PHYSICAL REVIEW D **74,** 085016 (2006)

Stability of the normal vacuum in multi-Higgs-doublet models

A. Barroso,[1] P. M. Ferreira,[1] R. Santos,[1] and João P. Silva[2,3]

[1]*Centro de Física Teórica e Computacional, Faculdade de Ciências, Universidade de Lisboa, Avenida Professor Gama Pinto 2, 1649-003 Lisboa, Portugal*

[2]*Instituto Superior de Engenharia de Lisboa, Rua Conselheiro Emídio Navarro, 1900 Lisboa, Portugal*

[3]*Centro de Física Teórica de Partículas, Instituto Superior Técnico, P-1049-001 Lisboa, Portugal*

(Received 28 August 2006; published 16 October 2006)

We show that the vacuum structure of a generic multi-Higgs-doublet model shares several important features with the vacuum structure of the two and three Higgs-doublet model. In particular, one can still define the usual charge breaking, spontaneous *CP* breaking, and normal (charge and *CP* preserving) stationary points. We analyze the possibility of charge or spontaneous *CP* breaking by studying the relative depth of the potential in each of the possible stationary points.

DOI: 10.1103/PhysRevD.74.085016 PACS numbers: 11.30.Er, 11.30.Ly, 12.60.Fr, 14.80.Cp

I. INTRODUCTION

Most features of the standard model (SM) of electroweak interactions have been probed to a very high precision. Still, the Higgs sector remains largely untested and new physics is certainly possible. In particular, one might have more than one Higgs, as required, for example, by supersymmetry. Multi-Higgs models are also appealing for a variety of theoretical reasons related to *CP* violation: (i) if the Higgs potential conserves *CP*, this symmetry could be spontaneously broken by the vacuum; (ii) if there are three or more Higgs doublets, then there might be *CP* violation in the mixing matrix of the charged Higgs; and (iii) the new sources of *CP* violation have the potential to explain baryogenesis.

A drawback of these models is that they involve a large number of parameters. For example, the scalar potential of the most general two-Higgs-doublet model (2HDM) involves 14 real parameters, while the potential which explicitly preserves *CP* involves 10 parameters (these may be reduced to 11 and 9, respectively, through suitable basis choices). Given the large number of parameters present in these models, a variety of methods have been developed in order to restrict the parameter space, some related to the vacuum structure of the scalar potential. Recently, some interesting features of the vacuum structure have been obtained for the particular case of the 2HDM [1]. Namely, it was shown that, whenever a normal-charge and *CP* conserving minimum exists in the 2HDM, the global minimum of that potential is the normal one. Moreover, it was shown that the depth of the potential at a stationary point that breaks charge or *CP*, relative to the normal minimum, is related with the squared mass matrix of the charged or pseudoscalar Higgs (evaluated at the normal minimum), respectively. Recent work on these subjects may be found in [2]. In this work we will analyze how these conclusions may, or may not, be generalized to the case of a potential with *N* Higgs doublets.

The paper is organized as follows. In Sec. II we introduce our notation and prove one of our main results: that the vacuum structure of a generic multi-Higgs-doublet model may be reduced to vacua involving only two or three doublets. This is accomplished through a series of basis transformations, for which the potential is invariant, even if its parameters are not. We discover that the study of possible charge-breaking (CB) vacua is more easily done in a basis where only three out of the *N* doublets have nonvanishing vacuum expectation values (vevs). For *CP* violation, the appropriate basis is even simpler: only two doublets are nonzero. In Sec. III we compute the values of the potential at the charge-breaking and normal vacua and compare their values. We study whether it is possible to obtain charge-breaking minima deeper than a normal minimum. In Sec. IV we repeat this procedure, but now for *CP* breaking vacua. We present our conclusions in Sec. V. Appendix A provides a basis and gauge independent definition of the CB vacuum, while Appendix B contains a specific example of a three Higgs-doublet model for which the CB vacuum lies below the normal vacuum.

II. THE SCALAR SECTOR OF A GENERIC *N*-HIGGS-DOUBLET MODEL

A. The scalar potential

In this article we follow closely the notation of Refs. [3,4], where more details may be found—see also [5–9]. Let us consider a SU(2) ⊗ U(1) gauge theory with *N* Higgs doublets with the same hypercharge $y = 1/2$, denoted by

$$\Phi_i = \begin{pmatrix} \phi_i^u \\ \phi_i^d \end{pmatrix} = v_i + \varphi_i = \begin{pmatrix} v_i^u \\ v_i^d \end{pmatrix} + \begin{pmatrix} \varphi_i^u \\ \varphi_i^d \end{pmatrix}, \tag{1}$$

where v_i are their vacuum expectation values (vevs), and i runs from 1 to N. In all that follows, we will use the standard definition for the electric charge: $Q = T_3 + Y$, meaning that all vevs in the lower components of the doublets are electrically neutral. With this definition, a vacuum with all upper components of the vevs equal to zero, $v_i^u = 0$, does not break the charge symmetry.

The scalar potential may be written as

$$V_H = \mu_{ij}(\Phi_i^\dagger \Phi_j) + \lambda_{ij,kl}(\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l), \tag{2}$$

where Hermiticity implies

$$\mu_{ij} = \mu_{ji}^*, \qquad \lambda_{ij,kl} \equiv \lambda_{kl,ij} = \lambda_{ji,lk}^*. \tag{3}$$

Under a unitary basis transformation of the Higgs fields, their kinetic terms remain the same but the coefficients μ_{ij} and $\lambda_{ij,kl}$ are transformed in such a way that the potential remains invariant. Using Eq. (1), the scalar potential becomes

$$\begin{aligned} V_H =\ & \mu_{ij}(v_i^\dagger v_j) + \lambda_{ij,kl}(v_i^\dagger v_j)(v_k^\dagger v_l) + v_i^\dagger[\mu_{ij} + 2\lambda_{ij,kl}v_k^\dagger v_l]\varphi_j + \varphi_i^\dagger[\mu_{ij} + 2\lambda_{ij,kl}v_k^\dagger v_l]v_j + \mu_{ij}(\varphi_i^\dagger \varphi_j) \\ & + 2\lambda_{ij,kl}(\varphi_i^\dagger \varphi_j)(v_k^\dagger v_l) + 2\lambda_{il,kj}(\varphi_i^\dagger v_l)(v_k^\dagger \varphi_j) + \lambda_{ij,kl}(\varphi_i^\dagger v_j)(\varphi_k^\dagger v_l) + \lambda_{ij,kl}(v_i^\dagger \varphi_j)(v_k^\dagger \varphi_l) + 2\lambda_{ij,kl}(\varphi_i^\dagger \varphi_j)(\varphi_k^\dagger v_l) \\ & + 2\lambda_{ij,kl}(\varphi_i^\dagger \varphi_j)(v_k^\dagger \varphi_l) + \lambda_{ij,kl}(\varphi_i^\dagger \varphi_j)(\varphi_k^\dagger \varphi_l). \end{aligned} \tag{4}$$

Requiring in Eq. (4) that the linear terms in φ_i vanish gives us the stationarity conditions

$$[\mu_{ij} + 2\lambda_{ij,kl}v_k^\dagger v_l]v_j = 0 \qquad (\text{for } i = 1, \ldots, N). \tag{5}$$

Multiplying by $v_i^\dagger$ leads to

$$\mu_{ij}(v_i^\dagger v_j) = -2\lambda_{ij,kl}(v_i^\dagger v_j)(v_k^\dagger v_l). \tag{6}$$

The value of the potential at a stationary point is found from Eq. (4) by setting all $\varphi_i = 0$. Using Eq. (6), this may be written in the following three forms:

$$V_H^{\text{stationary point}} = \mu_{ij}(v_i^\dagger v_j) + \lambda_{ij,kl}(v_i^\dagger v_j)(v_k^\dagger v_l) \tag{7}$$

$$= \tfrac{1}{2}\mu_{ij}(v_i^\dagger v_j) \tag{8}$$

$$= -\lambda_{ij,kl}(v_i^\dagger v_j)(v_k^\dagger v_l). \tag{9}$$

B. A simple basis to study charge breaking in the *N*-Higgs-doublet model (NHDM)

After spontaneous symmetry breaking the Higgs fields acquire the vevs

$$\begin{pmatrix} v_1^u \\ v_1^d \end{pmatrix}, \quad \begin{pmatrix} v_2^u \\ v_2^d \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} v_{N-1}^u \\ v_{N-1}^d \end{pmatrix}, \quad \begin{pmatrix} v_N^u \\ v_N^d \end{pmatrix}. \tag{10}$$

An analysis of the potential with such a complicated vev structure would be too difficult to perform. We will now show how, using the freedom to choose a basis for the Higgs doublets, one manages to simplify immensely this study. We start by performing a unitary transformation on the last two Higgs fields according to

$$\begin{pmatrix} \Phi_{N-1}' \\ \Phi_N' \end{pmatrix} = \frac{1}{\sqrt{|v_{N-1}^u|^2 + |v_N^u|^2}} \begin{pmatrix} v_{N-1}^{u*} & v_N^{u*} \\ -v_N^u & v_{N-1}^u \end{pmatrix} \begin{pmatrix} \Phi_{N-1} \\ \Phi_N \end{pmatrix}. \tag{11}$$

With this transformation, the vevs of the last two fields become

$$\begin{aligned} \langle \Phi_{N-1}' \rangle &= \frac{1}{\sqrt{|v_{N-1}^u|^2 + |v_N^u|^2}} \begin{pmatrix} |v_{N-1}^u|^2 + |v_N^u|^2 \\ v_{N-1}^u v_{N-1}^d + v_N^u v_N^d \end{pmatrix}, \\ \langle \Phi_N' \rangle &= \frac{1}{\sqrt{|v_{N-1}^u|^2 + |v_N^u|^2}} \begin{pmatrix} 0 \\ -v_N^u v_{N-1}^d + v_{N-1}^u v_N^d \end{pmatrix}, \end{aligned} \tag{12}$$

respectively. We have thus succeeded in removing the upper component of the vev of the last Higgs field. Moreover, the upper component of $\langle \Phi_{N-1}' \rangle$ became real and positive. We can continue with similar transformations, applied to successive pairs of Higgs fields, until the corresponding vevs become

$$\begin{pmatrix} v_1^u \\ v_1^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ v_2^d \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} 0 \\ v_{N-1}^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ v_N^d \end{pmatrix}. \tag{13}$$

Notice that v_1^u and v_i^d in Eq. (13) are *not* the same as in Eq. (10), but rather the values obtained after the successive transformations of the type shown in Eq. (11). Similarly, we keep the notation for the fields, although they have been transformed through a series of basis changes. At the end of the process outlined, v_1^u is real and positive.

We may now repeat the exercise with the lower components. Indeed, through the transformation

$$\begin{pmatrix} \Phi_{N-1}' \\ \Phi_N' \end{pmatrix} = \frac{1}{\sqrt{|v_{N-1}^d|^2 + |v_N^d|^2}} \begin{pmatrix} v_{N-1}^d & v_N^d \\ -v_N^d & v_{N-1}^d \end{pmatrix} \begin{pmatrix} \Phi_{N-1} \\ \Phi_N \end{pmatrix}, \tag{14}$$

we can change the vevs of the last two Higgs fields in Eq. (13) into

$$\langle \Phi_{N-1}' \rangle = \begin{pmatrix} 0 \\ \sqrt{|v_{N-1}^d|^2 + |v_N^d|^2} \end{pmatrix}, \qquad \langle \Phi_N' \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{15}$$

respectively. This eliminates the lower component of $\langle \Phi_N' \rangle$ and makes the lower component of $\langle \Phi_{N-1}' \rangle$ real and positive. We may continue with the other down components, until we reach the following vev structure:

$$\begin{pmatrix} |v_1^u| \\ v_1^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ |v_2^d| \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16}$$

Notice that $|v_2^d|$, which is real and positive due to Eq. (15), cannot be removed without implying the appearance of an upper component on the second vev. We have thus reached a simple but remarkable result. Indeed, although there are many parameters involved in the general N-Higgs-doublet model, its vacuum [10] structure can be brought into a much simpler form, through a suitable basis choice.

If, after all these basis transformations, we are left with a vev structure for which $v_1^u \neq 0$, then the vacuum breaks electric charge. As in the 2HDM, we may now utilize the gauge freedom in order to bring the vevs into the final form: [11]

$$\begin{pmatrix} \alpha \\ v_{c1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ v_{c2}e^{i\delta_c} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{17}$$

where δ_c is a phase, while α, v_{c1}, and v_{c2} are positive real numbers. This, then, is the simplest form one can find for a CB vacuum.

However, we are interested in comparing the value of the potential at the CB vacuum with its value at the normal vacuum. Therefore, we must find out the form of the most general normal vacuum, in the basis in which Eq. (17) is written. Clearly, given our definition of electric charge, it will have all $v_i^u = 0$. A generic charge-preserving vacuum, then, will have the form

$$\begin{pmatrix} 0 \\ v_1^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ v_2^d \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} 0 \\ v_{N-1}^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ v_N^d \end{pmatrix}. \tag{18}$$

We emphasize that these v_i^d are *not* the same as those appearing in Eq. (10). However, we can now apply the same method we used previously to bring the normal vacuum to a more manageable form. Through a transformation analogous to that of Eq. (14), we can set the last doublet to zero. Notice that this basis change does not involve the first two doublets, so the charge-breaking vacuum structure, Eq. (17), remains unaffected. Successive basis transformations may be applied that do not change Eq. (17) but set to zero the lower component vevs of Eq. (18), until one is left with the final normal vacuum structure,

$$\begin{pmatrix} 0 \\ v_1^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ v_2^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ |v_3^d| \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{19}$$

If we try to perform another basis change to set $|v_3^d|$ to zero, we will destroy the simple form for the charge-breaking vevs, Eq. (17). We denote by the "B-basis" the basis where the charge-breaking vevs have the simple form of Eq. (17) and the normal vacuum vevs, the form given by Eq. (19). The B-basis is appropriate to study the possibility of charge-breaking vacua. This result shows that the study of charge breaking for an N-doublet potential is reduced to the analysis of the three-doublet situation.

C. A simple basis to study *CP* breaking in the NHDM with explicit *CP* conservation

We are also interested in the possibility of CP being spontaneously broken in N-Higgs-doublets models. In this case we cannot simply choose the most general NHDM potential—we must make sure that that potential does not break CP explicitly. In Appendix A of [9], Gunion and Haber invoke CPT and $\mathcal{T}^2 = 1$ (where $\mathcal{T}$ is the time-reversal operator) to show that: "The Higgs potential is explicitly CP-conserving if and only if a basis exists in which all Higgs potential parameters are real." We therefore consider one such basis for our NHDM potential: all of its parameters are real and it explicitly preserves CP.

Given our definition of electric charge, the most general charge-preserving vacuum (CP violating or not) will be of the same form as Eq. (18). Our starting point, however, is not the basis in which we wrote Eq. (18), but rather a generic basis for which the parameters of the potential are all real. It is now convenient to ensure that all basis changes do not introduce complex parameters in the potential. This means that we are restricted to orthogonal basis transformations. Even with this restriction we are still able to simplify immensely the study of the NHDM potential. This is accomplished through two series of steps:

(1) We start with an orthogonal transformation on the last two Higgs fields according to

$$\begin{pmatrix} \Phi'_{N-1} \\ \Phi'_N \end{pmatrix} = \frac{1}{\sqrt{\mathrm{Im}^2(v_{N-1}^d) + \mathrm{Im}^2(v_N^d)}} \times \begin{pmatrix} -\mathrm{Im}(v_{N-1}^d) & -\mathrm{Im}(v_N^d) \\ -\mathrm{Im}(v_N^d) & \mathrm{Im}(v_{N-1}^d) \end{pmatrix} \times \begin{pmatrix} \Phi_{N-1} \\ \Phi_N \end{pmatrix}. \tag{20}$$

This eliminates the imaginary part of the vev of the last Higgs field. We can continue with similar transformations, applied to successive pairs of Higgs fields, until the corresponding vevs reach a structure of the type

$$\begin{pmatrix} 0 \\ v_1^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \mathrm{Re}(v_2^d) \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} 0 \\ \mathrm{Re}(v_{N-1}^d) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \mathrm{Re}(v_N^d) \end{pmatrix}. \tag{21}$$

Notice that the v_i^d in Eq. (21) are *not* the initial ones, but rather the values obtained after the successive transformations of the type shown in Eq. (20). Similarly, we keep the notation for the fields, although they have been transformed through a series of basis changes. After these steps $\mathrm{Im}(\nu_1^d) < 0$.

(2) We continue with the orthogonal transformation on the last two Higgs fields

$$\begin{pmatrix} \Phi'_{N-1} \\ \Phi'_N \end{pmatrix} = \frac{1}{\sqrt{\mathrm{Re}^2(\nu_{N-1}^d) + \mathrm{Re}^2(\nu_N^d)}} \times \begin{pmatrix} \mathrm{Re}(\nu_{N-1}^d) & \mathrm{Re}(\nu_N^d) \\ -\mathrm{Re}(\nu_N^d) & \mathrm{Re}(\nu_{N-1}^d) \end{pmatrix} \begin{pmatrix} \Phi_{N-1} \\ \Phi_N \end{pmatrix}. \tag{22}$$

This eliminates the vev $\langle \Phi'_N \rangle$, simultaneously making the lower component of $\langle \Phi'_{N-1} \rangle$ real and positive. We can continue with similar transformations, applied to successive pairs of Higgs fields, until the corresponding vevs reach a structure of the type

$$\begin{pmatrix} 0 \\ \nu_1^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \mathrm{Re}(\nu_2^d) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{23}$$

where $\mathrm{Re}(\nu_2^d) \geq 0$.

Hence, when the scalar potential conserves *CP*, it is possible to choose a basis in which only the first two doublets have vevs, while keeping all parameters in the potential real. At this point, we distinguish two physically distinct scenarios. If $\mathrm{Im}(\nu_1^d) = 0$, then the vacuum is a normal one and preserves *CP*; if not, it spontaneously breaks that symmetry—we call it a *CP* violating (CPV) vacuum.

Let us then suppose we started with a normal vacuum, and that we employed the basis transformations described above until the vacuum structure was reduced to Eq. (23), with real vevs $\nu_1^d = v_1$ and $\nu_2^d = v_2$. Because the remaining vevs are real, we can perform a final basis transformation on the first two fields

$$\begin{pmatrix} \Phi'_1 \\ \Phi'_2 \end{pmatrix} = \frac{1}{\sqrt{v_1^2 + v_2^2}} \begin{pmatrix} v_1 & v_2 \\ -v_2 & v_1 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{24}$$

bringing their vevs into the form

$$n_1 = \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad n_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{25}$$

where $v = \sqrt{v_1^2 + v_2^2}$, and all remaining doublets are zero. This is known as the "Higgs basis" for the normal vacuum in the 2HDM [3,4].

However, we are interested in comparing the value of the potential at the normal vacuum with its value at a CPV vacuum. Therefore, we must find out what is the form of the most general CPV vacuum, in the basis in which Eq. (25) is written and with the definition of electric charge we have adopted. It will be of the form of Eq. (18), with *new* vevs $\tilde{\nu}_i^d$,

$$\begin{pmatrix} 0 \\ \tilde{\nu}_1^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \tilde{\nu}_2^d \end{pmatrix}, \quad \cdots \quad \begin{pmatrix} 0 \\ \tilde{\nu}_{N-1}^d \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \tilde{\nu}_N^d \end{pmatrix}. \tag{26}$$

Because the normal vacuum has been reduced to the form of Eq. (25), where only the first doublet is different from zero, we can apply the steps 1 and 2 detailed above for this new vacuum and again reduce the CPV vacuum to the form

$$s_1 = \begin{pmatrix} 0 \\ z_1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \tag{27}$$

with all remaining doublets zero. In this equation, z_1 and z_2 are complex numbers. The sequence of steps that led us from Eq. (26) and (27) did *not* change the normal vacuum of Eq. (25), because none of those steps involved the first doublet. We cannot further remove the imaginary or real parts of z_2 because that operation would involve the first doublet and, thus, take us away from a basis in which the form of the normal vacuum of Eq. (25) remains valid. The basis for which the normal and the CPV stationary points have the simple form given by Eqs. (25) and (27) will be called the "*S*-basis."

The *S*-basis is very useful because, when using it for the normal minimum, the Goldstone bosons are isolated as the components of φ_1, while the other φ_i $(i = 2, \ldots, N)$ contain other charged and neutral scalars fields [3–5]. Indeed, in the *S*-basis

$$\varphi_1 = \begin{pmatrix} G^+ \\ (H^0 + iG^0)/\sqrt{2} \end{pmatrix}, \tag{28}$$

$$\varphi_i = \begin{pmatrix} H_i^+ \\ (R_i + iI_i)/\sqrt{2} \end{pmatrix}, \qquad (\text{for } i = 2, \ldots, N), \tag{29}$$

where G^+ and G^0 are the Goldstone bosons (which, in the unitary gauge, become the longitudinal components of the W^+ and of the Z^0); H^0 couples to fermions proportionally to their masses (in the fermion mass basis); and H_i^+, R_i, and I_i $(i = 2, \ldots, N)$ are the charged and neutral scalars fields. Notice that these are not the physical particles; those will be obtained by diagonalizing the squared mass matrix of the charged Higgs, and the squared mass matrix of the neutral Higgs (including H^0, R_i, and I_i). These important properties will become obvious below.

D. The mass terms at the normal vacua

We now wish to study the quadratic terms in Eq. (4), when the vevs are taken to coincide with those at a normal vacuum. Since the basis transformations do not mix the upper and lower components, the normal vacua have $\nu_i^u = 0$ for all i, in any basis (as long as no gauge transformations are made). As a result, the quadratic terms of Eq. (4) evaluated at a normal stationary point may be written, in any basis, as

$$(M^2)^{\rm N}_{ij} = (M^2_\pm)^{\rm N}_{ij}\varphi_i^{u*}\varphi_j^u + (M^2_R)^{\rm N}_{ij}\,{\rm Re}(\varphi_i^d)\,{\rm Re}(\varphi_j^d) + (M^2_I)^{\rm N}_{ij}\,{\rm Im}(\varphi_i^d)\,{\rm Im}(\varphi_j^d) + (M^2_{RI})^{\rm N}_{ij}\,{\rm Re}(\varphi_i^d)\,{\rm Im}(\varphi_j^d) + (M^2_{IR})^{\rm N}_{ij}\,{\rm Im}(\varphi_i^d)\,{\rm Re}(\varphi_j^d), \quad (30)$$

where we identify

$$(M^2_\pm)^{\rm N}_{ij} = \mu_{ij} + 2\lambda_{ij,kl}v_k^{d*}v_l^d, \quad (31)$$

$$(M^2_R)^{\rm N}_{ij} = {\rm Re}[\mu_{ij} + 2\lambda_{ij,kl}v_k^{d*}v_l^d + 2\lambda_{ik,lj}v_k^d v_l^{d*} + 2\lambda_{ik,jl}v_k^d v_l^d], \quad (32)$$

$$(M^2_I)^{\rm N}_{ij} = {\rm Re}[\mu_{ij} + 2\lambda_{ij,kl}v_k^{d*}v_l^d + 2\lambda_{ik,lj}v_k^d v_l^{d*} - 2\lambda_{ik,jl}v_k^d v_l^d], \quad (33)$$

$$(M^2_{RI})^{\rm N}_{ij} = -{\rm Im}[\mu_{ij} + 2\lambda_{ij,kl}v_k^{d*}v_l^d + 2\lambda_{ik,lj}v_k^d v_l^{d*} - 2\lambda_{ik,jl}v_k^d v_l^d], \quad (34)$$

$$(M^2_{IR})^{\rm N}_{ij} = {\rm Im}[\mu_{ij} + 2\lambda_{ij,kl}v_k^{d*}v_l^d + 2\lambda_{ik,lj}v_k^d v_l^{d*} + 2\lambda_{ik,jl}v_k^d v_l^d], \quad (35)$$

and the superscript N indicates that these mass matrices have been evaluated at the normal vacuum.

Using Eqs. (3), one can show that the matrix $(M^2_\pm)^{\rm N}$ is Hermitian, while the real matrices $(M^2_R)^{\rm N}$ and $(M^2_I)^{\rm N}$ are symmetric. The remaining two matrices are real and related by $(M^2_{RI})^{\rm N}_{ji} = (M^2_{IR})^{\rm N}_{ij}$. This implies that the $2N \times 2N$ matrix,

$$\begin{pmatrix} (M^2_R)^{\rm N} & (M^2_{RI})^{\rm N} \\ (M^2_{IR})^{\rm N} & (M^2_I)^{\rm N} \end{pmatrix}, \quad (36)$$

is symmetric. Moreover, the matrix $(M^2_\pm)^{\rm N}$ behaves like a second rank tensor under a basis transformation of the Higgs fields, but the other matrices do not.

As we mentioned, these expressions are valid for normal vacua in any basis. For the B-basis, where only $\{v_1^d, v_2^d, v_3^d\}$ are different from zero, the indices $\{k, l\}$ in the Eqs. (31)–(35) run only from 1 to 3. In the S-basis, where only the first doublet has a nonzero vev, the mass matrices are simplified considerably. In what follows the vevs, the parameters μ_{ij}, and $\lambda_{ij,kl}$ are all written in the S-basis; it is important to understand that changing the basis would change the vevs, but also the parameters μ_{ij} and $\lambda_{ij,kl}$ [3]. We find

$$(M^2_\pm)^{\rm N}_{ij} = \mu_{ij} + 2v^2\lambda_{ij,11}, \quad (37)$$

$$(M^2_R)^{\rm N}_{ij} = {\rm Re}[\mu_{ij} + 2v^2(\lambda_{ij,11} + \lambda_{i1,1j} + \lambda_{i1,j1})], \quad (38)$$

$$(M^2_I)^{\rm N}_{ij} = {\rm Re}[\mu_{ij} + 2v^2(\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1})], \quad (39)$$

$$(M^2_{RI})^{\rm N}_{ij} = -{\rm Im}[\mu_{ij} + 2v^2(\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1})], \quad (40)$$

$$(M^2_{IR})^{\rm N}_{ij} = {\rm Im}[\mu_{ij} + 2v^2(\lambda_{ij,11} + \lambda_{i1,1j} + \lambda_{i1,j1})]. \quad (41)$$

Using the parametrization of the normal stationary point in the S-basis, shown in Eq. (25), on the stationarity conditions of Eq. (5), we find

$$\mu_{i1} + 2v^2\lambda_{i1,11} = 0. \quad (42)$$

But this coincides with the definition of $(M^2_\pm)^{\rm N}_{i1}$, in Eq. (37) and, since this is a Hermitian matrix, we conclude that the first row and the first column of $(M^2_\pm)^{\rm N}$ have zero in every entry,

$$(M^2_\pm)^{\rm N}_{i1} = 0 = (M^2_\pm)^{\rm N}_{1i} \qquad (\text{for } i = 1, \ldots, N). \quad (43)$$

This shows that, indeed, φ_1^u in the S-basis coincides with the charged Goldstone boson, in accordance with Eq. (28).

Also, using Eq. (39) with $j = 1$ and Eq. (42),

$$(M^2_I)^{\rm N}_{i1} = {\rm Re}(\mu_{i1} + 2v^2\lambda_{i1,11}) + 2v^2\,{\rm Re}(\lambda_{i1,11} - \lambda_{i1,11}) = 0. \quad (44)$$

Since $(M^2_I)^{\rm N}$ is symmetric, we find

$$(M^2_I)^{\rm N}_{i1} = 0 = (M^2_I)^{\rm N}_{1i} \qquad (\text{for } i = 1, \ldots, N). \quad (45)$$

To simplify, let us now consider for a moment the case in which all μ_{ij} and all $\lambda_{ij,kl}$ are real. As we explained above, a CP-conserving NHDM potential falls under this category. In that case, $(M^2_{RI})^{\rm N} = 0 = (M^2_{IR})^{\rm N}$, and the matrix in Eq. (36) becomes block diagonal. In addition, $(M^2_R)^{\rm N}$ and $(M^2_I)^{\rm N}$ are the squared mass matrices of the scalars and pseudoscalars, respectively. Thus, Eq. (45) shows that in the S-basis ${\rm Im}(\varphi_1^d)$ coincides with the neutral, pseudoscalar Goldstone boson, in accordance with Eq. (28).

III. THE CHARGE BREAKING VERSUS THE NORMAL STATIONARY POINTS

Throughout this section we will work in the B-basis, although it will be obvious that our final results hold in any basis. We assume that both the N and CB stationary points exist. We will now compute the values of the potential at each of those stationary points. To that effect, we first recall the results of Sec. II, where we showed that, in the B-basis, the vev structure of both stationary points is given by

$$c_1 = \begin{pmatrix} \alpha \\ v_{c1} \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ v_{c2} e^{i\delta_c} \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{Charge breaking vacuum (CB)},$$
$$n_1 = \begin{pmatrix} 0 \\ \nu_1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 0 \\ \nu_2 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0 \\ v_3 \end{pmatrix}, \quad \text{Normal vacuum (N)}, \tag{46}$$

with all remaining doublets having vevs equal to zero. The parameter δ_c is a phase, while α, v_{ci}, and v_3 are real, positive numbers. In general, ν_1 and ν_2 are complex. From Eq. (5), we obtain the stationarity conditions for the CB stationary point in the B-basis,

$$(\mu_{i1} + 2\lambda_{i1,kl} c_k^\dagger c_l)\alpha = 0 \tag{47}$$

$$(\mu_{i1} + 2\lambda_{i1,kl} c_k^\dagger c_l) v_{c1} + (\mu_{i2} + 2\lambda_{i2,kl} c_k^\dagger c_l) v_{c2} e^{i\delta_c} = 0, \tag{48}$$

for $i = 1, \ldots, N$ and $k, l = 1, 2$. Since we assume that the CB stationary point exists, $\alpha \neq 0$ and its coefficient in Eq. (47) must equal zero. From Eq. (48), then, the coefficient of $v_{c2} e^{i\delta_c}$ is also zero. As a result, the stationarity conditions at the CB stationary point may be written as

$$\mu_{ij_1} + 2\lambda_{ij_1,kl} c_k^\dagger c_l = 0 \qquad (i = 1 \ldots, N;\ j_1 = 1, 2). \tag{49}$$

Let us now contract the indices $\{i, j_1\}$ with $n_i^\dagger n_{j_1}$. This gives

$$(\mu_{ij_1} + 2\lambda_{ij_1,kl} c_k^\dagger c_l) n_i^\dagger n_{j_1} = 0 \qquad (i = 1, 2, 3;\ j_1 = 1, 2), \tag{50}$$

and from here it is trivial to obtain

$$\mu_{ij} n_i^\dagger n_j + 2\lambda_{ij,kl} n_i^\dagger n_j c_k^\dagger c_l - (\mu_{i3} + 2\lambda_{i3,kl} c_k^\dagger c_l) n_i^\dagger n_3 = 0 \qquad (i, j = 1, \ldots, 3). \tag{51}$$

Notice the appearance of the term $\mu_{ij} n_i^\dagger n_j$ which, according to Eq. (8), equals twice the value of the potential at the normal stationary point, V_H^N.

Now, from Eq. (31), the mass matrix for the charged scalars at the N vacuum in the B-basis is given by

$$(M_\pm^2)^{\rm N}_{kl} = \mu_{kl} + 2\lambda_{kl,ij} n_i^\dagger n_j. \tag{52}$$

Contracting the indices $\{k, l\}$ with $c_k^\dagger c_l$ we obtain

$$\begin{aligned}(M_\pm^2)^{\rm N}_{kl} c_k^\dagger c_l &= \mu_{kl} c_k^\dagger c_l + 2\lambda_{kl,ij} n_i^\dagger n_j c_k^\dagger c_l \\ &= 2V_H^{\rm CB} + 2\lambda_{ij,kl} n_i^\dagger n_j c_k^\dagger c_l,\end{aligned} \tag{53}$$

where we have used Eq. (8) to identify $\mu_{kl} c_k^\dagger c_l$ as twice the value of the potential at the CB stationary point and the symmetries of the λ coefficients from Eq. (3). Comparing Eqs. (51) and (53), we can subtract them to find

$$V_H^{\rm CB} - V_H^N = \tfrac{1}{2}(M_\pm^2)^{\rm N}_{ij} c_i^\dagger c_j - \tfrac{1}{2} v_3 (\mu_{i3} + 2\lambda_{i3,kl} c_k^\dagger c_l) n_i. \tag{54}$$

This is our main result regarding the possibility of charge breaking in the NHDM. Although obtained in the B-basis, it is very simple to rewrite Eq. (54) in a basis invariant form. At this point it is important to recall the results obtained in Ref. [1] for CB in the case of the 2HDM. In the notation of this paper, the conclusions therein reached are written as

$$V_H^{\rm CB} - V_H^N = \tfrac{1}{2}(M_\pm^2)^{\rm N}_{ij} c_i^\dagger c_j. \tag{55}$$

When the normal stationary point is a minimum, the matrix $(M_\pm^2)^{\rm N}$ has, besides the Goldstone bosons, only positive eigenvalues, and it is very easy to prove [1] that one obtains $V_H^{\rm CB} - V_H^{\rm N} > 0$. Hence, if a normal minimum exists, the CB stationary point is *always* above it. No possibility of tunneling from the normal minimum to a deeper CB stationary point exists in the 2HDM.

The similarity with the NHDM case is clear, but the difference of the potential depths now contains an extra term, proportional to v_3. Let us consider that the normal vacuum in the NHDM is indeed a minimum. Then, as before, the term $(M_\pm^2)^{\rm N}_{ij} c_i^\dagger c_j$ is strictly positive [12]. However, there is no *a priori* reason for the second term in the right-hand side of Eq. (54) to be positive. In fact, depending on the values of the parameters μ and λ, it may well be negative, so much so that it overwhelms the positive contributions from $(M_\pm^2)^{\rm N}_{ij} c_i^\dagger c_j$.

As an example of this possibility, we undertook a study of CB in the 3HDM for generic values of the parameters of the potential. For simplicity we considered the 3HDM potential without explicit *CP* violation. Our conclusions are as follows:

(1) As in the case of the 2HDM, it is certainly possible to find combinations of $\{\mu, \lambda\}$ for which there are normal minima with a CB stationary point located *above* them.

(2) However, unlike the 2HDM situation, we have found combinations of $\{\mu, \lambda\}$ for which both the normal and charge-breaking stationary points are minima, *but* verify $V_H^{\rm CB} < V_H^N$.

In Appendix B we give a set of numerical values of $\{\mu, \lambda\}$ corresponding to this situation. In fact we obtain, from such parameter values,

$$\begin{aligned} V_H^{\rm CB} &= -2.6678 \times 10^9\ {\rm GeV}^4 < V_H^N \\ &= -2.2792 \times 10^9\ {\rm GeV}^4. \end{aligned} \tag{56}$$

A numerical minimization of the potential found no value below $V_H^{\rm CB}$.

To ensure that both CB and N are minima, we calculated the scalar squared mass matrices at both stationary points. Other than the expected zero eigenvalues (3 for the N minimum, 4 for the CB one) all the others are positive.

In conclusion, the study of charge-breaking vacua in the NHDM reduces itself to the study of a 3HDM potential. For this one—and unlike the 2HDM case—there is the possibility of CB minima which are *deeper* than a normal minimum.

IV. THE CPV VERSUS THE NORMAL STATIONARY POINTS

Let us consider a Higgs potential with explicit CP conservation and no CB stationary points. As we showed in Sec. II, it is possible, through a series of orthogonal transformations that preserve μ and λ as real (though changing its values), to reach what we called the S-basis, where a normal (CB and CP preserving) and CPV (CB conserving, CP violating) stationary points have vevs given by

$$\begin{aligned} s_1 &= \begin{pmatrix} 0 \\ z_1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \qquad CP\text{-violating vacuum (CPV)}, \\ n_1 &= \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad n_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{Normal vacuum (N)}, \end{aligned} \tag{57}$$

where v is real and at least one of $\{z_1, z_2\}$ is complex. Unlike the CB case, we are able to reduce the study of CP violation in the NHDM to the analysis of only two doublets. Throughout this section we will work in the S-basis. We now assume that both the N and CPV stationary points exist. Equation (42) shows the stationarity conditions of Eq. (5) applied to the normal stationary point and written in the S-basis. Similarly, using the parametrization of the CPV vevs in the S-basis, shown in Eq. (27), on the stationarity conditions of Eq. (5), we find

$$(\mu_{i1} + 2\lambda_{i1,kl} s_k^\dagger s_l) z_1 + (\mu_{i2} + 2\lambda_{i2,kl} s_k^\dagger s_l) z_2 = 0. \tag{58}$$

Specifying for $i = 1$ and rearranging the terms, we obtain

$$z_1 \mu_{11} + z_2 \mu_{12} = -2\lambda_{11,kl} s_k^\dagger s_l z_1 - 2\lambda_{12,kl} s_k^\dagger s_l z_2. \tag{59}$$

This can be viewed as a system of one complex (two real) equation in the two real unknowns μ_{11}, and μ_{12}. The solutions are easily obtained. One finds that

$$\begin{aligned} -\tfrac{1}{2}\mu_{11} &= \lambda_{11,11}|z_1|^2 + \lambda_{11,21}(z_1^* z_2 + z_2^* z_1) \\ &\quad + (\lambda_{11,22} - \lambda_{12,12} + \lambda_{12,21})|z_2|^2 \\ &= \lambda_{11,11}|z_1|^2 + \lambda_{12,11} z_1^* z_2 + \lambda_{21,11} z_2^* z_1 \\ &\quad + (\lambda_{22,11} + \lambda_{21,12} - \lambda_{21,21})|z_2|^2 \\ &= (\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1}) s_i^\dagger s_j. \end{aligned} \tag{60}$$

In addition, the stationarity condition at the normal minimum, Eq. (42), yields

$$\begin{aligned} -\tfrac{1}{2}\mu_{11} &= v^2 \lambda_{11,11} = \lambda_{ij,11} n_i^\dagger n_j \\ &= (\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1}) n_i^\dagger n_j. \end{aligned} \tag{61}$$

We conclude that

$$(\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1}) s_i^\dagger s_j = (\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1}) n_i^\dagger n_j. \tag{62}$$

We are now ready to calculate the difference between the value of the scalar potential at the CPV stationary point and the value of the scalar potential at the N stationary point. We start from the definition of the pseudoscalar mass matrix $(M_I^2)^{\rm N}$ in Eq. (39) and multiply it, respectively, by $s_i^\dagger s_j$ and $n_i^\dagger n_j$, to find

$$\begin{aligned} \tfrac{1}{2}(M_I^2)^{\rm N}_{ij} s_i^\dagger s_j &= \tfrac{1}{2}\mu_{ij} s_i^\dagger s_j + v^2(\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1}) s_i^\dagger s_j, \\ \tfrac{1}{2}(M_I^2)^{\rm N}_{ij} n_i^\dagger n_j &= \tfrac{1}{2}\mu_{ij} n_i^\dagger n_j + v^2(\lambda_{ij,11} + \lambda_{i1,1j} - \lambda_{i1,j1}) n_i^\dagger n_j. \end{aligned} \tag{63}$$

Subtracting both lines we find

$$V_H^{\rm CPV} - V_H^{\rm N} = \tfrac{1}{2}(M_I^2)^{\rm N}_{ij} s_i^\dagger s_j. \tag{64}$$

In obtaining this result we have used Eq. (62), and we noticed that, according to Eq. (8),

$$V_H^{\rm CPV} = \tfrac{1}{2}\mu_{ij} s_i^\dagger s_j, \qquad V_H^{\rm N} = \tfrac{1}{2}\mu_{ij} n_i^\dagger n_j. \tag{65}$$

Furthermore, we used the fact that Eqs. (25) and (45) imply that

$$(M_I^2)^{\rm N}_{ij} n_i^\dagger n_j = 0. \tag{66}$$

Equation (64) is the generalization of the results obtained in Ref. [1] for CP violation in the 2HDM.

It can be shown, using the general definition of $(M_I^2)^{\rm N}$ in Eq. (33), that Eq. (64) is invariant under orthogonal basis transformations, so that in Eq. (64) we can actually consider the indices $\{i, j\}$ going from 1 to N. For simplicity, we evaluate it in the S-basis. As before, when N is a minimum, we will have

$$V_H^{\rm CPV} - V_H^{\rm N} = \tfrac{1}{2}(M_I^2)^{\rm N}_{22}|z_2|^2 > 0. \tag{67}$$

The result is strictly positive because the only zero eigenvalue of the matrix $(M_I^2)^{\rm N}$ is in the first line/row; the remaining submatrix is definite positive. This implies that all of the elements of its diagonal—such as $(M_I^2)^{\rm N}_{22}$—are positive. This is another advantage of utilizing the Higgs basis.

Equation (67) generalizes the results obtained in Ref. [1] for the particular case of $N = 2$: whenever a normal minimum exists it is certainly deeper than any CPV stationary point.

Notice that one can obtain a CPV stationary point which is deeper than a normal stationary point N. This occurs for parameters such that $(M_I^2)^{\rm N}_{22} < 0$. However, in that case N is *not* a minimum (although it is a stationary point).

V. CONCLUSIONS

We have studied the vacuum structure of the most general N-Higgs-doublet model. We have shown that, in order to compare the depth of the potential at a normal minimum with its depth at a CB stationary point, a basis may be chosen such that the vacuum structure mimics that of the 3HDM. Similarly, in order to compare the depth of the potential at a normal minimum with its depth at a CPV stationary point, a basis may be chosen such that the vacuum structure mimics that of the 2HDM.

This great simplification allowed us to generalize the results of [1], showing that, whenever a normal minimum exists, it is certainly deeper than any CPV stationary point. However, we found one remarkable difference regarding CB: whereas in the 2HDM it is impossible to find CB minima below normal ones, that does not happen for the NHDM, with $N \geq 3$. This raises the possibility of finding charge-breaking bounds [13] for these potentials, which might improve their predictive power. Notice, however, that if the parameters of the potential are such that at the N minimum (in the B-basis) one has $v_3 = 0$, one recovers the 2HDM result for the NHDM potential: if such a normal minimum exists, it is certainly deeper than the CB one. This can be used as a sufficient condition to prevent CB from occurring in the NHDM.

It is interesting to note that Eq. (54) shows that the difference between the value of the potential at the CB stationary point and the value of the potential at the normal stationary point is related to the charged Higgs squared mass matrix. That relation is perfect for the 2HDM, but "spoiled" by the v_3 terms in Eq. (54) for the NHDM. Similarly, when the potential conserves CP, Eq. (64) shows that the difference between the value of the potential at the CPV stationary point and the value of the potential at the normal stationary point is related to the pseudoscalar squared mass matrix. Thus, the depth of a potential at a stationary point that breaks a given symmetry, relative to the normal minimum, is related to the squared mass matrix of the scalar particles directly linked with that symmetry.

ACKNOWLEDGMENTS

This work was supported by the Portuguese *Fundação para a Ciência e a Tecnologia* (FCT) under the Contracts No. PDCT/FP/FNU/50155/2003, No. POCI/FIS/59741/2004, and CFTP-Plurianual (777). In addition, P. M. F. is supported by FCT under Contract No. SFRH/BPD/5575/2001, and J. P. S. is supported in part by Project No. POCTI/37449/FNU/2001, approved by the Portuguese FCT and POCTI, and cofunded by FEDER.

APPENDIX A: CHARGE BREAKING—THE KINETIC TERMS

In this Appendix we present a basis and gauge independent definition of the charge-breaking vacua. This could be done by looking at the Goldstone bosons in the scalar mass matrix, but it is easier to look at the mass of the photon instead. The kinetic terms for the scalar fields are

$$\left|\left(i\partial_\mu - \frac{g}{2}\tau_a W_\mu^a - \frac{g'}{2}B_\mu\right)\Phi_i\right|^2, \tag{A1}$$

where W_μ^a $(a = 1, 2, 3)$ and B_μ are the SU$(2)_L$ and U$(1)_Y$ gauge bosons, respectively, and τ_a $(a = 1, 2, 3)$ are the Pauli matrices. After spontaneous symmetry breaking we obtain mass terms for the gauge fields given by

$$\frac{1}{4}\left|\begin{bmatrix} g'B_\mu + gW_\mu^3 & \sqrt{2}gW_\mu^p \\ \sqrt{2}gW_\mu^m & g'B_\mu - gW_\mu^3 \end{bmatrix}\begin{pmatrix} v_i^u \\ v_i^d \end{pmatrix}\right|^2, \tag{A2}$$

where

$$W_\mu^p = \frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}, \qquad W_\mu^m = \frac{W_\mu^1 + iW_\mu^2}{\sqrt{2}}. \tag{A3}$$

After some reorganization, the result is proportional to

$$\left(W_\mu^m, W_\mu^p, W_\mu^3, \frac{g'}{g}B_\mu\right)\mathcal{M}_i^{\rm GB}\begin{pmatrix} W^{p\mu} \\ W^{m\mu} \\ W^{3\mu} \\ \frac{g'}{g}B^\mu \end{pmatrix}, \tag{A4}$$

where

$$\mathcal{M}_i^{\rm GB} = \begin{bmatrix} |v_i^u|^2 + |v_i^d|^2 & 0 & 0 & \sqrt{2}v_i^{d*}v_i^u \\ 0 & |v_i^u|^2 + |v_i^d|^2 & 0 & \sqrt{2}v_i^{u*}v_i^d \\ 0 & 0 & |v_i^u|^2 + |v_i^d|^2 & |v_i^u|^2 - |v_i^d|^2 \\ \sqrt{2}v_i^{u*}v_i^d & \sqrt{2}v_i^{d*}v_i^u & |v_i^u|^2 - |v_i^d|^2 & |v_i^u|^2 + |v_i^d|^2 \end{bmatrix}, \tag{A5}$$

for each doublet Φ_i ($i = 1, \ldots, N$). Summing over all doublets and introducing the complex vectors

$$z_u = \{\nu_1^u, \nu_2^u, \cdots \nu_N^u\}, \qquad z_d = \{\nu_1^d, \nu_2^d, \cdots \nu_N^d\}, \tag{A6}$$

the mass matrix may be written as

$$\mathcal{M}^{\rm GB} = \sum_{i=1}^{N} \mathcal{M}_i^{\rm GB} = \begin{bmatrix} |z_u|^2 + |z_d|^2 & 0 & 0 & \sqrt{2} z_d \cdot z_u \\ 0 & |z_u|^2 + |z_d|^2 & 0 & \sqrt{2} z_u \cdot z_d \\ 0 & 0 & |z_u|^2 + |z_d|^2 & |z_u|^2 - |z_d|^2 \\ \sqrt{2} z_u \cdot z_d & \sqrt{2} z_d \cdot z_u & |z_u|^2 - |z_d|^2 & |z_u|^2 + |z_d|^2 \end{bmatrix}, \tag{A7}$$

with the notation

$$(z_u \cdot z_d)^* = z_d \cdot z_u \equiv \sum_{i=1}^{N} (\nu_i^d)^* \nu_i^u. \tag{A8}$$

The determinant becomes

$$\det \mathcal{M}^{\rm GB} = 4(|z_u|^2 + |z_d|^2)^2[|z_u|^2|z_d|^2 - |z_d \cdot z_u|^2]. \tag{A9}$$

This allows us to define the charge-breaking (CB) vacuum in a completely basis and gauge independent fashion. Indeed, in order for the vacuum to conserve charge, i.e., to conserve $U(1)_{\rm elmg}$, we need to have a massless photon. But that implies that the determinant in Eq. (A9) must vanish. Any combination of vevs $\{\nu_i^u, \nu_i^d\}$ for which this does not occur is therefore a CB stationary point. As is easily seen from Eq. (A9), we cannot get a CB vacuum with only one Higgs doublet, a well known result.

APPENDIX B: A THREE HIGGS-DOUBLET POTENTIAL WITHOUT EXPLICIT *CP* VIOLATION

In this Appendix we give a specific example of a 3HDM potential without explicit *CP* breaking for which one finds a CB minimum deeper than a normal one. Since no explicit *CP* breaking occurs, we work in a basis where all $\{\mu, \lambda\}$ are real. The values of the parameters are given in Tables I and II.

All remaining parameters are obtained from these using the symmetries of $\{\mu, \lambda\}$ expressed in Eq. (3). As mentioned this set of parameters gives us a normal minimum and a CB one. The values of the vevs obtained are given in

TABLE I. Values of the μ parameters (GeV^2).

μ_{11}	μ_{12}	μ_{13}	μ_{22}	μ_{23}	μ_{33}
-7.0655×10^4	1.6359×10^4	-2.0184×10^4	-1.2587×10^4	-1.7382×10^4	5.0687×10^4

TABLE II. Values of the λ parameters.

$\lambda_{11,11}$	$\lambda_{11,12}$	$\lambda_{11,13}$	$\lambda_{11,22}$	$\lambda_{11,23}$	$\lambda_{11,33}$	$\lambda_{12,12}$	$\lambda_{12,13}$	$\lambda_{12,21}$
0.6385	−0.4227	−0.0347	0.2500	0.3128	0.8696	−0.0987	0.2285	0.3917
$\lambda_{12,22}$	$\lambda_{12,23}$	$\lambda_{12,31}$	$\lambda_{12,32}$	$\lambda_{12,33}$	$\lambda_{13,13}$	$\lambda_{13,22}$	$\lambda_{13,23}$	$\lambda_{13,31}$
−0.3132	0.1735	0.1780	−0.0190	0.4370	0.1852	−0.1830	0.3268	0.1620
$\lambda_{13,32}$	$\lambda_{13,33}$	$\lambda_{22,22}$	$\lambda_{22,23}$	$\lambda_{22,33}$	$\lambda_{23,23}$	$\lambda_{23,32}$	$\lambda_{23,33}$	$\lambda_{33,33}$
−0.1566	−0.2230	0.2373	−0.2803	−0.1203	0.0536	0.4147	0.1545	0.5368

TABLE III. Values of the vevs for the normal and CB minima (GeV).

v_1	v_2	v_3	v_{c1}	v_{c2}	α
225.2135	−41.9564	89.6355	325.5199	343.9166	17.9887

Table III (we considered a CB minimum with the phase δ_c equal to zero).

With these vevs, one finds the values of the potential quoted in the main text. Using the methods developed in the first paper of [1], it is a simple matter to write down the squared mass matrices for the scalar fields. They are found to have, other than the expected Goldstone bosons, only positive eigenvalues for both the CB and N stationary points, thus proving that both are minima.

[1] P.M. Ferreira, R. Santos, and A. Barroso, Phys. Lett. B **603**, 219 (2004); **629**, 114(E) (2005); A. Barroso, P.M. Ferreira, and R. Santos, Phys. Lett. B **632**, 684 (2006); Proc. Sci., HEP2005 2006 337 [hep-ph/0512037].

[2] M. Maniatis, A. von Manteuffel, O. Nachtmann, and F. Nagel, hep-ph/0605184; I.P. Ivanov, hep-ph/0609018.

[3] F.J. Botella and J.P. Silva, Phys. Rev. D **51**, 3870 (1995).

[4] G.C. Branco, L. Lavoura, and J.P. Silva, *CP Violation* (Oxford University Press, Oxford, 1999).

[5] L. Lavoura and J.P. Silva, Phys. Rev. D **50**, 4619 (1994).

[6] J.F. Gunion, CPNSH, CERN, Switzerland, 2004 (unpublished).

[7] G.C. Branco, M.N. Rebelo, and J.I. Silva-Marcos, Phys. Lett. B **614**, 187 (2005).

[8] S. Davidson and H.E. Haber, Phys. Rev. D **72**, 035004 (2005); **72**, 099902(E) (2005).

[9] J.F. Gunion and H.E. Haber, Phys. Rev. D **72**, 095002 (2005).

[10] By "vacuum" we mean any vev configuration at a stationary point, not necessarily at a minimum. In fact many of these "vacua" may well end up being revealed as saddle points or maxima, as occurs in the 2HDM [1].

[11] Since in Eq. (16) ν_1^u and ν_2^d are real, we may change the notation into $\nu_1^u = \alpha$, $\nu_1^d = v_1 e^{-i\delta}$, and $\nu_2^d = v_2$. We may then apply an SU(2) gauge transformation combined with a (hypercharge) U(1) transformation: $e^{-i\delta\tau_3/2} e^{i\delta\mathbf{1}/2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}$, where $\mathbf{1}$ is the 2×2 unit matrix. This transforms Eq. (16) into Eq. (17).

[12] The demonstration is trivial, using the stationarity conditions for the normal minimum in Eq. (5), the form of the CB vevs given in Eq. (46), and the definition of $(M_\pm^2)^{\mathrm{N}}$ in Eq. (31).

[13] J.M. Frére, D.R.T. Jones, and S. Raby, Nucl. Phys. **B222**, 11 (1983).