17]

$$S_{A_\theta} = \frac{a}{2}\int\left[F^{\mu\nu}F_{\mu\nu}\left(1 + \frac{1}{2}\theta^{\mu\nu}F_{\mu\nu}\right) + L_{\theta^2}\right]d^3x, \tag{30}$$

with

$$L_{\theta^2} = -2\,\mathrm{tr}(\theta F\theta F^3) + \mathrm{tr}(\theta F^2\theta F^2) + \mathrm{tr}(\theta F)\,\mathrm{tr}(\theta F^3) - \tfrac{1}{8}\mathrm{tr}(\theta F)^2\,\mathrm{tr}(F^2) + \tfrac{1}{4}\mathrm{tr}(\theta F\theta F)\,\mathrm{tr}(F^2) \tag{31}$$

and $\mathrm{tr}(AB) = A_{\mu\nu}B^{\nu\mu}$, $\mathrm{tr}(ABCD) = A_{\mu\nu}B^{\nu\lambda}C_{\lambda\kappa}D^{\kappa\mu}$, etc.

Fortunately, in 3D space-time, the above expression can be considerably simplified. We have already used in Eq. (14) that $\mathrm{tr}(FF\theta F) = \frac{1}{2}\,\mathrm{tr}(FF)\,\mathrm{tr}(F\theta)$; with some reflection, this relation can be generalized to

$$\mathrm{tr}\,(AB_1AB_2...AB_n) = \left(\frac{1}{2}\right)^{n-1}\prod_{k=1}^{n}\mathrm{tr}(AB_k), \tag{32}$$

for any antisymmetric 2-rank tensors A, $\{B_k\}$. Therefore,

$$L_{\theta^2} = \tfrac{1}{4}\mathrm{tr}(FF)\,\mathrm{tr}(\theta F)^2. \tag{33}$$

The master action S_{M_θ} (18) can now be extended to second order in θ; this is achieved by adding $-a\int {}^\star F \wedge F\langle F, \theta\rangle^2$ to the first order expression. Thus,

$$\begin{aligned}{}^\star F = \frac{d\phi}{2a}\left(1 - \frac{\langle\theta, {}^\star d\phi\rangle}{2a} - 3\frac{\langle\theta, {}^\star d\phi\rangle^2}{4a^2} + \langle\theta, \theta\rangle\frac{\langle d\phi, d\phi\rangle}{8a^2}\right) \\ - {}^\star\theta\frac{\langle d\phi, d\phi\rangle}{8a^2}\left(1 - 5\frac{\langle\theta, {}^\star d\phi\rangle}{2a}\right)\end{aligned} \tag{34}$$

and

$$\begin{aligned}S_{\phi_\theta} = -\frac{1}{4a}\int d\phi \wedge {}^\star d\phi\Big(1 - \langle\tilde\theta, d\phi\rangle + 3\langle\tilde\theta, d\phi\rangle^2 \\ + \frac{1}{4}\langle\tilde\theta, \tilde\theta\rangle\langle d\phi, d\phi\rangle\Big),\end{aligned} \tag{35}$$

where $\tilde\theta = {}^\star\theta/2a$. Hence, in the scalar picture, at least to second order, $\tilde\theta$ is the Lorentz violation parameter and θ is unnecessary. Note that only through the employment of $\tilde\theta$ the coupling constant a of the original gauge theory appears in the dual picture as a global factor a^{-1}. *A priori*, one can even conjecture that $\tilde\theta$ is the fundamental parameter of the scalar picture, while θ is inferred by duality. Nevertheless, unless the slowly varying fields limit is employed, this is just an illusion of a nonexact symmetry.

Starting from the third order expansion in θ, terms with more derivatives than potentials appear in the Seiberg-Witten map of $\hat F$ and are present in L_{θ^3}, as we will show (any L_{θ^n} can only depend on A through F, but it can have more derivatives than A's). These factors spoil the last suggested symmetry. To infer these terms, we will use the following Seiberg-Witten differential equation [1]:

[4]This is just an additional observation; in this work we do not aim to directly deal with quantization issues of NC theories.

[5]Note that Ref. [3] uses a different convention in the differential form constant factors.

$$\delta\hat{F}_{\mu\nu}(\theta) = \tfrac{1}{4}\delta\theta^{\alpha\beta}[2\hat{F}_{\mu\alpha} * \hat{F}_{\nu\beta} + 2\hat{F}_{\nu\beta} * \hat{F}_{\mu\alpha} - \hat{A}_\alpha * (\hat{D}_\beta\hat{F}_{\mu\nu} + \partial_\beta\hat{F}_{\mu\nu}) - (\hat{D}_\beta\hat{F}_{\mu\nu} + \partial_\beta\hat{F}_{\mu\nu}) * \hat{A}_\alpha]. \tag{36}$$

Expanding $\hat{F}$ and $\hat{A}$ in powers of θ, to third order it reads

$$\delta\hat{F}^{(3)}_{\mu\nu}(\theta) = -\tfrac{1}{4}\delta\theta^{\alpha\beta}\theta^{\alpha'\beta'}\theta^{\alpha''\beta''}(\partial_{\alpha'}\partial_{\alpha''}F_{\mu\alpha}\partial_{\beta'}\partial_{\beta''}F_{\nu\beta} - \partial_{\alpha'}\partial_{\alpha''}A_\alpha\partial_{\beta'}\partial_{\beta''}\partial_\beta F_{\mu\nu}) + \ldots \tag{37}$$

where $F_{\mu\nu} = \hat{F}^{(0)}_{\mu\nu}$ and $A_\mu = A^{(0)}_\mu$. Only the terms with more derivatives than fields were written in the above expression. Inserting this result into Eq. (11), the only terms of L_{θ^3} which have more derivatives than fields are in the following expression[6]:

$$\theta^{\alpha\beta}\theta^{\alpha'\beta'}\,\mathrm{tr}(\partial_\alpha\partial_{\alpha'}F\theta\partial_\beta\partial_{\beta'}FF) - \tfrac{1}{4}\theta^{\alpha\beta}\theta^{\alpha'\beta'}\,\mathrm{tr}(F\theta)\,\mathrm{tr}(\partial_\alpha\partial_{\alpha'}F\partial_\beta\partial_{\beta'}F). \tag{38}$$

The contribution of these terms to the equations of motion is given by

$$\theta^{\alpha\beta}\theta^{\alpha'\beta'}\partial_\mu\Big[F^\mu_{\alpha\alpha'}\theta F^\kappa_{\beta\beta'} + \tfrac{1}{2}\mathrm{tr}(F_{\alpha\alpha'}\theta)F^{\kappa\mu}_{\beta\beta'} + F^{[\mu}_{\alpha\alpha'}F_{\beta\beta'}\theta^{\kappa]} + \tfrac{1}{4}\mathrm{tr}(F_{\alpha\alpha'}F_{\beta\beta'})\theta^{\kappa\mu}\Big]. \tag{39}$$

In the above, we introduced a compact notation: nonexplicit indices are contracted like in matrices, extra indices in F are derivatives, and $F^{[\mu}_{\alpha\alpha'}F_{\beta\beta'}\theta^{\kappa]} = F^\mu_{\alpha\alpha'}F_{\beta\beta'}\theta^\kappa - F^\kappa_{\alpha\alpha'}F_{\beta\beta'}\theta^\mu$. For instance, the first term in (39) reads $\partial_\alpha\partial_{\alpha'}F^\mu_\nu\theta^{\nu\lambda}\partial_\beta\partial_{\beta'}F^\kappa_\lambda$.

A careful analysis of the symmetries and antisymmetries of each term of (39) and their linear independence for arbitrary θ and $D \geq 4$ shows that (39) is not null. To directly assure unambiguously in any dimension ($D \geq 2$) that (39) is not the trivial identity [or that (38) is not a surface term or null], one may evaluate a particular case of (39); for instance, for $D \geq 3$, let $\kappa = 2$ and θ be equal to zero except for the components θ^{01} and θ^{10}, namely,

$$(\theta^{10})^3[\partial_\mu(\ddot{F}^{\mu[0}F''^{1]2} + F''^{\mu[0}\ddot{F}^{1]2} - 2\dot{F}'^{\mu[0}\dot{F}'^{1]2} + \ddot{F}^{10}F''^{2\mu} + +F''^{10}\ddot{F}^{2\mu} - 2\dot{F}'^{10}\dot{F}'^{2\mu}) + \partial_{[0}(\ddot{F}^{2\nu}F''_{1]\nu} + F''^{2\nu}\ddot{F}_{1]\nu} - 2\dot{F}'^{2\nu}\dot{F}'_{1]\nu})], \tag{40}$$

where each dot and each prime means, respectively, ∂_0 and ∂_1. The above expression is not identically null in any dimension (greater than 2). This result is in conflict with a certain proposition of Ref. [19]; see our Conclusions for more details.

The expressions (36)–(39) are valid for arbitrary space-time dimensions. Once again, in 3D space-time a considerable simplification is possible. Although the property (32), in that form, cannot be used in (38), a straightforward computation shows that an analogous result is valid. In 3D space-time, the expression (38) is equal to

$$\tfrac{1}{4}\theta^{\alpha\beta}\theta^{\alpha'\beta'}\,\mathrm{tr}(F_{\alpha\alpha'}F_{\beta\beta'})\,\mathrm{tr}(F\theta). \tag{41}$$

Adhering to the third order expansion, the contribution of the above expression to S_{ϕ_θ} (35) is obtained by the replacement $F \rightarrow {}^\star d\phi/(2a)$. Consequently, to third order in θ, S_{ϕ_θ} cannot be expressed only through $\tilde{\theta}$; θ is also necessary.[7] This violates the symmetry between θ and $\tilde{\theta}$ present in electromagnetic duality up to the second order in θ. Consequently, in the scalar picture, the constant a does not appear as a global a^{-1} and S-duality is broken (at least in regard to its usual form).

In the slowly varying fields limit, the terms in the Seiberg-Witten mapped action which depend on the derivatives of F are neglected; therefore, S_{ϕ_θ} to third order in θ can be solely expressed in terms of $\tilde{\theta}$. In this limit, since the Seiberg-Witten mapped Lagrangian is a function of F alone (without derivatives) [1], the Lagrangian is expressed as a function of $\mathrm{tr}(FF)$ and $\mathrm{tr}(F\theta)$ only [due to Eq. (32)]; therefore, the dual scalar action S_{ϕ_θ} to all orders in θ can be expressed by $\tilde{\theta}$, without reference to θ (or ${}^\star\tilde{\theta}$). Although the property (32) is, in general, false in the 4D space-time, the dual action can also be expressed by $\tilde{\theta}$ alone in the 4D space-time, to all orders in θ, if the slowly varying fields limit is used [4]. The relation (32) considerably simplifies the work in the 3D analysis.

V. CONCLUSIONS

In this paper we establish, to second order in θ, the scalar description of 3D NC electromagnetic theory, which is usually described by the gauge model in Eq. (11). We show that the rule $\theta \rightarrow \tilde{\theta} = {}^\star\theta g^2$, found in Ref. [3] in the context of 4D NC electromagnetic duality, can be extended to the 3D case up to second order in θ [Eqs. (30) and (35)]. With this rescaling of θ, the coupling constant of one model becomes the inverse of the other. This is indeed a curious relation between these dual models, but this relation is only approximately valid: starting from the third order θ expansion, in general it becomes false in both 3D and 4D cases. The coupling constant does not appear proportionally to θ, but to ϕ instead; so, to any order, it is possible to do the replacement $\phi \rightarrow \phi g^2$ and the final answer is a noninversion of the coupling constant. In the 4D case, *a priori* it is possible to think that somehow

[6]This solution can also be inferred by the results of Ref. [18], Sec. 3.2, in which the Seiberg-Witten map is expanded in powers of A.

[7]One may artificially insert ϵ's in order to change $\theta^{\alpha\beta}\partial_\alpha\partial_\beta$ to $\propto \tilde{\theta}_\mu\epsilon^{\mu\alpha\beta}\partial_\alpha\partial_\beta$. This procedure is innocuous since ${}^\star\tilde{\theta} \propto \theta$, but we are adopting the rule to always write θ, or $\tilde{\theta}$, never ${}^\star\tilde{\theta}$. Moreover, this procedure does not avoid the difficulties with S-duality, since $\tilde{\theta}$ will not occur proportionally to ϕ in the dual picture.

the coupling constant appears proportionally to θ; in the 3D case no such doubt occurs, for g^2 is dimensionful in this space and θ appears proportionally to $\partial\partial$ in some terms, like those in Eq. (38). Since, up to the subleading order in θ or in the slowly varying fields limit, the 4D duality connects two $U(1)$ theories, one with θ and the other with $\tilde{\theta}$ [3–5], it might appear that θ and $\tilde{\theta}$ could be used indistinguishably; however, a simple analysis of the 3D case shows this does not proceed. The 3D case clearly states that, if a theory has θ as its parameter, there is another equivalent one with a *different definition of the fields* which has the parameter $\tilde{\theta}$ [this is a direct interpretation of the duality map, e.g., Eq. (34)]. As a final remark of this duality to second order in θ, it is easy to see from the equations of motion and the interchange between θ and $\tilde{\theta}$ that the 3D NC duality preserves spacial isotropy (i.e., if one of the dual models is isotropic, the other is also) and, if a spacial anisotropy is present, duality rotates the preferential direction by $\pi/2$.

Currents can be easily inserted in this duality, along the lines of Sec. II, if we assume a θ nondependent coupling like $A \wedge {}^\star J$ in the mapped action. Nevertheless, this violates correspondence with the $U_*(1)$ theory, which asserts the coupling $\hat{A} \wedge_* {}^\star \hat{J}$, whose map was found in Ref. [20].

In Sec. IV, we proved, by means of a straightforward calculation valid in any dimension greater than 2, that the Seiberg-Witten mapped Lagrangian of the NC electromagnetic theory (L_{A_θ}) depends on F and its derivatives.[8] Up to the second order in θ, the derivatives on F can be combined with the fields A to produce another F (eliminating all the explicit dependence on the A's). Nevertheless, the Seiberg-Witten differential equation (36) leads to the appearance of terms with more derivatives than fields in the third order expansion. These terms were applied to the NC electromagnetic Lagrangian ($L_{\hat{A}_*}$) and the resulting terms were stated in (38). Perhaps surprisingly, these terms are not null nor are they surface terms, as we have shown subsequently.[9] This result is not in agreement with the first part of a proposition in Ref. [19]. We think our result should be considered as a counter-example to it. Indeed, the first part of Proposition 3.1 does not seem to be correct in general [22]. However, it should be stressed that it clearly holds in the slowly varying fields limit and, in this limit, it is compatible with our results; moreover, any results which depend on that proposition are perfectly valid in that limit. There are some other interesting consequences which we are now evaluating [23].

As previously stated, this work does not aim to resolve string S-duality issues in the presence of a magnetic background, like Ref. [8] does. However, a certain exchange of θ with ${}^\star\theta g^2$, among other similarities, occurs in both cases. According to our result, this exchange only occurs to all orders in θ in the slowly varying fields limit. At the moment, it is not clear to us if our result has consequences to the string S-duality of NC theories since, among other possibilities, we may have come across a pathological feature of the Seiberg-Witten map [23].

We think further developments of the NC electromagnetic duality can prove useful to construct effective models and to understand NC theories in general.

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[8]Although it was not explicitly shown in Sec. IV, it is not hard to evaluate that the particularization of (38) to $D = 2$ is also different from zero.

[9]In general, the Seiberg-Witten map is not unique [21]; nevertheless, the additional terms do not influence this analysis.

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