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Bulk viscosity of high-temperature QCD

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We compute the bulk viscosity ζ of high-temperature QCD to leading order in powers of the running coupling $\alpha_s(T)$. We find that it is negligible compared to shear viscosity η for any α_s that might reasonably be considered small. The physics of bulk viscosity in QCD is a bit different than in scalar ϕ^4 theory. In particular, unlike in scalar theory, we find that an old, crude estimate of $\zeta \simeq 15(\frac{1}{3} - v_s^2)^2\eta$ gives the correct order of magnitude, where v_s is the speed of sound. We also find that leading-log expansions of our result for ζ are not accurate except at very small coupling.

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I. INTRODUCTION AND RESULTS

Studies of collective flow at RHIC [1], particularly of elliptic flow, seem to be well described by nearly ideal hydrodynamics [2]. In fact, it has recently been claimed that these experiments prove that the quark-gluon plasma is the most nearly ideal fluid known, with a viscosity close to the conjectured lower bound on viscosities in any system [3,4]. Such startling claims should be tested in any way we have available. This requires studying flow in heavy ion collisions using nonideal hydrodynamics, that is, hydrodynamics including viscous effects [5]. It also would be valuable to know as much as possible about the theoretical expectations for viscosity in the quark-gluon plasma.

In an ideal hydrodynamical treatment, the evolution of the plasma is determined by stress-energy conservation, $\partial_\mu T^{\mu\nu} = 0$, together with an equilibrium equation of state which relates the pressure to the energy density, $P = P_{\rm eq}(\epsilon)$. This should work whenever the system is locally in equilibrium, which is the case in the limit of arbitrarily slowly varying flow velocity $u_i(x)$. When $u_i(x)$ varies somewhat in space, the fluid will not be precisely in local equilibrium, which will modify the stress tensor. For slowly varying $u_i(x)$, the corrections to the stress tensor T_{ij} can be expanded in gradients of u_i. The leading order corrections are parametrized by two quantities, the shear viscosity η and the bulk viscosity ζ:

$$T_{ij} = P_{\rm eq}(\epsilon)\delta_{ij} - \eta(\partial_i u_j + \partial_j u_i - \tfrac{2}{3}\delta_{ij}\partial_k u_k) - \zeta\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{1.1}$$

where the expression is implicitly written in the instantaneous local rest frame (where $T_{0i} = 0$).

While we are really interested in the viscosities η and ζ of the quark-gluon plasma at temperatures $T \sim 200$ MeV, where the theory is far from weakly coupled, we only possess reliable tools for computing dynamical properties such as viscosities at weak coupling.[1] Hopefully, extrapolating these results to strong coupling should give the right ballpark for the same quantities at moderately strong coupling, with uncertainties of perhaps a factor of a few. This motivates investigating η and ζ at weak coupling.

In a relativistic system, on dimensional grounds, both η and ζ must scale as $\eta, \zeta \propto T^3$. A great deal of study has gone into the shear viscosity in QCD. It has been known for 20 years that the parametric behavior is $\eta \sim T^3/(\alpha_s^2 \log[1/\alpha_s])$ [9,10]; the leading coefficient was closely estimated in 1990 [11], and complete results now exist both at leading logarithmic order [12] and full leading order [13] in the QCD coupling α_s. On the other hand, the calculation of the bulk viscosity has been completely neglected. To our knowledge, no paper in the literature even correctly states what *power* of α_s it is proportional to. The purpose of this paper is to fill this gap, by computing the bulk viscosity in weakly coupled QCD at leading order in α_s, using kinetic theory. We will only consider the case of vanishing (or negligible) chemical potential, $\mu = 0$.

In the next section, we will review the relevant physics of bulk viscosity, explaining why the parametric behavior is

$$\begin{aligned} \zeta &\sim \frac{\alpha_s^2 T^3}{\log[1/\alpha_s]} \quad (m_0 \ll \alpha_s T); \\ \zeta &\sim \frac{m_0^4}{T\alpha_s^2 \log[1/\alpha_s]} \quad (\alpha_s T \ll m_0 \ll T). \end{aligned} \tag{1.2}$$

Here m_0 refers to the heaviest zero-temperature (current) quark mass which is smaller than or of order the temperature T. We use the subscript zero to emphasize that m_0

[1]The lattice is a rigorous nonperturbative tool for studying thermodynamic properties of the quark-gluon plasma at strong coupling, but dynamical properties such as viscosities are hard to study on the lattice; see for instance, Refs. [6–8].

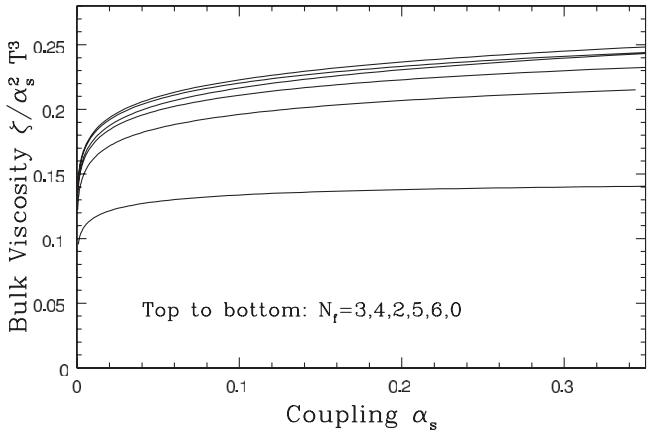

FIG. 1. Bulk viscosity for massless QCD at several values of N_f, as a function of the coupling α_s.

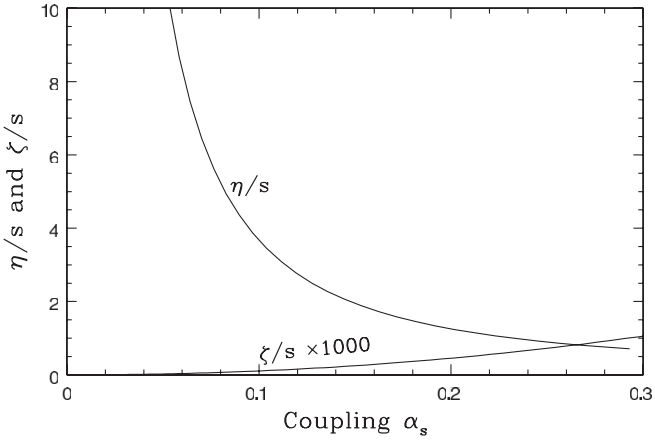

FIG. 3. Shear versus bulk viscosity: η/s and ζ/s (s the entropy density) as a function of α_s, for $N_f = 3$ QCD, neglecting quark masses. Bulk viscosity ζ has been rescaled by a factor of 1000.

represents a zero-temperature mass and not a finite-temperature effective quasiparticle mass. We will see that the physics of bulk viscosity is much richer than that of shear viscosity. In particular, the conformal anomaly (i.e. scaling violations) and the corrections to quasiparticle dispersion relations due to interactions, both irrelevant for shear viscosity, are both essential pieces of physics for bulk viscosity. Particle number changing interactions also play a much larger role in bulk than in shear viscosity. These qualitative points have been anticipated by the pioneering work of Jeon and Yaffe [14,15] on bulk viscosity in relativistic ϕ^4 theory. However, we shall see later that there are some significant qualitative differences between bulk viscosity in ϕ^4 theory and in QCD.

Section III will present the details of the calculation of bulk viscosity. Our discussion will at times be abbreviated, referring back to previous papers [12,13], where much of the technology has already been presented. We will end with a discussion in section IV. However, for the impatient reader, we now present our main results. The coefficients, missing in Eq. (1.2), are presented in Figs. 1 and 2. Here, N_f is the number of flavors of quarks. In Fig. 1, all quark flavors are assumed to be massless ($m_0 \ll \alpha_s T$); in Fig. 2, all but one flavor is assumed to be massless, with that one flavor's mass in the range $\alpha_s T \ll m_0 \ll \alpha_s^{1/2} T$. A comparison of bulk viscosity and shear viscosity for three massless flavors is given in Fig. 3 as a function of α_s. The figure makes clear that neglecting bulk viscosity in favor of shear viscosity is actually quite a good approximation, not only at weak coupling but probably also at moderately strong, physically interesting couplings. Figure 4 shows the ratio $\zeta/\alpha_s^4\eta$, which at very small α_s approaches a constant with corrections given by powers of $(\log(1/\alpha_s))^{-1}$. The dashed line shows an old, crude estimate of the ratio of bulk to shear viscosity which will be discussed in Sec. IV.

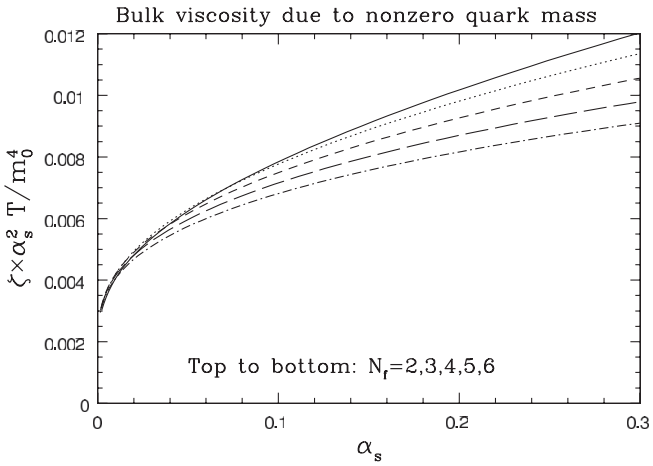

FIG. 2. Bulk viscosity when it is dominated by a single quark flavor's mass, as a function of α_s, for several values of N_f.

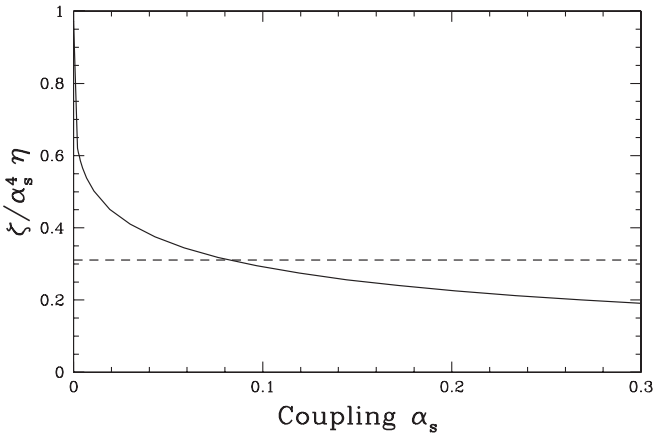

FIG. 4. The ratio $\zeta/\alpha_s^4\eta$ for $N_f = 3$ QCD, neglecting quark masses. The dashed line shows the crude estimate of (4.1) with (3.32). As $\alpha_s \to 0$ (and leading-log approximations to the leading-order result become applicable), the ratio approaches the limit $\zeta/\alpha_s^4\eta \to 0.973$.

Throughout this paper, we will not attempt to project our leading-order results to coupling higher than $\alpha_s \simeq 0.3$. In previous studies of diffusion constants [13], it was found that this is where different formulations of the effective kinetic theory, which were equivalent at leading-order in coupling, no longer agreed within a factor of 2, suggesting a complete breakdown of the perturbative treatment.[2]

II. PHYSICS OF BULK VISCOSITY

A. Basic picture

When a fluid is uniformly compressed, it leaves equilibrium. The energy density rises, but the pressure temporarily rises by more than what is predicted by the equation of state.[3] Under uniform rarefaction, the pressure temporarily falls further than is predicted by the fall in the density and the equation of state. The bulk viscosity quantifies the time integral of this extra shift in the pressure (per e-folding of expansion).

The change in pressure occurs because the fluid leaves equilibrium. The time scale for weakly coupled QCD to relax towards equilibrium is set by the rate $\Gamma \sim \alpha_s^2 T \log[1/\alpha]$ for a typical particle ($p \sim T$) to randomize its momentum $\boldsymbol{p}$. The faster the fluid equilibrates, the nearer to equilibrium it remains, so the smaller the shift in the pressure; therefore the viscosity should be proportional to $\epsilon/\Gamma \sim T^3/\alpha_s^2 \log[1/\alpha]$. This naive estimate turns out to be parametrically correct for shear viscosity.

However, it is wrong for bulk viscosity. The reason is that QCD (at high temperatures and away from mass thresholds) is a nearly conformal theory, and the bulk viscosity vanishes in a conformal theory, *for two reasons*. First, uniform compression or rarefaction is the same as a dilatation transformation. In a conformal theory, a dilatation transformation is a symmetry, and so the fluid will not leave equilibrium. Therefore, ζ must be proportional to the breaking of conformal invariance.

Furthermore, in a conformal theory, even if the fluid *is* out of equilibrium, the pressure still does not deviate from the value given by the equation of state, which for a conformal theory is exactly $P = \epsilon/3$. This is just the tracelessness of the stress-energy tensor in a conformal theory. For instance, consider massless $\lambda\phi^4$ theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{24}\phi^4, \tag{2.1}$$

at finite λ [with $(-+++)$ metric convention]. The Euler-Lagrange equation is,

$$\partial^2\phi - \frac{\lambda}{6}\phi^3 = 0, \tag{2.2}$$

and the stress tensor and its trace are

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}, \qquad T^\mu_\mu = -(\partial\phi)^2 - \frac{\lambda}{6}\phi^4. \tag{2.3}$$

Multiplying the Euler-Lagrange equation by ϕ shows that T^μ_μ vanishes up to a total derivative, which averages to zero. This argument is only flawed because of the conformal anomaly, which arises because of the running of λ with scale. The bulk viscosity coefficient will therefore contain another power of the smallness of conformal invariance breaking

Thus, in a nearly conformal theory, the bulk viscosity coefficient ζ vanishes as the *second* power of the departure from conformality: one power because the departure from equilibrium is small, and another power because any departure from equilibrium has a small impact on the pressure. For massless QCD, conformal symmetry is broken by the running of the coupling, $\beta(\alpha_s) \sim \alpha_s^2$, and so the bulk viscosity is

$$\zeta \sim \frac{T^3}{\alpha_s^2\log[1/\alpha_s]} \times (\alpha_s^2)^2 \sim \frac{\alpha_s^2 T^3}{\log[1/\alpha_s]}, \tag{2.4}$$

as claimed before. The presence of quark masses also constitutes a breaking of conformal invariance provided $m_0 \lesssim T$ (otherwise there are no quarks in the thermal bath and the influence of the quark can be neglected). In this case the pressure deviates from the massless value by a relative amount $\sim m_0^2/T^2$, and

$$\zeta \sim \frac{T^3}{\alpha_s^2\log[1/\alpha_s]} \times \left(\frac{m_0^2}{T^2}\right)^2 \sim \frac{m_0^4}{T\alpha_s^2\log[1/\alpha_s]}. \tag{2.5}$$

For future reference, note that if one formally defines the pressure as $P = T_{ii}/3$ and linearizes the hydrodynamic formula (1.1) about global equilibrium $P = P_{\rm eq}(\epsilon)$, then the bulk viscosity parametrizes

$$\Delta P - v_s^2\Delta\epsilon = \zeta\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{2.6}$$

where v_s is the velocity of sound, given by $v_s^2 = \partial P_{\rm eq}/\partial\epsilon$, and ΔP and $\Delta\epsilon$ are the local deviations of pressure and energy density.

B. Number changing processes: Comparison with ϕ^4 theory

There is one detail which this brief discussion has brushed over. Viscosities are typically determined by the *slowest* process which is required for relaxation to equilib-

[2] See, in particular, Fig. 4 of Ref. [13] at $m_D/T = 2.4$ for 3-flavor QCD, which corresponds to $\alpha_s = 0.3$.

[3] That the pressure is higher during compression and lower during rarefaction is dictated by the second law of thermodynamics; if the pressure during compression were lower than in equilibrium, one could construct a perpetual motion machine of the second kind, which rapidly compressed a fluid (encountering a lower than thermodynamic pressure) and then slowly expanded a fluid (encountering the full thermodynamic pressure). This constraint, that ζ is positive, is another way of seeing that ζ must be proportional to the *second* power of the beta function (or other source of conformal invariance breaking), since the beta function can be of either sign.

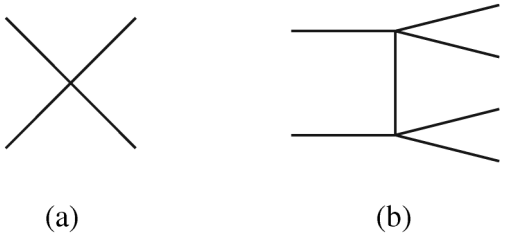

FIG. 5. Examples of (a) number conserving and (b) number changing processes in ϕ^4 theory.

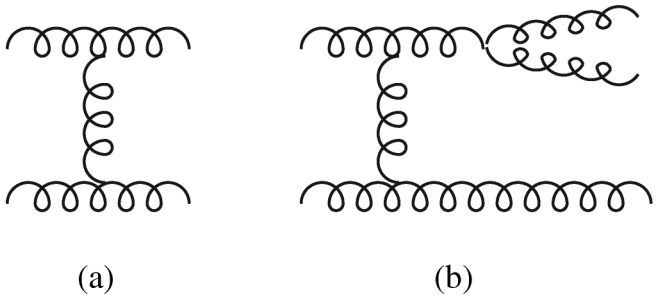

FIG. 6. Examples of (a) number conserving and (b) number changing processes in QCD.

rium. Certain departures from equilibrium can be very slow to equilibrate, due to the presence of almost-conserved quantities.

For instance, when considering bulk viscosity in scalar $\lambda\phi^4$ theory, Jeon found [15] that the total particle number equilibrates very slowly. The dominant process which randomizes momenta and determines the shear viscosity is shown in Fig. 5(a), with rate $\Gamma \sim \lambda^2 T$. In contrast, an example process which changes particle number, required for bulk viscosity, is shown in Fig. 5(b). One might naively expect that the particle number changing rate from such processes is $\Gamma \sim \lambda^4 T$, but this misses a soft enhancement. Number change primarily occurs between low energy excitations, where Bose stimulation increases the rate. The correct estimate is that the number of excitations relaxes at a rate $\Gamma \sim \lambda^3 T$, but this is still parametrically small compared to the $2 \to 2$ scattering processes of Fig. 5(a). This leads to the result $\zeta \propto \lambda T^3$ in ϕ^4 theory, up to logarithms [15]. In scalar theory, number-changing processes are the bottleneck for the relaxation to equilibrium characterized by bulk viscosity.

The same does not occur in QCD (at vanishing chemical potential[4]), however, because number changing processes are much more efficient in gauge theory.[5] The analog of Fig. 5 is Fig. 6. Number change is relatively fast even among hard particles and occurs by $1 \leftrightarrow 2$ splitting of a hard particle into two other hard particles during a small-angle collision, such as depicted by Fig. 6(b). The small-angle collision rate is of order $\alpha_s T$, and the nearly collinear emission from such scatterings costs one extra factor of α_s, giving a hard splitting rate $\Gamma \sim \alpha_s^2 T$. For comparison, the rate for a hard particle to randomize its momentum through $2 \leftrightarrow 2$ collisions is of order $\alpha_s^2 T \log[1/\alpha_s]$, which is larger by a logarithm. One might then suppose that number change is still the bottleneck process for bulk viscosity (by a logarithm), that the relevant rate is therefore $\alpha_s^2 T$ rather than $\alpha_s^2 T \log[1/\alpha_s]$, and that therefore there should be no logarithm in the parametric formula Eq. (1.2) for ζ. This turns out not to be the case, though, because $2 \leftrightarrow 2$ scattering processes exchange gluons between hard and soft momenta efficiently, and *soft* gluon number changing processes are efficient enough to prevent a particle number chemical potential from developing. In Sec. III E, we will show that, because of Bose stimulation enhancements for soft gluon emission from hard particles, the total rate of number-changing processes per particle is $O(\alpha_s^{3/2} T)$, which is parametrically faster than the $O(\alpha_s^2 T \log)$ rates discussed above. It is the latter, $O(\alpha_s^2 T \log)$ rates that will therefore be the bottleneck for equilibration and which will determine the QCD bulk viscosity.

III. DETAILS OF THE CALCULATION

A. Overview

We now proceed with the details of the calculation of bulk viscosity. Our general approach and notation will follow [12]. To begin, note that, at weak coupling, there are long lived quasiparticles, and a kinetic theory treatment should be valid. The plasma is well described by a phase space density for each particle type, $f(\boldsymbol{x}, \boldsymbol{p})$, which can be expanded about a local equilibrium distribution $f_{\rm eq}(\boldsymbol{x}, \boldsymbol{p})$ as

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{p}, t) &= f_{\rm eq}(\boldsymbol{x}, \boldsymbol{p}, t) + f_1(\boldsymbol{x}, \boldsymbol{p}, t), \\ f_{\rm eq}(\boldsymbol{x}, \boldsymbol{p}, t) &= (\exp[\beta(t)\gamma_u(E_{\boldsymbol{p}} - \boldsymbol{p}\cdot\boldsymbol{u}(\boldsymbol{x}))] \mp 1)^{-1}, \end{aligned} \tag{3.1}$$

with $\beta \equiv T^{-1}$ and $\gamma_u \equiv (1 - u^2)^{-1/2}$. The departure from equilibrium is determined by the Boltzmann equation,

$$\frac{\partial f}{\partial t} + \boldsymbol{v}_{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f = -C[f], \tag{3.2}$$

with $C[f]$ the collision integral. Above, $E_{\boldsymbol{p}}$ and

$$\boldsymbol{v}_{\boldsymbol{p}} \equiv \boldsymbol{\nabla}_{\boldsymbol{p}} E_{\boldsymbol{p}} \tag{3.3}$$

[4]For the Standard Model at finite baryon number chemical potential μ and finite quark mass m_0, the bulk viscosity would be very large. Compressing the system changes the temperature, which shifts how much of the baryon number is stored in each quark type, in equilibrium. The actual distribution of baryons between quark types approaches this equilibrium value only by weak interactions, leading to a bulk viscosity $\zeta \sim \mu^2 m_0^4 / G_{\rm F}^2 T^7$ for $\mu \lesssim T$ in cases where this is the rate-limiting process. Note however that in the early universe $\mu/T \sim 10^{-9}$ is negligible, while in a heavy ion collision weak interactions can be neglected entirely and one should take the numbers for each quark type to be separately conserved.

[5]See, for example, Ref. [16] and the related discussion of photon Bremsstrahlung in Ref. [17].

are the energy and velocity of a particle with momentum $\boldsymbol{p}$.[6] To study transport coefficients such as viscosity, we are interested in small departures from equilibrium in the hydrodynamic limit of slow variation in $\boldsymbol{x}$ and t. The left-hand side of (3.2) is explicitly small because of the derivatives, and so we may replace f by $f_{\rm eq}$ there. The collision term must be expanded to first order, $C[f] \propto f_1$, noting that $C[f_{\rm eq}] = 0$ by local detailed balance.

It is convenient to analyze the problem in a local region, choosing an approximate rest frame where $u(\boldsymbol{x})$ and the variation of $\beta(t)$ can be taken to be small. To first order in these small quantities, the derivatives appearing on the left-hand side of the Boltzmann equation are

$$\partial_\mu f_{\rm eq}(\boldsymbol{x}, \boldsymbol{p}, t) = -f_0(E_{\boldsymbol{p}})[1 \pm f_0(E_{\boldsymbol{p}})]\partial_\mu \times [\beta(t)(E_{\boldsymbol{p}} - \boldsymbol{p}\cdot\boldsymbol{u}(\boldsymbol{x}))]|_{\beta(t)=\beta,\boldsymbol{u}=0}, \quad (3.4)$$

where f_0 is the Bose or Fermi distribution

$$f_0(E) = (e^{\beta E} \mp 1)^{-1}. \quad (3.5)$$

The departure from equilibrium, in the case of bulk viscosity, arises because

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} \equiv X \neq 0. \quad (3.6)$$

In Sec. III B below, we will use the derivatives (3.4) and thermodynamic relations (in a treatment slightly generalizing that of Jeon and Yaffe [14]) to rewrite the left-hand side of the Boltzmann equation (3.2) in the form

$$\frac{\partial f^a_{\rm eq}}{\partial t} + \boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} f^a_{\rm eq} = \beta f_0(1 \pm f_0)X(\boldsymbol{x})q^a(\boldsymbol{p}) \quad (3.7)$$

for the case of isotropic compression or expansion, relevant to bulk viscosity. Here, a is a species label, and $q^a(\boldsymbol{p})$ represents how much a particle of type a and momentum $\boldsymbol{p}$ contributes to the $\Delta P - v_{\rm s}^2\Delta\epsilon$ of (2.6).

The departure f_1 from local equilibrium, at linearized order, must also be proportional to $X(\boldsymbol{x})$, and it is convenient to parametrize it as

$$f_1^a(\boldsymbol{x}, \boldsymbol{p}) = \beta^2 f_0(1 \pm f_0)X(\boldsymbol{x})\chi^a(|\boldsymbol{p}|). \quad (3.8)$$

The function $\chi(|\boldsymbol{p}|)$ will be a nontrivial function of the magnitude of momentum $p \equiv |\boldsymbol{p}|$, but (in the local rest frame) it is direction independent, because X is a scalar quantity. Defining

$$\mathcal{S}^a(\boldsymbol{p}) = -Tq^a(\boldsymbol{p})f_0(1 \pm f_0), \quad (3.9)$$

the Boltzmann equation can be written as

$$\mathcal{S}^a(\boldsymbol{p}) = [\mathcal{C}\chi]^a(\boldsymbol{p}), \quad (3.10)$$

with $\mathcal{C}$ the linearization of the collision integral, which we will give in Sec. III D.

The bulk viscosity is then determined as the shift in the pressure induced by the departure from equilibrium χ. As we shall discuss, this is an integral over $\boldsymbol{p}$ of χ^a times the same source $\mathcal{S}^a$ already introduced,

$$\zeta = \beta^3\sum_a \nu_a \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\mathcal{S}^a(p)\chi^a(p) \equiv (\mathcal{S}, \chi), \quad (3.11)$$

where ν_a is the multiplicity of species type a. The collision operator $\mathcal{C}$ is Hermitian under this inner product, and we may formally write,

$$\zeta = (\mathcal{S}, \mathcal{C}^{-1}\mathcal{S}). \quad (3.12)$$

This can then be treated variationally, by the techniques presented in Refs. [12,13].

B. General formula for $q^a(\boldsymbol{p})$

It remains to determine q^a, to establish the form of $\mathcal{C}$, and to explain how the integral equations will be solved. We will treat q^a first, since it is the most different from the problems already addressed in Refs. [12,13]. The second term on the left-hand side of Eq. (3.2) is

$$\boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} f = \beta f_0(1 \pm f_0)p_i v_{\boldsymbol{p},j}\nabla_j u_i \quad (3.13)$$

at linearized order. Specializing to isotropic compression or expansion, $\nabla_i u_j = (\delta_{ij}/3)\boldsymbol{\nabla}\cdot\boldsymbol{u}$, this becomes

$$\boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} f = \beta f_0(1 \pm f_0)\boldsymbol{\nabla}\cdot\boldsymbol{u}\frac{\boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{p}}{3}. \quad (3.14)$$

Unlike the case of shear viscosity, the term $\partial_t f$ is also nonzero; the compression or rarefaction of the fluid causes its density, and therefore its temperature, to change with time. By the chain rule,

$$\frac{\partial f_0}{\partial t} = \frac{d\beta}{dt}\frac{\partial f_0}{\partial\beta} = -\frac{d\beta}{dt}f_0(1 \pm f_0)\frac{\partial(\beta E_{\boldsymbol{p}})}{\partial\beta}. \quad (3.15)$$

Now, stress-energy conservation implies

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_t\epsilon = -(\epsilon + P)\boldsymbol{\nabla}\cdot\boldsymbol{u}. \quad (3.16)$$

By standard thermodynamic relations,

$$\epsilon + P = T\frac{dP}{dT} \quad (3.17)$$

(recall that $P = -F$ for a theory without chemical potentials), and by the chain rule,

$$\frac{d\epsilon}{dt} = \frac{dP}{dt}\frac{d\epsilon}{dP} = v_{\rm s}^{-2}\partial_t P. \quad (3.18)$$

Combining (3.16) through (3.18),

$$\frac{dP}{dt} = -v_{\rm s}^2\boldsymbol{\nabla}\cdot\boldsymbol{u}T\frac{dP}{dT} = v_{\rm s}^2\boldsymbol{\nabla}\cdot\boldsymbol{u}\beta\frac{dP}{d\beta}, \quad (3.19)$$

and since the dependence of P on t is through its β

[6] We use the general formula (3.3) [which can be understood as the group velocity of a wave packet] because we would like to make a general treatment of quasiparticles with some dispersion relation $E_{\rm p}$, and there is no need at this point to specialize, for example, to $E_{\boldsymbol{p}}^2 = p^2 + m^2$.

dependence, it follows that

$$\frac{d\beta}{dt} = \beta v_s^2 \boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{3.20}$$

Therefore, combining (3.14), (3.15), and (3.20), the full left-hand side of the Boltzmann equation is

$$(\partial_t + \boldsymbol{v_p}\cdot\boldsymbol{\nabla}_x) f_0 = \beta f_0(1 \pm f_0)\boldsymbol{\nabla}\cdot\boldsymbol{u}\left(\frac{\boldsymbol{p}\cdot\boldsymbol{v_p}}{3} - v_s^2\frac{\partial(\beta E_p)}{\partial\beta}\right). \tag{3.21}$$

Comparing to the definition (3.7) of $q^a(\boldsymbol{p})$, we determine

$$q^a(\boldsymbol{p}) = \frac{\boldsymbol{p}\cdot\boldsymbol{v_p}}{3} - v_s^2\frac{\partial(\beta E_p)}{\partial\beta}. \tag{3.22}$$

A nice property of this formula is that one can easily verify that the source vanishes in a conformal theory. In a conformal theory, the only dimensionful scale would be T, and so, by dimensional analysis, $E_{\boldsymbol{p}}$ must have the form $E_{\boldsymbol{p}} = pF(p/T)$ for some function F. Using (3.3) and (3.22), and the conformal result $v_s^2 = \frac{1}{3}$, one would then find $q^a(\boldsymbol{p}) = 0$.

Before finding explicit expressions for $E_{\boldsymbol{p}}$ and v_s^2 in QCD, let us briefly discuss the result for $q^a(\boldsymbol{p})$. The $\boldsymbol{p}\cdot\boldsymbol{v_p}$ term represents the change $v_{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_x f_{\rm eq}$ in the quasiparticle distribution function due to free propagation. For all other transport coefficients we have computed [12,13], this type of change was the appropriate "source" in the Boltzmann equation, and the collision integral was to be equated with it. But here the source has nonvanishing energy, and energy is conserved. The collision integral has an exact zero mode associated with energy conservation; therefore collisions will not erase the change in f, but will redistribute it until it looks like a shift in the temperature. The size of the temperature shift is fixed by energy conservation—that is, by the amount of energy the $\boldsymbol{p}\cdot\boldsymbol{v_p}$ term introduces. Therefore, the true departure from equilibrium is the difference between this $\boldsymbol{p}\cdot\boldsymbol{v_p}$ source term, and the temperature shift which carries the same total energy. This is the role of the second $v_s^2\partial(\beta E_{\boldsymbol{p}})/\partial\beta$ term. In other words, considering the linearized collision operator $\mathcal{C}$ as an operator on the space of departures from equilibrium δf, we must project the source $\boldsymbol{p}\cdot\boldsymbol{v_p}$ into the subspace orthogonal to the zero mode of $\mathcal{C}$ (since the eigenvector of the zero mode is not actually a departure from equilibrium).

As a check, we give a general demonstration in the appendix that the source term determined by (3.22) indeed carries no energy in the quasiparticle approximation we have used throughout. (One may also eschew generality and instead directly check with the explicit formulas for $q^a(\boldsymbol{p})$ given in the next section.) In the appendix, we also discuss in more detail why the quasiparticle approximation is justified for a leading-order calculation of the bulk viscosity.

With this in mind, we can see why it is this same $q^a(p)$ which is relevant in determining the pressure shift due to the departure from equilibrium $\propto \chi^a(p)$. Naively, the extra pressure due to a departure from equilibrium $f_1(\boldsymbol{p})$ should be $\frac{1}{3}\int_{\boldsymbol{p}} f_1^a(\boldsymbol{p})\boldsymbol{p}\cdot\boldsymbol{v_p}$. However, a general shift in the equilibrium distribution function by f_1 leads to a shift in the energy. Bulk viscosity involves the difference between the actual pressure, and the pressure determined by ϵ and thermodynamics, $P(\epsilon)$. Therefore, we must subtract off $(dP/d\epsilon)\delta\epsilon = v_s^2\delta\epsilon$, the shift in the pressure due to the extra energy density contributed by f_1. That is precisely what the second term in Eq. (3.22) does.[7]

C. Specific formula for $q^a(\boldsymbol{p})$

Now we will determine in detail the form $q^a(\boldsymbol{p})$ takes in QCD at weak coupling $\alpha_s \ll 1$. For simplicity we will also take quark masses $m_0 \ll T$, though nothing in principle stops us from considering the case of quark masses $m_0 \sim T$. We will assume that there is at most one quark species with non-negligible quark mass, which we denote M_0. In this case, the energy of a quasiparticle excitation of momentum $p \gg gT$, to first order in g^2, is given by

$$\begin{aligned} E_{\boldsymbol{p}}^2 &= p^2 + m_\infty^2, \\ m_{\infty,a}^2[\text{quark}] &= m_{0,a}^2 + \frac{C_{\rm F} g^2 T^2}{4} = m_{0,a}^2 + \frac{g^2T^2}{3}, \\ m_\infty^2[\text{gluon}] &= (C_{\rm A} + N_{\rm f} t_{\rm F})\frac{g^2T^2}{6} = \frac{6+N_{\rm f}}{12} g^2T^2, \end{aligned} \tag{3.23}$$

where $m_{0,a}$ is the mass of quark species a. The masses m_∞ here are the corrections to the large p dispersion relations.[8] We have written these expressions in terms of group Casimirs so that they can be evaluated for a general group, and have also given the specialization to QCD, where the adjoint Casimir $C_{\rm A} = 3$, and the fermions are in a representation with Casimir $C_{\rm F} = 4/3$ and trace normalization $t_{\rm F} = 1/2$. Here $N_{\rm f}$ is the number of light Dirac fermions, or half the number of Weyl fermions.

Using these expressions, q^a becomes

$$q^a = \frac{1}{E_{\boldsymbol{p}}}\left(\frac{\boldsymbol{p}^2}{3} - v_s^2(\boldsymbol{p}^2 + \tilde{m}_a^2)\right), \tag{3.24}$$

[7] This subtraction is technically unnecessary if one has already projected the source to be orthogonal to the zero mode, since then no shift in the energy would be produced. However, it is convenient, because it allows a symmetric treatment of the source and the pressure shift, as is manifested by the symmetric appearance of S in (3.12).

[8] Their relation to frequently-used zero-momentum masses are $m_\infty^2 = m_{\rm D}^2/2 = 3m_{\rm pl}^2/2$ for gluons and $m_\infty^2 = 2m_{\rm F}^2$ for massless quarks, where $m_{\rm D}$ is the Debye mass, $m_{\rm F}$ is the analogous screening mass for quark exchange, and $m_{\rm pl}$ is the plasma frequency. For further details, see, for example, Refs. [18,19].

$$\tilde{m}_a^2 \equiv m_{\infty,a}^2 - \frac{d(m_{\infty,a}^2)}{d(\ln T^2)}, \tag{3.25}$$

which coincides with the results of Jeon and Yaffe [14].

The speed of sound can be determined by writing out the temperature dependence of the pressure. At order g^2 and M_0^2, the pressure of the QCD plasma is [20]

$$\begin{aligned} P &= (a + bg^2[\mu^2 = T^2])T^4 + cM_0^2T^2, \\ a &= \frac{\pi^2}{180}(4d_A + 7N_f d_F) = \frac{\pi^2}{180}(32 + 21N_f), \\ b &= \frac{-1}{288}(2d_A C_A + 5N_f d_F C_F) = \frac{-1}{288}(48 + 20N_f), \\ c &= \frac{-1}{12}d_F = \frac{-1}{4}, \end{aligned} \tag{3.26}$$

where $d_A = 8$ and $d_F = 3$ are the dimensions of the adjoint and fermion color representations. Using $\epsilon = TdP/dT - P$, one finds

$$v_s^2 = \frac{dP}{d\epsilon} = \frac{dP/dT}{d\epsilon/dT} = \frac{1}{3} - \frac{2b}{9a}\beta(g^2) + \frac{cM_0^2}{9aT^2}, \tag{3.27}$$

up to $O(g^5)$, $O(m_0^2 g^2/T^2)$, and $O(m_0^4/T^4)$ corrections. Here,

$$\begin{aligned} \beta(g^2) \equiv \frac{\mu^2 dg^2}{d[\mu^2]} &= \frac{g^4}{16\pi^2}\left(\frac{4N_f t_F - 11C_A}{3}\right) \\ &= \frac{g^4}{16\pi^2}\left(\frac{2N_f - 33}{3}\right) \end{aligned} \tag{3.28}$$

is the beta function of QCD, which enters on taking the temperature dependence of g^2 into account.[9] Similarly, the quantities $\tilde{m}^2$ introduced earlier involve m_0^2 and $\beta(g^2)$, and are

$$\begin{aligned} \tilde{m}^2[\text{quark}] &= m_{0,a}^2 - \frac{C_F T^2}{4}\beta(g^2), \\ \tilde{m}^2[\text{gluon}] &= -\frac{(C_A + N_f t_F)T^2}{6}\beta(g^2). \end{aligned} \tag{3.29}$$

Collecting these results, and making the approximation

$$q^a = \frac{1}{E_p}\left(\frac{\boldsymbol{p}^2}{3} - v_s^2(\boldsymbol{p}^2 + \tilde{m}_a^2)\right) \simeq \frac{1}{p}\left(\left(\frac{1}{3} - v_s^2\right)\boldsymbol{p}^2 - \frac{1}{3}\tilde{m}_a^2\right), \tag{3.30}$$

valid for $m_0^2/T^2 \ll 1$ and $\beta(g^2) \ll 1$, we find

$$q^{q,a} = |\Delta v_s^2| p + \left[\frac{C_F}{12}\beta(g^2)T^2 - \frac{m_{0,a}^2}{3}\right]p^{-1}, \tag{3.31a}$$

$$q^g = |\Delta v_s^2| p + \left[\frac{C_A d_A + N_f d_F C_F}{18 d_A}\beta(g^2)T^2\right]p^{-1}, \tag{3.31b}$$

where[10]

$$|\Delta v_s^2| = \frac{-5(2d_A C_A + 5d_F C_F N_f)\beta(g^2) + 60 d_F M_0^2/T^2}{36\pi^2(4d_A + 7d_F N_f)}. \tag{3.32}$$

Here, the M_0^2 in (3.32) appears in the q for every species, but the $m_{0,a}^2$ in the second term of (3.31a) only contributes to the possibly massive quark, for which $m_{0,a} = M_0$. Note that, as promised, q^a is proportional to the source of conformal invariance violation, either the beta function or the current quark mass. Because $\mathcal{S} \propto q$ enters quadratically in Eq. (3.12), we see that ζ will depend quadratically on the size of conformal invariance breaking, as claimed.

D. Variational method and the collision integral

It remains to specify the form of the collision integral, and to explain how it will be inverted to establish ζ using Eq. (3.12). Since the details here are rather similar to the previous literature [12,13], we will be somewhat brief in our discussion. First, define an inner product as in Eq. (3.11) (summation over species label and integration over momenta). Then the Boltzmann equation and bulk viscosity can be formulated variationally; define

$$Q(\chi) \equiv (\chi, \mathcal{S}) - \tfrac{1}{2}(\chi, \mathcal{C}\chi), \tag{3.33}$$

and observe that $\delta Q/\delta\chi = 0$ when χ satisfies the Boltzmann equation (3.10). Furthermore, the value (3.12) of ζ is $2Q$ evaluated at this extremum:

$$\zeta = 2Q_{\max}. \tag{3.34}$$

A variational *Ansatz* for χ will give a lower bound on the value of the extremum which will improve rapidly as the variational basis is increased. Therefore, we write a multiparameter, linear Ansatz for $\chi(p)$, in terms of a set of basis functions. As we will discuss momentarily, $\chi(p) \propto p$ at small momenta, and χ grows no faster than $\sim p^2/T$ at large p. Therefore, we use a slight modification of the basis functions considered in [12],

$$\phi_m(p) = \frac{p^m T^{K-m-1}}{(T + p)^{K-2}}, \qquad m = 1 \ldots K. \tag{3.35}$$

The function $\chi^a(p)$ is then assumed to be of form,

[9] We are implicitly taking Td/dT holding μ/T fixed. But we would get the same answer if we performed the derivative holding μ fixed; in writing $g^2[\mu^2 = T^2]$ in Eq. (3.26), what we really mean is that there is explicit μ dependence in the g^4 term, of form $B\beta(g^2)\log(T^2/\mu^2)T^4$. Holding μ fixed, $\beta(g^2)$ arises from the T derivative of this logarithm.

[10] Our notation $|\Delta v_s^2|$ is simply short-hand notation for $\frac{1}{3} - v_s^2$. It is positive for normal, asymptotically free QCD, but it can be negative in other theories, such as massless QED.

$$\chi^a(p) = \sum_m \tilde{\chi}^a_m \phi_m(p). \tag{3.36}$$

Within this variational Ansatz, the required inner products for Q are

$$(\chi, \mathcal{S}) = \sum_{a,m} \tilde{\chi}^a_m \tilde{S}^a_m, \qquad (\chi, \mathcal{C}\chi) = \sum_{abmn} \tilde{\chi}^a_m \tilde{C}^{ab}_{mn} \tilde{\chi}^b_n, \tag{3.37}$$

where

$$\begin{aligned} \tilde{S}^a_m &\equiv \nu_a \int_{\boldsymbol{p}} \phi_m(p) \mathcal{S}^a(p), \\ \tilde{C}^{ab}_{mn} &\equiv \nu_a \int_{\boldsymbol{p}} \phi_m(p) [\mathcal{C}^{ab} \phi_n](p), \end{aligned} \tag{3.38}$$

where $\mathcal{C}^{ab}$ means the collision integral for species a when species b is out of equilibrium by the amount indicated by χ^b. Considering $\tilde{S}^a_m$ to be a rank NK column vector $\tilde{S}$ and $\tilde{C}^{ab}_{mn}$ to be a $NK \times NK$ matrix, where N is the number of possibilities for the species index a, the bulk viscosity (3.12) is

$$\zeta = \tilde{S} \tilde{C}^{-1} \tilde{S}. \tag{3.39}$$

In practice, $N = 2$ (quarks vs gluons) if all quarks are massless, and $N = 3$ (massive quark vs massless quarks vs gluons) if one quark species is massive.

The detailed form of $\tilde{C}$ is given in Ref. [13],[11] which we summarize here for completeness:

$$\begin{aligned} \tilde{C}^{ab}_{mn} \equiv{}& \frac{\beta^3}{8} \sum_{cdef} \int_{\boldsymbol{pkp'k'}} |\mathcal{M}^{cd}_{ef}(\boldsymbol{p}, \boldsymbol{k}; \boldsymbol{p'}, \boldsymbol{k'})|^2 (2\pi)^4 \delta^{(4)}(P + K - P' - K') f^c_0(p) f^d_0(k) [1 \pm f^e_0(p')][1 \pm f^f_0(k')] \\ &\times [\phi_m(p)\delta^{ac} + \phi_m(k)\delta^{ad} - \phi_m(p')\delta^{ae} - \phi_m(k')\delta^{af}][\phi_n(p)\delta^{bc} + \phi_n(k)\delta^{bd} - \phi_n(p')\delta^{be} - \phi_n(k')\delta^{bf}] \\ &+ \frac{\beta^3}{2} \sum_{cde} 4\pi \int_0^\infty dp' dp dk \gamma^c_{de}(p'; p, k) \delta(p' - p - k) f^c_0(p') [1 \pm f^d_0(p)][1 \pm f^e_0(k)] \\ &\times [\phi_m(p')\delta^{ac} - \phi_m(p)\delta^{ad} - \phi_m(k)\delta^{ae}][\phi_n(p')\delta^{bc} - \phi_n(p)\delta^{bd} - \phi_n(k)\delta^{be}]. \end{aligned} \tag{3.40}$$

All factors of the number of degrees of freedom of each species are implicitly included in these sums.[12] The detailed expressions for the $2 \leftrightarrow 2$ amplitude $\mathcal{M}$ and the effective $1 \leftrightarrow 2$ splitting function γ fill two appendices of Ref. [13] and will not be reproduced here. In treating the kinematics of these processes, we have neglected the masses of all external states, which is consistent with our approximation, $m_0 \ll T$. In principle there is no obstacle to treating the case $m_0 \sim T$, but we have not done so, primarily out of laziness.

Besides the difference in the source, which we have already stressed, the other difference between bulk and shear viscosity calculations is in the angular dependence of χ in the collision integral. For shear viscosity, it was not $\phi_m(p)$ which appeared above, but $\phi_m(p)\hat{p}_i\hat{p}_j$. (See, for example, Ref. [13] for a discussion in the conventions of this paper.) When suitably averaged over the indices ij, this led to angular factors of $P_2(\cos\theta_{\boldsymbol{pk}})$ in the cross-term between $\chi_m(p)$ and $\chi_n(k)$, for instance, where $P_2(x) = (3x^2 - 1)/2$ is the second Legendre polynomial. Since bulk viscosity arises due to $X = \boldsymbol{\nabla} \cdot \boldsymbol{u}$, a scalar quantity, this angular dependence is absent. This makes the calculation of the collision integral somewhat simpler, but it does add two complications involving zero modes of the collision operator, to which we now turn.

E. Zero modes of C

The first term in the collision integral (3.40), corresponding to $2 \leftrightarrow 2$ processes, has two exact zero modes, corresponding to all $\chi^a(p) \propto 1$ and all $\chi^a(p) \propto p$, corresponding to particle number conservation and energy conservation, respectively. The second term, corresponding to collinear $1 \leftrightarrow 2$ processes, breaks particle number but still has the zero mode corresponding to energy conservation. Therefore, the collision matrix $\tilde{C}$ will have a zero mode, and can potentially have a second approximate zero mode to the extent that the $2 \leftrightarrow 2$ term is larger than the $1 \leftrightarrow 2$ term. Since the collision integral must be inverted in evaluating Eq. (3.39), we must address the exact zero mode. We will see that making a leading-log expansion of bulk viscosity (if such is desired) requires treating the $2 \leftrightarrow 2$ term as larger, by a logarithm, than the $1 \leftrightarrow 2$ term for $p \sim T$. In order to understand why number-changing processes are not a bottleneck for equilibration, and to understand expansions in $[\log(1/\alpha_s)]^{-1}$, we will need to address the approximate zero mode as well. Both of these zero modes

[11] See specifically Eqs. (2.22) and (2.23) of Ref. [13] with $\chi^a_{i\ldots j}$ replaced by $\chi^a(p)$ to specialize to the isotropic ($l = 0$) angular dependence relevant to bulk viscosity, and then define $\tilde{C}^{ab}_{mn}$ as in (3.38) of this paper.

[12] In the convention of Ref. [13], the sums (no averages) over all initial and final colors are included in $|\mathcal{M}|^2$ and γ, and each of the indices $cdef$ in the explicit sums above denote gluons vs different flavors of quarks vs different flavors of antiquarks.

are specific to the case of isotropic $\chi(p)$, and neither is relevant to the analysis of other standard transport coefficients such as shear viscosity and flavor diffusion constants.

The presence of an exact zero mode in the collision integral is not problematic, precisely because the source $\mathcal{S}$ carries precisely zero energy, and so is orthogonal to the zero eigenvector. Therefore, our previous expressions should be understood as valid in the subspace orthogonal to the zero mode of $\mathcal{C}$. In practice our basis of functions ϕ_m^a are not restricted to this orthogonal subspace. But the collision integral can be rendered invertible without changing its behavior in the orthogonal subspace by adding a constant times the projection operator for the pure temperature fluctuation (the zero mode);

$$\tilde{C}^{ab}_{mn} \to \tilde{C}^{ab}_{mn} + \lambda\left(\nu_a \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \phi_m(p) f_0^a(1 \pm f_0^a)\right) \times \left(\nu_b \int \frac{d^3k}{(2\pi)^3} E_{\boldsymbol{k}} \phi_n(k) f_0^b(1 \pm f_0^b)\right), \tag{3.41}$$

for any positive λ. This renders $\tilde{C}$ invertible; and while $\tilde{C}^{-1}$ is λ dependent, $\tilde{C}^{-1}\tilde{S}$ is not, since $\tilde{S}$ has zero projection onto the modified direction. In our numerical evaluations we have checked explicitly that the determined value of ζ has no sensitivity to the added value of λ.

Next, consider the possible approximate zero mode, $\chi^a(p)$ a constant, corresponding to a chemical potential for particle number.[13] First note that the constant value must be the same for fermionic and bosonic species, because the set of $2 \leftrightarrow 2$ processes includes fermionic pair annihilation to gluons, which contributes at leading logarithmic order. However, no elastic $2 \leftrightarrow 2$ scattering processes will drive a common chemical potential for both quark and gluon number to zero. For the case of bulk viscosity in $\lambda\phi^4$ theory it was found that this played a major role in setting the bulk viscosity [14]. In that theory, $\chi \propto 1$ is an approximate zero mode of the full collision operator: $(\chi|\mathcal{C}\chi)$ for $\chi = 1$ is parametrically small compared to typical hard collision rates.[14]

However, for the bulk viscosity of QCD, this would-be zero mode actually plays no role: the expectation $(\chi|\mathcal{C}\chi)$ for $\chi = 1$ is parametrically large rather than small compared to typical hard scattering rates. The reason is that, while number changing collinear processes [the second term in Eq. (3.40)] are subdominant to $2 \leftrightarrow 2$ processes at generic momenta, they are very fast at producing and destroying soft gluons. To see this, let us estimate the total rate for a hard particle to produce a soft gluon of momentum k by Bremsstrahlung. Combine (i) the $O(g^2T)$ rate for small-angle scattering, as in Fig. 6(a), times (ii) a factor of g^2 for absorbing or emitting the additional gluon in Fig. 6(b), times (iii) an initial or final state factor of $f(k)$ or $1 + f(k)$ for that gluon, and (iv) a momentum integral dk/k (responsible for the logarithmically large rate of soft bremsstrahlung emission in vacuum[15]). $f(k) \sim T/k$ for small k, and the result for the number changing rate $\Gamma^{\text{total}}_{1\to 2}$ is then[16]

$$\Gamma^{\text{total}}_{1\leftrightarrow 2} \sim g^4 T \int \frac{dk}{k} f(k) \sim g^4 T^2 \int \frac{dk}{k^2}. \tag{3.42}$$

The infrared divergence of the integral will be cut off by the effective thermal mass $m \sim gT$ of the emitted gluon, so that

$$\Gamma^{\text{total}}_{1\leftrightarrow 2} \sim \frac{g^4 T^2}{m} \sim g^3 T. \tag{3.43}$$

As discussed earlier, this is parametrically faster than the $O(g^4 T \log)$ rate to redistribute momenta between soft and hard particles, which is the bottleneck which determines bulk viscosity. The total rate $\Gamma^{\text{total}}_{1\leftrightarrow 2}$ for creating or absorbing soft particles can therefore be taken as formally infinite for the purpose of a leading-order calculation of bulk viscosity.

The same result can also be obtained, with some difficulty, from Eq. (3.40) of this paper together with eqs. (B1–B6) of Ref. [13], which determine the splitting functions γ. In particular, the $k \ll p$ behavior of $\gamma^{\text{g}}_{\text{gg}}(p'; pk)$ and $\gamma^{\text{q}}_{\text{qg}}(p'; pk)$ is $\gamma \sim (g^4T)p^2/k$. If we substitute $\phi_m = \phi_n = \chi = 1$ into the $1 \leftrightarrow 2$ term in Eq. (3.40), the k integration for $p \sim T$ then gives the linear divergence $\int dk/k^2$ of (3.42).

This means that a chemical potential is actually very rapidly thermalized by number changing processes. Any

[13] By "particle number," we mean the sum of quark, antiquark, and gluon numbers, not a difference like quark minus antiquark number.

[14] This permits a simplification in ϕ^4 theory whereby one can avoid solving an integral equation and instead determine the leading-order bulk viscosity from a simple expectation value $(\chi|\mathcal{C}\chi)$ for $\chi \propto 1$ [14].

[15] In vacuum, there is an additional logarithmic factor for bremsstrahlung from an ultrarelativistic particle—a collinear logarithm $\int d^2k_\perp/k_\perp^2 \sim \ln(q/m)$, where q is the momentum-transfer in the underlying $2 \to 2$ collision. In our case, the most frequent collisions are the small-angle ones, whose impact parameter is limited by Debye screening, and $q \sim m \sim gT$ so that there is no collinear log enhancement.

[16] For $k \ll T$, the Landau-Pomeranchuk-Migdal (LPM) effect plays no role in gluon emission, as discussed qualitatively in Sec. 5.2 of Ref. [19]. This is different than the case of soft *photon* emission due to the $O(gT)$ thermal mass and scattering of the emitted gluon, either of which, for $k \ll T$, causes loss of the multiple-collision coherence that produces the LPM effect. Here is a quick argument: For small k, the internal hard particle line in Fig. 6(b) is off-shell in energy by an amount of order $\Delta E = E_{\boldsymbol{p}+\boldsymbol{k}} - E_{\boldsymbol{p}} - E_{\boldsymbol{k}} \sim (m_{\text{g}}^2 + k_\perp^2)/(2k)$. The formation time of the gluon is therefore of order $(\Delta E)^{-1} \lesssim k/m_{\text{g}}^2 \sim k/(g^2T^2)$, which is small compared to the time $1/(g^2T)$ between collisions when $k \ll T$.

$\chi(k)$ which falls more weakly than $\chi(k) \propto k$ at small k will lead to a divergent collision rate, meaning that such departures from equilibrium are so efficiently equilibrated that we need not consider them. Therefore we should restrict our Ansatz for χ to only functions which are linear or higher powers of k in the soft region. This justifies our choice in Eq. (3.35). Within this subspace of functions χ, the $2 \leftrightarrow 2$ part of the collision integral has only one zero mode, that associated with energy conservation, which we have already discussed. Therefore the small α_s behavior will indeed be $\zeta \propto \alpha_s^2 T^3 / \log[1/\alpha_s]$, and one can perform an expansion in logarithms of the coupling if desired.

F. Expansion in $\log[1/\alpha_s]$

In Ref. [13] it was shown that an expansion in inverse powers of $\ln[1/\alpha_s]$ works surprisingly well at small values of α_s, if it is carried to next-to-leading order. As we have just seen, there is no obstacle to making a similar expansion here. We have done so, by following the procedure described in detail in Ref. [13], but we find that the expansion works much less well than in the case of shear viscosity and number diffusion. The reason is that the dominant physics in shear viscosity and number diffusion is *angle change*. The charge q^a in that case is 1 or $|\boldsymbol{p}|$ times a nontrivial function of angle. The departure from equilibrium, $\chi(\boldsymbol{p})$, has nontrivial angular dependence, but turns out to have very simple $|\boldsymbol{p}|$ dependence, so a one parameter Ansatz works very well. In a next-to-leading log treatment, one fixes the $|\boldsymbol{p}|$ dependence of $\chi(\boldsymbol{p})$ using the leading-log part of the $2 \leftrightarrow 2$ processes and evaluates the collision integral using this fixed form of $\chi(\boldsymbol{p})$. This works because this functional form of $\chi(\boldsymbol{p})$ is essentially correct, whatever collision processes are involved.

For bulk viscosity, on the other hand, the charge q^a changes sign as a function of the particle's momentum, as the $1/p$ and p terms in Eq. (3.31) change relative importance. The $1/p$ term is also larger for gluons than for quarks, due to their larger thermal masses; therefore, over most of the momentum range the quarks and gluons display opposite departures from equilibrium. In QCD, the physics of bulk viscosity is primarily the physics of rearranging the $|\boldsymbol{p}|$ dependence of particle distributions. This is what the number changing $1 \leftrightarrow 2$ processes do best; so they play a much larger role in bulk viscosity than in shear. Indeed, unlike the case of shear, dropping the $2 \leftrightarrow 2$ processes and retaining only the number changing ones would still give a finite answer for ζ—which in fact turns out to be within a factor of 2 of the leading-order answer over most of the range of α_s we have considered. However, the detail of how they rearrange the momentum distributions is different than for the elastic processes. Therefore the detailed p dependence of $\chi(p)$ is quite different if only the leading-log $2 \leftrightarrow 2$ processes are considered, than if the full collision integral is used. We illustrate this difference in Fig. 7. This limits the range of validity of the expansion in logs to the regime where the $2 \leftrightarrow 2$ processes are much faster. But as we just said, the $1 \leftrightarrow 2$ processes are more important to bulk viscosity than to shear, so this requires the logarithm actually to be large. Therefore the expansion in logs works poorly and should not be used in treating bulk viscosity.

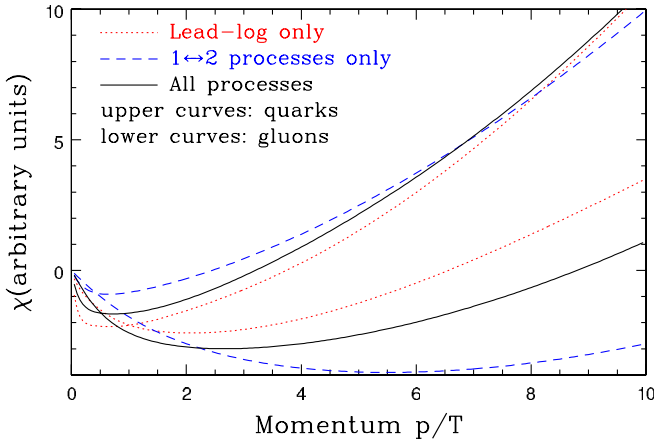

FIG. 7 (color online). Functional form of $\chi(p)$ as a function of p, shown for quarks and gluons in massless $N_f = 3$ QCD at $\alpha_s = .053$ (or $m_D/T = 1$). The three curves are the functional form using the leading-log $2 \leftrightarrow 2$ processes only, using the number changing processes only, and using all processes.

Another consequence of the quite nontrivial form of $\chi(p)$ is that several basis functions must be used to get accurate numerical values of ζ. For instance, we need at least 5 basis functions to get 0.1% accuracy, something accomplished with two basis functions for shear viscosity. For this reason, the results presented in Figs. 1 and 2 are "only" good to about 0.1%.

For completeness, Table I lists the first two coefficients in an expansion in leading logs for massless QCD:

$$\zeta = \frac{A\alpha_s^2 T^3}{\ln[\mu^*/m_D]}, \qquad m_D^2 = 2m_\infty^2[\text{gluon}] = (1 + \tfrac{1}{6}N_f)g^2T^2. \tag{3.44}$$

TABLE I. Next-to-leading log bulk viscosity, $\zeta = A\alpha_s^2 T^3/\ln[\mu^*/m_D]$, and ζ calculated using only number changing collinear processes, $\zeta_{1\leftrightarrow 2}$. All N_f quarks are taken to be massless.

QCD, $N_f =$	Leading-log A	NLL μ^*/T	$\zeta_{1\leftrightarrow 2}/\alpha_s^2 T^3$
0	0.443	7.14	.151
2	0.638	7.57	.282
3	0.657	7.77	.286
4	0.650	7.93	.279
5	0.622	8.06	.263
6	0.577	8.17	.242

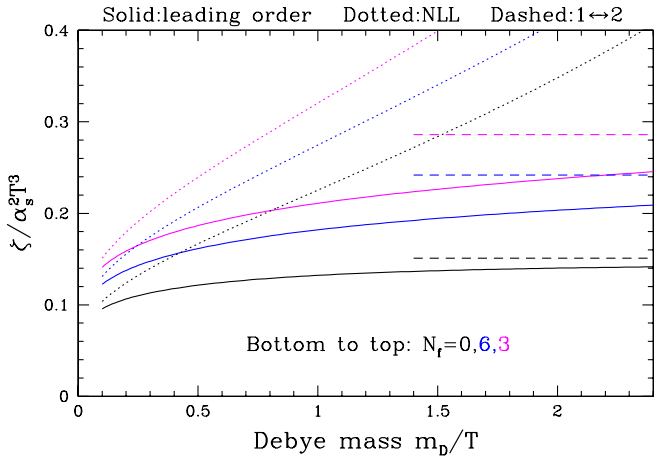

FIG. 8 (color online). Bulk viscosity ζ, plotted against $m_{\rm D}$ rather than $\alpha_{\rm s}$ for massless QCD. The dotted curves are the leading-log results; the dashed line on the right is the result neglecting everything but number changing collinear processes.

The table also contains the coefficient $\zeta = C\alpha_{\rm s}^2 T^3$ we would obtain if we ignored *all* $2 \leftrightarrow 2$ processes and considered only the number changing processes. To display the futility of using the next-to-leading log results, we compare them with the leading order results in Fig. 8. The failure of the next-to-leading log approximation by a factor of at least 1.5 for $N_{\rm f} = 3$ or 6 at $m_{\rm D} \geq 1.1T$ corresponds to $\alpha_{\rm s} \geq 0.05$.

IV. DISCUSSION

The physics of bulk viscosity in QCD is very interesting. The QCD plasma leaves equilibrium under uniform compression or rarefaction only due to conformal symmetry breaking, and the bulk viscosity depends quadratically on the size of conformal symmetry violation (either through quark masses or the beta function). To find the departure from equilibrium one must include the forward-scattering corrections to dispersion relations, and must account carefully for the shift in the plasma temperature. The departure from equilibrium due to compression is of opposite sign for high and low momentum excitations, and of opposite sign at intermediate momenta $p \sim \pi T$ for quarks versus gluons. Collinear splitting processes actually dominate the equilibration of the plasma except at very small coupling, although in the formal weak coupling limit, equilibration should be logarithmically dominated by $2 \leftrightarrow 2$ scattering, annihilation, and Compton processes—with the proviso that soft gluon bremsstrahlung is also included, since it prevents the development of a chemical potential for particle number. Putting this physics all together, one finds that the bulk viscosity at leading-log order (i.e. for exceptionally small coupling) is $\zeta \sim \alpha_{\rm s}^2 T^3/\log[1/\alpha_{\rm s}]$, with a leading coefficient of about 1. More practically, however, one can see from Fig. 1 that the complete result to leading order in powers of $\alpha_{\rm s}$ is roughly $\zeta \simeq 0.2\alpha_{\rm s}^2 T^3$ for any reasonable perturbative value of $\alpha_{\rm s}$ ($0.02 \lesssim \alpha_{\rm s} \lesssim 0.3$).

The practical import of bulk viscosity in QCD is very limited, however, in the regime where a perturbative treatment has any hope of applicability. We find that even for $\alpha_{\rm s} = 1/3$, the bulk viscosity is hundreds of times smaller than the shear viscosity. In practice, this means that bulk viscosity can be neglected, whenever shear viscosity plays a role. For instance, the decay of a sound wave depends on the combination $\zeta + 4\eta/3$; so one may drop the ζ term to a very good approximation. The expanding QCD plasma in an ultrahigh energy heavy ion collision is expected to be quite anisotropic, so shear viscosity again plays a role and bulk viscosity can be ignored. Similarly, while the expansion of the QCD plasma in the early universe should have been nearly isotropic, any flows in the presence of a phase interface—the only circumstances where nonequilibrium behavior may leave records in the early universe—are expected to be quite anisotropic, and again shear viscosity will be more important than bulk.

Besides the quite elegant physics involved in the bulk viscosity of QCD, it also provides a nice example of the dangers of interpreting scalar field theory as a toy model for gauge theory, with λ playing the role of g^2. In massless $\lambda\phi^4$ theory, Jeon and Yaffe showed [14,15] that the shear viscosity behaves as $\eta \sim T^3/\lambda^2$, while bulk viscosity behaves as $\zeta \sim \lambda T^3 \log^2[1/\lambda]$. For shear viscosity, the scalar theory provides a successful toy model, missing only the logarithmic dependence. For bulk viscosity, although some of the physics is the same, scalar field theory is a misleading guide to gauge theory, getting even the power of the coupling wrong. The difference arises because number-changing processes in scalar theory are slow compared to processes which redistribute hard momenta (rate $\lambda^3 T$ vs $\lambda^2 T$); in QCD, they are fast ($\alpha_{\rm s}^{3/2} T$ vs $\alpha_{\rm s}^2 T$).

One consequence of slow particle number changing rates for scalar theory, observed by Jeon and Yaffe, was that the bulk viscosity did not match the crude estimate

$$\zeta \approx 15\eta(\tfrac{1}{3} - v_{\rm s}^2)^2 \tag{4.1}$$

that had previously been made for scalar theory in the literature [21].[17] (This same relation was found earlier by Weinberg for a photon gas coupled to hot matter [22].) However, these same estimates turn out to be *parametrically* correct for QCD, reproducing (1.2). In QCD, the bottleneck rate is the same for both shear and bulk viscosity,

$$v_{\rm s}^2 - \tfrac{1}{3} = O(\beta(g^2)) + O(m_0^2/T^2) \tag{4.2}$$

[17] A similar estimate was made by Ref. [9] but differs by a factor of 2. The difference is likely due to the incorrect identification of the shear viscosity η, by a factor of 2, in Eq. (2.39) of Ref. [9].

is a measure of the deviation from conformal symmetry, and this deviation is squared, just as discussed in section II A. One could reproduce (4.1) from the derivation of bulk viscosity in this paper and of shear viscosity in Ref. [13] by (i) keeping only the $|\Delta v_s^2|$ term in the source term (3.31), and (ii) making a relaxation-time approximation of the collision operator as a rate $\Gamma(\boldsymbol{p})$ that is the same for bulk viscosity and shear viscosity.[18] In Fig. 4, the estimate (4.1) is shown by the dashed line for the leading-order result (3.32) for $\frac{1}{3} - v_s^2$. It does reasonably well at estimating the order of magnitude of our result for bulk viscosity.

It is interesting that there are studies of certain strongly coupled but nearly conformal theories which find a parametrically different dependence on $\frac{1}{3} - v_s^2$ than (4.1). In certain theories with gravity duals that make them amenable to calculation,[19] Ref. [24] finds $\zeta \sim \eta(\frac{1}{3} - v_s^2)$. This result is difficult to understand from the picture of viscosity developed in weakly-coupled field theories and provides an interesting conceptual puzzle for understanding bulk viscosity in strongly-coupled but nearly-conformal theories.

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APPENDIX: ORTHOGONALITY OF $\mathcal{S}$ TO THE ENERGY ZERO MODE

In this appendix, we verify that the source derived in this paper, given by (3.9) and (3.22), is orthogonal to the energy-changing zero mode $\chi(p) \propto E_{\boldsymbol{p}}$ discussed in section III E. Specifically, we show that $(\mathcal{S}, E_{\boldsymbol{p}}) = 0$ at the order of our calculation. That is,

$$\sum_a \nu_a \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} f_0(1 \pm f_0)\left(\frac{\boldsymbol{p}\cdot\boldsymbol{v}_{\boldsymbol{p}}}{3} - v_s^2 \frac{\partial(\beta E_{\boldsymbol{p}})}{\partial\beta}\right) E_{\boldsymbol{p}} = 0. \tag{A1}$$

This can be checked directly using the QCD-specific formulas of Sec. III C, but it is instructive to give a more general argument.

[18] Specifically, the ratio of the sources in the two cases then becomes $q_\zeta/q_\eta = |\Delta v_s^2|$. The relaxation-time approximation is $[\mathcal{C}\chi](\boldsymbol{p}) = \Gamma(\boldsymbol{p})\chi(\boldsymbol{p})$. Using $\zeta = 2Q_{\max}|_{l=0,q=q_\zeta}$ from (3.34) and $\eta = \frac{2}{15}Q_{\max}|_{l=2,q=q_\eta}$ from Ref. [13], one then obtains $\zeta/\eta = 15q_\zeta^2/q_\eta^2 = 15|\Delta v_s^2|^2$.

[19] For other bulk viscosity results in strongly interacting theories, see Ref. [23].

Use $\partial f_0 = -f_0(1 \pm f_0)\partial(\beta E_{\boldsymbol{p}})$ and $\boldsymbol{v}_{\boldsymbol{p}} = \boldsymbol{\nabla}_{\boldsymbol{p}} E_{\boldsymbol{p}}$ to rewrite the orthogonality condition as

$$\sum_a \nu_a \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{T}{3} E_{\boldsymbol{p}} \boldsymbol{p}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}} f_0 = v_s^2 \sum_a \nu_a \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}} \partial_\beta f_0. \tag{A2}$$

We then need the following two, slightly subtle equilibrium relations, which we will discuss below:

$$\partial_\beta P = \sum_a \nu_a \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{T}{3} E_{\boldsymbol{p}} \boldsymbol{p}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}} f_0, \tag{A3}$$

$$\partial_\beta \epsilon = \sum_a \nu_a \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}} \partial_\beta f_0. \tag{A4}$$

The orthogonality relation is then equivalent to the equilibrium relation

$$\partial_\beta P - v_s^2 \partial_\beta \epsilon = 0, \tag{A5}$$

which is satisfied because $v_s^2 = dP/d\epsilon = (\partial_\beta P)/(\partial_\beta \epsilon)$.

For the rest of this appendix, we will use the shorthand notation $\int$ to stand for $\sum_a \nu_a \int (d^3p)/(2\pi)^3$.

Deriving general relations for pressure and energy density and their derivatives in a gas of quasiparticles is slightly subtle because the effective energies $E_{\boldsymbol{p}}$ of the quasiparticles depend on temperature and include the effects of interactions with other quasiparticles. The energy density is not simply $\epsilon = \int E_{\boldsymbol{p}} f_0$, for example, because this expression suffers the usual Hartree problem of double-counting the interaction energy. [And, if ϵ actually were $\int E_{\boldsymbol{p}} f_0$, we would not get (A4) because there would be an additional term where the ∂_β hit the $E_{\boldsymbol{p}}$.] As discussed in Refs. [25,26], one simple solution to this problem is to start with the entropy density S rather than P or ϵ. Up to higher-order corrections which we shall review in a moment, the entropy density of a quasiparticle gas *is* given by the naive ideal gas formula,

$$S = S_{\text{ideal}} = \beta(P_{\text{ideal}} + \epsilon_{\text{ideal}}) = \beta \int \left(\frac{1}{3}\boldsymbol{p}\cdot\boldsymbol{v}_{\boldsymbol{p}} + E_{\boldsymbol{p}}\right) f_0$$
$$= -\beta \int \frac{1}{3} E_{\boldsymbol{p}} \boldsymbol{p}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}} f_0, \tag{A6}$$

where the last step follows by integrating the term involving $\boldsymbol{v}_{\boldsymbol{p}} = \boldsymbol{\nabla}_{\boldsymbol{p}} E_{\boldsymbol{p}}$ by parts. Starting from this formula for the entropy, we can then use the thermodynamic relation $S = \partial_T P$ to write $\partial_\beta P = -T^2 S$ and obtain (A3).

To get the formula for $\partial_\beta \epsilon$, it is convenient to first use $\boldsymbol{v}_{\boldsymbol{p}} f_0 = \mp T \boldsymbol{\nabla}_{\boldsymbol{p}} \ln(1 \pm f_0)$ and integrate by parts to rewrite (A6) as

$$S = \int [\pm \ln(1 \pm f_0) + \beta E_{\boldsymbol{p}} f_0]. \tag{A7}$$

Then use the the thermodynamic relations $\epsilon = TS - P$ and $\partial_\beta P = -T^2 S$ to write

$$\partial_\beta \epsilon = \partial_\beta (TS) - \partial_\beta P = T \partial_\beta S. \tag{A8}$$

Use of (A7) for S then produces the desired formula (A4).

It remains only to discuss the approximations that have been used in this analysis. In evaluating the entropy, the treatment of the system as an ideal gas of on-shell propagating quasiparticles breaks down at order g^3 and above. (See, for instance, the analysis in Ref. [26].) But it is adequate to obtain the $O(g^2)$ and the $O(m_0^2)$ terms in the entropy. For massless QCD, that might sound inadequate, because the breaking of conformal invariance is an $O(g^4)$ effect. For example, the effective energy of a hard quark is given by

$$\begin{aligned} E_p^2 &\simeq p^2 + \tfrac{1}{3} g^2(T) T^2 \\ &= p^2 + \tfrac{1}{3} g^2(\mu) T^2 + \tfrac{1}{3} \beta_0 g^4(\mu) T^2 \ln(T/\mu) + \cdots, \end{aligned} \tag{A9}$$

and it is the last term which breaks conformal invariance. However, this $O(g^4)$ conformal-breaking log is determined by knowledge of the $O(g^2)$ contribution; any $O(g^3)$ or additional $O(g^4)$ contributions to thermodynamic quantities will be conformal, up to corrections of $O(g^5)$, and so will not contribute to the leading-order bulk viscosity.

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