PHYSICAL REVIEW D **74,** 085012 (2006)

Heavy quark diffusion in strongly coupled $\mathcal{N} = 4$ Yang-Mills theory

Jorge Casalderrey-Solana[1] and Derek Teaney[2]

[1]*Department of Physics & Astronomy, SUNY at Stony Brook, Stony Brook, New York 11764, USA*

[2]*Department of Chemistry & Physics, Arkansas State University, State University, Arkansas 72467, USA*

(Received 20 June 2006; published 11 October 2006)

We express the heavy quark diffusion coefficient as the temporal variation of a Wilson line along the Schwinger-Keldysh contour. This generalizes the classical formula for diffusion as a force-force correlator to a non-Abelian theory. We use this formula to compute the diffusion coefficient in strongly coupled $\mathcal{N} = 4$ Yang-Mills theory by studying the fluctuations of a string in $AdS_5 \times S_5$. The string solution spans the full Kruskal plane and gives access to contour correlations. The diffusion coefficient is $D = 2/\sqrt{\lambda}\pi T$ and is therefore parametrically smaller than momentum diffusion, $\eta/(e + p) = 1/4\pi T$. The quark mass must be much greater than $T\sqrt{\lambda}$ in order to treat the quark as a heavy quasiparticle. The result is discussed in the context of the Relativistic Heavy Ion Collider (RHIC) experiments.

DOI: 10.1103/PhysRevD.74.085012 PACS numbers: 11.15.Me, 11.10.Wx, 12.38.Mh

I. INTRODUCTION

The experimental relativistic heavy ion program has produced a variety of evidences which suggest that a quark gluon plasma (QGP) has been formed at the Relativistic Heavy Ion Collider (RHIC) [1,2]. One of the most exciting observables is the medium modifications of heavy quarks. In particular the electron spectrum from the semileptonic decays of heavy quarks is substantially suppressed relative to scaled proton-proton collisions [3,4]. Furthermore preliminary measurements indicate that the heavy quark elliptic flow is significant although less than the light hadron elliptic flow [5].

A variety of phenomenological models have estimated how the transport mean free path of heavy quarks in the medium is ultimately reflected in the suppression factor and elliptic flow [6–8]. The result of these model studies is best expressed in terms of the heavy quark diffusion coefficient. (In a relaxation time approximation the diffusion coefficient is related to the equilibration time, $\tau_R^{\text{heavy}} = \frac{M}{T} D$.) There is a sense from the models that if the diffusion coefficient of the heavy quark is greater than

$$D \gtrsim \frac{1}{T},$$

the heavy quark medium modifications will be small and probably in contradiction with current data. This interpretation of the RHIC results is perhaps too naive since the diffusion coefficient dictates the dynamics of nonrelativistic heavy quarks. Diffusion may be irrelevant for the dynamics of the mildly relativistic heavy quarks measured at RHIC where radiative energy loss may be significant [9,10]. Nevertheless, the diffusion coefficient is a fundamental parameter of the plasma and is essential to any discussion of the RHIC heavy flavor data.

Unfortunately, for the experimentally relevant range of energy densities the QGP is not weakly coupled, and it is not easy to determine this transport coefficient. Ideally, the diffusion coefficient should be measured on the lattice, but this is difficult [11,12]. Most theoretical works either compute the diffusion coefficient in perturbation theory and subsequently extrapolate to strong coupling (see e.g. [6]), or develop models for the strongly interacting QGP [13]. In this work we will compute the diffusion coefficient in strongly coupled $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory where rigorous computations are possible if the anti-de Sitter (AdS)/conformal field theory (CFT) conjecture is accepted. Although this theory is not QCD, computations in this theory serve as a foil to the extrapolations based on weak coupling.

This conjecture states that for a large number of colors, strongly coupled $\mathcal{N} = 4$ SYM theory is dual to classical type IIB supergravity on an $AdS_5 \times S_5$ background [14–16]. (For reviews and lectures see Refs. [17,18].) The physics of heavy quarks has been studied using semiclassical strings [19–24]. On the gauge theory side a heavy charge is realized by breaking the $U(N)$ gauge theory to $U(N-1) \times U(1)$ through the Higgs mechanism. The resulting W boson transforms in the fundamental and is heavy if the scalar expectation value is large. On the gravity side this corresponds to placing one of the $D3$ branes far from the remaining $N - 1$ branes. The dynamics of the heavy quark is dictated by the classical dynamics of the Nambu-Goto string stretching in the $AdS_5 \times S_5$ background. The first thing computed was the expectation value of the Wilson loop to find the heavy quark potential [19]. The result was immediately extended to finite temperature [21–24]. Since then there have been numerous studies of the other properties of Wilson loops both at zero and nonzero temperature [17].

One of the most interesting transport properties of the $\mathcal{N} = 4$ plasma is the heavy quark diffusion coefficient. Various transport properties have been computed using the correspondence leading to the remarkable conjecture that the shear viscosity to entropy ratio is bounded from below by $1/4\pi$ [25–28]. This bound is influencing the interpretation of heavy ion results. Generally these calculations of transport in $\mathcal{N} = 4$ use the supergravity approximation. In contrast the computation of heavy quark diffusion will utilize the classical string theory directly. The diffusion computation in $\mathcal{N} = 4$ SYM theory is therefore experimentally and theoretically significant. For other applications of the AdS/CFT correspondence motivated by RHIC physics, see Refs. [29,30].

The computation proceeds as follows. In Sec. II we review the Langevin dynamics of the heavy quark. In Sec. III we show that the strength of the noise in the Langevin process is determined by real time electric field correlators once the heavy quark has been integrated out. These electric field correlators can be expressed as fluctuations of a Wilson line running along the Schwinger-Keldysh contour. These results generalize the classical formula for the diffusion as a force-force correlator to non-Abelian theories. In Sec. IV we compute the fluctuations of the string configuration which corresponds to this Wilson line running along the contour; the relevant string spans the Kruskal plane. Finally in Sec. V, we summarize our results and discuss the implications for the RHIC experiments.

II. LANGEVIN DYNAMICS

In this section we will discuss the predictions of the Langevin process for determining the retarded correlator. We will assume that the Langevin equations provide a good macroscopic description of the thermalization of heavy particles

$$\frac{dx_i}{dt} = \frac{p_i}{M}, \qquad \frac{dp_i}{dt} = \xi_i(t) - \eta_D p_i, \qquad \langle \xi_i(t)\xi_j(t')\rangle = \kappa\delta_{ij}\delta(t-t'). \tag{2.1}$$

The drag and fluctuation coefficients are related by the Einstein relation

$$\eta_D = \frac{\kappa}{2MT}. \tag{2.2}$$

The drag coefficient η_D can be related in turn to the diffusion coefficient

$$D = \frac{T}{M\eta_D} = \frac{2T^2}{\kappa}. \tag{2.3}$$

For a brief review of the Langevin equation and a derivation of these results, see Ref. [6].

The relevant time scale for medium correlations in a relativistic plasma is $\sim D$ which is short compared to the relaxation time of the heavy particle $(M/T)D$. Furthermore, over the time scale of medium correlations the quark moves a negligible distance, $\sqrt{T/M}D$. Thus, for the purposes of calibrating the noise (κ), the mass may be taken to infinity and the heavy quark may be considered fixed.

More generally, the microscopic equations of motion for a heavy particle in the medium are

$$\frac{dx^i}{dt} = \frac{p^i}{M}, \tag{2.4}$$

$$\frac{dp^i}{dt} = \mathcal{F}^i(t). \tag{2.5}$$

Compare the response of the Langevin process to the microscopic theory. Over a time which is long compared to medium correlations D, but short compared to the time scale of equilibration $(M/T)D$, the drag term may be dropped, and we find

$$\int dt \int dt' \langle \xi_i(t)\xi_j(t')\rangle = (\text{time}) \times \kappa\delta_{ij} = \int dt \int dt' \langle \mathcal{F}_i(t)\mathcal{F}_j(t')\rangle. \tag{2.6}$$

Taking the force in the y-direction for instance, we have

$$\kappa = \int dt \langle \mathcal{F}_y(t)\mathcal{F}_y(0)\rangle. \tag{2.7}$$

We will drop the "y" in what follows. In QED this analysis relates the Langevin noise to an electric field correlator [31,32]. The next section generalizes this result to gauge theories.

III. MOMENTUM DIFFUSION IN GAUGE THEORIES

We study the propagation of a heavy quark in a thermal bath of SU(N) gauge fields. After a Foldy-Wouthuysen transformation that reduces the heavy quark field to a two-dimensional spinor field Q [33], the heavy quark effective Lagrangian is

$$\mathcal{L} = Q^\dagger(i\partial_t - M - A_0)Q. \tag{3.1}$$

From this Lagrangian it is natural to identify the force in the Langevin equation with the operator

$$\mathcal{F} \equiv \int d^3x Q^\dagger(t,\mathbf{x})T^a Q(t,\mathbf{x})E_a(t,x), \tag{3.2}$$

where E is the chromo-electric field in the $\hat{y}$ direction. The momentum diffusion coefficient κ is given by Eq. (2.7)

$$\kappa = \int dt \langle \mathcal{F}(t)\mathcal{F}(0)\rangle_{\mathrm{HQ}}, \tag{3.3}$$

where the average should be understood as a thermal average in the presence of a heavy quark—the meaning of this average is clarified below and in the appendix.

In the real time formalism of thermal field theory (see e.g. [34,35]), the quantization of the fields is performed by extending the time coordinates into the complex plane along the Schwinger-Keldysh contour—see Fig. 1. There are two types of quantum fields: type 1 fields, evaluated on the real time axis and type 2 fields, which are evaluated on the second $-i\beta/2$ time axis. We therefore define the correlation functions

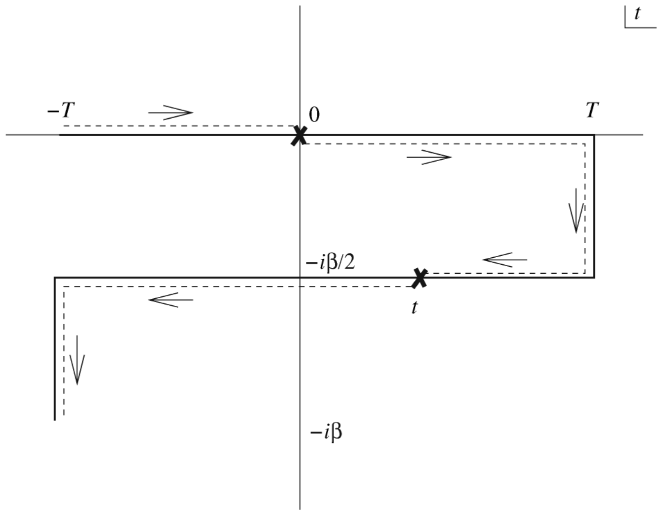

FIG. 1. The Schwinger-Keldysh contour (with $T \to \infty$). Fields evaluated along the real axes are labeled as type 1, while fields evaluated on the $-i\beta/2$ axis are labeled as type 2. The crosses indicate the insertion points of the color singlet force operator $\mathcal{F}$ (Eq. (3.2)). After integrating out the heavy quark, the crosses indicate insertions of the electric field, and the dotted lines indicate the path of the corresponding links. The electric field insertion may be rewritten as a variation (at times 0 and t) of the Wilson line running along the whole contour.

$$iG_{11}(t, t') = \langle T\mathcal{F}_1(t)\mathcal{F}_1(t')\rangle_{\rm HQ}, \tag{3.4}$$

$$iG_{12}(t, t') = \langle \mathcal{F}_2(t')\mathcal{F}_1(t)\rangle_{\rm HQ}, \tag{3.5}$$

$$iG_{21}(t, t') = \langle \mathcal{F}_2(t)\mathcal{F}_1(t')\rangle_{\rm HQ}, \tag{3.6}$$

$$iG_{22}(t, t') = \langle \tilde{T}\mathcal{F}_2(t)\mathcal{F}_2(t')\rangle_{\rm HQ}, \tag{3.7}$$

where T and $\tilde{T}$ denote time and antitime ordering, respectively, and

$$\mathcal{F}_1(t) = e^{+iHt}\mathcal{F}(0)e^{-iHt}, \tag{3.8}$$

$$\mathcal{F}_2(t) = e^{+iH(t-i\beta/2)}\mathcal{F}(0)e^{-iH(t-i\beta/2)}. \tag{3.9}$$

Temporal correlators of the operator $\mathcal{F}$ in the presence of an external heavy quark verify the Kubo-Martin-Schwinger (KMS) relations, provided single particle states may be considered complete, i.e. gauge fields are not so strong as to create heavy quark-antiquark pairs and the density of heavy quarks is small—see the appendix. In particular we define the retarded Green function

$$iG_R(t) = \theta(t)\langle[\mathcal{F}(t), \mathcal{F}(0)]\rangle_{\rm HQ}, \tag{3.10}$$

and relate the remaining Green functions by inserting complete sets of states

$$iG_{11}(\omega) = i\,{\rm Re}G_R(\omega) - \coth\left(\frac{\omega}{2T}\right){\rm Im}G_R(\omega), \tag{3.11}$$

$$iG_{12}(\omega) = iG_{21}(\omega) = -\frac{2e^{-\beta\omega/2}}{1-e^{-\beta\omega}}{\rm Im}G_R(\omega), \tag{3.12}$$

$$iG_{22}(\omega) = -i\,{\rm Re}G_R(\omega) - \coth\left(\frac{\omega}{2T}\right){\rm Im}G_R(\omega). \tag{3.13}$$

Using the fluctuation dissipation theorem, κ is related to zero frequency limit of the retarded correlator

$$\kappa = \lim_{\omega\to 0} -\frac{2T}{\omega}{\rm Im}G_R(\omega). \tag{3.14}$$

This can be determined from $iG_{12}(\omega)$ for instance.

For a Yang-Mills theory with external heavy quark the partition function is [36]

$$Z_{\rm HQ} = \int d^3x\Big\langle \frac{1}{\det[i\partial_{t_C} - M - A_0 - i\epsilon]} \times \int [DQ][DQ^\dagger]e^{i\int_C Q^\dagger(i\partial_{t_C}-M-A_0-i\epsilon)Q} \times Q_i(\mathbf{x}, -i\beta)Q_i^\dagger(\mathbf{x}, -T)\Big\rangle_{\rm YM}, \tag{3.15}$$

where the integration is performed along the Schwinger-Keldysh contour.

The heavy fermion may be integrated out of this expression of the partition function. The propagator of the heavy quark in a fixed gauge background is computed from the Green's function

$$\left(i\frac{\partial}{\partial t_C} - M - A_0 - i\epsilon\right)G(\mathbf{x}, t_C; \mathbf{x}', t'_C) = i\delta(\mathbf{x}-\mathbf{x}')\delta(t_C - t'_C). \tag{3.16}$$

With this propagator, the partition function of the heavy quark yields the Polyakov loop along the contour[1]

$$Z_{\rm HQ} = V_{\rm \mathbf{ps}}e^{-\beta M}\langle W_C[0]\rangle. \tag{3.18}$$

Usually the contour of the heavy quark partition function is taken straight down the imaginary axis to $-i\beta$, yielding the usual Polyakov loop. However this is unnecessary and the answers are the same.

Similarly, the heavy fermion fields may be integrated out of contour ordered force-force correlators

$$\langle T_C[\mathcal{F}(t_C)\mathcal{F}(0)]\rangle_{\rm HQ} = \frac{1}{\langle W_C[0]\rangle} \times \langle {\rm tr}[U(-T-i\beta, t_C)E(t_C)U(t_C, 0) \times E(0)U(0, -T)]\rangle, \tag{3.19}$$

where the transporters $U(t_f, t_i)$ are calculated along the Schwinger-Keldysh contour with t_i and t_f the initial and

[1] $V_{\rm \mathbf{ps}}$ is the phase space volume.

$$V_{\rm \mathbf{ps}} = V\int\frac{d^3p}{(2\pi)^3} = V\delta^3(\mathbf{0}). \tag{3.17}$$

final contour times. The denominator $\langle W_C[0]\rangle$ stems from the partition function Eq. (3.18). This correlator of electric fields can be generated by starting with a Wilson loop running along the contour and inserting small notches at contour times $t = 0$ and $t = t_C$ as shown in Fig. 1.

Generalizing this procedure, we first introduce a source $\delta y(t_C)$ for the force operator

$$\frac{1}{Z_{\rm HQ}[0]} Z_{\rm HQ}[\delta y] = \langle T_C e^{i\int_C dt \delta y(t)\mathcal{F}(t)}\rangle_{\rm HQ}. \qquad (3.20)$$

This is accounted for by modifying the heavy quark Lagrangian

$$\mathcal{L}_{\rm HQ} = Q^\dagger(i\partial_{t_C} - M - A_0 - \delta y(t_C)E(t_C))Q. \qquad (3.21)$$

As in Eq. (3.19), we integrate the partition function in the presence of a heavy quark

$$Z_{\rm HQ}[\delta y] = V_{\bf ps} e^{-\beta M}\langle W_C[\delta y]\rangle, \qquad (3.22)$$

where

$$W_C[\delta y] = T_C \exp\left\{-i\int_C dt(A_0 + \delta y(t)E(t))\right\}. \qquad (3.23)$$

This object can be easily expressed as[2]

$$W_C[\delta y] = T_C \exp\left\{-i\int_C dt A_0\right\} T_C \exp\left\{i\int_C dt\delta y(t)\tilde{E}(t)\right\}, \qquad (3.24)$$

where $\tilde{E}$ is the dressed field strength

$$\tilde{E}(t_C) = U(-T, t_C)E(t_C)U(t_C, -T), \qquad (3.25)$$

i.e. the transporter starts at an initial time $-T$, runs along the Schwinger-Keldysh contour from the initial point up to the contour time t_C, and returns to the initial time, $-T$. By means of the non-Abelian Stokes theorem [37] or some thought, it can be shown that Eq. (3.24) is the Wilson line along the time contour with a deformation in the $\hat{y}$ direction given by $\delta y(t_C)$

$$W_C[\delta y] = T_C \exp\left\{-i\int_C dt(A_0 + \delta\dot{y}A_y)\right\}. \qquad (3.26)$$

In summary, the path $\delta y(t)$ of a Wilson loop running around the Schwinger-Keldysh contour is the source for contour ordered force operators $\mathcal{F}$. In real time thermal field theory it is customary to break up the source into 1 type sources and 2 type sources with the understanding that variations of the vertical part are not considered. The source $\delta y_1(t)$ and $\delta y_2(t)$ are variations of the Wilson loop on the 1 and 2 axis, respectively,

$$\langle Te^{i\int dt\delta y_1(t)\mathcal{F}_1(t)}\tilde{T}e^{-i\int dt'\delta y_2(t')\mathcal{F}_2(t')}\rangle_{\rm HQ} = \frac{1}{\langle W_C[0,0]\rangle}\langle W_C[\delta y_1, \delta y_2]\rangle. \qquad (3.27)$$

The momentum diffusion coefficient may then be written

$$\kappa = \lim_{\omega\to 0}\int dt e^{+i\omega t}\frac{1}{\langle W_C[0,0]\rangle}\left\langle\frac{\delta^2 W_C[\delta y_1, \delta y_2]}{\delta y_2(t)\delta y_1(0)}\right\rangle. \qquad (3.28)$$

$\mathcal{N} = 4$ super Yang-Mills theory

The previous discussion was performed in a SU(N) gauge theory. However, we are interested in studying the interactions of a heavy W boson in a $\mathcal{N} = 4$ U(N) gauge theory broken to U($N-1$) $\times$ U(1). The heavy W boson propagator is given by the following Wilson line [19,20]

$$U = T_C \exp\left\{-i\int ds(A_\mu\dot{x}^\mu + |\dot{x}|\Theta^I X^I)\right\}, \qquad (3.29)$$

where in addition to the gauge fields the adjoint scalars X^I couple to the W boson through a Yukawa type interaction. Here Θ^I is the vacuum expectation value (VEV) angle.

The extension to $\mathcal{N} = 4$ SYM theory amounts to changing the transporters and Wilson lines in accord with Eq. (3.29). In particular, using the non-Abelian Stokes theorem [37] the $\mathcal{N} = 4$ analog of Eq. (3.24)

$$W_C[\delta y] = T_C \exp\left\{-i\int_C dt(A_0 + \Theta^I X^I)\right\} T_C \times \exp\left\{i\int_C dt\delta y(t)\tilde{\mathcal{E}}_{\rm SYM}(t)\right\}. \qquad (3.30)$$

Here we have defined

$$\mathcal{E}_{\rm SYM} = E + D_t(\Theta^I X^I) - D_y(\Theta^I X^I), \qquad (3.31)$$

and dressed it with transporters as in Eq. (3.25). The necessity of the scalar derivatives is obvious after computing a plaquette with the links defined in Eq. (3.29). In the low frequency limit the time derivatives of the scalars can be dropped and the QCD-like field strength is modified by gradients of the scalar fields.

IV. THE ADS/CFT CORRESPONDENCE

Fundamental Wilson loops were first computed by Maldacena whose work we recall [19]. On the gauge theory side heavy gauge bosons are introduced through the Higgs mechanism. Specifically the U(N) gauge group is broken to U($N-1$) $\times$ U(1) by giving a large expectation value to one of the scalars in the theory. The resulting W boson transforms in the fundamental representation of the remaining U($N-1$) gauge group. The equation of motion for the W boson is

[2] δy is assumed to be small.

$$(i\partial_t - M - A_0 - \theta^I X^I)W = 0, \quad (4.1)$$

and naturally leads to the Wilson loops considered in the previous section.

On the gravity side this Higgs construction corresponds to placing one of the $D3$ branes far from the remaining $N-1$ $D3$ branes. The propagation of the heavy W bosons is represented by semiclassical strings which are governed by the Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-\det G_{MN}\partial_a X^M \partial_b X^N}. \quad (4.2)$$

In the original Maldacena computation, the two parallel Wilson lines corresponded to a string world sheet that circumscribed the trajectory of a $w(\mathbf{x})$ and $\bar{w}(\mathbf{y})$ bosons on the boundary.

Finite temperature is introduced by inserting a black hole into the AdS space. When the black hole is viewed in Kruskal coordinates there are two boundaries which are asymptotically AdS. These boundaries correspond to the type "1" and "2" axis of the real time partition function [38,39]. Naturally the thermal description of a broken gauge theory at finite temperature also has w bosons running along the 1 and 2 axis.

Since we are considering a single heavy gauge boson which propagates along the full Schwinger-Keldysh contour, there is a single Wilson line on the 1 axis and a single Wilson line on the 2 axis. It is natural to look for string solutions in the full Kruskal plane. Such solutions connect the type 1 w boson at the boundary of the right quadrant with the type 2 boson at the boundary of the left quadrant—see Fig. 2.

To start, we look for string solutions in the first quadrant. The metric is

$$ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + d\mathbf{x}_\parallel^2] + \frac{R^2}{f(r)r^2}dr^2 + R^2 d\Omega_5^2, \quad (4.3)$$

where $f(r) = 1 - (\frac{r_o}{r})^4$, R is the AdS radius, and r_o is related to the Hawking temperature $\pi T R^2 = r_0$. We define the scaled units, $\pi T t = \bar{t}$, $\pi T x = \bar{x}$, and $\bar{r} = r/r_o$ and the metric reads

$$\frac{ds^2}{R^2} = -\bar{r}^2 f(\bar{r})d\bar{t}^2 + \bar{r}^2 d\bar{\mathbf{x}}_\parallel^2 + \frac{d\bar{r}^2}{f(\bar{r})\bar{r}^2} + d\Omega_5^2. \quad (4.4)$$

In what follows we will drop the "bar" until Eq. (4.19).

Parametrizing the string as

$$(\tau, \sigma) \mapsto (t = \tau, \mathbf{x}_\parallel(\tau, \sigma), r = \sigma, \Omega_5(\tau, \sigma)),$$

it is straightforward to show and well known that

$$\mathbf{x}_\parallel = \text{Const} \quad \text{and} \quad \Omega_5 = \text{Const},$$

is a solution to the equations of motion. This solution extends to the full Kruskal plane and is the string solution which we are looking for.

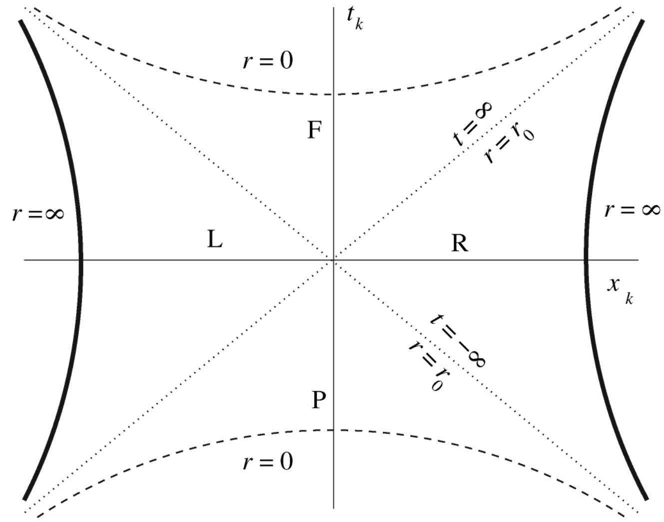

FIG. 2. Kruskal diagram for the AdS black hole. The coordinates (t, r) span the right (R) quadrant. The dotted lines and the dashed hyperbolas represent the future and past horizons and the singularities, respectively. The thick hyperbolas on the sides of the two quadrants are the boundaries at $r = \infty$. The Wilson line running along the Schwinger-Keldysh contour runs along the 1 axis (the R boundary) and 2 axis (the L boundary). This corresponds to a string whose endpoints follow these boundaries. The minimal surface with these boundary conditions is the full Kruskal plane.

To see this we first write the metric in Kruskal coordinates[3]

$$\frac{ds^2}{R^2} = g_k(-dt_k^2 + dx_k^2) + r^2 d\mathbf{x}_\parallel^2 + d\Omega_5^2, \quad (4.5)$$

where the Kruskal scale factor is $g_k = (1/4)r^2 f(r)e^{-4r_*}$. Parametrizing the string as

$$(\tau, \sigma) \mapsto (t_k = \tau, \mathbf{x}_\parallel(\tau, \sigma), x_k = \sigma, \Omega_5(\tau, \sigma)), \quad (4.6)$$

we determine the induced metric

$$\mathcal{L} = \sqrt{-\det h} = R^2[g_k^2 + g_k(r^2(\mathbf{x}'_\parallel)^2 - r^2(\dot{\mathbf{x}}_\parallel)^2 + (\Omega'_5)^2 - (\dot{\Omega}_5)^2)]^{1/2}, \quad (4.7)$$

where, for instance, $\dot{\mathbf{x}}_\parallel = \partial_\tau \mathbf{x}_\parallel$ and $\mathbf{x}'_\parallel = \partial_\sigma \mathbf{x}_\parallel$. Then we compute the canonical momenta

$$\frac{\delta\mathcal{L}}{\delta(\dot{\mathbf{x}}_\parallel)} = -\frac{g_k r^2 \dot{\mathbf{x}}_\parallel}{\sqrt{-\det h}}, \quad (4.8)$$

[3]Our notation closely follows Ref. [40], in the appendix. Define $r_* \equiv \int_0^r \frac{dr}{r^2 f(r)} + i\frac{\pi}{4}$ so that r^* is real in the right quadrant. The Kruskal coordinates are $-e^{4r_*} = t_k^2 - x_k^2$ and $2t = \tanh^{-1}(t_k/x_k)$.

$$\frac{\delta\mathcal{L}}{\delta(\mathbf{x}'_{\|})} = \frac{g_k r^2 \mathbf{x}'_{\|}}{\sqrt{-\det h}}, \tag{4.9}$$

and we see that the equations of motion

$$\partial_\tau \frac{\delta\mathcal{L}}{\delta\dot{\mathbf{x}}} + \partial_\sigma \frac{\delta\mathcal{L}}{\delta\mathbf{x}'_{\|}} = 0,$$

are satisfied trivially by $\mathbf{x}_{\|} = \text{Const}$. The canonical momenta vanish. A similar remark applies to the Ω_5 coordinates. Thus

$$\mathbf{x}_{\|}(t_k, x_k) = \text{Const}, \qquad \Omega_5(t_k, x_k) = \text{Const}, \tag{4.10}$$

is a solution to the string equations of motion throughout the Kruskal plane. This represents a Wilson line along the 1 time axis and back along the 2 axis of the Schwinger-Keldysh contour.

It may disturb some readers that the string crosses the event horizon. However, an observer at spatial infinity (using e.g. the t, r coordinates in the right-coordinate patch) will not be able tell whether or not the string actually crosses the horizon. Signals sent to detect this crossing will not return from the black hole. Thus, from the perspective of this observer, the string exists in the R-coordinate patch only. A similar remark applies to the L coordinate patch. Nevertheless, another intrepid observer traveling through the event horizon in a space ship will follow the trajectory of the string across this coordinate singularity without difficulty.

Having constructed the string solution representing a W boson propagating all along the Schwinger-Keldysh contour we will proceed to fluctuate the shape of this string solution. Following the general philosophy of the AdS/CFT correspondence we equate the exponential of the classical action of the source to the generating functional

$$\frac{1}{e^{iS_{\rm NG}[0,0]}} e^{iS_{\rm NG}[\delta y_1, \delta y_2]} = \frac{1}{\langle W[0,0]\rangle}\langle W[\delta y_1, \delta y_2]\rangle. \tag{4.11}$$

Now that the problem has been formulated along the Schwinger-Keldysh contour and KMS is satisfied, the results of Son and Herzog [38] may be adopted *mutatis mutandis*. (For related work, see Refs. [39,41,42].) For the retarded correlator,

$$G_R(\omega) = -i\int_0^\infty dt\, e^{+i\omega t}\theta(t)\langle[\mathcal{F}(t), \mathcal{F}(0)]\rangle_{\rm HQ}, \tag{4.12}$$

the procedure is the following. In the right Kruskal quadrant, solve the classical equations of motion for the source $\delta y(t, u)$ with infalling boundary conditions at the horizon. The retarded correlator may be found by evaluating the classical action of the string and differentiating twice with respect to the boundary source, $\delta y(t, u = 0)$.

Returning to the first quadrant with t, $\mathbf{x}_{\|}$, $u \equiv \frac{1}{r^2}$ coordinates, the relevant string action with small fluctuations in the y-direction is

$$S_{\rm NG} = \frac{R^2}{2\pi\alpha'}\int \frac{dt du}{2u^{3/2}}\left[1 - \frac{1}{2}\left(\frac{\dot{y}_{\|}^2}{f} - 4fu(y'_{\|})^2\right)\right]. \tag{4.13}$$

Notice that the infinite action corresponding to the unperturbed string is to be subtracted since it appears in the numerator and denominator of Eq. (4.11). Fluctuating in the y-direction, we define

$$y(t, u) = \int e^{-i\mathfrak{w}t} y(\mathfrak{w}) Y_{\mathfrak{w}}(u)\frac{d\mathfrak{w}}{2\pi}, \tag{4.14}$$

and the equation of motion of the fluctuating string is

$$\partial_u^2 Y_{\mathfrak{w}} - \frac{(2 + 6u^2)}{4uf}\partial_u Y_{\mathfrak{w}} + \frac{\mathfrak{w}^2}{4uf^2} Y_{\mathfrak{w}} = 0. \tag{4.15}$$

This equation has regular singular points at the boundary $u = 0$ and at the horizon $u = 1$. Near the horizon the solution behaves as

$$Y_{\mathfrak{w}}(u) = (1 - u)^{\pm i\mathfrak{w}/4}. \tag{4.16}$$

Remembering that $u \equiv 1/r^2$, these solutions correspond to outgoing and infalling fluctuations. For the retarded propagator,

$$G_R(\omega) = -i\int_0^\infty dt\, e^{+i\omega t}\theta(t)\langle[\mathcal{F}(t), \mathcal{F}(0)]\rangle_{\rm HQ}, \tag{4.17}$$

incoming boundary conditions must be selected for the corresponding source, $Y_{\mathfrak{w}}(u)$. We therefore substitute $Y_{\mathfrak{w}}(u) = (1 - u^2)^{-i\mathfrak{w}/4} F_{\mathfrak{w}}(u)$ where $F_{\mathfrak{w}}(u)$ is regular at the horizon. Since $F_{\mathfrak{w}}(u)$ is regular at the horizon we may expand in a power series in $\mathfrak{w}$ and solve order by order[4]

$$\begin{aligned} Y_{\mathfrak{w}}(u) = (1 - u^2)^{-i(\mathfrak{w}/4)}\Big[1 &+ \frac{i\mathfrak{w}}{8}\{\pi - 4\tan^{-1}(\sqrt{u}) \\ &- 6\log 2 + 4\log(1 + \sqrt{u}) + 2\log(1 + u)\} \\ &+ O(\mathfrak{w}^2)\Big]. \end{aligned} \tag{4.18}$$

The insertion of this solution into the action reduces to a boundary term. After reinserting units

$$y_\omega(\omega) = (\pi T)^2 \bar{y}_{\mathfrak{w}}\left(\mathfrak{w} = \frac{\omega}{\pi T}\right),$$

$$Y_\omega(\omega, u) = \bar{Y}_{\mathfrak{w}}\left(\mathfrak{w} = \frac{\omega}{\pi T}, u\right),$$

and denoting $A(u)$ as the coefficient in front of the kinetic term as in Refs. [28,38] we have

$$\begin{aligned} G_R(\omega) &= -A(u) Y_{-\omega}(u)\partial_u Y_\omega(u)|_{u\to 0}, \\ A(u) &= \frac{R^2(\pi T)^2}{\pi\alpha'}\frac{f}{u^{1/2}}. \end{aligned} \tag{4.19}$$

Substituting the solution Eq. (4.18) into this expression we

[4] We thank A. Starinets for providing us this expression.

obtain

$$\kappa = \lim_{\omega\to 0} \frac{-2T}{\omega} \mathrm{Im} G_R(\omega), \quad (4.20)$$

$$= \sqrt{\lambda} T^3 \pi. \quad (4.21)$$

The divergence which arises as $u \to 0$ does not affect the imaginary part. In the last step we have used the relation $R^2/\alpha' = \sqrt{\lambda}$. The significance of the diffusion computation will be discussed in the next section.

V. DISCUSSION

In summary we have computed the heavy quark diffusion coefficient in $\mathcal{N} = 4$ SYM theory by exploiting the AdS/CFT correspondence. From Eqs. (2.3) and (4.20) the diffusion coefficient is

$$D = \frac{2T^2}{\kappa} \quad (5.1)$$

$$= \frac{2}{\pi T} \frac{1}{\sqrt{\lambda}}, \quad (5.2)$$

where $\lambda = g_{\mathrm{YM}}^2 N_c$. This result is interesting both theoretically and phenomenologically.

An immediate observation is that the heavy quark diffusion coefficient D is parametrically small compared to the momentum diffusion coefficient $\eta/(e+p) = 1/(4\pi T)$ [27], i.e. D depends on the peculiar, but characteristic, $1/\sqrt{\lambda}$. This should be contrasted with perturbation theory where all transport scales are the same order of magnitude, $1/(\lambda^2 T)\log(\lambda^{-1})$.

Theoretically, the diffusion computation proceeded as follows. A Wilson line running along the Schwinger-Keldysh contour is the partition function of a single heavy quark.[5] There is a Wilson line on the 1 axis and a Wilson line on the 2 axis. In a Minkowski metric the corresponding semiclassical string solution must span the full Kruskal plane if the right (left) quadrant of the Kruskal diagram is to be identified with 1 (2) fields in the real time path integral. We find this semiclassical solution in Sec. IV, and it is a single straight string in $\mathrm{AdS}_5 \times S_5$ which extends from the boundary to the event horizon when viewed from an observer in the right quadrant. Force-force correlators in the presence of a heavy quark are electric field correlators with links running along the contour after the heavy quark fields have been integrated out. These same correlators may be obtained by variations of the unperturbed Wilson loop. Thus by varying the end point of the relevant string solution in $\mathrm{AdS}_5 \times S_5$ one may compute real time force-force correlators, i.e. the diffusion coefficient.

[5] The shape of the contour is irrelevant. Taking a contour which runs straight down the imaginary time axis to $-i\beta$ gives the Polyakov loop. This is the more common but equivalent definition of the heavy quark partition function.

Given the diffusion coefficient in Eq. (5.1), the quark must be sufficiently heavy for the Langevin theory to apply. Since the Langevin theory is classical, consistency demands that the relaxation time be large compared to the inverse temperature

$$\frac{M}{T} D \gg \frac{1}{T}. \quad (5.3)$$

This leads to the following constraint

$$M \gg \frac{\pi T}{2}\sqrt{\lambda}. \quad (5.4)$$

For a string of length L stretching outward from the horizon we have, $M \sim L/\alpha'$, $\pi T = r_o/R_{\mathrm{AdS}}^2$, and $\sqrt{\lambda} = R_{\mathrm{AdS}}^2/\alpha'$. This Langevin constraint is satisfied by the gravity computation whenever

$$L \gg r_o, \quad (5.5)$$

i.e. whenever the length of the string is long enough. Clearly mass or energy of order $T\sqrt{\lambda}$ sets a boundary for a quasiparticle picture to be valid. In the gravity computation this corresponds to a string of length $L \sim r_o$.

Phenomenologically the diffusion coefficient is an extremely interesting number. Bearing in mind that $\mathcal{N} = 4$ SYM theory is not QCD, the authors still want to substitute numbers

$$D \simeq \frac{0.9}{2\pi T}\left(\frac{1.5}{\alpha_{\mathrm{SYM}} N}\right)^{1/2}. \quad (5.6)$$

This could be compared to extrapolations of weak coupling to QCD [6,43]

$$D \simeq \frac{6}{2\pi T}\left(\frac{1.5}{\alpha_s N_c}\right)^2. \quad (5.7)$$

Of equal importance is the mass scale where we expect the heavy quark theory to apply

$$M \gg 1.7\ \mathrm{GeV}\left(\frac{T}{0.250\ \mathrm{GeV}}\right)\left(\frac{\alpha_{\mathrm{SYM}} N}{1.5}\right)^{1/2}. \quad (5.8)$$

This suggests that the Langevin process might not be applicable to charm quarks, though of course there is an unknown proportionality factor in this equation.

The RHIC data favor a diffusion coefficient of about $3/(2\pi T)$. However, relativistic effects obscure the relation between the diffusion coefficient and the semileptonic data. The role of radiative and collisional energy loss remains unclear [9,10]. The next step from the perspective of the AdS/CFT correspondence is to consider quarks with finite velocity—work is in progress in this direction [44]. The framework set up here should allow a computation of the real time transport properties of these quarks.

ACKNOWLEDGMENTS

We wish to thank Andrei Starinets and Edward Shuryak for fruitful discussions. J. C. thanks M. Kulaxizi, S. Giombi, R. Ricci, and D. Trancanelli. We especially thank Ismail Zahed for challenging us to find a clean formulation of the problem. This work was supported by grants from the U.S. Department of Energy, No. DE-FG02-88ER40388 and No. DE-FG03-97ER4014.

Note Added.—In the few days surrounding this preprint three similar studies of heavy quark energy loss appeared. The first of these computed the transverse momentum diffusion of an ultra-relativistic quark [46]. The second paper, denoted HKKKY after the authors, computed the drag on a quark moving with finite velocity [47]. A third paper [48] independently computed the drag using methods similar to HKKKY and obtained the same answer. The setup in these papers differs significantly from this work.

The HKKKY paper is particularly instructive. Consider the Langevin process of a heavy quark

$$\frac{dp_i}{dt} = \xi_i(t) - \eta_D p_i, \qquad \langle \xi_i(t)\xi_j(t')\rangle = \kappa\delta_{ij}\delta(t-t'), \tag{5.9}$$

where the drag, fluctuations, and diffusion are related by Einstein relations

$$\eta_D = \frac{\kappa}{2MT}, \qquad D = \frac{2T^2}{\kappa}.$$

In this paper we considered times short compared to the relaxation time $(M/T)D$, and computed the strength of the noise κ, taking the mass to infinity. The mass is not needed to compute the diffusion coefficient or $dE/dx = T/D$.

HKKKY computed the drag by considering the change in the average momentum over a time long compared to the time scale of noise correlations, $\sim 1/T$. The Langevin process (see e.g. [6]) predicts the probability that a quark with momentum $\mathbf{p}_0$ at time 0 will arrive with momentum $\mathbf{p}$ at later time t

$$P(\mathbf{p}, t|\mathbf{p}_0, 0) = \frac{1}{\sqrt[3]{2\pi MT(1-e^{-2\eta_D t})}} \times \exp\left[-\frac{(\mathbf{p}-\mathbf{p}_0 e^{-\eta_D t})^2}{2MT(1-e^{-2\eta_D t})}\right].$$

The average momentum obeys

$$\langle \mathbf{p}(t)\rangle = \mathbf{p}_0 e^{-\eta_D t}, \tag{5.10}$$

while the width obeys

$$\langle(\Delta\mathbf{p})^2\rangle = 3MT(1-e^{-2\eta_D t}). \tag{5.11}$$

HKKKY considered an atypical quark, with momentum $p_0 \gg \sqrt{MT}$, and computed the change in the average momentum over a time interval of order, $1/\eta_D = (M/T)D$. This determines the drag, η_D. For an *atypical* quark the fluctuations are negligible over this time interval. They also compute the mass and then are able to deduce the diffusion coefficient through the Einstein relations. The result agrees with Eq. (5.1); the AdS/CFT correspondence is neatly consistent with the fluctuation dissipation theorem. It would be quite interesting to see the full structure of the Langevin Green function emerge from the string theory. In essence this computation has been performed already through an amalgamation of our works.

HKKKY also computed the heavy quark energy loss at finite velocity. The bending string solution they obtained agrees with our preliminary computations.

APPENDIX: THE HEAVY QUARK PARTITION FUNCTION

In this section we clarify several aspects concerning the heavy quark partition function. We follow closely the discussion of McLerran and Svetitsky [36]. We introduce the operators $Q(\mathbf{x}, t)$, $Q^\dagger(\mathbf{x}, t)$ which create and annihilate static quarks at point $\mathbf{x}$ and time t. These fields satisfy the anticommutation relations

$$\{Q_i(\mathbf{x}, t), Q_j^\dagger(\mathbf{x}', t)\} = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}'). \tag{A1}$$

From the Lagrangian equation (3.1) the time evolution of quark fields is given by

$$Q(\mathbf{x}, t) = e^{-iM(t-t_0)}U(t, t_0)Q(\mathbf{x}, t_0), \tag{A2}$$

where both t and t_0 are in the Schwinger-Keldysh contour and $U(t, t_0)$ is the transporter between these two points at fixed position $\mathbf{x}$. As in Ref. [36], the partition of function in the presence of a heavy quark is

$$Z_{\rm HQ} = \sum_s \langle s|e^{-\beta H}|s\rangle = \int d^3x \sum_{s'}\langle s'|Q(\mathbf{x}, -T)e^{-\beta H}Q^\dagger(\mathbf{x}, -T)|s'\rangle, \tag{A3}$$

$$= \int d^3x \sum_{s'}\langle s'|e^{-\beta H}Q(\mathbf{x}, -T-i\beta)Q^\dagger(\mathbf{x}, -T)|s'\rangle, \tag{A4}$$

in which $|s\rangle$ is a state of the system with only one heavy quark, and $|s'\rangle$ is a state with no heavy quarks (i.e. $Q|s'\rangle = 0$). By means of Eq. (A2) we obtain Eq. (3.18). With this definition of the partition function, the force-force correlator is

$$D^>(t) = \langle \mathcal{F}(t)\mathcal{F}(0)\rangle_{\rm HQ} = \frac{1}{Z_{\rm HQ}}\sum_s \langle s|e^{-\beta H}\mathcal{F}(t)\mathcal{F}(0)|s\rangle. \tag{A5}$$

We introduce a complete set of states between the two force operators. We notice that, since we are assuming weak fields (which cannot create heavy quarks) and since the force operator does not change the number of quarks, it is enough to consider a complete set of one particle heavy quark states plus bath. Following Ref. [34] we obtain the KMS relation,

$$D^{>}(t) = \langle \mathcal{F}(0)\mathcal{F}(t + i\beta)\rangle_{\rm HQ} = D^{<}(t + i\beta). \tag{A6}$$

Since the force is a Hermitian operator local in time, the definition of the spectral density $\rho(t) = D^{>}(t) - D^{<}(t)$ leads to the standard relations between correlators, Eq. (3.11), (3.12), and (3.13).

We conclude by remarking that, since the states $|s'\rangle$ can be considered as the $|0_A\rangle$ Fock states for the heavy quarks in a gauge field background, the partition function is

$$Z_{\rm HQ} = \int d^3x\, {\rm Tr}[\langle 0_A|Q(\mathbf{x}, -T - i\beta)Q^{\dagger}(\mathbf{x}, -T)|0_A\rangle], \tag{A7}$$

where the Tr represents the thermal trace over the bath (the Yang-Mills fields). Thus, representing the vacuum expectation value in Eq. (A7) as a path integral we arrive at expression Eq. (3.15). The effect of the vacuum average is to introduce the $i\epsilon$ prescription in Eq. (3.16) as in zero temperature field theory [45].

[1] J. Adams *et al.* (STAR Collaboration), Nucl. Phys. **A757**, 102 (2005).
[2] K. Adcox *et al.* (PHENIX Collaboration), Nucl. Phys. **A757**, 184 (2005).
[3] S. S. Adler *et al.* (PHENIX Collaboration), Phys. Rev. Lett. **96**, 032301 (2006).
[4] J. Bielcik (STAR Collaboration), Nucl. Phys. **A774**, 697 (2006).
[5] Y. Akiba (PHENIX Collaboration), Nucl. Phys. **A774**, 403 (2006).
[6] G. D. Moore and D. Teaney, Phys. Rev. C **71**, 064904 (2005).
[7] D. Molnar, J. Phys. G **31**, S421 (2005).
[8] B. Zhang, L. W. Chen, and C. M. Ko, Phys. Rev. C **72**, 024906 (2005).
[9] M. Djordjevic, M. Gyulassy, R. Vogt, and S. Wicks, Phys. Lett. B **632**, 81 (2006).
[10] N. Armesto, M. Cacciari, A. Dainese, C. A. Salgado, and U. A. Wiedemann, Nucl. Phys. **A774**, 589 (2006).
[11] P. Petreczky and D. Teaney, Phys. Rev. D **73**, 014508 (2006).
[12] G. Aarts and J. M. Martinez Resco, Nucl. Phys. B, Proc. Suppl. **119**, 505 (2003).
[13] H. van Hees and R. Rapp, Phys. Rev. C **71**, 034907 (2005).
[14] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[15] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[16] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[17] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[18] E. D'Hoker and D. Z. Freedman, hep-th/0201253.
[19] J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998).
[20] N. Drukker, D. J. Gross, and H. Ooguri, Phys. Rev. D **60**, 125006 (1999).
[21] A. Brandhuber, N. Itzhaki, J. Sonnenschein, and S. Yankielowicz, Phys. Lett. B **434**, 36 (1998).
[22] A. Brandhuber, N. Itzhaki, J. Sonnenschein, and S. Yankielowicz, J. High Energy Phys. 06 (1998) 001.
[23] S. J. Rey, S. Theisen, and J. T. Yee, Nucl. Phys. **B527**, 171 (1998).
[24] S. J. Rey and J. T. Yee, Eur. Phys. J. C **22**, 379 (2001).
[25] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
[26] A. Buchel, Phys. Lett. B **609**, 392 (2005).
[27] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).
[28] G. Policastro, D. T. Son, and A. O. Starinets, J. High Energy Phys. 09 (2002) 043.
[29] S. J. Sin and I. Zahed, Phys. Lett. B **608**, 265 (2005).
[30] E. Shuryak, S. J. Sin, and I. Zahed, hep-th/0511199.
[31] D. Forster, *Hydrodynamics, Fluctuations, Broken Symmetry, and Correlation Functions* (Perseus Books, New York, 1990).
[32] J. P. Boon and S. Yip, *Molecular Hydrodynamics* (McGraw-Hill, New York, 1980).
[33] L. S. Brown and W. I. Weisberger, Phys. Rev. D **20**, 3239 (1979).
[34] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[35] F. Gelis, Z. Phys. C **70**, 321 (1996).
[36] L. D. McLerran and B. Svetitsky, Phys. Rev. D **24**, 450 (1981).
[37] N. E. Bralic, Phys. Rev. D **22**, 3090 (1980).
[38] C. P. Herzog and D. T. Son, J. High Energy Phys. 03 (2003) 046.
[39] S. F. Ross, hep-th/0502195 and references therein.
[40] L. Fidkowski, V. Hubeny, M. Kleban, and S. Shenker, J. High Energy Phys. 02 (2004) 014.
[41] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[42] W. Israel, Phys. Lett. A **57**, 107 (1976).
[43] E. Braaten and M. H. Thoma, Phys. Rev. D **44**, R2625 (1991). **44**, 1298 (1991).
[44] D. Teaney and J. Casalderrey-Solana (unpublished).
[45] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 2002), Vol. I.
[46] Hong Liu, Krishna Rajagopal, and Urs Achim Wiedemann, hep-ph/0605178.
[47] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz, and L. G. Yaffe, J. High Energy Phys. 07 (2006) 013.
[48] S. S. Gubser, hep-th/0605182.