Renormalized effective actions in radially symmetric backgrounds: Partial wave cutoff method

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The computation of the one-loop effective action in a radially symmetric background can be reduced to a sum over partial-wave contributions, each of which is the logarithm of an appropriate one-dimensional radial determinant. While these individual radial determinants can be evaluated simply and efficiently using the Gel'fand-Yaglom method, the sum over all partial-wave contributions diverges. A renormalization procedure is needed to unambiguously define the finite renormalized effective action. Here we use a combination of the Schwinger proper-time method, and a resummed uniform DeWitt expansion. This provides a more elegant technique for extracting the large partial-wave contribution, compared to the higher-order radial WKB approach which had been used in previous work. We illustrate the general method with a complete analysis of the scalar one-loop effective action in a class of radially separable SU(2) Yang-Mills background fields. We also show that this method can be applied to the case where the background gauge fields have asymptotic limits appropriate to uniform field strengths, such as, for example, in the Minkowski solution, which describes an instanton immersed in a constant background. Detailed numerical results will be presented in a sequel.

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I. INTRODUCTION

In the study of quantum field theories, one is often led to consider the one-loop effective action in nontrivial background fields. While the renormalization counterterm structure of one-loop effective actions can readily be exhibited for general backgrounds, the explicit evaluation of its full finite part in an interesting specific background still constitutes a highly nontrivial problem. For gauge theories, explicit analytic results are known only for very special backgrounds: the Euler-Heisenberg effective action for a background with constant Abelian field strength [1–3], its generalization to a non-Abelian covariantly constant background [4–7], and a special solvable Abelian background [8–11]. For applications in both continuum and lattice field theory, one would like to enlarge this set of backgrounds for which we have accurate computations of the finite renormalized effective action.

In a series of recent publications (together with Hyunsoo Min) [12], we presented a new method for computing the renormalized one-loop effective action in a radially symmetric non-Abelian background, and used it to evaluate explicitly the QCD single-instanton determinant for arbitrary quark mass values. The related computation in the massless limit was performed in a classic paper of 't Hooft [13], while the heavy quark mass limit was studied in [14,15]. The new method in [12] works for any quark mass, not relying on small or large mass expansions, and the result interpolates smoothly and precisely between these two extremes. In this paper we present the general formalism, and we introduce a simplified analysis based on a uniform Schwinger-DeWitt expansion, replacing the higher-order radial WKB analysis used in [12,16]. This approach has also been used to evaluate the exact determinant prefactor in false vacuum decay (or nucleation) [17]. Related techniques have been developed for a variety of field theoretic applications, including sphalerons [18,19] and false vacuum decay [20,21], and the relation between the renormalization procedure in our partial-wave cutoff method and these Feynman diagrammatic approaches is explained in [22], where a zeta function approach to the determinant of a radially symmetric Schrödinger operator is also given. Finally, we note that another related method has recently been applied to the two-dimensional chiral Higgs model [23].

The starting idea is very simple. If the background field is radially symmetric, the effective action Γ can be expressed formally in terms of one-dimensional functional determinants of radial differential operators for various partial waves. Explicitly, writing J for all quantum numbers specific to a given partial wave, it has the general structure

$$\Gamma \sim \sum_{J=0}^{\infty} \ln\left(\frac{\det(\mathcal{H}_J + m^2)}{\det(\mathcal{H}_J^{\text{free}} + m^2)}\right). \tag{1.1}$$

Here $\mathcal{H}_J$, the radial Schrödinger-type differential operator for the Jth partial wave, contains a nontrivial (background-dependent) potential term, while $\mathcal{H}_J^{\text{free}}$ is the corresponding free operator. In general, there will also be appropriate

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degeneracy factors in the sum. For a given partial wave J, the individual determinant is finite once we divide by the corresponding "free" contribution. These finite one-dimensional determinants can be evaluated easily using the Gel'fand-Yaglom method [24–30], which reduces the computation to a trivial (numerical) integration with initial value boundary conditions.

The nontrivial aspect of this approach is that the sum over partial waves J diverges (in spacetime dimension $d \geq 2$). This is of course because the formal sum in (1.1) neglects renormalization. We solve this problem by introducing a partial-wave cutoff J_L, and we isolate the divergence of the sum in a form that can be absorbed via renormalization. Specific details are presented in the body of this paper, but the general structure is that we write the sum as

$$\Gamma = \sum_{J=0}^{J_L} \ln\left(\frac{\det(\mathcal{H}_J + m^2)}{\det(\mathcal{H}_J^{\rm free} + m^2)}\right) + \sum_{J>J_L}^{\infty} \ln\left(\frac{\det(\mathcal{H}_J + m^2)}{\det(\mathcal{H}_J^{\rm free} + m^2)}\right) + (\text{counterterm}). \quad (1.2)$$

The finite, renormalized effective action is then evaluated as follows:

(i) The first sum is evaluated numerically using the Gel'fand-Yaglom method.

(ii) The second sum is evaluated analytically in the large J_L limit, after regulating the determinants. This step uses our uniform Schwinger-DeWitt expansion and Euler-Maclaurin summation.

(iii) The analytic large J_L behavior of the regulated determinants leads to the correct renormalization counterterm, and moreover cancels exactly the numerical divergences of the first sum as $J_L \to \infty$. This produces a finite renormalized answer.

The technically difficult part of the computation is the analytic computation of the large J_L behavior of the second sum in (1.2). This was achieved in [12] using second-order radial WKB, based on Dunham's formula [31]. While this is very general, it can be quite cumbersome for complicated background fields. In this paper we present a simpler method to implement this part of the computation. The analysis reduces to simple algebraic manipulations, and can be more readily generalized. With this new approach we can now evaluate exactly the finite renormalized effective action for a very general class of radially symmetric backgrounds. In fact, with a background field involving an *unspecified* radial function, the large partial-wave contribution to the renormalized effective action can now be evaluated explicitly. This will be important to discuss the background-field dependence of the effective action and also to test various approximation schemes.

This paper is organized as follows. In Sec. II we define the renormalized one-loop effective action for the case of a scalar field in a class of spherically symmetric Yang-Mills background fields, assuming four-dimensional Euclidean spacetime. Its Schwinger proper-time representation [2] is given. The full amplitude is then expressed using partial-wave amplitudes, and we also elaborate here on the role of the two kinds of proper-time Green functions, one related to the quadratic differential operator given in 4D spacetime and the other for the radial quadratic differential operator. In Sec. III we explain how the large-l partial-wave contribution to the full effective action can be evaluated explicitly using a generalized DeWitt WKB expansion for the radial proper-time Green function (i.e., the $\frac{1}{l}$-expansion) and the Euler-Maclaurin summation method. This allows us to present the renormalized one-loop effective action (in the class of spherically symmetric backgrounds) in a form amenable to direct numerical analysis. In Sec. IV we show that our $\frac{1}{l}$-expansion formula can be applied to the calculation of the large partial-wave contribution even when background gauge fields do not fall off at large distance but approach those of uniform field strength. Also given here is the exact partial-wave-based treatment of the effective action in the background corresponding to strictly uniform self-dual field strengths. In Sec. V we conclude with some relevant discussions and comments. There are several appendices which contain supplementary materials and some technical details. In Appendix A the explicit form of the free radial proper-time Green function in n spacetime dimension is considered. In Appendix B we study the coefficient functions in the $\frac{1}{l}$-expansion when the potential is matrix-valued. Appendix C contains a brief account of the Euler-Maclaurin summation formula, and some explicit results obtained using this formula in connection with our problem. In a sequel we will address matrix-valued problems in more detail, and present detailed numerical results for the general radial cases for which the formalism is developed in this current paper.

II. EFFECTIVE ACTION IN RADIALLY SYMMETRIC BACKGROUNDS

A. Renormalized effective action

We choose for our field theory model an SU(2) Euclidean Yang-Mills theory with a complex scalar matter field (in the fundamental representation), in four-dimensional spacetime. As far as our general methodology is concerned, the model choice is not crucial; but, by choosing this case, we are able to crosscheck readily the findings of the present work against those of Refs. [12,16]. The case with Dirac fields is quite similar if one works with the *squared* Dirac operator. Also, by considering a gauge theory (rather than the much simpler scalar field theory), we can demonstrate the *gauge invariance* of our calculational method for the renormalized effective action.

Consider a generic Yang-Mills background: $A_\mu(\mathbf{x}) \equiv A_\mu^a(\mathbf{x})\frac{\tau^a}{2}$ ($\mu = 1, 2, 3, 4$, $a = 1, 2, 3$; τ's denote 2×2 Pauli matrices). The Pauli-Villars regularized one-loop effective action associated with scalar field fluctuations can be represented by

$$\Gamma_\Lambda(A;m) = \ln\left[\frac{\det(-D^2+m^2)\det(-\partial^2+\Lambda^2)}{\det(-\partial^2+m^2)\det(-D^2+\Lambda^2)}\right], \quad (2.1)$$

where m is the scalar mass, Λ a heavy regulator mass, and D^2 the covariant Laplacian operator

$$D^2 \equiv D_\mu D_\mu, \qquad (D_\mu = \partial_\mu - iA_\mu(\mathbf{x})). \quad (2.2)$$

The Schwinger proper-time representation for the form (2.1) is

$$\Gamma_\Lambda(A;m) = -\int_0^\infty \frac{ds}{s}(e^{-m^2 s} - e^{-\Lambda^2 s})F(s), \quad (2.3)$$

with $F(s)$ given by (s is the proper-time variable)

$$F(s) = \int d^4\mathbf{x}\, \mathrm{tr}\langle \mathbf{x}|\{e^{-s(-D^2)} - e^{-s(-\partial^2)}\}|\mathbf{x}\rangle. \quad (2.4)$$

The proper-time Green's function

$$\Delta(\mathbf{x},\mathbf{x}';s) \equiv \langle \mathbf{x}|e^{-s(-D^2)}|\mathbf{x}'\rangle, \quad (2.5)$$

admits an asymptotic expansion, the DeWitt (or heat kernel) expansion [32,33]:

$$\langle \mathbf{x}|e^{-s(-D^2)}|\mathbf{x}'\rangle \sim \frac{1}{(4\pi s)^2} e^{-|\mathbf{x}-\mathbf{x}'|^2/4s}\left\{\sum_{n=0}^\infty s^n a_n(\mathbf{x},\mathbf{x}')\right\}, \quad \text{for } s\to 0+. \quad (2.6)$$

The expansion coefficients, $a_n(\mathbf{x},\mathbf{x}')$ ($n = 0, 1, 2, \ldots$), and especially the coincidence limits $a_n(\mathbf{x},\mathbf{x})$ of the first few terms, can be found most simply using recurrence relations satisfied by the $a_n(\mathbf{x},\mathbf{x}')$'s. The divergence structure of $\Gamma_\Lambda(A;m)$ as $\Lambda\to\infty$ is governed by the values of $\mathrm{tr}\, a_1(\mathbf{x},\mathbf{x})$ and $\mathrm{tr}\, a_2(\mathbf{x},\mathbf{x})$, and in our case we have [32,33]

$$\begin{aligned} \mathrm{tr}\, a_1(\mathbf{x},\mathbf{x}) &= 0, \\ \mathrm{tr}\, a_2(\mathbf{x},\mathbf{x}) &= -\frac{1}{12}\mathrm{tr}[F_{\mu\nu}(\mathbf{x})F_{\mu\nu}(\mathbf{x})], \end{aligned} \quad (2.7)$$

where $F_{\mu\nu} \equiv F^a_{\mu\nu}\frac{\tau^a}{2} = i[D_\mu, D_\nu]$ is the field strength. Then the renormalized one-loop effective action in the minimal subtraction scheme is defined as

$$\Gamma_{\mathrm{ren}}(A;m) = \lim_{\Lambda\to\infty}\left[\Gamma_\Lambda(A;m) - \frac{1}{12}\frac{1}{(4\pi)^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right) \times \int d^4\mathbf{x}\,\mathrm{tr}[F_{\mu\nu}(\mathbf{x})F_{\mu\nu}(\mathbf{x})]\right], \quad (2.8)$$

where μ is the renormalization scale.

B. Radial backgrounds

It is a very difficult problem to explicitly evaluate this renormalized effective action (2.8). For a generic background gauge field $A_\mu(\mathbf{x})$, there is currently no known method leading to an *exact* evaluation of the one-loop effective action. On the other hand, there are many interesting physical applications (e.g., vortices, monopoles, instantons, ...) where the gauge background is radially symmetric. In this paper we show that this radial symmetry is strong enough to permit the computation of the renormalized effective action (2.8).

A large class of such radial backgrounds is covered by the ansatz form:

$$A_\mu(\mathbf{x}) = 2\eta_{\mu\nu a}x_\nu f(r)\frac{\tau^a}{2} + 2\eta_{\mu\nu 3}x_\nu g(r)\frac{\tau^3}{2}, \quad (r \equiv |\mathbf{x}| = \sqrt{x_\mu x_\mu}), \quad (2.9)$$

where the radial functions $f(r)$ and $g(r)$ are left unspecified, and $\eta_{\mu\nu a}$ ($a = 1, 2, 3$) denote the standard 't Hooft symbols [13]. With $A_\mu(\mathbf{x})$ of the form (2.9), the covariant Laplacian operator $-D^2$ becomes (here $\frac{\tau^a}{2} \equiv T_a$)

$$\begin{aligned} -D^2 = {} & -\partial_\mu\partial_\mu + 4if(r)\eta_{\mu\nu a}T_a x_\nu\partial_\mu \\ & + 4ig(r)\eta_{\mu\nu 3}T_3 x_\nu \partial_\mu \\ & + 4f(r)^2\eta_{\mu\nu a}\eta_{\mu\lambda b}T_aT_b x_\nu x_\lambda \\ & + 4g(r)^2\eta_{\mu\nu 3}\eta_{\mu\lambda 3}T_3^2 x_\nu x_\lambda \\ & + 8f(r)g(r)\eta_{\mu\nu a}\eta_{\mu\lambda 3}T_aT_3x_\nu x_\lambda. \end{aligned} \quad (2.10)$$

We may then define [13] the operators $L_a \equiv -\frac{i}{2}\eta_{\mu\nu a}x_\mu\partial_\nu$ (satisfying angular-momentum commutation relations $[L_a, L_b] = i\epsilon_{abc}L_c$) and use the relations

$$\eta_{\mu\nu a}\eta_{\mu\lambda b} = \delta_{ab}\delta_{\nu\lambda} + \epsilon_{abc}\eta_{\nu\lambda c}, \qquad T_aT_a = \frac{3}{4}\mathbf{1},$$
$$-\partial_\mu\partial_\mu = -\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4}{r^2}\vec{L}^2, \qquad (\vec{L}^2 \equiv L_aL_a) \quad (2.11)$$

to recast the expression (2.10) as

$$\begin{aligned} -D^2 = {} & -\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4}{r^2}\vec{L}^2 + 8f(r)\vec{T}\cdot\vec{L} \\ & + 8g(r)T_3L_3 + r^2\{3f(r)^2 + g(r)^2 + 2f(r)g(r)\}. \end{aligned} \quad (2.12)$$

Based on this form, we may associate an infinite number of partial-wave radial differential operators with the given system. We distinguish between three important cases.

1. Case 1: $g(r) \equiv 0$, but $f(r) \neq 0$

Suppose that $g(r) \equiv 0$, but $f(r) \neq 0$. This is the form relevant to the instanton computation in [12,13]. Then $A_\mu(\mathbf{x})$ is given by the first piece only on the right-hand side of (2.9). Then, noting that there exists another set of angular-momentum-like operators $\bar{L}_a \equiv -\frac{i}{2}\bar{\eta}_{\mu\nu a}x_\mu\partial_\nu$ (satisfying $[L_a, \bar{L}_b] = 0$ and $\bar{L}_a\bar{L}_a = L_aL_a \equiv \vec{L}^2$) [13], partial waves can be specified by the quantum numbers $J_1 \equiv (l, j, j_3, \bar{l}_3)$, where

$$(\vec{L}^2)' = l(l+1), \qquad l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots;$$
$$(\vec{J}^2)' = j(j+1), \qquad (\text{with } J_a \equiv L_a + T_a), \qquad j = |l \pm \tfrac{1}{2}|;$$
$$(J_3)' \equiv j_3 = -j, -j+1, \ldots, j;$$
$$(\bar{L}_3)' \equiv \bar{l}_3 = -l, -l+1, \ldots, l. \tag{2.13}$$

The radial differential operator, representing $-D^2$ in the given partial-wave sector, thus assumes the form

$$\mathcal{H}_{J_1} \equiv -D^2_{(l,j)} = -\partial^2_{(l)} + 4f(r)[j(j+1) - l(l+1) - \tfrac{3}{4}] + 3r^2 f(r)^2, \tag{2.14}$$

where $\partial^2_{(l)}$ is the partial-wave form of the free Laplacian $\partial_\mu \partial_\mu$:

$$\partial^2_{(l)} \equiv \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{4}{r^2} l(l+1). \tag{2.15}$$

2. Case 2: $f(r) = 0$, but $g(r) \neq 0$

The system with $f(r) = 0$, but $g(r) \neq 0$, is simpler: here, partial waves are specified by the quantum numbers $J_2 \equiv (l, l_3, t_3, \bar{l}_3)$, where $(L_3)' \equiv l_3 = -l, -l+1, \ldots, l$ and $(T_3)' \equiv t_3 = \pm\frac{1}{2}$. In this case, the radial differential operator becomes

$$\mathcal{H}_{J_2} \equiv -D^2_{(l,l_3,t_3)} = -\partial^2_{(l)} + 8g(r) l_3 t_3 + r^2 g(r)^2. \tag{2.16}$$

3. Case 3: both $f(r)$ and $g(r)$ nonvanishing

The situation is somewhat more complicated if both $f(r)$ and $g(r)$ are nonvanishing, since $-D^2$ will then be non-diagonal in either basis considered above. This case can be treated by allowing partial-wave sectors themselves to be finite-dimensional vector spaces. Explicitly, taking a partial wave specified by the quantum numbers $J_3 \equiv (l, j_3, \bar{l}_3)$, we can represent the operator $T_3 L_3$ according to

$$T_3 L_3 \leftrightarrow \begin{pmatrix} \frac{j_3^2}{2l+1} - \frac{1}{4} & -\frac{j_3}{2l+1}\sqrt{(l+\frac{1}{2})^2 - j_3^2} \\ -\frac{j_3}{2l+1}\sqrt{(l+\frac{1}{2})^2 - j_3^2} & -\frac{j_3^2}{2l+1} - \frac{1}{4} \end{pmatrix}, \qquad \text{if } j_3 = -l + \frac{1}{2}, \cdots, l - \frac{1}{2}$$
$$T_3 L_3 \leftrightarrow \frac{1}{2} l, \qquad \text{if } j_3 = \pm\left(l + \frac{1}{2}\right), \tag{2.17}$$

where the 2×2 matrix, appearing when $|j_3| \neq l + \frac{1}{2}$, is defined relative to the basis vectors $|j = l \pm \frac{1}{2}\rangle$. This allows us to represent $-D^2$ in the given partial wave by the (matrix) radial differential operator

$$\mathcal{H}_{J_3} \equiv -D^2_{(l,j_3)} = \begin{cases} -\partial^2_{(l)} + W(r) + Z_{(l,j_3)}, & \text{if } j_3 = -l + \frac{1}{2}, \ldots, l - \frac{1}{2} \\ -\partial^2_{(l)} + W(r) + 4lf(r) + 4lg(r), & \text{if } j_3 = \pm(l + \frac{1}{2}), \end{cases} \tag{2.18}$$

where

$$W(r) = r^2\{3f(r)^2 + g(r)^2 + 2f(r)g(r)\}, \tag{2.19}$$

$$Z_{(l,j_3)} = \begin{pmatrix} 4lf(r) + 8(\frac{j_3^2}{2l+1} - \frac{1}{4})g(r) & -\frac{8j_3}{2l+1}\sqrt{(l+\frac{1}{2})^2 - j_3^2}\, g(r) \\ -\frac{8j_3}{2l+1}\sqrt{(l+\frac{1}{2})^2 - j_3^2}\, g(r) & -4(l+1)f(r) - 8(\frac{j_3}{2l+1} + \frac{1}{4})g(r) \end{pmatrix}. \tag{2.20}$$

The Gel'fand-Yaglom method has a straightforward generalization [29] to matrix-valued operators, so the numerical part of the computation follows as before. Such matrix-valued radial operators have in fact been considered in [23], and also occur naturally when considering fluctuations of a Dirac-spinor matter field in a radially symmetric background.

An interesting subclass of these radial backgrounds consists of those that are *self-dual* (or *anti-self-dual*). Such gauge fields satisfy automatically the classical Yang-Mills field equations, and as such they are of particular importance. With our potential form in (2.9), such self-dual or anti-self-dual configurations are obtained if certain special functional forms are chosen for $f(r)$ and $g(r)$. Explicitly, for self-dual configurations, the following choices can be made:

(i) $f(r) = \frac{1}{r^2+\rho^2}$, $g(r) = 0$ (i.e., $A_\mu = \eta_{\mu\nu a}\frac{x_\nu}{r^2+\rho^2}\tau^a$) for a single-instanton solution in the regular gauge;

(ii) $f(r) = 0$, $g(r) = -\frac{B}{2} = \text{const}$ (i.e., $A_\mu = -\eta_{\mu\nu a}x^\nu \frac{B}{2}\tau^a$) for a uniform self-dual field strength background;

(iii) $f(r) = \frac{b}{\sinh[b(r^2+\rho^2)]}$, $g(r) = b\tanh[\frac{b}{2}(r^2+\rho^2)]$ for the so-called Minkowski solution [34] which describes a single instanton immersed in a uniform background.

Note that the Minkowski solution reduces to the case (i) or (ii) in appropriate limits. Anti-self-dual solutions may be obtained for the choice $f(r) = \frac{\rho^2}{r^2(r^2+\rho^2)}$, and $g(r) = 0$ (i.e.,

$A_\mu = \eta_{\mu\nu a}\frac{x_\nu \rho^2}{r^2(r^2+\rho^2)}$); this corresponds to a single anti-instanton in the singular gauge. With any of these classical solutions chosen as the background, our discussion above tells us that the operator $-D^2$ can be written in the partial-wave expanded form. In particular, in the case of the Minkowski solution for which both $f(r)$ and $g(r)$ are nonvanishing, the related partial-wave differential operator will take a 2×2 matrix form.

But, in the following analysis, it will be sufficient to assume that our background potentials are just of the radial form (2.9)—i.e., they *do not* have to satisfy classical field equations.

C. Partial-wave decomposition of effective action

Taking advantage of this radial symmetry, we can make a partial-wave decomposition in (2.4):

$$F(s) = \sum_J F_J(s), \tag{2.21}$$

where

$$F_J(s) = \int_0^\infty dr\, \mathrm{tr}\{\tilde{\Delta}_J(r,r;s) - \tilde{\Delta}_J^{\mathrm{free}}(r,r;s)\}. \tag{2.22}$$

The proper-time *radial* Green's function for each partial wave J is defined as

$$\tilde{\Delta}_J(r,r';s) \equiv \langle r|e^{-s\tilde{\mathcal{H}}_J}|r'\rangle, \tag{2.23}$$

in terms of the radial operator

$$\tilde{\mathcal{H}}_J \equiv \frac{1}{r^{3/2}}\mathcal{H}_J r^{3/2} = -\frac{d^2}{dr^2} + V_J(r). \tag{2.24}$$

Note that we have extracted a measure factor $r^{3/2}$ in writing $\tilde{\mathcal{H}} = r^{-3/2}\mathcal{H} r^{3/2}$. The form of the (possibly matrix valued) radial potential $V_J(r)$ depends on the specific form of the gauge field entering the covariant Laplacian operator. In each case, the effective radial partial-wave potential $V_J(r)$ contains a centrifugal term, having the structure

$$V_J(r) = \frac{4l(l+1)+\frac{3}{4}}{r^2} + U_J(r), \tag{2.25}$$

where l is the half-integer valued quantum number in (2.13).

The partial-wave-based representation of the regularized effective action is

$$\Gamma_\Lambda(A;m) = -\sum_J \int_0^\infty \frac{ds}{s}(e^{-m^2 s} - e^{-\Lambda^2 s}) \times \int_0^\infty dr\, \mathrm{tr}\{\tilde{\Delta}_J(r,r;s) - \tilde{\Delta}_J^{\mathrm{free}}(r,r;s)\}. \tag{2.26}$$

Then, for the explicit evaluation of the renormalized effective action given by (2.8), it is convenient to separate the partial-wave sum into two parts [12,16]: (i) the sum over partial waves with $J \le J_L$ (here, J_L is chosen such that it may refer to some large l-value, $l = L$); and (ii) the remaining infinite sum involving all $J > J_L$ terms. In the first contribution involving the finite J-sum, the regulator plays no role in the limit $\Lambda \to \infty$, and so may be removed from this sum. Based on this procedure, we can now write

$$\Gamma_{\mathrm{ren}}(A;m) = \Gamma_{J\le J_L}(A;m) + \Gamma_{J>J_L}(A;m), \tag{2.27}$$

with

$$\begin{aligned}\Gamma_{J\le J_L}(A;m) &= -\sum_{J\le J_L}\int_0^\infty \frac{ds}{s}e^{-m^2 s}\int_0^\infty dr \times \mathrm{tr}\{\tilde{\Delta}_J(r,r;s) - \tilde{\Delta}_J^{\mathrm{free}}(r,r;s)\}\\ &= \sum_{J\le J_L} \mathrm{ln}\det\left(\frac{\tilde{\mathcal{H}}_J + m^2}{\tilde{\mathcal{H}}_J^{\mathrm{free}} + m^2}\right),\end{aligned} \tag{2.28}$$

$$\Gamma_{J>J_L}(A;m) = -\sum_{J>J_L}\int_0^\infty \frac{ds}{s}(e^{-m^2 s} - e^{-\Lambda^2 s})F_J(s) - \frac{1}{12}\frac{1}{(4\pi)^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\, \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}). \tag{2.29}$$

On the right-hand side of (2.28), $\mathrm{ln}\det[(\tilde{\mathcal{H}}_J + m^2)/(\tilde{\mathcal{H}}_J^{\mathrm{free}} + m^2)]$ may well be replaced by $\mathrm{ln}\det[(\mathcal{H}_J + m^2)/(\mathcal{H}_J^{\mathrm{free}} + m^2)]$, the two being the same. In our model cases, we notice that not all quantum numbers in J are relevant for our radial Green function $\Delta_J(r,r';s)$. In view of this, for the three possible forms of $\mathcal{H}_J$ considered above [see (2.14), (2.16), and (2.18)], we may express the decomposition formula (2.21) in the more explicit forms:

$$\text{Case 1: } F(s) = \sum_{(i,j)}(2j+1)(2l+1)F_{(l,j)}(s) = \sum_{l=0,1/2,1,\ldots}(2l+1)(2l+2)[F_{(l,j=l+1/2)}(s) + F_{(l+1/2,j=(l+1/2)-(1/2))}(s)], \tag{2.30a}$$

$$\text{Case 2: } F(s) = \sum_{l=0,1/2,1,\ldots}\sum_{l_3=-l}^{l}\sum_{t_3=\pm 1/2}(2l+1)F_{(l,l_3,t_3)}(s), \tag{2.30b}$$

$$\text{Case 3: } F(s) = \sum_{l=0,1/2,1,\ldots}\sum_{j_3=-(l+1/2)}^{l+1/2}(2l+1)F_{(l,j_3)}(s). \tag{2.30c}$$

Note that the degeneracy factors here contain a common factor $(2l+1)$ from the $\bar{l}_3$-sum. Here, as the notations of (2.30a)–(2.30c) are used, the designation $J \le J_L$ or $J > J_L$ may be identified with the appropriate division in the values of the quantum number l, i.e., $J \le J_L$ when $l \le L$ (L is some, arbitrarily chosen, large value) and $J > J_L$ when $l > L$. For example, the low partial-wave sum in (2.28) is expressed explicitly as

$$\text{Case 1: } \Gamma_{J\le J_L}(A,m) = \sum_{l=0}^{L}(2l+1)(2l+2)\bigg\{\ln\frac{\det(-D^2_{(l,l+1/2)}+m^2)}{\det(-\partial^2_{(l)}+m^2} + \ln\frac{\det(-D^2_{(l+1/2,l)}+m^2)}{\det(-\partial^2_{(l+1/2)}+m^2)}\bigg\} \tag{2.31a}$$

$$\text{Case 2: } \Gamma_{J\le J_L}(A,m) = \sum_{l=0}^{L}\sum_{l_3=-l}^{l}\sum_{t_3=\pm 1/2}(2l+1)\ln\frac{\det(-D^2_{(l,l_3,t_3)}+m^2)}{\det(-\partial^2_{(l)}+m^2)} \tag{2.31b}$$

$$\text{Case 3: } \Gamma_{J\le J_L}(A,m) = \sum_{l=0}^{L}\sum_{j_3=-(l+1/2)}^{l+1/2}(2l+1)\ln\frac{\det(-D^2_{(l,l_3,t_3)}+m^2)}{\det(-\partial^2_{(l)}+m^2)}. \tag{2.31c}$$

The low partial-wave contribution, $\Gamma_{J\le J_L}$, in (2.28), may be determined numerically. On the other hand, the large partial-wave contribution, $\Gamma_{J>J_L}$, in (2.29), is calculated analytically for large L, to the desired accuracy in powers of $1/L$.

D. Low partial-wave contribution

To evaluate the low partial-wave contribution $\Gamma_{J\le J_L}$, we use the Gel'fand-Yaglom technique [24–30], which can be summarized as follows. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ denote two second-order radial differential operators on the interval $r \in [0,\infty)$. Then the ratio of the determinants is given by

$$\frac{\det\mathcal{M}_1}{\det\mathcal{M}_2} = \lim_{R\to\infty}\left(\frac{\Phi_1(R)}{\Phi_2(R)}\right), \tag{2.32}$$

where $\Phi_i(r)$ $(i=1,2)$ satisfy the *initial value* problems:

$$\mathcal{M}_i\Phi_i(r) = 0; \qquad \Phi_i(r) \sim r^{2l} \quad \text{as } r\to 0. \tag{2.33}$$

Here l is the index in the centrifugal term in (2.25). Since an initial value problem is extremely simple to solve numerically, this provides an efficient calculational method for the individual radial determinants. Here we take $\mathcal{M}_1 = \mathcal{H}_J + m^2$, and $\mathcal{M}_2 = \mathcal{H}_J^{\text{free}} + m^2$. Thus, $\Phi_2(r)$ is in fact known analytically. Then better numerical results are obtained [12] by considering directly the initial value problem with the second-order differential equation derived for the ratio function $S(r) = \ln(\Phi_1(r)/\Phi_2(r))$. This method has been implemented successfully in [12] for the instanton determinant computation, and for the false vacuum decay problem in both flat [17] and curved [35] spacetime. Furthermore, this method of calculating radial determinants can be generalized to the case when the second-order differential operator $\mathcal{M}$ in question contains a matrix-type potential [29], as in our case 3. Explicit numerical results for the three radial cases discussed above will be presented in the sequel.

III. LARGE PARTIAL-WAVE CONTRIBUTIONS AND RENORMALIZATION

The large partial-wave contribution cannot be evaluated numerically because of the need to remove the heavy mass regulator, Λ. Instead we compute *analytically* the large L behavior of $\Gamma^\Lambda_{J>J_L}$. To determine the large partial-wave contribution $\Gamma^\Lambda_{J>J_L}$ [given by (2.29)], one needs the large-l (i.e., $l \gg L$) behavior of the function $F_J(s)$. To that end, in Refs. [12,16] we used the scattering phase shift representation of $F_J(s)$, and then the radial WKB approximation up to second order. Here we present a much simpler approach, introducing a new "uniform" DeWitt expansion for the radial proper-time Green's function $\tilde{\Delta}_J(r,r;s)$, which remains valid when l becomes large. As we shall see, this new approach gives rise to results in complete agreement with those from our earlier method, with much less labor.

A. Uniform DeWitt expansion

In the presence of the effective radial potential, $V(r) = \{[4l(l+1)+\frac{3}{4}]/r^2\} + \mathcal{V}(r)$, we seek a large-$l$ asymptotic representation of the related proper-time Green's function $\tilde{\Delta}(r,r';s)$:

$$(\partial_s - \partial_r^2 + V(r))\tilde{\Delta}(r,r';s) = 0, \qquad (\text{for } s>0) \tag{3.1a}$$

$$s\to 0+: \tilde{\Delta}(r,r';s) \to \delta(r-r'). \tag{3.1b}$$

We take $\mathcal{V}(r)$ to be a typical smooth potential. Now, as l becomes very large, the presence of the large centrifugal potential, $[4l(l+1)+\frac{3}{4}]/r^2$, has the consequence that our Green's function $\tilde{\Delta}(r,r';s)$ acquires a totally negligible amplitude for $s \gg A/l^2$ [where A is an $O(1)$ constant]. Because of this property, given a certain quantity which involves the integral of this function over s [such as, for example, $\Gamma_{J>J_L}(A;m)$, given by (2.29)], it will suffice to use an accurate representation of $\tilde{\Delta}(r,r';s)$ for s satisfying the condition $0 < sl^2 \lesssim O(1)$, and having the property that it becomes exponentially small for $s \gg A/l^2$. Although only small-s values are relevant, the usual small-s DeWitt expansion [the one-dimensional analogue of (2.6)] cannot serve this purpose since it fails to account for the effect of the large centrifugal potential term. There is a conflict between the small s limit and the large l limit. To see this problem more clearly, consider the behavior of the function $\tilde{\Delta}(r,r';s)$ with $\mathcal{V}(r)$ set to zero. For this free case, denoted $\tilde{\Delta}^{\rm free}(r, r'; s)$, we have a closed-form expression in general n spacetime dimensions (see Appendix A). As l becomes large, this function admits a uniform approximation of the form [see (A10)]

$$\tilde{\Delta}^{\rm free}(r, r'; s) \sim \frac{1}{\sqrt{4\pi s}} e^{-((r-r')^2/4s)-([4l(l+1)+(3/4)]/rr')s} \times \{1 + O(s^2)\}, \tag{3.2}$$

valid as long as s is such that $0 < sl^2 \lesssim O(1)$. The naive small-s DeWitt expansion is not adequate for our purpose since it effectively replaces the exponential factor $e^{-\{[4l(l+1)+(3/4)]/rr'\}s}$ [which can be $O(1)$ for $s \sim \frac{A}{l^2}$] by the first few terms of its Taylor series in s.

To obtain the desired large-l expansion of our radial proper-time Green's function, it is convenient to set

$$V(r) = l^2 U(r), \tag{3.3}$$

[so that $U(r)$ remains finite as $l \to \infty$], and introduce the rescaled proper-time variable

$$t = l^2 s. \tag{3.4}$$

Now the situation for the large-l limit of $\tilde{\Delta}(r, r'; \frac{t}{l^2})$ is actually the same as that appropriate to the so-called $\frac{1}{\Lambda}$-expansion of the proper-time Green function considered previously (for a different purpose) in Ref. [36], identifying Λ with l^2. Thus, based on the result of [36], we may immediately write the $\frac{1}{l}$-expansion of $\tilde{\Delta}(r, r'; \frac{t}{l^2})$, having the structure

$$\tilde{\Delta}\left(r, r'; \frac{t}{l^2}\right) = \frac{l}{\sqrt{4\pi t}} e^{-\{[l^2(r-r')^2]/4t\}} \left\{\sum_{k=0}^{\infty} b_k(r, r'; t)\left(\frac{1}{l^2}\right)^k\right\}, \tag{3.5}$$

with suitable coefficient functions $b_k(r, r'; t)$ which are regular near $r = r'$. Inserting this form in (3.1a), we see that the coefficient functions $b_k(r, r'; t)$ must satisfy

$$l^2 U(r) \sum_{k=0}^{\infty} b_k(r, r'; t)\left(\frac{1}{l^2}\right)^k + l^2 \sum_{k=0}^{\infty} \partial_t b_k(r, r'; t)\left(\frac{1}{l^2}\right)^k + \frac{l^2(r-r')}{t} \sum_{k=0}^{\infty} \partial_r b_k(r, r'; t)\left(\frac{1}{l^2}\right)^k - \sum_{k=0}^{\infty} \partial_r^2 b_k(r, r'; t)\left(\frac{1}{l^2}\right)^k = 0. \tag{3.6}$$

We can here regard $U(r)$ to be strictly of order $(\frac{1}{l^2})^0$, i.e., disregard the fact that it might contain terms with $\frac{1}{l}$-suppression, to simplify the presentation of our result. Then, (3.6) gives rise to recurrence relations satisfied by the coefficient functions $b_k(r, r'; t)$:

$$O(l^2)\colon\ U(r) b_0(r, r'; t) + \partial_t b_0(r, r'; t) + \frac{r-r'}{t} \partial_r b_0(r, r'; t) = 0, \tag{3.7a}$$

$$O(l^{2-2k})\colon\ U(r) b_k(r, r'; t) + \partial_t b_k(r, r'; t) + \frac{r-r'}{t} \partial_r b_k(r, r'; t) - \partial_r^2 b_{k-1}(r, r'; t) = 0, \qquad (k = 1, 2, 3, \ldots). \tag{3.7b}$$

Further, because of the boundary condition (3.1b), we must have $b_k(r, r'; t = 0) = \delta_{k0}$.

To simplify the analysis of the recurrence relations, we may introduce a new variable u (instead of r) by setting $r = r' + tu$ and define a new set of functions,

$$\tilde{b}_k(u, r', t) = e^{1/u \int_{r'}^{r'+tu} U(w)dw} b_k(r' + tu, r'; t). \tag{3.8}$$

[Here we have restricted our attention to the case when $U(r)$ is not a matrix-valued potential. The case with a matrix-valued potential is discussed in Appendix B.] Then the above recurrence relations can be recast as

$$O(l^2)\colon\ \frac{\partial}{\partial t}\tilde{b}0(u, r'; t) = 0, \tag{3.9a}$$

$$O(l^{2-2k})\colon\ \frac{\partial}{\partial t}\tilde{b}k(u, r'; t) = \frac{1}{t^2}\left\{\frac{\partial^2}{\partial u^2}\tilde{b}_{k-1}(u, r'; t) - 2g'(u, r'; t)\frac{\partial}{\partial u}\tilde{b}_{k-1}(u, r'; t) + [g'(u, r'; t)^2 - g''(u, r'; t)]\tilde{b}_{k-1}(u, r'; t)\right\}, \quad (k = 1, 2, \ldots), \tag{3.9b}$$

where $g(u, r'; t) \equiv \frac{1}{u}\int_{r'}^{r'+tu} U(w)dw$, $g'(u, r'; t) \equiv \frac{\partial}{\partial u} g(u, r'; t)$, and $g''(u, r'; t) \equiv \frac{\partial^2}{\partial u^2} g(u, r'; t)$. Since $b_0(r, r'; t = 0) = 1$, and so $\tilde{b}_0(u, r'; t = 0) = 1$, we now immediately conclude from (3.9a) that $\tilde{b}_0(u, r'; t) = 1$ for any $t > 0$. This in turn tells us that

$$b_0(r, r'; t) = e^{-[t/(r-r')]\int_r^{r'} U(w)dw}. \tag{3.10}$$

As a check, we note that if we choose the form $U(r) = [4l(l+1) + \frac{3}{4}]/(l^2 r^2)$ [appropriate to the free case with $\mathcal{V}(r) = 0$], then (3.10) reduces to $b_0(r, r'; t) = e^{-\{[4l(l+1)+(3/4)]/rr'\}(t/l^2)}$, producing the correct exponential factor in (3.2), related to the centrifugal potential term. Clearly, in the coincidence limit of $r' = r$, we have

$$b_0(r, r; t) = e^{-tU(r)}. \tag{3.11}$$

Using $\tilde{b}_0(u, r'; t) = 1$ in the $k = 1$ case of (3.9b), we obtain

$$\frac{\partial}{\partial t}\tilde{b}_1(u, r'; t) = \frac{1}{t^2}\{g'(u, r'; t)^2 - g''(u, r'; t)\}. \tag{3.12}$$

Then, to find the coincidence limit of $b_1(r, r'; t)$, i.e., the expression for $r' = r$, we may set $u = 0$ in (3.12) [together with the easily obtained expressions $g'(0, r'; t) = \frac{1}{2}t^2U'(r)$ and $g''(0, r'; t) = \frac{1}{3}t^3U''(r')$] to obtain

$$\frac{\partial}{\partial t}\tilde{b}_1(u = 0, r; t) = \frac{1}{4}t^2U'(r)^2 - \frac{1}{3}tU''(r). \tag{3.13}$$

This immediately leads to the expression

$$b_1(r, r; t) = e^{-tU(r)}\left\{\frac{1}{12}t^3U'(r)^2 - \frac{1}{6}t^2U''(r)\right\}. \tag{3.14}$$

Higher-order coefficients can be found similarly; for instance, for $b_2(r, r; t)$ we find

$$b_2(r, r; t) = e^{-tU(r)}\left\{\frac{1}{288}t^6U'(r)^4 - \frac{11}{360}t^5U'(r)^2U''(r) + \frac{1}{40}t^4U''(r)^2 + \frac{1}{30}t^4U'(r)U^{(3)}(r) - \frac{1}{60}t^3U^{(4)}(r)\right\}. \tag{3.15}$$

We can now exhibit the desired $\frac{1}{l}$-expansion structure for our radial proper-time Green's function in the coincidence limit. Returning to the notations using $V(r)$ and s, it takes, based on the results of (3.11), (3.12), (3.13), and (3.14), the following form:

$$\tilde{\Delta}(r, r; s) = \frac{1}{\sqrt{4\pi s}}e^{-sV(r)}\left\{1 + \left(\frac{1}{12}s^3V'(r)^2 - \frac{1}{6}s^2V''(r)\right) + O\left(\frac{1}{l^4}\right)\right\}. \tag{3.16}$$

[See Appendix A for the explicit verification that this gives rise to a correct large-l expansion for $\tilde{\Delta}^{\text{free}}(r, r; s)$]. In effect we have resummed all nonderivative terms in the standard DeWitt expansion. These nonderivative terms are all of the form $(sV)^k$ for some k, and recalling that V depends quadratically on the partial-wave index l, we see that all these terms are of $O(1)$ for $sl^2 \sim O(1)$. On the other hand, the remaining terms in the expansion (3.16) go like $s^3l^4 \sim s(sl^2)^2$, and $s^2l^2 \sim s(sl^2)$, etc., and so remain small in this uniform limit of small s and large l.

This modified expansion may be used even when $V(r)$ contains, apart from $O(l^2)$ terms, some subleading terms as with the case $V(r) = l^2U(r) + lT(r) + Q(r)$; in this case, one can also generate an equally valid $\frac{1}{l}$-expansion starting from the form (3.16) by having the exponential of the subleading terms, i.e., $e^{-(t/l^2)(lT(r)+Q(r))}$ expanded (partly or wholly) in powers of $\frac{1}{l}$. This can be justified when r is restricted to the range in which $T(r)$ and $Q(r)$ remain bounded. This trivial rearrangement can, in fact, be incorporated within our $\frac{1}{l}$-expansion ansatz (3.5) by allowing the power series development in the ansatz to have also odd-power terms in $\frac{1}{l}$.

B. Explicit large partial-wave contributions

We may use the form (3.16), with the formulas (2.22) and (2.29), to determine explicitly the large partial-wave contribution to the effective action. To facilitate this calculation, we follow Refs. [12,16] by trading the regulator mass Λ for a dimensional regularization parameter ϵ. This is achieved by demanding that

$$-\int_0^\infty \frac{ds}{s}(e^{-m^2s} - e^{-\Lambda^2 s})F(s) \sim -\int_0^\infty \frac{ds}{s}e^{-m^2s}s^\epsilon F(s). \tag{3.17}$$

Since $F(s) = (\text{finite constant}) + O(s)$ for small s, we then see that (3.17) requires

$$-\ln\left(\frac{\Lambda^2}{m^2}\right) + O\left(\frac{1}{\Lambda^2}\right) = -\frac{1}{\epsilon} + (\gamma + 2\ln m) + O(\epsilon), \tag{3.18}$$

where $\gamma = 0.5772\ldots$ is Euler's constant. Thus, the relation between ϵ and Λ is given by

$$\epsilon \leftrightarrow \frac{1}{\gamma + \ln\Lambda^2}. \tag{3.19}$$

Note that this is only to simplify our calculations; all the s-integrations appearing below can also be carried out within the original Pauli-Villars regularization framework.

With this preparation, we now proceed to the calculation of $\Gamma_{J>J_L}(A; m)$ for our case 1 and case 2. Case 3 will be considered in the sequel.

1. Case 1

With $\mathcal{H}_{J_1}$ given in (2.14), we have the radial potential

$$V_{(l,j)}(r) = \frac{4l(l+1) + \frac{3}{4}}{r^2} + 4f(r)\left\{j(j+1) - l(l+1) - \frac{3}{4}\right\} + 3r^2f(r)^2 \tag{3.20}$$

which may be used in (3.16). Then, from (2.22), $F_J(s)$ for large enough l will follow. Further, if we represent $F(s)$ by the form (2.30a) and use the correspondence (3.19), it is possible to express the first part of (2.29) [the contribution to $\Gamma_{J>J_L}(A; m)$ other than the renormalization counterterm] as

$$\Gamma^{\epsilon}_{J>J_L}(A;m) = \int_0^\infty dr \int_0^\infty ds \bigg[-\frac{1}{s} e^{-m^2 s} s^\epsilon \sum_{l=L+1/2}^{\infty} (2l+1)(2l+2) \{ \tilde{\Delta}_{(l,j=l+1/2)}(r,r;s) + \tilde{\Delta}_{(l+1/2,j=l)}(r,r;s) - \tilde{\Delta}^{\text{free}}_{(l)}(r,r;s) - \tilde{\Delta}^{\text{free}}_{(l+1/2)}(r,r;s) \} \bigg], \tag{3.21}$$

where we placed the s-integral before the r-integral, in order to give an explicit result for $\Gamma_{J>J_L}(A;m)$ for general fields. Using (3.16) in (3.21) with $V_{(l,j)}(r)$ given by (3.20), the right-hand side of (3.21) can be expressed as

$$\begin{aligned} \Gamma^{\epsilon}_{J>J_L}(A;m) = \int_0^\infty dr \int_0^\infty ds \bigg[-\frac{1}{s} e^{-m^2 s} s^\epsilon \sum_{l=L+1/2}^{\infty} (2l+1)(2l+2) \frac{1}{\sqrt{4\pi s}} \\ \times \bigg\{ e^{-sV_{(l,l+1/2)}(r)} \bigg(1 + \bigg[\frac{1}{12} s^3 V_{(l,l+1/2)}{}'(r)^2 - \frac{1}{6} s^2 V_{(l,l+1/2)}{}''(r) \bigg] + O\bigg(\frac{1}{l^4}\bigg)\bigg) \\ + e^{-sV_{(l+1/2,l)}(r)} \bigg(1 + \bigg[\frac{1}{12} s^3 V_{(l+1/2,l)}{}'(r)^2 - \frac{1}{6} s^2 V_{(l+1/2,l)}{}''(r) \bigg] + O\bigg(\frac{1}{l^4}\bigg)\bigg) \\ - e^{-sV^{\text{free}}_{(l)}(r)} \bigg(1 + \bigg[\frac{1}{12} s^3 V^{\text{free}\prime}_{(l)}(r)^2 - \frac{1}{6} s^2 V^{\text{free}\prime\prime}_{(l)}(r) \bigg] + O\bigg(\frac{1}{l^4}\bigg)\bigg) \\ - e^{-sV^{\text{free}}_{(l+1/2)}(r)} \bigg(1 + \bigg[\frac{1}{12} s^3 V^{\text{free}\prime}_{(l+1/2)}(r)^2 - \frac{1}{6} s^2 V^{\text{free}\prime\prime}_{(l+1/2)}(r) \bigg] + O\bigg(\frac{1}{l^4}\bigg)\bigg) \bigg\} \bigg], \end{aligned} \tag{3.22}$$

where $V^{\text{free}}_{(l)}(r) \equiv [4l(l+1) + \frac{3}{4}]/r^2$, and we recall from (2.13) that the l summation is over integer and half-integer values. We remark that, if we consider the total of all explicitly kept terms in the integrand of (3.22), the neglected terms would at most be $O(\frac{1}{l^5})$; this happens because the leading, i.e., order-$\frac{1}{l^4}$ terms coming from the four pieces denoted $O(\frac{1}{l^4})$ in (3.22) [which are given in terms of $b_2(r,r;t)$] necessarily cancel, as the leading terms of the potential V match those of V^{free}. The above expression is fully equivalent to that found using the second-order radial WKB approximation for phase shifts in Refs. [12,16]. (Actually, in Refs. [12,16], the WKB approximation was used with the Langer-modified radial potential [37]; but this is inessential for large partial-wave contributions as the difference corresponds to a trivial rearrangement of our $\frac{1}{l}$-expansion series.) It is not difficult to see that this equivalence between the result based on the radial WKB approximation and our present approach using the $\frac{1}{l}$-expansion persists to even higher orders also. But our new $\frac{1}{l}$-expansion for the radial proper-time Green function is considerably simpler.

The l-sum in (3.22) can be performed with the help of the Euler-Maclaurin summation formula [adapted to the l summation over integer and half-integer values; recall (2.13)]

$$\sum_{l=L+1/2}^{\infty} f(l) = 2\int_L^\infty dl f(l) - \frac{1}{2} f(L) - \frac{1}{24} f'(L) + \cdots. \tag{3.23}$$

All terms in this expansion, including the integral term, can be computed analytically. The result of this calculation, which is rather lengthy, is given in Appendix C. We thus obtain an explicit double-integral representation for $\Gamma^{\epsilon}_{J>J_L}(A;m)$ [see the expressions given in (C9)–(C11)]. Then the integration over the proper-time variable s can be performed in a straightforward manner. After carrying out these s-integrations, we find that the quantity $\Gamma^{\epsilon}_{J>J_L}(A;m)$ for sufficiently large L is given explicitly by the form

$$\begin{aligned} \Gamma^{\epsilon}_{J>J_L}(A;m) = \frac{1}{8\epsilon} \int_0^\infty dr r^3 [4h(r)^2 + (2f(r) + rf'(r))^2] + \int_0^\infty dr \bigg[-2r(L+2)h(r)\sqrt{4L^2 + m^2 r^2} \\ - \frac{r^3}{8} \bigg\{ 2\ln\bigg(\frac{4L}{r}\bigg) + \gamma \bigg\} \{4h(r)^2 + (2f(r) + rf'(r))^2\} \\ + \frac{Lr}{12\sqrt{4L^2 + m^2 r^2}} \{ h(r)(24m^2 r^2 + 44h(r) r^2 - 75) + 39 f(r)^2 r^2 + (6r^2 f'(r)^2 + f''(r)) r^2 \\ + f(r)(-2f''(r) r^4 + 24 f'(r) r^3 - 9) \} \bigg] + O\bigg(\frac{1}{L}\bigg) + O(\epsilon), \end{aligned} \tag{3.24}$$

where $h(r) \equiv f(r)[r^2 f(r) - 1]$.

For the expression of $\Gamma_{J>J_L}(A;m)$ we must subtract the renormalization counterterm from the expression (3.24). Using (3.19), the renormalization counterterm appearing in the definition of the renormalized effective action (2.8) is

$$\frac{1}{12}\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma-\ln\mu^2\right)\int d^4x\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}). \tag{3.25}$$

In this radial ansatz case, $F_{\mu\nu}$ is equal to

$$F_{\mu\nu} = -2\eta_{\mu\nu a}f(r)(r^2f(r)-1)\tau^a + \frac{x_\lambda}{r}(x_\mu\eta_{\nu\lambda a}-x_\nu\eta_{\mu\lambda a})(2rf(r)^2+f'(r))\tau^a. \tag{3.26}$$

Hence the counterterm reads

$$\frac{1}{8}\left(\frac{1}{\epsilon}-\gamma-\ln\mu^2\right)\int_0^\infty dr r^3[4r^4f(r)^4-8r^2f(r)^3+8f(r)^2 + 4rf'(r)f(r)+r^2f'(r)^2]. \tag{3.27}$$

Comparing the expression (3.27) with (3.24), we see that the divergence terms as $\epsilon\to 0$ match precisely between the two. This also verifies that our renormalization procedure is a *gauge-invariant* one. The full large partial-wave contribution to the effective action, the sum of the expression in (3.24) and minus the result in (3.27), now becomes

$$\Gamma_{J>J_L}(A;m) = \int_0^\infty dr\left[\frac{r^3}{4}\ln\left(\frac{\mu r}{4L}\right)\{4h(r)^2+(2f(r)+rf'(r))^2\} - 2r(L+2)h(r)\sqrt{4L^2+m^2r^2} + \frac{Lr}{12\sqrt{4L^2+m^2r^2}}\{h(r)(24m^2r^2+44h(r)r^2-75)+39f(r)^2r^2+(6r^2f'(r)^2+f''(r))r^2 + f(r)(-2f''(r)r^4+24f'(r)r^3-9)\}\right]+O\left(\frac{1}{L}\right). \tag{3.28}$$

This generalizes the result of Refs. [12,16] where the calculation was performed assuming the special form $f(r)=\frac{1}{r^2+\rho^2}$ (i.e., the single-instanton solution).

For any given radial function $f(r)$, one can then consider the sum of this analytic expression (3.28) for the large partial-wave contribution, and the numerically determined result for $\Gamma_{J\le J_L}(A;m)$ [based on (2.28) and the relation (2.32)] to determine the corresponding full renormalized effective action. Each has quadratic, linear, and log divergences for large L. For large L, the L-dependence in the two expressions cancels, as was originally found in [12]. In fact, to improve the numerical efficiency of the full effective action calculation, one may extend the large L expression for $\Gamma_{J>J_L}(A;m)$ in (3.28), to include also terms up to $O(\frac{1}{L^2})$. This can be done with the help of our $\frac{1}{l}$-expansion, using the explicit expression for $b_2(r,r;t)$ in (3.15).

2. Case 2

With $\mathcal{H}_{J_2}$ given in (2.16), we have the radial potential

$$V_{(l,l_3,t_3)}(r) = \frac{4l(l+1)+\frac{3}{4}}{r^2}+8g(r)l_3t_3+r^2g(r)^2. \tag{3.29}$$

If we represent $F(s)$ by the form (2.30b) and use the correspondence (3.19), the first part of (2.29) can be expressed by

$$\Gamma^\epsilon_{J>J_L}(A;m) = \int_0^\infty dr\int_0^\infty ds\left[-\frac{1}{s}e^{-m^2s}s^\epsilon\sum_{l=L+1/2}^\infty(2l+1)\sum_{l_3=-l}^{l}\sum_{t_3=\pm1/2}\{\tilde\Delta_{(l,l_3,t_3)}(r,r;s)-\tilde\Delta^{\mathrm{free}}_{(l)}(r,r;s)\}\right]. \tag{3.30}$$

For the function $\tilde\Delta_{(l,l_3,t_3)}(r,r;s)$ [or $\tilde\Delta^{\mathrm{free}}_{(l)}(r,r;s)$] on the right-hand side, we may use the $\frac{1}{l}$-expansion result in (3.16) with $V(r)$ taken to be equal to the radial potential in (3.29) [the radial potential $V^{\mathrm{free}}_{(l)}(r)=[4l(l+1)+\frac{3}{4}]/r^2$]. The l_3-sum and t_3-sum can be done explicitly, using the formulas

$$\sum_{l_3=-l}^{l}\sum_{t_3=\pm1/2}e^{-8l_3t_3g(r)s} = \frac{2\{e^{4(l+1)g(r)s}-e^{-4lg(r)s}\}}{e^{4g(r)s}-1}, \tag{3.31a}$$

$$\sum_{l_3=-l}^{l}\sum_{t_3=\pm1/2}l_3t_3e^{-8l_3t_3g(r)s} = \frac{1}{(e^{4g(r)s}-1)^2}\{le^{-4lg(r)s}-le^{4(l+2)g(r)s}+(l+1)e^{4(l+1)g(r)s}-(l+1)e^{-4(l-1)g(r)s}\}. \tag{3.31b}$$

We then perform the l-sum with the help of the Euler-Maclaurin summation formula; this produces a double-integral representation in which the integration over the proper-time variable s can be executed without too much difficulty. The result of these manipulations is

$$\Gamma^{\epsilon}_{J>J_L}(A;m) = \frac{1}{24\epsilon}\int_0^\infty drr^3[8g(r)^2 + 4rg'(r)g(r) + r^2g'(r)^2] + \int_0^\infty dr\bigg[-r^3g(r)^2\sqrt{4L^2+m^2r^2} - \frac{r^3}{24}\bigg\{2\ln\bigg(\frac{4L}{r}\bigg) + \gamma\bigg\}\{8g(r)^2 + 4rg'(r)g(r) + r^2g(r)^2\} + \frac{Lr^3}{90\sqrt{4L^2+m^2r^2}}\{6r^4g(r)^4 + (20-240L)g(r)^2 - 5r(rg''(r) - 12g'(r))g(r) + 15r^2g'(r)^2\}\bigg] + O\bigg(\frac{1}{L}\bigg) + O(\epsilon). \tag{3.32}$$

The field strength appropriate to this case 2 reads

$$F_{\mu\nu} = -2\eta_{\mu\nu3}g(r)\tau^3 + \frac{x_\lambda}{r}(x_\mu\eta_{\nu\lambda3} - x_\nu\eta_{\mu\lambda3})g'(r)\tau^3, \tag{3.33}$$

and hence the renormalization counterterm (3.25) is given by

$$\frac{1}{24}\bigg(\frac{1}{\epsilon} - \gamma - \ln\mu^2\bigg)\int_0^\infty drr^3[8g(r)^2 + 4rg'(r)g(r) + r^2g'(r)^2]. \tag{3.34}$$

Again we see that the divergent terms as $\epsilon \to 0$ in (3.32) match precisely those of the renormalization counterterm. The full large partial-wave contribution, including the renormalization term, is thus given by

$$\Gamma_{J>J_L}(A;m) = \int_0^\infty dr\bigg[\frac{r^3}{12}\ln\bigg(\frac{\mu r}{4L}\bigg)\{8g(r)^2 + 4rg'(r)g(r) + r^2g'(r)^2\} - r^3g(r)^2\sqrt{4L^2+m^2r^2} + \frac{Lr^3}{90\sqrt{4L^2+m^2r^2}}\{6r^4g(r)^4 + (20-240L^2)g(r)^2 - 5r(rg''(r) - 12g'(r))g(r) + 15r^2g'(r)^2\}\bigg] + O\bigg(\frac{1}{L}\bigg). \tag{3.35}$$

This expression can now be combined with the numerical low partial-wave contribution to determine the finite renormalized effective action in a radial Yang-Mills background of the form $A_\mu(x) = 2\eta_{\mu\nu3}x_\nu g(r)\frac{\tau^3}{2}$.

IV. CASES WITH ASYMPTOTICALLY UNIFORM FIELD STRENGTHS

In Ref. [36] the basis for the validity of the $\frac{1}{l}$-expansion structure (3.5) was a perturbative argument: i.e., in the effective potential $V(r) = [4l(l+1) + \frac{3}{4}]/r^2 + \mathcal{V}(r)$, $\mathcal{V}(r)$ can be treated as a perturbation to the centrifugal potential term. We then saw in the previous section that the resulting explicit expression in (3.16), if used to calculate the large partial-wave contribution of the effective action, yields a result completely equivalent to that obtained from using the quantum-mechanical WKB approximation with scattering phase shifts [12,16]. It is not immediately clear what happens if the potential is unbounded as $r \to \infty$, in which case in the WKB language one should use a bound state analysis rather than a scattering analysis. This is precisely the potential form that arises when the gauge-field strength approaches a nonzero constant value at infinity. Indeed, in our case 2 with $g(r) = -\frac{B}{2}$ (leading to uniform self-dual field strengths), we have the quadratic potential [see (3.29)]

$$\mathcal{V}_{(l,l_3,t_3)}(r) = -4Bl_3t_3 + \frac{B^2}{4}r^2. \tag{4.1}$$

Thus, the eigenstates of the corresponding radial operator $\mathcal{H}_{J_2}$ consist of only bound states. In principle, for the large partial-wave contribution to the effective action, one might still try to use the quantum-mechanical WKB approximation; but, the WKB approximation for bound states has a rather different structure from that for scattering states.

It is thus an important issue to know whether or not the $\frac{1}{l}$-expansion for the radial proper-time Green function, considered in the previous section, retains its validity even when the potential $\mathcal{V}(r)$ blows up for large r, as above. In fact, our $\frac{1}{l}$-expansion-based formula in (3.16) *does* describe the correct asymptotic form valid for all $r \in (0, \infty)$ even with such an unbounded potential; with the *proviso* that in (3.16), the exponential prefactor $e^{-sV(r)}$ cannot be replaced by its (truncated) power series in s, for the form to be valid even for very large r. That is, we also have *infrared* physics captured correctly by our large-l limit form of the radial proper-time Green function. We demonstrate this below through the treatment of an important special case, that of uniform self-dual field strengths for which the quadratic potential (4.1) is relevant. Note that the effective action in a uniform self-dual field strength background has been studied by many authors before

[7,38,39], but not using the partial-wave-based proper-time formalism. This analysis will also provide a useful comparison when we consider, in the sequel to this paper, the effective action in the inhomogeneous background described by the form (2.9) with $f(r) = 0$ and $g(r) = -\frac{B}{2} \times \tanh(r/r_0)$.

In our procedure the calculation of the one-loop effective action in the gauge-field background $A_\mu(\mathbf{x}) = -\eta_{\mu\nu a} x^\nu \frac{B}{2} \tau^a$ starts from the study of the radial proper-time Green function $\tilde{\Delta}_{(l,l_3,t_3)}(r, r'; s)$, satisfying the equation

$$\left(\partial_s - \partial_r^2 + \frac{4l(l+1) + \frac{3}{4}}{r^2} - 4Bl_3t_3 + \frac{B^2}{4} r^2\right)\tilde{\Delta}_{(l,l_3,t_3)}(r, r'; s) = 0, \qquad (s > 0). \tag{4.2}$$

Just like the free radial proper-time Green function discussed in Appendix A, in this case it is possible to find the corresponding radial Green function in a closed form, by using (for example) the method of quantum canonical transformations [40,41]. It is given by an expression involving the modified Bessel function

$$\tilde{\Delta}_{(l,l_3,t_3)}(r, r'; s) = \frac{B\sqrt{rr'}}{2\sinh(Bs)} e^{-(1/4)B\coth(Bs)(r^2+r'^2)+4Bl_3t_3 s} I_{2l+1}\left(\frac{Brr'}{2\sinh(Bs)}\right). \tag{4.3}$$

We check the large-l limit of the expression (4.3) directly against our general formula (3.16), as the potential is specialized to the form $V(r) = [4l(l+1) + \frac{3}{4}]/r^2 - 4Bl_3t_3 + \frac{B^2}{4} r^2$. As we explained earlier, we want our large-l limit expression to have validity for s satisfying the condition $0 < sl^2 \lesssim O(1)$. But no restriction follows on the range of the radial coordinate r, and so, for the given potential (which diverges quadratically for large r), we want our large-l limit form to be faithful with the true behavior for an arbitrarily large value of $r^2 s$. With this point kept in mind, we may set $s = t/l^2$ and use the uniform asymptotic expansion for large orders for the modified Bessel function in Appendix A [see (A8)] with our expression (4.3), in the coincident limit, to obtain the large-l limit form

$$\tilde{\Delta}_{(l,l_3,t_3)}\left(r, r; \frac{t}{l^2}\right) = \frac{Br}{2\sinh(\frac{Bt}{l^2})} \frac{e^{-(1/2)Br^2\coth(Bt/l^2)+4Bl_3t_3(t/l^2)+\nu\{\sqrt{1+\tilde{z}^2}+\ln[\tilde{z}/(\sqrt{1+\tilde{z}^2}+1)]\}}}{\sqrt{2\pi\nu}(1+\tilde{z}^2)^{1/4}}\left(1 + \frac{3x - 5x^3}{24\nu} + O\left(\frac{1}{\nu^2}\right)\right), \tag{4.4}$$

where $\nu = 2l + 1$, $\tilde{z} \equiv Br^2/[2\nu\sinh(\frac{Bt}{l^2})]$, and $x \equiv \frac{1}{\sqrt{1+\tilde{z}^2}}$.

For l very large and $0 < t \lesssim O(1)$ (but with no restriction on the value of r), it is possible to simplify the complicated exponential term in (4.4) by keeping in its exponent only the piece $-\frac{4t}{r^2} - \frac{1}{4}B^2r^2\frac{t}{l^2}$, namely, the leading terms of the given exponent when l becomes very large (but for an unrestricted value of r), and expand all the other terms in increasing powers of $\frac{1}{l}$. Since $\tilde{z}$ is $O(l)$, this also implies expansion in powers of $\frac{1}{\tilde{z}}$. Then, from (4.4), we obtain the following expansion

$$\tilde{\Delta}_{(l,l_3,t_3)}\left(r, r; \frac{t}{l^2}\right) = \frac{le^{-(4t/r^2)-B^2r^2(t/4l^2)}}{\sqrt{4\pi t}}\left\{1 - \frac{4t}{r^2}\frac{1}{l} + \left(\frac{16t^3}{3r^6} + \frac{4t^2}{r^4} + 4Bl_3t_3t - \frac{3t}{4r^2}\right)\frac{1}{l^2} + O\left(\frac{1}{l^3}\right)\right\}. \tag{4.5}$$

It should be noted that, in the same limit, the leading terms of our potential, $V(r) = [4l(l+1) + \frac{3}{4}]/r^2 - 4Bl_3t_3 + \frac{B^2}{4} r^2$, are given by $\frac{4l^2}{r^2} + \frac{B^2}{4} r^2$. Now, according to the same kind of reasoning as discussed after (3.16), we may replace (4.5) by another expansion in which the exponential factor at front is assumed by $e^{-V(r)t/l^2}$; this rearrangement is harmless for arbitrarily large values of $r^2\frac{t}{l^2}$ here. The result of this rearrangement is to make (4.5) turn into the structure predicted by our $\frac{1}{l}$-expansion formula (3.16), with all factors precisely equal.

We will now show that our radial proper-time Green function (4.3) can be used to rederive the known expression for the effective action. In a uniform self-dual field strength background, it is known that [38,39]

$$\mathrm{tr}\langle \mathbf{x}s|\mathbf{x}\rangle = \frac{2}{(4\pi s)^2}\frac{(Bs)^2}{\sinh^2(Bs)}. \tag{4.6}$$

Thus we have, as this form is used with (2.3), (2.4), and (2.8),

$$\Gamma_{\rm ren}(A; m) = -\frac{1}{(4\pi)^2}\frac{2}{3}B^2\ln\left(\frac{m^2}{\mu^2}\right) - 2\int d^4\mathbf{x}\int_0^\infty \frac{ds}{s}e^{-m^2 s}\frac{1}{(4\pi s)^2} \times \left[\frac{(Bs)^2}{\sinh^2(Bs)} - 1 + \frac{1}{3}(Bs)^2\right]. \tag{4.7}$$

Here, in view of (2.4) and (2.22), it will suffice to show that

$$\sum_{l=0,1/2,1,\cdots}\sum_{l_3=-l}^{l}\sum_{t_3=\pm 1/2}(2l+1)\tilde{\Delta}_{(l,l_3,t_3)}(r, r; s) = 2\pi^2 r^3\,\mathrm{tr}\langle \mathbf{x}s|\mathbf{x}\rangle \tag{4.8}$$

since $d^4\mathbf{x} = 2\pi^2 r^3 dr$ after the angular integration.

Inserting the form (4.3) for $\tilde{\Delta}_{(l,l_3,t_3)}(r, r; s)$ into the left-hand side of (4.8) and carrying out the l_3 and t_3 sums yields the expression

$$\sum_{l=0,1/2,1,\ldots} \frac{(2l+1)Br}{\sinh^2(Bs)} \sinh((2l+1)Bs) e^{-(1/2)Br^2 \coth(Bs)} I_{2l+1}\left(\frac{Br^2}{2\sinh(Bs)}\right) = \frac{Br}{\sinh^2(Bs)} e^{-(1/2)Br^2 \coth(Bs)} \sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} \nu \sinh(\nu Bs) \frac{z^{2k+\nu}}{\Gamma(k+\nu+1)k!}, \tag{4.9}$$

where we set $\nu = 2l + 1$ and $z = \frac{Br^2}{4\sinh(Bs)}$, aside from using the power series representation of the modified Bessel function. If we change the summation over (k, ν) to those over $(k, n = 2k + \nu)$ and use the relation

$$\sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} \nu \sinh(\nu Bs) \frac{z^{2k+\nu}}{\Gamma(k+\nu+1)k!} = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{k_m} \frac{(n-2k)\sinh((n-2k)Bs)}{\Gamma(n-k+1)k!} = \sum_{n=1}^{\infty} z^n \frac{2^{n-1}\sinh(Bs)\cosh^{n-1}(Bs)}{(n-1)!} = \frac{1}{4}Br^2 \sum_{n=1}^{\infty} \frac{[\frac{1}{2}Br^2 \coth(Bs)]^{n-1}}{(n-1)!} = \frac{1}{4}Br^2 e^{1/2Br^2 \coth(Bs)} \tag{4.10}$$

[here $k_m = \frac{n-1}{2}(\frac{n}{2})$ if n is odd (even)], we then see that the expression in (4.9) reduces to

$$\frac{B^2 r^3}{4\sinh^2(Bs)}. \tag{4.11}$$

This coincides with the expression for $2\pi^2 r^3 \operatorname{tr}\langle \mathbf{x} s | \mathbf{x} \rangle$ when (4.6) is used for $\operatorname{tr}\langle \mathbf{x} s | \mathbf{x} \rangle$. Hence we have the relation (4.8) established.

V. CONCLUSIONS

In this work we have simplified significantly the calculational method for the one-loop effective action developed in Refs. [12,16], so that any radially symmetric background case may now be studied with calculational efficiency. The computation is split into two parts: the contribution from low partial waves is calculated numerically using the Gel'fand-Yaglom technique, and the contribution from high partial waves has been computed analytically using a modified DeWitt expansion. It is no longer necessary to invoke the results of higher-order quantum-mechanical WKB approximation explicitly—this is now automatically accounted for by using the $\frac{1}{l}$-expansion for the radial proper-time Green function. The main results are contained in the expressions (3.28) and (3.35) for the analytic behavior of the large partial-wave contribution to the renormalized effective action, for two general classes of radially symmetric gauge fields. Our approach observes gauge invariance, and can be used for any mass value for the associated quantum fluctuations. It can also be applied to the case with nonvanishing asymptotic backgrounds. In the sequel, we shall report an extensive analysis of the Yang-Mills one-loop effective action (not only for scalar matter but also for fermion fields as well), taking the radial gauge-field background form of the present work. We can then use these results to check for instance the range of validity of the derivative expansion [16,39].

In this work we have used the $\frac{1}{l}$-expansion to calculate the large partial-wave contribution to the effective action. This expansion could alternatively be used to calculate *approximately* the lower partial-wave contributions as well. Aside from the Langer modification [37] which can easily be incorporated in our $\frac{1}{l}$-expansion, this is effectively what we have done in Ref. [16] with the Yang-Mills instanton background; there, the instanton determinant was found to be good to 5% accuracy. One might be somewhat surprised by this success. But it need not be so surprising; observe that the $\frac{1}{l}$-expansion as given in (3.16) also serves to generate a systematic derivative expansion. In fact, using the expansion (3.16), we have studied several cases of one-dimensional functional determinants (including powerlike potentials and the case of $V(x) \propto \operatorname{sech}^2 x$), to find that the deviation from the exact value is typically not more than 5%. This observation could potentially be used to obtain simple approximate estimates for general radial background fields.

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APPENDIX A: THE FREE RADIAL PROPER-TIME GREEN FUNCTION IN GENERAL SPACETIME DIMENSION

In this Appendix we find the explicit form of the free radial proper-time Green's function in n spacetime dimension and then discuss its large angular-momentum limit (to facilitate the application of our approach in problems with spacetime dimension not equal to four). In n dimensions, the consideration of the Laplacian $\partial_\mu \partial_\mu$ in generalized spherical coordinates leads to the radial differential operator

$$\partial^2_{\kappa/2} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{\kappa(\kappa+n-2)}{r^2}, \quad \text{(A1)}$$

where $r = \sqrt{x_1^2 + \cdots + x_n^2}$, and $\kappa = 0, 1, 2, \ldots$. [From this form, our expression (2.15) is recovered upon setting $n = 4$ and $\kappa = 2l$]. Then, noting that $d^n x = r^{n-1} d^{n-1}\Omega$, we require the free radial proper-time Green's function $\Delta^{\text{free}}_{\kappa/2}(r, r'; s)$ to satisfy the conditions

$$\left\{\frac{\partial}{\partial s} - \frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\kappa(\kappa+n-2)}{r^2}\right\}\Delta^{\text{free}}_{\kappa/2}(r, r'; s) = 0, \quad (\text{for } s > 0) \quad \text{(A2a)}$$

$$s \to 0+: \Delta^{\text{free}}_{\kappa/2}(r, r'; s) \to \frac{1}{r^{n-1}}\delta(r - r'). \quad \text{(A2b)}$$

We introduce the modified radial proper-time Green's function $\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s)$ according to

$$\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s) = r^{(n-1)/2}\Delta^{\text{free}}_{\kappa/2}(r, r'; s) r'^{(n-1)/2}. \quad \text{(A3)}$$

In terms of this function, (A2a) and (A2b) can be rewritten as

$$\left\{\frac{\partial}{\partial s} - \frac{\partial 2}{\partial r 2} + V^{\text{free}}_{(\kappa,n)}(r)\right\}\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s) = 0, \quad (\text{for } s > 0) \quad \text{(A4a)}$$

$$s \to 0+: \tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s) \to \delta(r - r'), \quad \text{(A4b)}$$

where the centrifugal potential is

$$V^{\text{free}}_{(\kappa,n)}(r) = \left\{\kappa(\kappa+n-2) + \frac{(n-1)(n-3)}{4}\right\}\frac{1}{r^2} \equiv \frac{g^2}{r^2}. \quad \text{(A5)}$$

To obtain the explicit form of $\tilde{\Delta}^{\text{free}}_{\kappa/2}$, one can resort to a variety of methods (developed to find the Green function of the one-dimensional Schrödinger equation especially). A particularly elegant method is the one utilizing quantum canonical transformations, as detailed in Refs. [40,41]. As it turns out, for $\tilde{\Delta}^{\text{free}}_{\kappa/2}$, we have a simple closed-form expression involving the modified Bessel function:

$$\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s) = \frac{\sqrt{rr'}}{2s} e^{-(1/4s)(r^2+r'^2)} I_{\kappa+(n/2)-1}\left(\frac{rr'}{2s}\right). \quad \text{(A6)}$$

Since $I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}[1 + O(\frac{1}{|z|})]$ for large $|z|$, the $s \to 0+$ limit of this expression is

$$s \to 0+: \tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s) \to \frac{1}{\sqrt{4\pi s}} e^{-(1/4s)(r-r')^2}\{1 + O(s)\}. \quad \text{(A7)}$$

The large-κ limiting form of $\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s)$ is of interest. Then, due to the large centrifugal potential term in (A5), we expect that the function $\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s)$ be significant (i.e., acquire not-too-small amplitude) only when s lies in the range $0 < s \lesssim \frac{A}{\kappa^2}$, A denoting a constant of $O(1)$. Now, for some large given value of κ, suppose that we wish to obtain a systematic approximation of $\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s)$ which can be used for any s satisfying the condition $0 < s\kappa^2 \lesssim O(1)$ (this incidentally implies that $s\kappa \ll 1$). Then, to study the expression in (A6), we use the known large-order asymptotic expansion of the modified Bessel function [42]

$$\nu\text{large: } I_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu(\sqrt{1+(z/\nu)^2} + \ln(z/\nu/\sqrt{1+(z/\nu)^2}+1))}}{\{1 + (z/\nu)^2\}^{1/4}} \times \left[1 + \frac{3x - 5x^3}{24\nu} + O\left(\frac{1}{\nu^2}\right)\right], \quad \left(x \equiv \frac{1}{\sqrt{1 + (z/\nu)^2}} \in (0, 1]\right) \quad \text{(A8)}$$

with $z = \frac{rr'}{2s}$ and $\nu = \kappa + \frac{n}{2} - 1$. [Note that the expansion (A8) holds *uniformly* with respect to z (i.e., for any small or large z), and in the limit $|z| \to \infty$ (for fixed ν) goes back to the asymptotic form given earlier.] Since we are interested in the case $0 < s\kappa^2 \lesssim O(1)$, we may further take the limit $|\frac{z}{\nu}| = |(rr')/[2s(k + \frac{n}{2} - 1)]| \to \infty$ with the formula (A8) (i.e., consider an expansion in powers of $|\frac{\nu}{z}|$) and then use it in (A6). After some straightforward calculations, we then obtain the large-κ expansion of the form

$$\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s) \sim \frac{1}{\sqrt{4\pi s}} e^{-(1/4s)(r-r')^2 - ((\kappa+(n/2)-1)^2/rr')s} \times \left\{1 + \frac{1}{4rr'}s - \frac{(\kappa + \frac{n}{2} - 1)^2}{(rr')^2}s^2 + \frac{1}{3}\frac{(\kappa + \frac{n}{2} - 1)^4}{(rr')^3}s^3 + O(\kappa^{-3})\right\}. \quad \text{(A9)}$$

We remark that the form (A9) may be used to evaluate a certain quantity which involves, say, the integration of $\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s)$ over the full s-range [i.e., over $s \in (0, \infty)$], as long as κ is constrained to be large. This is because, when κ is large, (i) $\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s)$ becomes very small unless $s\kappa^2 \lesssim O(1)$ [this is also manifest in our form (A9)] and (ii) for s satisfying the condition $0 < s\kappa^2 \lesssim O(1)$ we can exploit the expansion of the form (A9) for $\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s)$. In view of this, the same purpose can be served by rewriting

(A9) as

$$\tilde{\Delta}^{\text{free}}_{\kappa/2}(r, r'; s) \sim \frac{1}{\sqrt{4\pi s}} e^{-(1/4s)(r-r')^2-(g^2/rr')s} \times \left\{1 - \frac{g^2}{(rr')^2} s^2 + \frac{1}{3}\frac{g^4}{(rr')^3} s^3 + \cdots\right\}, \tag{A10}$$

where $g^2 \equiv \kappa(\kappa + n - 2) + \frac{(n-1)(n-3)}{4} = (\kappa + \frac{n}{2} - 1)^2 - \frac{1}{4}$ [see (A5)]. Note that this series is fully consistent with our formula (3.16) if we set $V(r)$ to be equal to $\frac{g^2}{r^2}$.

APPENDIX B: THE $\frac{1}{l}$-EXPANSION WITH A MATRIX-VALUED POTENTIAL

The $\frac{1}{l}$-expansion of $\tilde{\Delta}(r, r; s)$ given in (3.16) is valid when the potential $V(r)$ is not a matrix type. In this Appendix we shall find a more general form which can be used when $V(r)$ and hence also the Green function $\tilde{\Delta}(r, r'; s)$ are matrix valued. The coefficient matrices $b_k(r, r'; t)$ in the $\frac{1}{l}$-expansion will now have to satisfy the matrix equations in (3.7a) and (3.7b). Choosing a new independent variable u (instead of r) by setting $r = r' + tu$ and writing $b_k(r' + tu, r'; t) \equiv \bar{b}_k(u, r'; t)$, we may recast these equations as

$$O(l^2){:}\ \partial_t \bar{b}_0(u, r'; t) + U(r' + tu)\bar{b}_0(u, r'; t) = 0, \tag{B1a}$$

$$O(l^{2-2k}){:}\ \partial_t \bar{b}_k(u, r'; t) + U(r' + tu)\bar{b}_k(u, r'; t) - \frac{1}{t^2}\partial_\mu^2 \bar{b}_{k-1}(u, r'; t) = 0, \qquad (k = 1, 2, 3, \ldots). \tag{B1b}$$

The solution of (B1a), satisfying the boundary condition $\bar{b}_0(u, r'; t = 0) = 1$, is

$$\bar{b}_0(u, r'; t) = P[e^{-\int_0^t U(r'+t_1 u)dt_1}], \tag{B2}$$

where $P[\cdots]$ denotes the t-ordering. Setting $u = 0$ in (B2) then gives

$$b_0(r, r; t) = e^{-tU(r)}, \tag{B3}$$

i.e., our formula (3.1) for the coincidence limit of $b_0(r, r'; t)$ holds even when $U(r)$ is matrix valued.

To solve (B1b) for higher-order coefficient $\bar{b}_k(u, r'; t)$ ($k = 1, 2, \ldots$), one may follow the steps similar to (3.8) and (3.9b)—rewrite the equations as those for the matrix functions $\tilde{b}_k(u, r'; t)$ which are obtained through multiplying $\bar{b}_k(u, r'; t)$ from the left by the inverse of the t-ordered exponential matrix in (B2). But what we need here is only the coincidence limits, i.e., $b_k(r, r; t) \equiv \bar{b}_k(u = 0, r' = r; t)$ for small k, and for the latter it is actually simpler to obtain the desired expressions by considering the $u = 0$ limits of our differential Eqs. (B1a) and (B1b) and of their derivative relations [36]. Specifically, for $\bar{b}_1(u = 0, r, t)$, we have the equation [by setting $u = 0$ with (B1b)]

$$\partial_t \bar{b}_1(0, r; t) + U(r)\bar{b}_1(0, r; t) - \frac{1}{t^2}[\partial_u^2 \bar{b}_0(u, r; t)]|_{u=0} = 0, \tag{B4}$$

which can readily be integrated [with the "initial" condition $\bar{b}_1(u, r; t = 0) = 0$] only if the expression for $[\partial_u^2 \bar{b}_0(u, r; t)]|_{u=0}$ is known. Then, setting $u = 0$ in the relations obtained after single and twice differentiations of (B1a) with respect to u, we have

$$\partial_t[\partial_u \bar{b}_0(u, r; t)]|_{u=0} + U(r)[\partial_u \bar{b}_0(u, r; t)]|_{u=0} + tU'(r)e^{-tU(r)} = 0, \tag{B5}$$

$$\partial_t[\partial_u^2 \bar{b}_0(u, r; t)]|_{u=0} + U(r)[\partial_u^2 \bar{b}_0(u, r; t)]|_{u=0} + 2tU'(r)[\partial_u \bar{b}_0(u, r; t)]|_{u=0} + t^2 U''(r)e^{-tU(r)} = 0, \tag{B6}$$

where the result in (B3) has been used. From (B5) it follows that

$$[\partial_u \bar{b}_0(u, r; t)]|_{u=0} = e^{-tU(r)} \int_0^t dt_1 t_1 e^{t_1 U(r)} U'(r) e^{-t_1 U(r)}. \tag{B7}$$

Using this result, we can go on to integrate (B6) to obtain

$$[\partial_u^2 \bar{b}_0(u, r; t)]|_{u=0} = e^{-tU(r)}\left\{2\int_0^t dt_1 t_1 e^{t_1 U(r)} U'(r) e^{-t_1 U(r)} \times \int_0^{t_1} dt_2 t_2 e^{t_2 U(r)} U'(r) e^{-t_2 U(r)} - \int_0^t dt_1 t_1^2 e^{t_1 U(r)} U''(r) e^{-t_1 U(r)}\right\}. \tag{B8}$$

Now, by using this result in (B4) and integrating the resulting equation, we find the expression for the coincidence limit $b_1(r, r; t) (= \bar{b}_1(0, r; t))$:

$$b_1(r, r; t) = e^{-tU(r)} \int_0^t dt_1 \frac{1}{t_1^2}\left\{2\int_0^{t_1} dt_2 t_2 e^{t_2 U(r)} U'(r) e^{-t_2 U(r)} \times \int_0^{t_2} dt_3 t_3 e^{t_3 U(r)} U'(r) e^{-t_3 U(r)} - \int_0^{t_1} dt_2 t_2^2 e^{t_2 U(r)} U''(r) e^{-t_2 U(r)}\right\}. \tag{B9}$$

The desired $\frac{1}{l}$-expansion, which generalizes (3.16) to the case of a matrix-valued potential, follows upon using the results (B3) and (B9) with (3.5). It has the following structure:

$$\tilde{\Delta}(r,r;s) = \frac{1}{\sqrt{4\pi s}} e^{-sV(r)} \bigg[1 + \int_0^s ds_1 \frac{1}{s_1^2} \times \bigg\{ 2\int_0^{s_1} ds_2 s_2 e^{s_2 V(r)} V'(r) e^{-s_2 V(r)} \times \int_0^{s_2} ds_3 s_3 e^{s_3 V(r)} V'(r) e^{-s_3 V(r)} - \int_0^{s_1} ds_2 s_2^2 e^{s_2 V(r)} V''(r) e^{-s_2 V(r)} \bigg\} + O\Big(\frac{1}{l^4}\Big) \bigg]. \tag{B10}$$

APPENDIX C: USE OF THE EULER-MACLAURIN SUMMATION FORMULA

In this Appendix we explain how the Euler-Maclaurin summation formula [42,43] can be used to sum the various partial-wave contributions to the effective action. First, we present the related mathematical theory. Let $f(x)$ be a function with continuous derivatives up to order $2m+2$ for $x \in [a,b]$, where a and b are integers. Then, for the sum

$$\sum_{n=a}^{b} f(n) \equiv f(a) + f(a+1) + \cdots + f(b-1) + f(b), \tag{C1}$$

we have the Euler-Maclaurin summation formula

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \frac{1}{2}[f(a)+f(b)] + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(b) - f^{(2k-1)}(a)] + R_m, \tag{C2}$$

where B_j are the Bernoulli numbers ($B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots$) and the remainder term is

$$R_m = \frac{(b-a)B_{2m+2}}{(2m+2)!} f^{(2m+2)}(\bar{\theta}), \qquad \text{for some } \bar{\theta} \in (a,b). \tag{C3}$$

This formula will be particularly useful to evaluate the sum of slowly varying terms with decreasing derivatives. (A nice treatment on this formula, including the derivation, is given in Ref. [43].)

As for the partial-wave sums in our work, we may use (C1) in the form (here $l = \frac{n}{2}$, n being integers)

$$\begin{aligned} \sum_{l=L+1/2}^{\infty} f(l) &= \sum_{n=2L}^{\infty} f\Big(\frac{n}{2}\Big) - f(L) \\ &= \bigg\{ \int_{2L}^{\infty} f\Big(\frac{x}{2}\Big)dx + \frac{1}{2}f(L) - \frac{1}{12}\bigg[\frac{d}{dx} f\Big(\frac{x}{2}\Big)\bigg]\bigg|_{x=2L} + \cdots \bigg\} - f(L) \\ &= 2\int_L^{\infty} f(l)dl - \frac{1}{2}f(L) - \frac{1}{24}f'(L) + \cdots, \end{aligned} \tag{C4}$$

assuming $f(\infty) = f'(\infty) = 0$, etc. To deal with various l-sums appearing in (3.22), we may here take

$$f(l) = e^{-s(a_2 l^2 + a_1 l + a_0)}(b_0 + b_1 l + b_2 l^2 + \cdots), \qquad (a_2 > 0). \tag{C5}$$

Then, to perform the l-integral $\int_L^\infty f(l)dl$, we change the integration variable from l to t by setting

$$l = \frac{t}{\sqrt{2a_2 s}} - \frac{a_1}{2a_2}. \tag{C6}$$

This will put the function (C5) in a simpler form, i.e.,

$$f(l) \to \tilde{f}(t) = e^{-t^2}(\tilde{b}_0 + \tilde{b}_1 t + \tilde{b}_2 t^2 + \cdots) \tag{C7}$$

and the resulting integrals can be done by using the formula [42]

$$\int_T^\infty dt e^{-t^2} t^n = \begin{cases} \frac{1}{2}(\frac{n-1}{2})! e^{-T^2} \sum_{k=0}^{n-1/2} \frac{T^{2k}}{k!}, & (n = \text{odd integer}) \\ \frac{1}{2}[(\frac{n-1}{2})! \mathrm{erfc}(T) + T e^{-T^2} \sum_{k=0}^{(n/2)-1} (k+\frac{3}{2})_{-k+(n/2)-1} T^{2k}], & (n = \text{even integer}) \end{cases} \tag{C8}$$

where $(a)_n \equiv a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol. Now the l-sums in (3.22) can be performed explicitly. If we discard contributions vanishing for sufficiently large L, these sums for the first two terms in the right-hand side of (3.22) read

$$
\begin{aligned}
&\sum_{l=L+1/2}^{\infty}(2l+1)(2l+2)\tilde{\Delta}_{(l,j=l+1/2)}(r,r;s)\\
&\quad = e^{s(1/4r^2+2f(r)-2r^2f(r)^2)}\mathrm{erfc}\bigg(\frac{\sqrt{s}}{r}(2L+1+r^2f(r))\bigg)\\
&\qquad\times\bigg\{\frac{r^3}{8s^2}+\frac{r}{32s}(-1-8r^2f(r)+8r^4f(r)^2)+\frac{1}{16}(f(r)(r-4r^4f'(r))-5r^3f(r)^2-r^5f'(r)^2)\bigg\}\\
&\qquad+\frac{1}{\sqrt{4\pi s}}e^{-s\{[4L(L+1)+3/4]/r^2+4Lf(r)+3r^2f(r)^2\}}\bigg\{\frac{r^2L}{s}+\bigg(-2L^2+\frac{r^2}{s}-\frac{r^4}{2s}f(r)\bigg)+\bigg(-\frac{43L}{12}+\frac{2sL^3}{3r^2}+\frac{16s^2L^5}{3r^4}\bigg)\\
&\qquad+\bigg(-\frac{37}{24}-\frac{r^2}{8}f(r)-\frac{2r^3}{3}f'(r)-\frac{r^4}{6}f''(r)+\frac{sL^2}{r^2}-\frac{sL^2}{3}f(r)-\frac{8rsL^2}{3}f'(r)-\frac{2r^2sL^2}{3}f''(r)+\frac{24s^2L^4}{r^4}\\
&\qquad-\frac{8s^2L^4}{3r^2}f(r)-\frac{16s^2L^4}{3r}f'(r)-\frac{32s^3L^6}{3r^6}\bigg)+\bigg(\frac{sL}{2r^2}-\frac{sL}{6}f(r)-3r^2sLf(r)^2-\frac{r^4sL}{2}f'(r)^2-\frac{8rsL}{3}f'(r)\\
&\qquad-\frac{14r^3sL}{3}f(r)f'(r)-\frac{2r^2sL}{3}f''(r)-\frac{2r^4sL}{3}f(r)f''(r)+\frac{122s^2L^3}{3r^4}-\frac{20s^2L^3}{3}f(r)^2+\frac{4r^2s^2L^3}{3}f'(r)^2\\
&\qquad-\frac{20s^2L^3}{3r^2}f(r)-\frac{32s^2L^3}{3r}f'(r)-\frac{16rs^2L^3}{3}f(r)f'(r)+\frac{4s^2L^3}{3}f''(r)-\frac{48s^3L^5}{r^6}+\frac{32s^3L^5}{3r^3}f'(r)+\frac{64s^4L^7}{9r^8}\bigg)\bigg\}\\
&\quad\equiv \mathcal{F}_1(r,s), \qquad\qquad (\text{C9})
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l=L+1/2}^{\infty}(2l+1)(2l+2)\tilde{\Delta}_{(l+1/2,j=l)}(r,r;s)\\
&\quad = e^{s((1/4r^2)+2f(r)-2r^2f(r)^2)}\mathrm{erfc}\bigg(\frac{\sqrt{s}}{r}(2L+2-r^2f(r))\bigg)\\
&\qquad\times\bigg\{\frac{r^3}{8s^2}+\frac{r}{32s}(-1-8r^2f(r)+8r^4f(r)^2)+\frac{1}{16}(f(r)(r-4r^4f'(r))-5r^3f(r)^2-r^5f'(r)^2)\bigg\}\\
&\qquad+\frac{1}{\sqrt{4\pi s}}e^{-s\{[4L(L+2)+(15/4)]/r^2-(4L+6)f(r)+3r^2f(r)^2\}}\bigg\{\frac{r^2L}{s}+\bigg(-2L^2+\frac{r^2}{2s}-\frac{r^4}{2s}f(r)\bigg)+\bigg(-\frac{43L}{12}+\frac{2sL^3}{3r^2}+\frac{16s^2L^5}{3r^4}\bigg)\\
&\qquad+\bigg(-\frac{4}{3}+\frac{r^2}{8}f(r)+\frac{2r^3}{3}f'(r)+\frac{r^4}{6}f''(r)+\frac{2sL^2}{r^2}+\frac{sL^2}{3}f(r)+\frac{8rsL^2}{3}f'(r)+\frac{2r^2sL^2}{3}f''(r)+\frac{32s^2L^4}{r^4}\\
&\qquad+\frac{8s^2L^4}{3r^2}f(r)+\frac{16s^2L^4}{3r}f'(r)-\frac{32s^3L^6}{3r^6}\bigg)+\bigg(\frac{2sL}{r^2}+\frac{5sL}{6}f(r)-3r^2sLf(r)^2-\frac{r^4sL}{2}f'(r)^2+\frac{16rsL}{3}f'(r)\\
&\qquad-\frac{14r^3sL}{3}f(r)f'(r)-\frac{4r^2sL}{3}f''(r)-\frac{2r^4sL}{3}f(r)f''(r)+\frac{218s^2L^3}{3r^4}-\frac{20s^2L^3}{3}f(r)^2+\frac{4r^2s^2L^3}{3}f'(r)^2\\
&\qquad+\frac{28s^2L^3}{3r^2}f(r)+\frac{64s^2L^3}{3r}f'(r)-\frac{16rs^2L^3}{3}f(r)f'(r)-\frac{4s^2L^3}{3}f''(r)-\frac{208s^3L^5}{3r^6}-\frac{32s^3L^5}{3r^3}f'(r)+\frac{64s^4L^7}{9r^8}\bigg)\bigg\}\\
&\quad\equiv \mathcal{F}_2(r,s). \qquad\qquad (\text{C10})
\end{aligned}
$$

If we set $f(r)=0$ in these expression, they represent the l-sums for the last two terms of (3.22) (i.e., for those containing the factors $e^{-sV^0_{(l)}(r)}$ and $e^{-sV^0_{(l+1/2)}(r)}$). In this way we obtain the explicit double-integral representation for $\Gamma^{\epsilon}_{J>J_L}(A;m)$ of the form

$$
\Gamma^{\epsilon}_{J>J_L}(A;m) = -\int_0^{\infty}dr\int_0^{\infty}ds\frac{e^{-m^2s}}{s}s^{\epsilon}[\mathcal{F}_1(r,s)+\mathcal{F}_2(r,s)-\mathcal{F}_1(r,s)|_{f(r)=0}-\mathcal{F}_2(r,s)|_{f(r)=0}]. \qquad (\text{C11})
$$

The l-sums in (3.30) can be performed in a similar manner.

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