

Perturbative zero-point energy for a cylinder of elliptical section

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We examine the Casimir effect for a perfectly conducting cylinder of elliptical section, taking as reference the known case of circular section. The zero-point energy of this system is evaluated by the mode summation method, using the ellipticity as a perturbation parameter. Mathieu function techniques are applied.

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I. INTRODUCTION

Vacuum fluctuations of quantum fields caused by the presence of boundaries produce changes in the zero-point energy, which give rise to the Casimir effect. Interest in quantum vacuum manifestations, including this phenomenon, has been propelled by new theoretical and experimental advances [1–3]. Hypothetical prospects of technological applications make even more desirable the knowledge of fundamental aspects of the theory, such as the value of the vacuum energy or its dependence on any of the problem conditions (even the sign of the effect is hard to predict, although for interactions between dielectric bodies some progress has been made in Ref. [4]).

Particularly striking is the modification of the zero-point energy caused by a change in boundary shape, even at an infinitesimal level. As Casimir energies prove to be very sensitive to purely geometrical modifications, the subject deserves further consideration. This question has already been addressed for a boundary departing from spherical, which has implications for QCD flux tube models [5].

On the other hand, cylindrical boundaries of circular section have been object of attention under a variety of settings: the perfectly conducting case [6–8], dielectric media with or without light-velocity conservation [9–17], dispersion [18–20], semitransparent boundary [21], varying Robin conditions [22], finite temperature¹ [24], coaxial surfaces [25], cosmic strings [26], etc. However, deviations from circular section constitute a largely uncharted land. In the present work, we make an incursion into this territory by considering a perfectly conducting and infinitely long cylindrical surface of elliptical section, which slightly deviates from circular shape.²

In Sec. II the solutions to the Maxwell equations in terms of Mathieu functions are considered, and the boundary conditions which determine the eigenmode set are estab-

lished. The summation of these eigenmodes is studied in Sec. III, thereby obtaining the zero-point energy of the electromagnetic field as a perturbative expansion in the ellipticity. Section IV offers, as an alternative view, the spectrum modification on the basis of a conformal transformation relating elliptical and circular cases. The conclusions follow in Sec. V. Relevant properties and results about Mathieu functions have been included in an appendix. Natural units ($\hbar = c = 1$) are used throughout.

II. SOLUTIONS TO MAXWELL'S EQUATIONS WITH PERFECT CONDUCTOR BOUNDARY CONDITIONS

We will consider the classic problem: find the zero-point energy of an electromagnetic field due to a perfectly conducting, neutral surface \mathcal{S} . In the absence of any charge or current density and with time dependence given by $\exp(-i\omega t)$, Maxwell's equations in Heaviside-Lorentz units are

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = 0, \quad (1)$$

$$\nabla \cdot \mathbf{B}(\mathbf{x}) = 0, \quad (2)$$

$$\nabla \times \mathbf{E}(\mathbf{x}) = i\omega \mathbf{B}(\mathbf{x}), \quad (3)$$

$$\nabla \times \mathbf{B}(\mathbf{x}) = -i\omega \mathbf{E}(\mathbf{x}), \quad (4)$$

where $\mathbf{E}(\mathbf{x})$ is the spatial part of the electric field and $\mathbf{B}(\mathbf{x})$ is the spatial part of the magnetic field. Both vector fields satisfy the vector Helmholtz equation, that is,

$$(\nabla^2 + \omega^2)\mathbf{A}(\mathbf{x}) = \mathbf{0}, \quad (5)$$

where $\mathbf{A}(\mathbf{x})$ can be $\mathbf{E}(\mathbf{x})$ or $\mathbf{B}(\mathbf{x})$. The vector Laplacian is

$$\nabla^2 \mathbf{A}(\mathbf{x}) = \nabla \nabla \cdot \mathbf{A}(\mathbf{x}) - \nabla \times \nabla \times \mathbf{A}(\mathbf{x}). \quad (6)$$

The boundary conditions are

$$\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{x})|_{\mathbf{x} \in \mathcal{S}} = \mathbf{0}, \quad (7)$$

$$\hat{\mathbf{n}} \cdot \mathbf{B}(\mathbf{x})|_{\mathbf{x} \in \mathcal{S}} = 0, \quad (8)$$

¹For a general discussion about the classical limit in temperature-dependent systems see Ref. [23].

²Yet, there are works [27,28] where the elliptical coordinates from scattering theory play a role in the study of General Relativity (averaged) null energy conditions for a Casimir system.

where $\hat{\mathbf{n}}$ is the outward-pointing, unit normal vector to the surface \mathcal{S} . In particular, we take \mathcal{S} to be an infinitely long cylinder, along the z -axis, with an elliptical section given (in Cartesian coordinates) by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (9)$$

where a is the semimajor axis and b is the semiminor axis. We introduce elliptic cylindrical coordinates, (ξ, η, z) , a left-handed coordinate system related to Cartesian coordinates by

$$x = f \cosh(\xi) \cos(\eta), \quad (10)$$

$$y = f \sinh(\xi) \sin(\eta), \quad (11)$$

where $0 \leq \xi < \infty$, $0 \leq \eta < 2\pi$ and f is the focal length given by

$$f = \sqrt{a^2 - b^2}. \quad (12)$$

The ellipticity is defined to be

$$e = \frac{f}{a}. \quad (13)$$

To avoid confusion with the exponential function, its full form, \exp , will be used. In the elliptic cylindrical coordinate system the ellipse (9) adopts the form

$$\xi_0 = \cosh^{-1}(1/e). \quad (14)$$

The boundary conditions become

$$E_\eta(\xi = \xi_0) = 0, \quad (15)$$

$$E_z(\xi = \xi_0) = 0, \quad (16)$$

$$B_\xi(\xi = \xi_0) = 0. \quad (17)$$

The vector Helmholtz equation can be solved by separation of variables in the elliptic cylindrical coordinate system [29]. Furthermore, since the media inside and outside \mathcal{S} are the same, and thus the speed of light is equal on both sides; we may split the solutions into transverse electric (TE) and transverse magnetic (TM) modes [13–16]. Each vector component can be written in terms of Mathieu and modified Mathieu functions.

A. Interior field

1. TM modes ($B_z = 0$)

It is enough to start considering the solution for one of the nonvanishing field components, which we choose to be E_z . The general form will be [30,31]:

$$E_z(\xi, \eta, z) = \begin{cases} \sum_{n=0}^{\infty} \int \frac{dk}{2\pi} C_n(k) \text{Mc}_n^{(1)}(\xi, q) \text{ce}_n(\eta, q) \exp(ikz), \\ \sum_{n=1}^{\infty} \int \frac{dk}{2\pi} S_n(k) \text{Ms}_n^{(1)}(\xi, q) \text{se}_n(\eta, q) \exp(ikz), \end{cases} \quad (18)$$

where $(\text{Mc}_n^{(1)})$ and $(\text{Ms}_n^{(1)})$ ce_n and se_n are the even and odd (modified) Mathieu functions of the first kind, respectively. Notice that there is no zeroth order odd Mathieu function. The coefficients $C_n(k)$ and $S_n(k)$ have no spatial dependence. Equation (16) implies that

$$\text{M c}_n^{(1)}(\xi_0, q_{n,p}^{\text{I, TM}}) = 0, \quad (19)$$

$$\text{M s}_n^{(1)}(\xi_0, \tilde{q}_{n,p}^{\text{I, TM}}) = 0, \quad (20)$$

where the superscripts refer to the interior region and TM mode, respectively. The second subscript indexes the p th zero. The tilde distinguishes the odd eigenmodes from the even ones. The dependence of the eigenmodes on the ellipticity has been suppressed. The eigenfrequencies are obtained through the relation

$$\omega = \sqrt{\frac{4q}{f^2} + k^2}. \quad (21)$$

The other two boundary conditions are automatically satisfied because E_η and B_ξ involve the same vanishing factors as in E_z [30].

2. TE modes ($E_z = 0$)

The TE mode is obtained by the principle of duality; B_z will have the same form as Eq. (18). Since

$$E_\eta \propto \frac{\partial}{\partial \xi} B_z, \quad (22)$$

Equation (15) implies that

$$\frac{\partial}{\partial \xi} B_z|_{\xi=\xi_0} = 0. \quad (23)$$

Therefore,

$$\text{M c}_n^{(1)'}(\xi_0, q_{n,p}^{\text{I, TE}}) = 0, \quad (24)$$

$$\text{M s}_n^{(1)'}(\xi_0, \tilde{q}_{n,p}^{\text{I, TE}}) = 0, \quad (25)$$

where the prime means differentiation with respect to ξ . The other boundary conditions are also satisfied; it follows that B_ξ vanishes on the boundary because $B_\xi \propto E_\eta$ and $E_z = 0$ by definition of a TE mode.

B. Exterior field

This part of the solution can be worked out in terms of functions akin to Hankel functions. The reason for doing so is the physical demand that, at large distances, the field components should behave like cylindrical waves. This is easily met by choosing the linear combinations

$$\text{M c}_n^{(3)}(\xi, q) = \text{M c}_n^{(1)}(\xi, q) + i\text{M c}_n^{(2)}(\xi, q), \quad (26)$$

$$\text{M c}_n^{(4)}(\xi, q) = \text{M c}_n^{(1)}(\xi, q) - i\text{M c}_n^{(2)}(\xi, q), \quad (27)$$

with corresponding relations for the odd modified Mathieu functions.

There are many parallels between modified Mathieu and Bessel functions. As with Bessel functions of the second kind, the modified Mathieu functions of the second kind are not regular in the interior region. This is why they were not included in the solution for the field inside the cylinder, but shall now be used for the field outside. As we want the waves to be outgoing, the modified Mathieu functions of the third kind are selected.

1. TM modes ($B_z = 0$)

The z component of the electric field will have the same form as Eq. (18) but with modified Mathieu functions of the third kind. The boundary conditions imply that

$$\text{M}c_n^{(3)}(\xi_0, q_{n,p}^{\text{II, TM}}) = 0, \quad (28)$$

$$\text{M}s_n^{(3)}(\xi_0, \tilde{q}_{n,p}^{\text{II, TM}}) = 0, \quad (29)$$

where the first superscript refers to the exterior region.

2. TE modes ($E_z = 0$)

Analogous to the interior TE modes, we find

$$\text{M}c_n^{(3)'}(\xi_0, q_{n,p}^{\text{II, TE}}) = 0, \quad (30)$$

$$\text{M}s_n^{(3)'}(\xi_0, \tilde{q}_{n,p}^{\text{II, TE}}) = 0. \quad (31)$$

III. REGULARIZED ZERO-POINT ENERGY

In natural units, the zero-point energy per lateral unit length amounts to half the sum (including integration for k) of all the eigenfrequencies given by Eq. (21). Since such a quantity is divergent, some form of regularization is called for; we use zeta-function regularization.

The even zeta-functions are

$$\zeta_n^{\mathcal{A}, \mathcal{B}}(s) = \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} [\omega_{n,p}^{\mathcal{A}, \mathcal{B}}]^{-s}, \quad (32)$$

where $\mathcal{A} \in \{\text{I, II}\}$ and $\mathcal{B} \in \{\text{TE, TM}\}$. Let

$$\frac{4q}{f^2} = \gamma^2, \quad (33)$$

then Eq. (21) becomes

$$\omega = \sqrt{\gamma^2 + k^2}. \quad (34)$$

Integrating over k leaves

$$\zeta_n^{\mathcal{A}, \mathcal{B}}(s) = \frac{1}{2\pi} \text{B}\left(\frac{1}{2}, \frac{s-1}{2}\right) \sum_{p=1}^{\infty} [\gamma_{n,p}^{\mathcal{A}, \mathcal{B}}]^{1-s}, \quad (35)$$

where B is the beta function. Following similar analysis, the odd zeta-functions are

$$\tilde{\zeta}_n^{\mathcal{A}, \mathcal{B}}(s) = \frac{1}{2\pi} \left(\frac{1}{2}, \frac{s-1}{2}\right) \sum_{p=1}^{\infty} [\tilde{\gamma}_{n,p}^{\mathcal{A}, \mathcal{B}}]^{1-s}. \quad (36)$$

There is no odd zeta-function for $n = 0$. The regularized zero-point energy per lateral unit length as a function of ellipticity is

$$\mathcal{E}_C(e) = \lim_{s \rightarrow -1} \frac{1}{2} \sum_{\mathcal{A}, \mathcal{B}} \left[\sum_{n=0}^{\infty} \zeta_n^{\mathcal{A}, \mathcal{B}}(s) + \sum_{n=1}^{\infty} \tilde{\zeta}_n^{\mathcal{A}, \mathcal{B}}(s) \right], \quad (37)$$

where the limit means analytical continuation to $s = -1$. As we are summing over the interior and exterior regions we expect the analytical continuation to be finite (see [7,32] and references therein). For this reason, neither arbitrary scales nor additional prescriptions have been introduced.

In terms of the dimensionless variable $z = a\gamma$, the even zeta-functions with $\mathcal{A} = \text{I}$ and $\mathcal{B} = \text{TM}$ are

$$\zeta_n^{\text{I, TM}}(s) = a^{s-1} \frac{1}{2\pi} \text{B}\left(\frac{1}{2}, \frac{s-1}{2}\right) \sum_{p=1}^{\infty} [z_{n,p}^{\text{I, TM}}]^{1-s}. \quad (38)$$

The summation over p can be written as a contour integral with the help of the argument principle (see e.g. [32–36]) as follows

$$\sum_{p=1}^{\infty} [z_{n,p}^{\text{I, TM}}]^{1-s} = \frac{s-1}{2\pi i} \int_{\partial\Omega} dz z^{-s} \ln[\text{Mc}^{(1)}(\xi_0, q)], \quad (39)$$

where, using Eqs. (13) and (33),

$$q = \frac{z^2 e^2}{4}. \quad (40)$$

The integration circuit is the boundary of a region Ω of the complex z -plane which contains all the wanted zeros and avoids the origin. Other \mathcal{A} and \mathcal{B} and the odd counterparts follow in a similar fashion.

Ellipticity expansion

The modified Mathieu functions appearing in the contour integrals can be expanded for small ellipticity in a formal series (see Appendix). Equation (39) becomes

$$\begin{aligned} \sum_{p=1}^{\infty} [z_{n,p}^{\text{I, TM}}]^{1-s} &= \frac{s-1}{2\pi i} \int_{\partial\Omega} dz z^{-s} \left[\ln(J_n(z)) - \frac{z}{4J_n(z)} \right. \\ &\quad \left. \times \left(J_n'(z) - \frac{\delta_{n1}}{2} J_0(z) \right) e^2 + \dots \right]. \end{aligned} \quad (41)$$

The corresponding odd sum is

$$\begin{aligned} \sum_{p=1}^{\infty} [\tilde{z}_{n,p}^{\text{I, TM}}]^{1-s} &= \frac{s-1}{2\pi i} \int_{\partial\Omega} dz z^{-s} \left[\ln(J_n(z)) - \frac{z}{4J_n(z)} \right. \\ &\quad \left. \times \left(J_n'(z) + \frac{\delta_{n1}}{2} J_0(z) \right) e^2 + \dots \right], \end{aligned} \quad (42)$$

which is identical (up to $\mathcal{O}(e^4)$) except for the sign in front

of the Kronecker delta. If the even and odd sums are added, then

$$\begin{aligned} & \sum_{p=1}^{\infty} [z_{n,p}^{\text{I, TM}}]^{1-s} + \sum_{p=1}^{\infty} [\bar{z}_{n,p}^{\text{I, TM}}]^{1-s} \\ &= d(n) \frac{s-1}{2\pi i} \int_{\partial\Omega} dz z^{-s} \left[\ln(J_n(z)) - \frac{z J_n'(z)}{4J_n(z)} e^2 + \dots \right], \end{aligned} \quad (43)$$

where to take into account that there is no odd sum for $n = 0$,

$$d(n) = \begin{cases} 1, & n = 0, \\ 2, & n \geq 1. \end{cases} \quad (44)$$

Integrating termwise (formally), the second term is proportional to the first since

$$\int_{\partial\Omega} dz z^{-s} \frac{z J_n'(z)}{J_n(z)} = (s-1) \int_{\partial\Omega} dz z^{-s} \ln(J_n(z)). \quad (45)$$

Thus, Eq. (43) simplifies to

$$\begin{aligned} & \sum_{p=1}^{\infty} [z_{n,p}^{\text{I, TM}}]^{1-s} + \sum_{p=1}^{\infty} [\bar{z}_{n,p}^{\text{I, TM}}]^{1-s} \\ &= d(n) \frac{s-1}{2\pi i} \int_{\partial\Omega} dz z^{-s} \ln(J_n(z)) \left[1 - \frac{s-1}{4} e^2 + \dots \right]. \end{aligned} \quad (46)$$

Using Eq. (38) and its odd counterpart,

$$\begin{aligned} & \sum_{n=0}^{\infty} \zeta_n^{\text{I, TM}}(s) + \sum_{n=1}^{\infty} \bar{\zeta}_n^{\text{I, TM}}(s) \\ &= a^{s-1} \frac{1}{2\pi} \left(\frac{1}{2}, \frac{s-1}{2} \right) \sum_{n=0}^{\infty} d(n) \frac{s-1}{2\pi i} \int_{\partial\Omega} dz z^{-s} \\ & \quad \times \ln(J_n(z)) \left[1 - \frac{s-1}{4} e^2 + \dots \right]. \end{aligned} \quad (47)$$

The prefactor is exactly what one gets in the circular cylindrical situation. The other combinations of \mathcal{A} and \mathcal{B} all have the same form as Eq. (47), but with their corresponding circular cylindrical prefactors. Thus, Eq. (37) gives

$$\mathcal{E}_C(e) = \mathcal{E}_C(0) \left[1 + \frac{1}{2} e^2 + \mathcal{O}(e^4) \right], \quad (48)$$

where $\mathcal{E}_C(0)$ is the regularized zero-point energy per lateral unit length of a circular cylinder of radius a . While it is doubtful that Eq. (48) converges for $0 \leq e < 1$, at worst it is an asymptotic series as $e \rightarrow 0$. In either case we may write the next term as $\mathcal{O}(e^4)$.

The numerical value of $\mathcal{E}_C(0)$ is [6–8]

$$\mathcal{E}_C(0) \approx -\frac{0.01356}{a^2}, \quad (49)$$

which, together with Eq. (48) gives

$$\mathcal{E}_C(e) \approx -\frac{0.01356}{a^2} \left[1 + \frac{1}{2} e^2 + \mathcal{O}(e^4) \right], \quad (50)$$

where a is the semimajor axis. Let

$$R = \frac{a+b}{2}, \quad (51)$$

then

$$a = \frac{2R}{1 + \sqrt{1-e^2}}, \quad (52)$$

and

$$\mathcal{E}_C(e) \approx -\frac{0.01356}{R^2} [1 + \mathcal{O}(e^4)]. \quad (53)$$

That is, the zero-point energy per lateral unit length in terms of R is the same as that for a circular cylinder with radius R , up to quartic corrections in the ellipticity.³ Eq. (53) can be related to the existence of a conformal mapping in the complex plane which transforms the ellipse into a circle. This subject is discussed in the next section.

IV. CONFORMAL TRANSFORMATION

A. Formulation

Some of the ideas in Refs. [37–39] suggest the use of an adequate conformal map. We shall employ a transformation taking the interior of the ellipse (9) to the interior of the circle $w = R \exp(i\varphi)$, with R given by definition (51). The required map $w: \mathbb{C} \rightarrow \mathbb{C}$ is [40]

$$w(z) = R\sqrt{k} \operatorname{sn} \left[\frac{2K(k)}{\pi} \sin^{-1} \left(\frac{z}{\sqrt{a^2 - b^2}} \right), k \right], \quad (54)$$

where sn is a Jacobi elliptic function, K is the complete elliptic integral of the first kind and k depends on the semiaxes through theta functions as follows:

$$k = \left(\frac{\vartheta_2(0, q)}{\vartheta_3(0, q)} \right)^2, \quad (55)$$

where

$$q = \left(\frac{a-b}{a+b} \right)^2. \quad (56)$$

The k and q variables used here should not be confused with those in other sections. The theta functions are given by

$$\vartheta_2(z, q) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos((2n+1)z), \quad (57)$$

$$\vartheta_3(z, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz). \quad (58)$$

In terms of e and R ,

$$q = \left(\frac{ae}{2R} \right)^4. \quad (59)$$

³This is not unique. For instance, $R = \sqrt{(a^2 + b^2)}/2$ would cause the same effect as $R = (a+b)/2$, since they only differ at $\mathcal{O}(e^4)$.

Using the definitions of the theta functions, for small ellipticity,⁴

$$k = \frac{a^2}{R^2} e^2 + \mathcal{O}(e^6). \quad (60)$$

With the help of

$$K(k) = \frac{\pi}{2} + \frac{\pi}{8} k^2 + \mathcal{O}(k^4), \quad (61)$$

$$\begin{aligned} \operatorname{sn}(u, k) = \sin(u) - \frac{[u - \sin(u) \cos(u)] \cos(u)}{4} k^2 \\ + \mathcal{O}(k^4), \end{aligned} \quad (62)$$

the small ellipticity expansion of Eq. (54) is

$$w(z) = z - \frac{z^3}{4R^2} e^2 + \mathcal{O}(e^4). \quad (63)$$

Therefore,

$$w'(z) = 1 - \frac{3z^2}{4R^2} e^2 + \mathcal{O}(e^4), \quad (64)$$

where the prime denotes differentiation with respect to z . For our purposes, we also need the leading term of the inverse of w , given by $z(w) = w + \mathcal{O}(e^2)$. With $w = r \exp(i\varphi)$, $z = r \exp(i\varphi) + \mathcal{O}(e^2)$. Using Eq. (64),

$$|w'|^2 = 1 - \frac{3r^2 \cos(2\varphi)}{2R^2} e^2 + \mathcal{O}(e^4). \quad (65)$$

Under any conformal mapping, the 2-D Laplacian operator transforms as

$$\nabla_z^2 = |w'|^2 \nabla_w^2. \quad (66)$$

We start from the 2-D Helmholtz equation in the w -plane

$$(\nabla_w^2 + \gamma_n^2) \mathcal{U}_n = 0, \quad (67)$$

where the chosen eigenvalue symbol comes from Eq. (33). The boundary conditions are set on the image of the ellipse, which is $w = R \exp(i\varphi)$, $0 \leq \varphi < 2\pi$. Since the boundary is circular, the eigenfunctions are of the form

$$\mathcal{U}_n = f_n(r) \exp(in\varphi), \quad (68)$$

where $f_n(r)$ is the suitably normalized radial function which satisfies the boundary conditions.

B. Rayleigh-Schrödinger expansion

Next, Dirac notation shall be temporarily adopted in order to take advantage of the Rayleigh-Schrödinger method. Supposing that the eigenvalues λ_n and eigenstates $|\mathcal{U}_n\rangle$ for a problem

$$L|\mathcal{U}_n\rangle + \lambda_n|\mathcal{U}_n\rangle = 0 \quad (69)$$

with given boundary conditions are known, one wonders which are the new eigenvalues $\bar{\lambda}_n$ and eigenstates $|\bar{\mathcal{U}}_n\rangle$ for

⁴In the opposite case one may wish to consider, e.g., $a \rightarrow \infty$ and fixed b . Using theta function properties, for this limit, $w(z) = R \tanh(\pi z/(4b))$, which is the transformation taking the region between the parallel lines $y = \pm ib$ to the interior of the circle. Of course, these lines can be viewed as the 2-D projection of parallel plates.

the ‘‘perturbed problem’’

$$L|\bar{\mathcal{U}}_n\rangle + \bar{\lambda}_n|\bar{\mathcal{U}}_n\rangle - \varepsilon_{\text{pert}} r_{\text{pert}} |\bar{\mathcal{U}}_n\rangle = 0 \quad (70)$$

under the same type of boundary condition, where $\varepsilon_{\text{pert}}$ is some small parameter. Including just modifications to first order in $\varepsilon_{\text{pert}}$, we make the Ansätze

$$|\bar{\mathcal{U}}_n\rangle = |\mathcal{U}_n\rangle + \varepsilon_{\text{pert}} |\mathcal{V}_n\rangle + \mathcal{O}(\varepsilon_{\text{pert}}^2), \quad (71)$$

$$\bar{\lambda}_n = \lambda_n + \varepsilon_{\text{pert}} \mu_n + \mathcal{O}(\varepsilon_{\text{pert}}^2), \quad (72)$$

and replace them into Eq. (70). At $\mathcal{O}(\varepsilon_{\text{pert}}^0)$ Eq. (69) is recovered, while the $\mathcal{O}(\varepsilon_{\text{pert}}^1)$ contribution yields

$$(L + \lambda_n I) |\mathcal{V}_n\rangle = (r_{\text{pert}} - \mu_n) |\mathcal{U}_n\rangle, \quad (73)$$

where I is the identity operator. After applying $\langle \mathcal{U}_n |$ on both sides, taking into account the adjoint of Eq. (69) (with $L^\dagger = L$ understood), we obtain

$$\mu_n = \langle \mathcal{U}_n | r_{\text{pert}} | \mathcal{U}_n \rangle, \quad (74)$$

$$= \int dg r_{\text{pert}} \mathcal{U}_n^* \mathcal{U}_n, \quad (75)$$

where dg denotes the integration measure making the $\{\mathcal{U}_n\}$ orthonormal.

For our studied case, in the notation of Eq. (69),

$$L = \nabla^2, \quad (76)$$

$$\lambda_n = \gamma_n^2, \quad (77)$$

and the integration in Eq. (75) will be on the type

$$\int dg = \int_0^{2\pi} d\varphi \int_{\mathcal{R}} dr r, \quad (78)$$

where \mathcal{R} is the radial range. Now, application of the inverse of transformation (54) to Eq. (69), and use of Eq. (66), give

$$\nabla_z^2 \bar{\mathcal{U}}_n + \bar{\gamma}_n^2 |w'|^2 \bar{\mathcal{U}}_n = 0. \quad (79)$$

Since the small parameter in the perturbed Eq. (70) is identified as

$$\varepsilon_{\text{pert}} = e^2, \quad (80)$$

comparison of Eqs. (70) and (79) leads to $\bar{\gamma}_n^2 |w'|^2 = \bar{\lambda}_n - \varepsilon_{\text{pert}} r_{\text{pert}}$, which, taking into account Eqs. (65), (76), (77), and (80), yields

$$r_{\text{pert}} = \bar{\gamma}_n^2 \frac{3r^2 \cos(2\varphi)}{2R^2}. \quad (81)$$

With this r_{pert} , we calculate the $\mathcal{O}(\varepsilon_{\text{pert}}^2)$ contribution to the n th eigenvalue using Eq. (75)

$$\mu_n \propto \int_0^{2\pi} d\varphi \cos(2\varphi), \quad (82)$$

$$= 0. \quad (83)$$

Hence, we conclude that, up to $\mathcal{O}(e^4)$, the eigenvalues will not change, and neither will the zero-point energy.

V. CONCLUSIONS

The main result in this work is (53), which shows that the Casimir energy per lateral unit length for an elliptical cylinder has the same value, up to quartic corrections in ellipticity, as for a circular cylinder with radius equal to the mean of the two semiaxes. A quadratic correction appears if the same energy is expressed in terms of one of the semiaxes.

This can be envisaged from a conformal transformation (54) which maps the ellipse onto the circle in question and, perturbatively speaking, yields no quadratic contribution. Such an infinitesimal symmetry preserves the Casimir energy if deformations from circular to elliptical sections (or vice versa) do not go beyond $\mathcal{O}(e^2)$.

Looking back at (51) and recalling the value of the ellipse length, say l , one sees that (48) or (53) may be rewritten as

$$\mathcal{E}_C(e) = \mathcal{E}_C(0) \left(\frac{2\pi a}{l} \right)^2 + \mathcal{O}(e^4). \quad (84)$$

Remarkably, for the spheroids considered in Ref. [5] the corresponding relation holds, i.e.,

$$E(e) = E(0) \left(\frac{4\pi R^2}{A} \right)^{1/2} + \mathcal{O}(e^4), \quad (85)$$

where E is the scalar zero-point energy, R is the radius of the sphere and A is the surface area of the spheroid of ellipticity e .

In a recent discussion offered by Ref. [21], the obtained result is viewed inside a wider context which includes other second-order vanishing effects for the cylindrical geometry, like those found in Refs. [9–11, 13–16].

Furthermore, there is a significant connection with Kvitsinsky's work [39]. The 2-D elliptical zeta-function is

$$\zeta^{2-D}(\sigma, e) = a^\sigma \sum_{\mathcal{A}, \mathcal{B}} \left[\sum_{p=1}^{\infty} [z_{n,p}^{\mathcal{A}, \mathcal{B}}]^{-\sigma} + \sum_{p=1}^{\infty} [\tilde{z}_{n,p}^{\mathcal{A}, \mathcal{B}}]^{-\sigma} \right], \quad (86)$$

where $\sigma = s - 1$. Following a similar analysis as in Sec. III,

$$\zeta^{2-D}(\sigma, e) = \zeta^{2-D}(\sigma, 0) \left[1 - \frac{\sigma}{4} e^2 + \mathcal{O}(e^4) \right], \quad (87)$$

where $\zeta^{2-D}(\sigma, 0)$ is the 2-D zeta-function for a circular boundary of radius a . If $a = 1 + e^2/2$ and $b = 1$, then

$$\zeta^{2-D}(\sigma, e) = \zeta^{2-D}(\sigma, 0) \left[1 + \frac{\sigma}{4} e^2 + \mathcal{O}(e^4) \right], \quad (88)$$

where the 2-D circular zeta-function now has unit radius. Comparison of the different notations leads to $e^2 \leftrightarrow 2\alpha$,

$\sigma \leftrightarrow 2p$, $\zeta^{2-D}(\sigma, e) \leftrightarrow \zeta(p; \text{ellipse})$ and $\zeta^{2-D}(\sigma, 0) \leftrightarrow \zeta(p; D)$, where the objects on the right are those in Ref. [39]. Then, our result coincides with the unnumbered formula below Eq. (23) of the referred paper, originally derived for $p = 2, 3, \dots$ and just field modes like in our 'I, TM' subset, suggesting that the expression in question could be valid beyond its initial settings.

From the viewpoint of Mathieu functions, one may argue that the obtained relation has been established by virtue of a variable change (51) in formulas (A7) and (A8). Unfortunately, the next order in ellipticity is significantly more complicated.

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APPENDIX: MODIFIED MATHIEU FUNCTIONS

The modified Mathieu differential equation is

$$\left[\frac{\partial^2}{\partial \xi^2} - a(q) + 2q \cosh(2\xi) \right] \Xi(\xi) = 0, \quad (A1)$$

Let $\zeta = 2\sqrt{q} \cosh(\xi)$, then

$$\left[(\zeta^2 - 4q) \frac{\partial^2}{\partial \zeta^2} + \zeta \frac{\partial}{\partial \zeta} - a(q) - 2q + \zeta^2 \right] \Xi(\zeta) = 0. \quad (A2)$$

When $q = 0$, Eq. (A2) reduces to the Bessel differential equation. The even and odd modified Mathieu functions of the first kind are, respectively

$$\begin{aligned} \text{Mc}_n^{(1)}(\xi, q) &= \sum_{k=-\infty}^{\infty} d_{n,k}(q) J_{n+2k}(2\sqrt{q} \cosh(\xi)), \\ n &= 0, 1, 2, \dots, \end{aligned} \quad (A3)$$

$$\begin{aligned} \text{Ms}_n^{(1)}(\xi, q) &= \tanh(\xi) \sum_{k=-\infty}^{\infty} \tilde{d}_{n,k}(q) J_{n+2k}(2\sqrt{q} \cosh(\xi)), \\ n &= 1, 2, 3, \dots \end{aligned} \quad (A4)$$

They are normalized to have the same asymptotic form as the Bessel functions

$$\text{Mc}_n^{(1)}(\xi, q) \stackrel{\xi \rightarrow \infty}{\sim} \text{Ms}_n^{(1)}(\xi, q) \stackrel{\xi \rightarrow \infty}{\sim} J_n(\zeta). \quad (A5)$$

The expressions in Eq. (A3) and (A4) are absolutely convergent for $|\cosh(\xi)| > 1$ [30].

Consider the even modified Mathieu functions of the first kind. With $\xi_0 = \cosh^{-1}(1/e)$ and $q = z^2 e^2/4$, Eq. (A3) is

$$\text{Mc}_n^{(1)}(\xi_0, q) = \sum_{k=-\infty}^{\infty} d_{n,k}(z^2 e^2/4) J_{n+2k}(z). \quad (A6)$$

Expanding the coefficients in a formal power series and using Bessel function recurrence relations, we get

$$M c_n^{(1)}(\xi_0, q) = J_n(z) - \frac{z}{4} \left(J_n'(z) - \frac{\delta_{n1}}{2} J_0(z) \right) e^2 + \mathcal{O}(e^4). \quad (\text{A7})$$

By similar analysis, the ellipticity expansion for the odd modified Mathieu functions of the first kind is

$$M s_n^{(1)}(\xi_0, q) = J_n(z) - \frac{z}{4} \left(J_n'(z) + \frac{\delta_{n1}}{2} J_0(z) \right) e^2 + \mathcal{O}(e^4). \quad (\text{A8})$$

These formulas may also be derived from the set of expressions supplied in Ref. [31]. The same procedure is repeated for the modified Mathieu functions of the third kind and for the required derivatives.

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- [1] K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero-Point Energy* (World Scientific, Singapore, 2001).
- [2] K. A. Milton, *J. Phys. A* **37**, R209 (2004).
- [3] M. Bordag, U. Mohideen, and V. M. Mostepanenko, *Phys. Rep.* **353**, 1 (2001).
- [4] O. Kenneth and I. Klich, [quant-ph/0601011](#).
- [5] A. R. Kitson and A. I. Signal, *J. Phys. A* **39**, 6473 (2006).
- [6] L. L. DeRaad, Jr. and K. A. Milton, *Ann. Phys. (N.Y.)* **136**, 229 (1981).
- [7] P. Gosdzinsky and A. Romeo, *Phys. Lett. B* **441**, 265 (1998).
- [8] G. Lambiase, V. V. Nesterenko, and M. Bordag, *J. Math. Phys. (N.Y.)* **40**, 6254 (1999).
- [9] K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, *Phys. Rev. D* **59**, 105009 (1999).
- [10] V. V. Nesterenko, G. Lambiase, and G. Scarpetta, *Riv. Nuovo Cimento* **27**, 1 (2004).
- [11] I. Klich and A. Romeo, *Phys. Lett. B* **476**, 369 (2000).
- [12] M. Bordag and I. G. Pirozhenko, *Phys. Rev. D* **64**, 025019 (2001).
- [13] I. Cavero-Peláez and K. A. Milton, *Ann. Phys. (N.Y.)* **320**, 108 (2005).
- [14] I. Cavero-Peláez and K. A. Milton, *J. Phys. A* **39**, 6225 (2006).
- [15] A. Romeo and K. A. Milton, *Phys. Lett. B* **621**, 309 (2005).
- [16] A. Romeo and K. A. Milton, *J. Phys. A* **39**, 6703 (2006).
- [17] M. Schaden, [hep-th/0604119](#).
- [18] I. Brevik and G. H. Nyland, *Ann. Phys. (N.Y.)* **230**, 321 (1994).
- [19] G. Barton, *J. Phys. A* **34**, 4083 (2001).
- [20] I. Brevik and A. Romeo, [hep-th/0601211](#).
- [21] I. Cavero-Peláez, K. A. Milton, and K. Kirsten, [hep-th/0607154](#).
- [22] A. Romeo and A. A. Saharian, *Phys. Rev. D* **63**, 105019 (2001).
- [23] J. Feinberg, A. Mann, and M. Revzen, *Ann. Phys. (N.Y.)* **288**, 103 (2001).
- [24] M. Bordag, V. V. Nesterenko, and I. G. Pirozhenko, *Phys. Rev. D* **65**, 045011 (2002).
- [25] A. A. Saharian and A. S. Tarloyan, [hep-th/0603144](#).
- [26] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and A. S. Tarloyan, *Phys. Rev. D* **74**, 025017 (2006).
- [27] N. Graham and K. D. Olum, *Phys. Rev. D* **72**, 025013 (2005).
- [28] N. Graham, *J. Phys. A* **39**, 6423 (2006).
- [29] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
- [30] N. W. McLachlan, *Theory and Application of Mathieu Functions* (Clarendon Press, Oxford, 1947).
- [31] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
- [32] S. Leseduarte and A. Romeo, *Ann. Phys. (N.Y.)* **250**, 448 (1996).
- [33] N. G. van Kampen, B. R. A. Nijboer, and K. Schram, *Phys. Lett. A* **26**, 307 (1968).
- [34] S. Leseduarte and A. Romeo, *J. Phys. A* **27**, 2483 (1994).
- [35] A. Romeo, *Phys. Rev. D* **52**, 7308 (1995).
- [36] A. Romeo, *Phys. Rev. D* **53**, 3392 (1996).
- [37] M. V. Berry and M. Robnik, *J. Phys. A* **19**, 649 (1986).
- [38] M. V. Berry, *J. Phys. A* **19**, 2281 (1986).
- [39] A. A. Kvitsinsky, *J. Phys. A* **29**, 6379 (1996).
- [40] H. Kober, *Dictionary of Conformal Representations* (Dover Publications, New York, 1957).