Induced quantum numbers of a magnetic monopole at finite temperature

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A Dirac electron field is quantized in the background of a Dirac magnetic monopole, and the phenomenon of induced quantum numbers in this system is analyzed. We show that, in addition to electric charge, also squares of orbital angular momentum, spin, and total angular momentum are induced. The functional dependence of these quantities on the temperature and the *CP*-violating vacuum angle is determined. Thermal quadratic fluctuations of charge and squared total angular momentum, as well as the correlation between them and their correlations with squared orbital angular momentum and squared spin, are examined. We find the conditions when charge and squared total angular momentum at zero temperature are sharp quantum observables rather than mere quantum averages.

I. INTRODUCTION

The interaction of quantized Dirac fields with classical background fields of nontrivial topology can give rise to quantum states with rather unusual eigenvalues [1–6]. In particular, the ground state of a Dirac electron in the background of a pointlike magnetic monopole acquires nonzero electric charge, and this results in the monopole becoming a *CP* symmetry violating dyon [7–9]. The effect persists when thermal fluctuations of the quantized Dirac electron field are taken into account, and this yields temperature dependence of the induced charge [10,11].

The aim of the present paper is to show that, in addition to charge, also other quantum numbers are induced in the magnetic monopole background both at zero and nonzero temperatures. We find relationships between all these quantum numbers and discuss, which of them become sharp quantum observables rather than quantum averages and also when this happens. At nonzero temperature all quantum numbers are not sharp observables, but, instead, are thermal averages; and, appropriately, the thermal quadratic fluctuations are nonvanishing. If a quadratic fluctuation vanishes at zero temperature, then a corresponding quantum number at zero temperature becomes a sharp observable. We find out, in particular, that induced charge and squared total angular momentum at zero temperature are sharp observables for almost all values of the vacuum angle with the exception of the one corresponding to zero energy of the bound state in the one-particle electron spectrum.

A configuration of a pointlike monopole with magnetic charge g at the origin is given by the field strength in the form

$$\mathbf{B}(\mathbf{r}) = g\frac{\mathbf{r}}{r^3}, \qquad \boldsymbol{\partial} \cdot \mathbf{B}(\mathbf{r}) = 4\pi g\,\delta^3(\mathbf{r}). \qquad (1)$$

Although in the space outside the monopole (i.e. the punctured space that is characterized by the nontrivial second

homotopy group, $\pi_2 = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integer numbers), the magnetic field satisfies the usual sourceless equation, due to some cohomological obstacles the gauge vector potential can be defined only *locally*. When attempting to extend the local potential to a global single-valued one, a singularity on a half line or otherwise (so-called Dirac string) is inevitably encountered [12]. Namely, the condition of unobservability of the Dirac string yields quantization of monopole charge g.

It should be noted that the Dirac quantization was obtained by Jackiw [13] in a different way, as a consequence of associativity of spatial translations in quantum mechanics. Thus, his derivation is completely gauge invariant, dispensing with reference to a vector potential; moreover, it demonstrates in addition that magnetic monopoles have to be structureless point objects.

Following Wu and Yang [14], one can introduce the patched gauge vector potential which is free of singularities. The punctured space is divided into two overlapping regions, R_a: $0 < \vartheta < \frac{\pi}{2} + \delta$, and R_b: $\frac{\pi}{2} - \delta < \vartheta < \pi$ ($0 \le \vartheta \le \pi$ stands for the azimuthal angle in spherical coordinates, $x = r\sin\vartheta\cos\phi$, $y = r\sin\vartheta\sin\phi$, $z = r\cos\vartheta$, and $0 < \delta < \frac{\pi}{2}$), and the vector potential is defined for each of the regions:

$$\begin{cases} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} = g(1 - \cos\vartheta)d\phi, & \mathbf{r} \in R_a, \\ \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} = -g(1 + \cos\vartheta)d\phi, & \mathbf{r} \in R_b, \end{cases} \qquad (2)$$

then $\boldsymbol{\partial} \times \mathbf{A} = \mathbf{B}$, where $\mathbf{B}$ is given by Eq. (1). In the overlap R_{ab}: $\frac{\pi}{2} - \delta < \vartheta < \frac{\pi}{2} + \delta$, the two potentials are related by gauge transformation

$$\mathbf{A}\big|_a = \mathbf{A}\big|_b + \frac{\mathrm{i}}{e}S_{ab}\boldsymbol{\partial}S_{ab}^{-1}, \qquad (3)$$

with

$$S_{ab} = \mathrm{e}^{2\mathrm{i}eg\phi}, \qquad (4)$$

e is the electron charge. Therefore, the vector potential

serves as a connection on a nontrivial U(1) bundle, and the electron wave function is a section of this bundle, i.e. wave function $\Psi(\mathbf{r}, t)$ is two valued with its values in the overlap R_{ab} related by gauge transformation

$$\Psi|_a = S_{ab}\Psi|_b. \tag{5}$$

Generating function S_{ab} (4) is existing (i.e. single valued) only when

$$eg = \tfrac{1}{2}n, \qquad n \in \mathbb{Z}, \tag{6}$$

which is the celebrated Dirac quantization condition [12] that has already attained its 75-year anniversary.

Schwinger [15] and Zwanziger [16] generalized condition (6) to allow for the possibility of particles that carry both electric and magnetic charges (dyons). A quantum-mechanical theory can have two particles of electric and magnetic charges (Q_1, g_1) and (Q_2, g_2) only if

$$Q_1 g_2 - Q_2 g_1 = \tfrac{1}{2}n. \tag{7}$$

Since the electron has no magnetic charge, the quantization condition says nothing about the electric charge of a dyon. The quantization condition does say something about the difference between the electric charges of two dyons. Given, for instance, two dyons of minimally allowed magnetic charge $g = (2e)^{-1}$ and of electric charges Q_1 and Q_2, one gets

$$Q_1 - Q_2 = ne, \tag{8}$$

but there is no restriction on Q_1 and Q_2 *separately*. If, however, the Dirac-Schwinger-Zwanziger quantization condition, Eq. (7), is supplemented by *CP* conservation, then the allowed values of the electric charge of a dyon are quantized and restricted to be either integer or half-integer in units of e. This is due to the fact that the electric charge is odd and the magnetic charge is even under *CP*.

The effect of *CP* violation was analyzed by Witten [7] in the framework of a spontaneously broken gauge theory at nonzero vacuum angle Θ. By introducing the Θ-term which causes *CP* violation in the Lagrangian,

$$\Delta L = \Theta \frac{e^2}{(8\pi)^2} \varepsilon^{\mu\nu\mu'\nu'} F_{\mu\nu} F_{\mu'\nu'} \tag{9}$$

(here $F_{\mu\nu}$ is the gauge field strength and $\varepsilon^{\mu\nu\mu'\nu'}$ is the totally antisymmetric tensor), he got the expression for the dyon charge:

$$Q = ne - \frac{e\Theta}{2\pi}. \tag{10}$$

On the other hand, the naive Dirac Hamiltonian for the electron in the background of a pointlike magnetic monopole appears to be non-self-adjoint, and an extra boundary condition at the location of the monopole is required for the lowest partial wave in order to implement a self-adjoint extension. The boundary condition depends on self-adjoint extension parameter Θ which violates *CP* invariance. By quantizing the electron field in the monopole background and considering the appropriate vacuum polarization effects, Grossman [8] and Yamagishi [9] got the expression for the induced vacuum charge:

$$Q = -2e|eg| \frac{1}{\pi} \arctan\left(\tan\frac{\Theta}{2}\right), \tag{11}$$

which in the case of the minimal monopole strength, $g = \pm(2e)^{-1}$ agrees with Eq. (10). Thus, in this approach the self-adjoint extension parameter plays the role of the vacuum angle in Witten's approach and the monopole becomes a dyon owing to the vacuum polarization effects.

In the present paper we proceed further and find other quantum numbers of the fermionic vacuum and of the fermionic system in thermal bath in the monopole background. Similar problems were considered for planar fermionic systems in the background of a pointlike magnetic vortex in Refs. [17,18].

II. OPERATORS OF PHYSICAL OBSERVABLES AND THEIR VACUUM AND THERMAL EXPECTATION VALUES

For a given classical static background field configuration, the second-quantized fermion field operator Ψ can be expanded in a complete set of eigenstates of the Dirac (one-particle) Hamiltonian H, see, e.g., Ref. [19],

$$\Psi(\mathbf{r}, t) = \sum_{(E_\lambda > 0)} e^{-iE_\lambda t} \langle \mathbf{r}|\lambda\rangle a_\lambda + \sum_{(E_\lambda < 0)} e^{-iE_\lambda t} \langle \mathbf{r}|\lambda\rangle b_\lambda^\dagger, \tag{12}$$

where

$$H\langle \mathbf{r}|\lambda\rangle = E_\lambda \langle \mathbf{r}|\lambda\rangle \tag{13}$$

is the stationary Dirac equation with eigenvalues of H denoted by E_λ, λ stands for the set of parameters (quantum numbers) specifying a one-particle state, and symbol $\sum$ means the summation over discrete and the integration (with an appropriate measure) over continuous values of λ; $a_\lambda^\dagger$ and a_λ ($b_\lambda^\dagger$ and b_λ) are the fermion (antifermion) creation and destruction operators obeying the anticommutation relations

$$[a_\lambda, a_{\lambda'}^\dagger]_+ = [b_\lambda, b_{\lambda'}^\dagger]_+ = \langle \lambda|\lambda'\rangle, \tag{14}$$

and ground state $|\text{vac}\rangle$ of the second-quantized theory is defined as

$$a_\lambda|\text{vac}\rangle = b_\lambda|\text{vac}\rangle = 0. \tag{15}$$

In the second-quantized theory, the operator of a dynamical variable (physical observable) is given by the integrated commutator,

$$\hat{O}_\Upsilon = \frac{1}{2} \int d^3r \, \text{tr}[\Psi^+(\mathbf{r}, t), \Upsilon\Psi(\mathbf{r}, t)]_-, \tag{16}$$

where Υ is the appropriate one-particle operator in the first-quantized theory, and tr denotes the trace over spinor

indices; in particular, $\hat{O}_H$ is the operator of energy, and $\hat{O}_I$ is the operator of fermion number, where I is the unity matrix in spinor indices. The vacuum expectation value of the observable corresponding to Eq. (16) can be presented as

$$\langle \text{vac}|\hat{O}_\Upsilon|\text{vac}\rangle = -\tfrac{1}{2}\text{Tr}\Upsilon\,\text{sgn}(H), \qquad (17)$$

where

$$\text{sgn}(u) = \{ \begin{matrix} 1, & u > 0 \\ -1, & u < 0 \end{matrix} \},$$

and Tr is the trace of an integrodifferential operator in the functional space: $\text{Tr}U = \int d^3\mathbf{r}\,\text{tr}\langle \mathbf{r}|U|\mathbf{r}\rangle$. The thermal expectation value of the observable is conventionally defined as (see, e.g., Ref. [20])

$$O_\Upsilon(T) = \langle \hat{O}_\Upsilon \rangle_\beta \equiv \frac{\text{Sp}\hat{O}_\Upsilon\exp(-\beta\hat{O}_H)}{\text{Sp}\exp(-\beta\hat{O}_H)}, \qquad \beta = (k_BT)^{-1}, \tag{18}$$

where T is the equilibrium temperature, k_B is the Boltzmann constant, and Sp is the trace or the sum over the expectation values in the Fock state basis in the second-quantized theory. Evidently, the zero-temperature limit of Eq. (18) coincides with Eq. (17):

$$O_\Upsilon(0) = \langle \text{vac}|\hat{O}_\Upsilon|\text{vac}\rangle. \tag{19}$$

Thus, Eq. (18) can be presented in a way similar to that of Eq. (17), i.e., through the functional trace of operators in the first-quantized theory, see, e.g., Ref. [21],

$$O_\Upsilon(T) = -\tfrac{1}{2}\text{Tr}\Upsilon\tanh(\tfrac{1}{2}\beta H). \tag{20}$$

The self-adjointness of the Dirac Hamiltonian ensures the conservation of energy in the second-quantized theory, and the corresponding operator is diagonal in creation and destruction operators,

$$\hat{O}_H = \sum_\lambda E_\lambda\left[a_\lambda^+ a_\lambda - b_\lambda^+ b_\lambda - \frac{1}{2}\text{sgn}(E_\lambda)\right]; \tag{21}$$

the operator of any other conserved observable is diagonal as well.

If at least one of two observables is conserved, then their thermal correlation,

$$\Delta(T;\hat{O}_{\Upsilon_1},\hat{O}_{\Upsilon_2}) = \langle \hat{O}_{\Upsilon_1}\hat{O}_{\Upsilon_2}\rangle_\beta - \langle \hat{O}_{\Upsilon_1}\rangle_\beta\langle \hat{O}_{\Upsilon_2}\rangle_\beta, \tag{22}$$

takes the form

$$\Delta(T;\hat{O}_{\Upsilon_1},\hat{O}_{\Upsilon_2}) = \tfrac{1}{4}\text{Tr}\Upsilon_1\Upsilon_2\text{sech}^2(\tfrac{1}{2}\beta H). \tag{23}$$

In particular, the thermal quadratic fluctuation of a conserved observable takes form

$$\Delta(T;\hat{O}_\Upsilon,\hat{O}_\Upsilon) = \tfrac{1}{4}\text{Tr}\Upsilon^2\text{sech}^2(\tfrac{1}{2}\beta H). \tag{24}$$

It is instructive to present Eqs. (20) and (23) in terms of contour integrals involving the resolvent of the Dirac Hamiltonian:

$$O_\Upsilon(T) = -\frac{1}{2}\int_C\frac{d\omega}{2\pi i}\tanh\left(\frac{1}{2}\beta\omega\right)\text{Tr}\Upsilon(H-\omega)^{-1} \tag{25}$$

and

$$\Delta(T;\hat{O}_{\Upsilon_1},\hat{O}_{\Upsilon_2}) = \frac{1}{4}\int_C\frac{d\omega}{2\pi i}\text{sech}^2\left(\frac{1}{2}\beta\omega\right)\text{Tr}\Upsilon_1\Upsilon_2(H-\omega)^{-1}, \tag{26}$$

where C is the contour consisting of two collinear straight lines, $(-\infty + i0, +\infty + i0)$ and $(+\infty - i0, -\infty - i0)$, in the complex ω-plane. Note that only the even part of $\text{Tr}\Upsilon(H-\omega)^{-1}$ contributes to thermal average $O_\Upsilon(T)$, and only the odd part of $\text{Tr}\Upsilon_1\Upsilon_2(H-\omega)^{-1}$ contributes to thermal correlation $\Delta(T;\hat{O}_{\Upsilon_1},\hat{O}_{\Upsilon_2})$. By deforming contour C to encircle poles of the $\tanh(\tfrac{1}{2}\beta\omega)$ and $\text{sech}^2(\tfrac{1}{2}\beta\omega)$ functions, which occur along the imaginary axis, one gets

$$O_\Upsilon(T) = -\frac{1}{\beta}\sum_{n\in\mathbb{Z}}\text{Tr}\Upsilon(H-i\omega_n)^{-1} \tag{27}$$

and

$$\Delta(T;\hat{O}_{\Upsilon_1},\hat{O}_{\Upsilon_2}) = -\frac{1}{\beta^2}\sum_{n\in\mathbb{Z}}\text{Tr}\Upsilon_1\Upsilon_2(H-i\omega_n)^{-2}, \tag{28}$$

where $\omega_n = (2n+1)\pi/\beta$. Alternatively, by deforming contour C around poles and cuts of the spectrum of H, which lie on the real axis, one gets

$$O_\Upsilon(T) = -\frac{1}{2}\int_{-\infty}^{\infty}dE\tau_\Upsilon(E)\tanh\left(\frac{1}{2}\beta E\right) \tag{29}$$

and

$$\Delta(T;\hat{O}_{\Upsilon_1},\hat{O}_{\Upsilon_2}) = \frac{1}{4}\int_{-\infty}^{\infty}dE\tau_{\Upsilon_1\Upsilon_2}(E)\text{sech}^2\left(\frac{1}{2}\beta E\right), \tag{30}$$

where

$$\tau_\Upsilon(E) = \pm\frac{1}{\pi}\text{Im}\,\text{Tr}\Upsilon(H-E\mp i0)^{-1} \tag{31}$$

and

$$\tau_{\Upsilon_1\Upsilon_2}(E) = \pm\frac{1}{\pi}\text{Im}\,\text{Tr}\Upsilon_1\Upsilon_2(H-E\mp i0)^{-1} \tag{32}$$

are the appropriate spectral densities. Expressions (29) and (30) can be regarded as the Sommerfeld-Watson transforms of the infinite sum expressions, Eqs. (27) and (28). Note that only the odd part of $\tau_\Upsilon(E)$ contributes to $O_\Upsilon(T)$ and only the even part of $\tau_{\Upsilon_1\Upsilon_2}(E)$ contributes to $\Delta(T;\hat{O}_{\Upsilon_1},\hat{O}_{\Upsilon_2})$.

The Dirac Hamiltonian in the background of a static magnetic monopole takes the form

$$H = -\boldsymbol{\alpha}\cdot(i\boldsymbol{\partial} + e\mathbf{A}) + \gamma^0 M, \tag{33}$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, and γ^0, $\boldsymbol{\gamma}$ are the Dirac matrices, M is the electron mass, and $\mathbf{A}$ is given by Eq. (2). The magnetic

monopole background is rotationally invariant and three generators of rotations are identified with the components of vector **J**—the operator of total angular momentum in the first-quantized theory,

$$\mathbf{J} = \mathbf{\Lambda} + \mathbf{\Sigma}, \tag{34}$$

where

$$\mathbf{\Lambda} = -\mathbf{r} \times (i\boldsymbol{\partial} + e\mathbf{A}) - eg\frac{\mathbf{r}}{r} \tag{35}$$

is its orbital part, and

$$\mathbf{\Sigma} = \frac{1}{4i}\boldsymbol{\alpha} \times \boldsymbol{\alpha} \tag{36}$$

is its spin part; note that the last term in Eq. (35) is necessary in order to ensure the correct commutation relations:

$$[J^j, J^k]_- = i\varepsilon^{jkl}J^l.$$

The nonvanishing of any component of the vector vacuum expectation value in the second-quantized theory,

$$O_{\mathbf{J}}(0) = -\frac{1}{2}\mathrm{Tr}\mathbf{J}\,\mathrm{sgn}(H),$$

would point at spontaneous breaking of the rotational invariance (nonuniqueness of the ground state). Even if $O_{\mathbf{J}}(0)$ is vanishing, it may happen that quantity

$$O_{\mathbf{J}^2}(0) = -\frac{1}{2}\mathrm{Tr}\mathbf{J}^2\,\mathrm{sgn}(H) \tag{37}$$

is nonvanishing, which is compatible with the uniqueness of the ground state preserving the rotational invariance in the second-quantized theory. Note also that in the first-quantized theory operators H, $\mathbf{J}^2$, and any component of vector $\mathbf{J}$ are commuting, therefore the corresponding operators in the second-quantized theory can be diagonalized. On the contrary, operators $\hat{O}_{\mathbf{\Lambda}^2}$ and $\hat{O}_{\mathbf{\Sigma}^2}$ are not diagonalizable, and, consequently, quantities

$$O_{\mathbf{\Lambda}^2}(0) = -\frac{1}{2}\mathrm{Tr}\mathbf{\Lambda}^2\mathrm{sgn}(H) \tag{38}$$

and

$$O_{\mathbf{\Sigma}^2}(0) = -\frac{1}{2}\mathrm{Tr}\mathbf{\Sigma}^2\mathrm{sgn}(H) \tag{39}$$

have to be regarded as vacuum averages rather than sharp quantum observables. As to quantity $O_{\mathbf{J}^2}(0)$ (37), one might anticipate that it is a sharp observable, which is substantiated by the fact that its thermal quadratic fluctuation,

$$\Delta(T; \hat{O}_{\mathbf{J}^2}, \hat{O}_{\mathbf{J}^2}) = \frac{1}{4}\mathrm{Tr}\mathbf{J}^4\mathrm{sech}(\frac{1}{4}\beta H), \tag{40}$$

tends to zero in the limit $T \to 0$ ($\beta \to \infty$). However, we shall find out special circumstances when the fluctuation is nonzero at zero temperature and squared total angular momentum is not a sharp observable even at zero temperature.

III. SOLUTIONS TO THE DIRAC EQUATION IN THE MONOPOLE BACKGROUND

The usual spherical harmonics $Y_{lm}(\vartheta, \phi)$ are replaced by the (two-valued with two different values in R_a and R_b—see Sec. I) monopole harmonics $Y_{q,l,m}(\vartheta, \phi)$ [22], since orbital angular momentum, see Eq. (35), differs from the usual one:

$$Y_{q,l,m}(\vartheta, \phi) = M_{qlm}(1 - \cos\vartheta)^{\alpha/2}$$
$$\times (1 + \cos\vartheta)^{\beta/2}P_{l+m}^{\alpha,\beta}(\cos\vartheta)\mathrm{e}^{\mathrm{i}(m\pm q)\phi},$$
$$q = eg, \qquad \alpha = -q - m, \qquad \beta = q - m, \tag{41}$$
$$M_{qlm} = 2^m\sqrt{\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l-q)!(l+q)!}},$$
$$l = |q|, |q| + 1, \ldots, \qquad m = -l, -l+1, \ldots, l,$$

where

$$P_n^{\alpha,\beta}(u) = \frac{(-1)^n}{2^n n!}(1 - u)^{-\alpha}(1 + u)^{-\beta}\frac{d^n}{du^n}$$
$$\times [(1 - u)^{\alpha+n}(1 + u)^{\beta+n}]$$

are the Jacobi polynomials, see, e.g., Ref. [23]. The plus sign is chosen in R_a and the minus sign is chosen in R_b. The nontrivial nature of wave functions is completely embedded in the monopole harmonics.

The eigensections of $\mathbf{J}^2$ and J_z with eigenvalues equal to $j(j + 1)$ and m correspondingly are [24]

$$\varphi_{jm}^{(1)}(\vartheta, \phi) = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}Y_{q,j-(1/2),m-(1/2)}(\vartheta, \phi) \\ \sqrt{\frac{j-m}{2j}}Y_{q,j-(1/2),m+(1/2)}(\vartheta, \phi) \end{pmatrix},$$
$$\varphi_{jm}^{(2)}(\vartheta, \phi) = \begin{pmatrix} -\sqrt{\frac{j-m+1}{2j+2}}Y_{q,j+(1/2),m-(1/2)}(\vartheta, \phi) \\ \sqrt{\frac{j+m+1}{2j+2}}Y_{q,j+(1/2),m+(1/2)}(\vartheta, \phi) \end{pmatrix}, \tag{42}$$

where $j \geq |q| + \frac{1}{2}$ for $\varphi_{jm}^{(1)}$, and $j \geq |q| - \frac{1}{2}$ for $\varphi_{jm}^{(2)}$. One chooses the following linear combinations in the case of $j \geq |q| + \frac{1}{2}$:

$$\xi_{jm}^{(1)}(\vartheta, \phi) = c_j\varphi_{jm}^{(1)}(\vartheta, \phi) - s_j\varphi_{jm}^{(2)}(\vartheta, \phi),$$
$$\xi_{jm}^{(2)}(\vartheta, \phi) = s_j\varphi_{jm}^{(1)}(\vartheta, \phi) + c_j\varphi_{jm}^{(2)}(\vartheta, \phi),$$
$$c_j = \frac{\mathrm{sgn}(q)(\sqrt{2j+1+2q} + \sqrt{2j+1-2q})}{2\sqrt{2j+1}}, \tag{43}$$
$$s_j = \frac{\mathrm{sgn}(q)(\sqrt{2j+1+2q} - \sqrt{2j+1-2q})}{2\sqrt{2j+1}},$$

which satisfy the system of equations

$$-\boldsymbol{\sigma}(\mathrm{i}\boldsymbol{\partial} + e\mathbf{A})h(r)\xi_{jm}^{(1)} = \mathrm{i}(\partial_r + r^{-1} - \mu r^{-1})h(r)\xi_{jm}^{(2)}$$
$$-\boldsymbol{\sigma}(\mathrm{i}\boldsymbol{\partial} + e\mathbf{A})h(r)\xi_{jm}^{(2)} = \mathrm{i}(\partial_r + r^{-1} + \mu r^{-1})h(r)\xi_{jm}^{(1)} \tag{44}$$

for an arbitrary $h(r)$; here and in the following σ^j are the Pauli matrices and

$$\mu = \sqrt{(j + \tfrac{1}{2})^2 - q^2}.\tag{45}$$

In the case of $j = |q| - \frac{1}{2}$ one defines

$$\eta_m(\vartheta, \phi) = \varphi_{jm}^{(2)}(\vartheta, \phi),\tag{46}$$

which satisfies

$$\boldsymbol{\sigma}(i\boldsymbol{\partial} + e\mathbf{A})h(r)\eta_m = i\,\text{sgn}(q)(\partial_r + r^{-1})h(r)\eta_m.\tag{47}$$

In the standard representation for the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$

the spin part of angular momentum (36) is of the block-diagonal form,

$$\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},\tag{48}$$

and the solutions to the Dirac equation (13) in the monopole background are constructed as follows:
type 1 ($j \geq |q| + \frac{1}{2}$):

$$\langle \mathbf{r}|E, j, m\rangle^{(1)}$$

$$= \begin{pmatrix} \sqrt{1 + \frac{M}{E}}\sqrt{\frac{k}{2r}}J_{\mu-(1/2)}(kr)\xi_{jm}^{(1)}(\vartheta, \phi) \\ -i\,\text{sgn}(E)\sqrt{1 - \frac{M}{E}}\sqrt{\frac{k}{2r}}J_{\mu+(1/2)}(kr)\xi_{jm}^{(2)}(\vartheta, \phi) \end{pmatrix},\tag{49}$$

type 2 ($j \geq |q| + \frac{1}{2}$):

$$\langle \mathbf{r}|E, j, m\rangle^{(2)}$$

$$= \begin{pmatrix} \sqrt{1 + \frac{M}{E}}\sqrt{\frac{k}{2r}}J_{\mu+(1/2)}(kr)\xi_{jm}^{(2)}(\vartheta, \phi) \\ i\,\text{sgn}(E)\sqrt{1 - \frac{M}{E}}\sqrt{\frac{k}{2r}}J_{\mu-(1/2)}(kr)\xi_{jm}^{(1)}(\vartheta, \phi) \end{pmatrix},\tag{50}$$

type 3 ($j = |q| - \frac{1}{2}$):

$$\langle \mathbf{r}|E, m\rangle_\Theta = \begin{pmatrix} f(r)\eta_m(\vartheta, \phi) \\ g(r)\eta_m(\vartheta, \phi) \end{pmatrix},\tag{51}$$

where $k = \sqrt{E^2 - m^2}$, $J_\rho(u)$ is the Bessel function of order ρ, and radial functions $f(r)$ and $g(r)$ are divergent, although square integrable, at the origin. That is why the type 3 solution is called irregular, in contrast to types 1 and 2 solutions which are regular at the origin. The procedure of the self-adjoint extension is implemented for the partial Hamiltonian with $j = |q| - \frac{1}{2}$, yielding the boundary con-

dition for the corresponding partial mode [25]:

$$\cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\lim_{r\to 0} rf(r) = i\,\text{sgn}(q)\sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\lim_{r\to 0} rg(r),\tag{52}$$

where Θ is the self-adjoint extension parameter. This gives the explicit form for the radial functions in Eq. (51) [9]

$$f(r) = \frac{i\,\text{sgn}(q)}{r\sqrt{\pi E(E - M\sin\Theta)}}\left[(E + M)\sin kr\cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\right.$$
$$\left. + k\cos kr\sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\right],$$

$$g(r) = \frac{1}{r\sqrt{\pi E(E - M\sin\Theta)}}\left[k\cos kr\cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\right.$$
$$\left. - (E - M)\sin kr\sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\right].\tag{53}$$

If $\cos\Theta < 0$, then there exists in addition a bound state with energy $E_{\text{BS}} = M\sin\Theta$:

$$\langle \mathbf{r}|E_{\text{BS}}, m\rangle_\Theta = \frac{1}{r}\begin{pmatrix} i\,\text{sgn}(q)\sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\eta_m(\vartheta, \phi) \\ \cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)\eta_m(\vartheta, \phi) \end{pmatrix}$$
$$\times \sqrt{-2M\cos\Theta}\,e^{Mr\cos\Theta}.\tag{54}$$

These solutions form a complete orthonormalized set, i.e.,

$$^{(i)}\langle E, j, m|E', j', m'\rangle^{(i')} = \delta_{ii'}\delta_{jj'}\delta_{mm'}\delta_{\text{sgn}(E),\text{sgn}(E')}$$
$$\times \delta(k - k'),$$
$$i, i' = 1, 2,$$
$$_\Theta\langle E, m|E', m'\rangle_\Theta = \delta_{mm'}\delta_{\text{sgn}(E),\text{sgn}(E')}\delta(k - k'),$$
$$_\Theta\langle E_{\text{BS}}, m|E_{\text{BS}}, m'\rangle_\Theta = \delta_{mm'}.\tag{55}$$

IV. INDUCED QUANTUM NUMBERS AT ZERO TEMPERATURE

The vacuum expectation values are calculated using formula (17) and the classical solutions from the previous section. Let us consider an observable which corresponds to an operator in the first-quantized theory of the block-diagonal form,

$$\Upsilon = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix},\tag{56}$$

and containing no derivatives in r:

$$\Omega h(r)\varphi(\vartheta, \phi) = h(r)\Omega\varphi(\vartheta, \phi).\tag{57}$$

For the contribution of the type 1 solutions, one gets

$$^{(1)}\langle E, j, m|\mathbf{r}\rangle\Upsilon\langle\mathbf{r}|E, j, m\rangle^{(1)} = \frac{k}{2r}\left[\left(1 + \frac{M}{E}\right)J^2_{\mu-(1/2)}(kr)\xi^{(1)\dagger}_{jm}\Omega\xi^{(1)}_{jm} + \left(1 - \frac{M}{E}\right)J^2_{\mu+(1/2)}(kr)\xi^{(2)\dagger}_{jm}\Omega\xi^{(2)}_{jm}\right]; \qquad (58)$$

for that of the type 2 solutions, one gets

$$^{(2)}\langle E, j, m|\mathbf{r}\rangle\Upsilon\langle\mathbf{r}|E, j, m\rangle^{(2)} = \frac{k}{2r}\left[\left(1 + \frac{M}{E}\right)J^2_{\mu+(1/2)}(kr)\xi^{(2)\dagger}_{jm}\Omega\xi^{(2)}_{jm} + \left(1 - \frac{M}{E}\right)J^2_{\mu-(1/2)}(kr)\xi^{(1)\dagger}_{jm}\Omega\xi^{(1)}_{jm}\right]. \qquad (59)$$

Summing Eqs. (58) and (59), one gets

$$\sum_{i=1}^{2} {}^{(i)}\langle E, j, m|\mathbf{r}\rangle\Upsilon\langle\mathbf{r}|E, j, m\rangle^{(i)}$$
$$= \frac{k}{r}[J^2_{\mu-(1/2)}(kr)\xi^{(1)\dagger}_{jm}\Omega\xi^{(1)}_{jm} + J^2_{\mu+(1/2)}(kr)\xi^{(2)\dagger}_{jm}\Omega\xi^{(2)}_{jm}], \qquad (60)$$

which is independent of the sign of E; thus, the overall contribution of the types 1 and 2 solutions to Eq. (17) is zero. For the contribution of the type 3 solution (continuous spectrum) to Eq. (17), one gets a nonzero result:

$$-\frac{1}{2}\sum_{\text{sgn}(E)}{}_{\Theta}\langle E, m|\mathbf{r}\rangle\Upsilon\langle\mathbf{r}|E, m\rangle_{\Theta}\text{sgn}(E)$$
$$= \frac{2\eta^\dagger_m\Omega\eta_m kM\sin\Theta}{\pi r^2|E|(k^2 + M^2\cos^2\Theta)}(k\cos2kr + M\sin2kr\cos\Theta). \qquad (61)$$

In order to perform integration over k, one can take parity into account and extend integration from $(0, \infty)$ to $(-\infty, \infty)$, replacing $k\sin kr \rightarrow -ike^{ikr}$, $\cos kr \rightarrow e^{ikr}$. Adding the contribution of the bound state and summing over m, one gets

$$-\frac{1}{2}\text{tr}\langle\mathbf{r}|\Upsilon\,\text{sgn}(H)|\mathbf{r}\rangle = -\frac{M}{2r^2}\sum_{m=-|q|}^{|q|}\eta^\dagger_m\Omega\eta_m$$
$$\times\left\{\cos\Theta[\text{sgn}(\sin2\Theta)\right.$$
$$-\text{sgn}(\sin\Theta)]e^{2Mr\cos\Theta} + \frac{\sin\Theta}{\pi}$$
$$\left.\times\int_{-\infty}^{\infty}dk\frac{ke^{2ikr}}{\sqrt{k^2 + M^2}(k + iM\cos\Theta)}\right\}. \qquad (62)$$

The contour of integration can be deformed to the upper half-plane of complex k to enclose a cut along the imaginary axis at $\text{Im}k > M$ and encircle a pole occurring in the case of $\cos\Theta < 0$ at $\text{Im}k = -M\cos\Theta$. The contribution of the pole cancels that of the bound state, and only the contribution from the cut survives. Averaging over the angular variables yields

$$\rho_\Upsilon(r) = \frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^\pi d\vartheta\sin\vartheta\left(-\frac{1}{2}\right)\text{tr}\langle\mathbf{r}|\Upsilon\text{sgn}(H)|\mathbf{r}\rangle$$
$$= -\sum_{m=-|q|}^{|q|}\int_0^{2\pi}d\phi\int_0^\pi d\vartheta\sin\vartheta\eta^\dagger_m\Omega\eta_m\frac{M\sin\Theta}{(2\pi)^2r^2}$$
$$\times\int_M^\infty d\kappa\frac{\kappa e^{-2\kappa r}}{\sqrt{\kappa^2 - M^2}(\kappa + M\cos\Theta)}, \qquad (63)$$

and the vacuum expectation value takes the form

$$O_\Upsilon(0) = 4\pi\int_0^\infty drr^2\rho_\Upsilon(r)$$
$$= -\sum_{m=-|q|}^{|q|}\int_0^{2\pi}d\phi\int_0^\pi d\vartheta\sin\vartheta\eta^\dagger_m\Omega\eta_m$$
$$\times\frac{1}{\pi}\arctan\left(\tan\frac{\Theta}{2}\right). \qquad (64)$$

In the case of $\Omega = I$, where I is the 2×2 unity matrix, using the orthonormality of η_m's, one gets

$$\sum_{m=-|q|}^{|q|}\int_0^{2\pi}d\phi\int_0^\pi d\vartheta\sin\vartheta\eta^\dagger_m\eta_m = 2|q|, \qquad (65)$$

and the vacuum expectation value of fermion number takes the form [8,9]

$$O_I(0) = -2|eg|\frac{1}{\pi}\arctan\left(\tan\frac{\Theta}{2}\right), \qquad (66)$$

where we have recalled relation $q = eg$. Multiplying Eq. (66) by e, one gets the induced vacuum charge (11). Note that Eq. (66) at $|eg| = \frac{1}{2}$ coincides with the expression for the fermion number which is induced in $2 + 1$-dimensional space-time in the vacuum by a pointlike magnetic vortex with flux $\pi \mod 2\pi$ [17].

It is straightforward to prove that angular momentum, as well as its spin and orbital parts separately, is not induced in the vacuum, and, consequently, rotational invariance is not spontaneously broken. Indeed, since $J_z\eta_m = m\eta_m$, one gets

$$O_{J_z}(0) = 0 \qquad (67)$$

due to summation over m. As to the z-component of orbital angular momentum, this issue is more intricate. Using Eqs. (42) and (46) and relation $\Lambda_z Y_{q,l,m} = mY_{q,l,m}$, one gets

$$\eta_m^\dagger \Lambda_z \eta_m = (2|q| + 1)^{-1}[(|q| + \tfrac{1}{2} - m)$$
$$\times (m - \tfrac{1}{2})|Y_{q,|q|,m-(1/2)}|^2$$
$$+ (|q| + \tfrac{1}{2} + m)(m + \tfrac{1}{2})|Y_{q,|q|,m+(1/2)}|^2].$$

Integating over angular variables yields

$$\int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \, \eta_m^\dagger \Lambda_z \eta_m = \frac{2(|q| + 1)}{2|q| + 1} m,$$

where the orthonormality of the monopole harmonics has been used. Summation over m results in zero, and the same considerations apply to the z-component of spin:

$$O_{\Lambda_z}(0) = O_{\Sigma_z}(0) = 0. \tag{68}$$

Similarly, one can show that components $O_{J_x \pm iJ_y}(0)$, $O_{\Lambda_x \pm i\Lambda_y}(0)$, $O_{\Sigma_x \pm i\Sigma_y}(0)$ vanish. Here again the orthonormality is crucial: roughly speaking, the diagonal matrix elements of raising and lowering operators are equal to zero.

Let us turn now to the vacuum expectation values of the squares of orbital angular momentum, spin, and total angular momentum. Using relation $\Lambda^2 Y_{q,l,m} = l(l + 1)Y_{q,l,m}$, where $l = |q|$, one gets

$$\sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \, \eta_m^\dagger \Lambda^2 \eta_m = 2q^2(|q| + 1). \tag{69}$$

Taking account for relation $\tfrac{1}{4}\sigma^2 = \tfrac{3}{4}I$, one gets immediately

$$\sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \, \eta_m^\dagger \tfrac{1}{4}\sigma^2 \eta_m = \frac{3}{2}|q|. \tag{70}$$

Using relation $\mathbf{J}^2 \eta_m = j(j + 1)\eta_m$, where $j = |q| - \tfrac{1}{2}$, one gets

$$\sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \, \eta_m^\dagger \mathbf{J}^2 \eta_m = 2|q|\left(q^2 - \frac{1}{4}\right). \tag{71}$$

Thus, we get the following expressions for the vacuum expectation values:

$$O_{\Lambda^2}(0) = |eg|(|eg| + 1)O_I(0), \tag{72}$$

$$O_{\Sigma^2}(0) = \tfrac{3}{4}O_I(0), \tag{73}$$

$$O_{\mathbf{J}^2}(0) = [(eg)^2 - \tfrac{1}{4}]O_I(0), \tag{74}$$

where induced vacuum fermion number $O_I(0)$ is given by Eq. (66).

V. SPECTRAL DENSITIES

An alternative and more refined way of treating the induced quantum numbers, which is especially adapted to the case of nonzero temperature, involves the use of spectral densities, see Eqs. (31) and (32). In general, the spectral density is decomposed as

$$\tau_\Upsilon(E) = \tau_\Upsilon^{(0)}(E) + \tau_\Upsilon^{\mathrm{ren}}(E), \tag{75}$$

where

$$\tau_\Upsilon^{(0)}(E) = \pm\frac{1}{\pi} \operatorname{Im} \operatorname{Tr} \Upsilon^{(0)}(H^{(0)} - E \mp i0)^{-1} \tag{76}$$

is the spectral density in the absence of interaction (the operators in the free first-quantized theory are denoted by $H^{(0)}$ and $\Upsilon^{(0)}$), and

$$\tau_\Upsilon^{\mathrm{ren}}(E) = \pm\frac{1}{\pi} \operatorname{Im}[\operatorname{Tr}\Upsilon(H - E \mp i0)^{-1}]_{\mathrm{ren}} \tag{77}$$

is the addition which is due to interaction with the background field; the subscript $^{\mathrm{ren}}$ in the right-hand side of Eq. (77) denotes the renormalization of the functional trace by subtraction:

$$[\operatorname{Tr}\Upsilon(H - \omega)^{-1}]_{\mathrm{ren}} = \operatorname{Tr}\Upsilon(H - \omega)^{-1} - \operatorname{Tr}\Upsilon^{(0)}(H^{(0)} - \omega)^{-1}. \tag{78}$$

To compute $\tau_\Upsilon^{(0)}(E)$, let us consider matrix element

$$\langle \mathbf{r}|\Upsilon^{(0)}(H^{(0)} - \omega)^{-1}|\mathbf{r}'\rangle = \Upsilon^{(0)} \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}$$
$$\times \frac{\boldsymbol{\alpha} \cdot \mathbf{p} + \gamma^0 M + \omega}{\mathbf{p}^2 - \omega^2 + M^2}.$$

Although the integral in the right-hand side of the last equation is divergent, its imaginary part at $\omega = E \pm i0$ in the case of $\mathbf{r}' = \mathbf{r}$ is finite:

$$\pm\operatorname{Im}\langle \mathbf{r}|\Upsilon^{(0)}(H^{(0)} - E \mp i0)^{-1}|\mathbf{r}\rangle$$
$$= \pi \operatorname{sgn}(E) \int \frac{d^3 p}{(2\pi)^3} \delta(\mathbf{p}^2 - E^2 + M^2)$$
$$\times \Upsilon^{(0)}(\boldsymbol{\alpha} \cdot \mathbf{p} + \gamma^0 M + E). \tag{79}$$

Taking $\boldsymbol{\Sigma}$ (36) and $\boldsymbol{\Lambda}^{(0)} = -i\mathbf{r} \times \boldsymbol{\partial}$ in the capacity of $\Upsilon^{(0)}$, one gets immediately

$$\pm \operatorname{Im} \operatorname{tr}\langle \mathbf{r}|\boldsymbol{\Sigma}(H^{(0)} - E \mp i0)^{-1}|\mathbf{r}\rangle$$
$$= \pm\operatorname{Im} \operatorname{tr}\langle \mathbf{r}|\boldsymbol{\Lambda}^{(0)}(H^{(0)} - E \mp i0)^{-1}|\mathbf{r}\rangle = 0,$$

and, consequently,

$$\tau_{\boldsymbol{\Sigma}}^{(0)}(E) = \tau_{\boldsymbol{\Lambda}}^{(0)}(E) = \tau_{\mathbf{J}}^{(0)}(E) = 0. \tag{80}$$

Taking $\boldsymbol{\Sigma}^2$ and $(\boldsymbol{\Lambda}^{(0)})^2$, one gets nonzero results

$$\pm \operatorname{Im} \operatorname{tr} \langle \mathbf{r} | \boldsymbol{\Sigma}^2 (H^{(0)} - E \mp i0)^{-1} | \mathbf{r} \rangle$$

$$= 4\pi \operatorname{sgn}(E) \int \frac{d^3 p}{(2\pi)^3} \delta(p^2 - E^2 + M^2) \frac{3}{4} E$$

$$= \frac{3}{4\pi} |E| (E^2 - M^2)^{1/2} \theta(E^2 - M^2),$$

$$\pm \operatorname{Im} \operatorname{tr} \langle \mathbf{r} | (\boldsymbol{\Lambda}^{(0)})^2 (H^{(0)} - E \mp i0)^{-1} | \mathbf{r} \rangle$$

$$= 4\pi \operatorname{sgn}(E) \int \frac{d^3 p}{(2\pi)^3} \delta(p^2 - E^2 + M^2) (\mathbf{r} \times \mathbf{p})^2 E$$

$$= \frac{2}{3\pi} |E| r^2 (E^2 - M^2)^{3/2} \theta(E^2 - M^2),$$

where $\theta(u) = \frac{1}{2}[1 + \operatorname{sgn}(u)]$. Consequently, we get

$$\tau_{\boldsymbol{\Sigma}^2}^{(0)}(E) = \frac{3}{4\pi^2} V |E| (E^2 - M^2)^{1/2} \theta(E^2 - M^2), \quad (81)$$

$$\tau_{\boldsymbol{\Lambda}^2}^{(0)}(E) = \frac{2}{5\pi^2} \left(\frac{3}{4\pi}\right)^{2/3} V^{5/3} |E| (E^2 - M^2)^{3/2} \theta(E^2 - M^2), \quad (82)$$

$$\tau_{\mathbf{J}^2}^{(0)}(E) = \frac{V}{\pi^2} \left[\frac{2}{5} \left(\frac{3V}{4\pi}\right)^{2/3} (E^2 - M^2) + \frac{3}{4} \right] \times |E| (E^2 - M^2)^{1/2} \theta(E^2 - M^2), \quad (83)$$

where $V = \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \int_0^R dr\, r^2$ is the volume of the spherical box of radius R. In a similar way, one can get

$$\tau_I^{(0)}(E) = \frac{V}{\pi^2} |E| (E^2 - M^2)^{1/2} \theta(E^2 - M^2), \quad (84)$$

$$\tau_{\boldsymbol{\Sigma}^2 \mathbf{J}^2}^{(0)}(E) = \frac{3V}{4\pi^2} \left[\frac{2}{5} \left(\frac{3V}{4\pi}\right)^{2/3} (E^2 - M^2) + \frac{3}{4} \right] \times |E| (E^2 - M^2)^{1/2} \theta(E^2 - M^2), \quad (85)$$

$$\tau_{\boldsymbol{\Lambda}^2 \mathbf{J}^2}^{(0)}(E) = \frac{V}{5\pi^2} \left[\frac{8}{7} \left(\frac{3V}{4\pi}\right)^{4/3} (E^2 - M^2) + \frac{3}{2} \left(\frac{3V}{4\pi}\right)^{2/3} \right] \times |E| (E^2 - M^2)^{3/2} \theta(E^2 - M^2), \quad (86)$$

$$\tau_{\mathbf{J}^4}^{(0)}(E) = \frac{V}{\pi^2} \left[\frac{8}{35} \left(\frac{3V}{4\pi}\right)^{4/3} (E^2 - M^2)^2 \right.$$
$$\left. + \left(\frac{3V}{4\pi}\right)^{2/3} (E^2 - M^2) + \frac{9}{16} \right]$$
$$\times |E| (E^2 - M^2)^{1/2} \theta(E^2 - M^2). \quad (87)$$

It should be noted that Eqs. (81)–(87) are even in E, and, thus, they do not contribute to the expectation values, while contributing to the appropriate correlations and quadratic fluctuations.

Let us turn now to the part of the spectral density, Eq. (77), which is due to interaction with the monopole background, and decompose it in the following way:

$$\tau_\Upsilon^{\text{ren}}(E) = \tau_\Upsilon^{\text{ren}\prime}(E) + \tau_\Upsilon^{(3)}(E), \quad (88)$$

where $\tau_\Upsilon^{(3)}(E)$ is taking account of only the contribution of the type 3 solutions, Eqs. (51), (53), and (54), while $\tau_\Upsilon^{\text{ren}\prime}(E)$ is including the contribution of the types 1 and 2 solutions and subtracted plane wave solutions. In the case of Υ in the block-diagonal form (56) with no derivatives in r (57), the total contribution of the types 1 and 2 solutions is even in E, see Eq. (60), and, thus, one gets

$$\tau_\Upsilon^{\text{ren}\prime}(E) = \tau_\Upsilon^{\text{ren}\prime}(-E). \quad (89)$$

As to the contribution of the type 3 solutions, one, following Ref. [11], obtains their contribution to the trace of resolvent

$$[\operatorname{Tr}\Upsilon(H - \omega)^{-1}]_{(3)} = -\frac{1}{2} \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta$$
$$\times \sin\vartheta \eta_m^\dagger \Omega \eta_m M$$
$$\times \frac{\omega \sin\Theta - M - i\sqrt{\omega^2 - M^2} \cos\Theta}{(\omega^2 - M^2)(\omega - M \sin\Theta)}, \quad (90)$$

where a physical sheet for square root is chosen as $0 < \arg\sqrt{\omega^2 - m^2} < \pi$ $(\operatorname{Im}\sqrt{\omega^2 - m^2} > 0)$. Consequently, we get

$$\tau_\Upsilon^{(3)}(E) = \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \eta_m^\dagger \Omega \eta_m \left[\theta(-\cos\Theta) \right.$$
$$\times \delta(E - M \sin\Theta) - \frac{1}{4} \delta(E - M) - \frac{1}{4} \delta(E + M)$$
$$\left. + \frac{\cos\Theta}{2\pi} \frac{\operatorname{sgn}(E)}{E - M \sin\Theta} \frac{M}{\sqrt{E^2 - M^2}} \theta(E^2 - M^2) \right]. \quad (91)$$

One can conclude that irregular modes contribute both to expectation values and to correlations and fluctuations, and their contribution is finite in the infinite volume limit. On the contrary, the ideal gas (i.e. plane waves) contribution to correlations and fluctuations diverges by power law as $V \to \infty$, see Eqs. (81)–(87).

Using Eq. (91), one gets the following expression for the vacuum expectation value:

$$O_\Upsilon(0) = -\frac{1}{2} \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \eta_m^\dagger \Omega \eta_m$$
$$\times \left[\theta(-\cos\Theta)\operatorname{sgn}(\sin\Theta) + \frac{\sin 2\Theta}{2\pi} \int_1^\infty \frac{dw}{\sqrt{w^2 - 1}} \right.$$
$$\left. \times \frac{1}{w^2 - \sin^2\Theta} \right], \quad (92)$$

which, after performing integration, yields Eq. (64), as it should be expected.

VI. INDUCED QUANTUM NUMBERS AT NONZERO TEMPERATURE, THERMAL CORRELATIONS, AND QUADRATIC FLUCTUATIONS

Using the results of the preceding section, we get the following expression for the thermal expectation value (29), compare with Eq. (92):

$$
O_\Upsilon(T) = -\frac{1}{2} \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \eta_m^\dagger \Omega \eta_m
$$
$$
\times \left[\theta(-\cos\Theta) \tanh\left(\frac{1}{2}\beta M \sin\Theta\right) \right.
$$
$$
\left. + \frac{\sin 2\Theta}{2\pi} \int_1^\infty \frac{dw}{\sqrt{w^2-1}} \frac{\tanh(\frac{1}{2}\beta M w)}{w^2 - \sin^2\Theta} \right]. \quad (93)
$$

Taking the inverse Sommerfeld-Watson transformation, see Eq. (27), we get the infinite sum representation of Eq. (93):

$$
O_\Upsilon(T) = - \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta \sin\vartheta \eta_m^\dagger \Omega \eta_m \beta M
$$
$$
\times \sin\Theta \sum_{n=0}^\infty [(2n+1)^2 \pi^2 + \beta^2 M^2
$$
$$
+ \beta M \cos\Theta \sqrt{(2n+1)^2 \pi^2 + \beta^2 M^2}]^{-1}. \quad (94)
$$

In the case of $\Upsilon = I$, one gets induced fermion number [10,11]

$$
O_I(T) = -|eg| \left[\theta(-\cos\Theta) \tanh\left(\frac{1}{2}\beta M \sin\Theta\right) \right.
$$
$$
\left. + \frac{\sin 2\Theta}{2\pi} \int_1^\infty \frac{dw}{\sqrt{w^2-1}} \frac{\tanh(\frac{1}{2}\beta M w)}{w^2 - \sin^2\Theta} \right]
$$
$$
= -2|eg|\beta M \sin\Theta \sum_{n=0}^\infty [(2n+1)^2 \pi^2 + \beta^2 M^2
$$
$$
+ \beta M \cos\Theta \sqrt{(2n+1)^2 \pi^2 + \beta^2 M^2}]^{-1}; \quad (95)
$$

note that Eq. (95) at $|eg| = \frac{1}{2}$ coincides with the expression for fermion number which is induced in $2 + 1$-dimensional space-time at nonzero temperature by a pointlike magnetic vortex with flux $\pi \bmod 2\pi$ [18].

All other quantum numbers are related to Eq. (95): squared orbital angular momentum

$$
O_{\Lambda^2}(T) = |eg|(|eg|+1)O_I(T), \quad (96)
$$

squared spin

$$
O_{\Sigma^2}(T) = \tfrac{3}{4}O_I(T), \quad (97)
$$

and squared total angular momentum

$$
O_{\mathbf{J}^2}(T) = [(eg)^2 - \tfrac{1}{4}]O_I(T); \quad (98)
$$

incidentally, one gets

$$
O_\Lambda(T) = O_\Sigma(T) = O_{\mathbf{J}}(T) = 0. \quad (99)
$$

Let us turn now to thermal correlations and quadratic fluctuations of observables. As it was shown in the previous section, the ideal gas contribution (denoted by superscript $^{(0)}$) is prevailing over the contribution (denoted by superscript $^{\text{ren}}$) which is due to interaction with the monopole background, since the former is increasing, while the latter is constant as the volume of the system increases. Using Eqs. (81)–(87), one gets:

quadratic fluctuation of fermion number

$$
\Delta(T; \hat{O}_I, \hat{O}_I) = \frac{1}{4\pi^2} \frac{V}{\beta^3} \int_{\beta^2 M^2}^\infty ds \frac{(s - \beta^2 M^2)^{1/2}}{\cosh^2(\frac{1}{2}\sqrt{s})}, \quad (100)
$$

correlation of fermion number and squared spin

$$
\Delta(T; \hat{O}_{\Sigma^2}, \hat{O}_I) = \frac{3}{16\pi^2} \frac{V}{\beta^3} \int_{\beta^2 M^2}^\infty ds \frac{(s - \beta^2 M^2)^{1/2}}{\cosh^2(\frac{1}{2}\sqrt{s})}, \quad (101)
$$

correlation of fermion number and squared orbital angular momentum

$$
\Delta(T; \hat{O}_{\Lambda^2}, \hat{O}_I) = \frac{1}{10\pi^2} \left(\frac{3}{4\pi}\right)^{2/3} \frac{V^{5/3}}{\beta^5}
$$
$$
\times \int_{\beta^2 M^2}^\infty ds \frac{(s - \beta^2 M^2)^{3/2}}{\cosh^2(\frac{1}{2}\sqrt{s})}, \quad (102)
$$

correlation of fermion number and squared total angular momentum

$$
\Delta(T; \hat{O}_{\mathbf{J}^2}, \hat{O}_I) = \frac{1}{10\pi^2} \left(\frac{3}{4\pi}\right)^{2/3} \frac{V^{5/3}}{\beta^5}
$$
$$
\times \int_{\beta^2 M^2}^\infty ds \frac{(s - \beta^2 M^2)^{3/2}}{\cosh^2(\frac{1}{2}\sqrt{s})}, \quad (103)
$$

correlation of squared total angular momentum and squared spin

$$
\Delta(T; \hat{O}_{\Sigma^2}, \hat{O}_{\mathbf{J}^2}) = \frac{3}{40\pi^2} \left(\frac{3}{4\pi}\right)^{2/3} \frac{V^{5/3}}{\beta^5}
$$
$$
\times \int_{\beta^2 M^2}^\infty ds \frac{(s - \beta^2 M^2)^{3/2}}{\cosh^2(\frac{1}{2}\sqrt{s})}, \quad (104)
$$

correlation of squared total angular momentum and squared orbital angular momentum

$$\Delta(T; \hat{O}_{\Lambda^2}, \hat{O}_{\mathbf{J}^2}) = \frac{2}{35\pi^2}\left(\frac{3}{4\pi}\right)^{4/3}\frac{V^{7/3}}{\beta^7}$$

$$\times \int_{\beta^2 M^2}^{\infty} ds \frac{(s - \beta^2 M^2)^{5/2}}{\cosh^2(\frac{1}{2}\sqrt{s})}, \quad (105)$$

quadratic fluctuation of squared total angular momentum

$$\Delta(T; \hat{O}_{\mathbf{J}^2}, \hat{O}_{\mathbf{J}^2}) = \frac{2}{35\pi^2}\left(\frac{3}{4\pi}\right)^{4/3}\frac{V^{7/3}}{\beta^7}$$

$$\times \int_{\beta^2 M^2}^{\infty} ds \frac{(s - \beta^2 M^2)^{5/2}}{\cosh^2(\frac{1}{2}\sqrt{s})}, \quad (106)$$

where only the leading powers of volume in the large volume limit are retained.

In the high-temperature limit induced quantum numbers tend to zero as inverse temperature:

$$O_Y(T \to \infty) = -\frac{1}{8}\beta M \sin\Theta \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi$$

$$\times \int_0^{\pi} d\vartheta \sin\vartheta \eta_m^{\dagger} \Omega \eta_m, \quad (107)$$

whereas fluctuations and correlations increase as powers of temperature:

$$\Delta(T \to \infty; \hat{O}_I, \hat{O}_I) = \frac{4}{3}\Delta(T \to \infty; \hat{O}_{\Sigma^2}, \hat{O}_I) = \frac{1}{3}\frac{V}{\beta^3}, \quad (108)$$

$$\Delta(T \to \infty; \hat{O}_{\Lambda^2}, \hat{O}_I) = \Delta(T \to \infty; \hat{O}_{\mathbf{J}^2}, \hat{O}_I)$$

$$= \frac{4}{3}\Delta(T \to \infty; \hat{O}_{\Sigma^2}, \hat{O}_{\mathbf{J}^2})$$

$$= \frac{7\pi}{5^2}\left(\frac{\pi}{6}\right)^{1/3}\frac{V^{5/3}}{\beta^5}, \quad (109)$$

$$\Delta(T \to \infty; \hat{O}_{\Lambda^2}, \hat{O}_{\mathbf{J}^2}) = \Delta(T \to \infty; \hat{O}_{\mathbf{J}^2}, \hat{O}_{\mathbf{J}^2})$$

$$= \frac{31\pi^3}{5 \times 7^2}\left(\frac{6}{\pi}\right)^{1/3}\frac{V^{7/3}}{\beta^7}. \quad (110)$$

Thermal expectation value (93) can be presented as

$$O_Y(T) = O_Y(0) + O_Y^{(\Delta)}(T), \quad (111)$$

where $O_Y(0)$ is given by Eq. (64), and

$$O_Y^{(\Delta)}(T) = \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^{\pi} d\vartheta \sin\vartheta \eta_m \Omega \eta_m$$

$$\times \left\{ \frac{\theta(-\cos\Theta)\, \mathrm{sgn}_0(\sin\Theta)}{\exp(\beta M|\sin\Theta|) + 1} + \frac{\beta M}{4\pi} \right.$$

$$\left. \times \int_1^{\infty} dw \frac{\arctan[(1 - w^{-2})^{1/2}\tan\Theta]}{\cosh^2(\frac{1}{2}\beta M w)} \right\}, \quad (112)$$

where

$$\mathrm{sgn}_0(u) = \left\{ \begin{array}{ll} \mathrm{sgn}(u), & u \neq 0 \\ 0, & u = 0 \end{array} \right\}.$$

One can verify that relation $O_Y^{(\Delta)}(T)|_{\Theta = \pi \bmod 2\pi} = 0$ holds, and, thus, Eq. (112) vanishes exponentially in the zero-temperature limit (as $e^{-\beta M}$ at $\beta \to \infty$) for all values of Θ.

At a first glance, one may anticipate that also thermal correlations and quadratic fluctuations vanish exponentially in this limit for all values of Θ, since the prevailing ideal gas contribution is Θ-independent. However, the bound state with zero energy [$E_{BS} = 0$, i.e., $\Theta = \pi \bmod 2\pi$, see Eq. (54)] in the one-particle spectrum reveals itself in a completely different manner, as compared to Eq. (112). In the zero-temperature limit, both the ideal gas contribution and the renormalized contribution of the types 1 and 2 solutions to correlations and fluctuations vanish exponentially, whereas the contribution of the type 3 solutions behaves otherwise: the bound state pole in spectral density (91) is not exponentially damped in this limit if the bound state energy is zero. In general, we get

$$\Delta(T \to 0; \hat{O}_{Y_1}, \hat{O}_{Y_2}) = \sum_{m=-|q|}^{|q|} \int_0^{2\pi} d\phi \int_0^{\pi} d\vartheta$$

$$\times \sin\vartheta \eta_m^{\dagger} \Omega_1 \Omega_2 \eta_m$$

$$\times \left\{ \begin{array}{ll} 0, & \Theta \neq \pi \bmod 2\pi \\ \frac{1}{4}, & \Theta = \pi \bmod 2\pi. \end{array} \right. \quad (113)$$

In particular, the zero-temperature limits of the quadratic fluctuations of fermion number and squared total angular momentum are

$$\Delta(T \to 0; \hat{O}_I, \hat{O}_I) = \left\{ \begin{array}{ll} 0, & \Theta \neq \pi \bmod 2\pi \\ \frac{1}{2}|eg|, & \Theta = \pi \bmod 2\pi \end{array} \right. \quad (114)$$

and

$$\Delta(T \to 0; \hat{O}_{\mathbf{J}^2}, \hat{O}_{\mathbf{J}^2})$$

$$= \left\{ \begin{array}{ll} 0, & \Theta, \neq \pi \bmod 2\pi \\ \frac{1}{2}|eg|[(eg)^2 - \frac{1}{4}]^2, & \Theta = \pi \bmod 2\pi \end{array} \right. \quad (115)$$

VII. SUMMARY

It is well known [7–11] that the vacuum and thermal fluctuations of the quantized Dirac electron field in the background of a pointlike magnetic monopole result in the monopole becoming a dyon with electric charge eO_I depending on the *CP* violating vacuum angle, see Eqs. (66) and (95). In the present study we find out that, in addition to charge, also other quantum numbers are induced in the monopole background. These comprise squares of orbital angular momentum, spin, and total angular momentum, and we show that they are related to charge, see Eqs. (72)–(74) and (96)–(98). The density of induced quantum numbers is considerable around a monopole in the region of order of the Compton size of the electron, decreasing exponentially at larger distances (as $r^{-5/2}e^{-2Mr}$ at $r \to \infty$), see Eq. (63).

The conserved observables are charge and squared total angular momentum; note that the latter vanishes in the case of the minimal monopole strength, $|eg| = \frac{1}{2}$. We analyze thermal correlations between conserved and nonconserved observables and thermal quadratic fluctuations of conserved observables, and find out that these quantities at nonzero temperature are given by the ideal gas expressions, see Eqs. (100)–(106), and, thus, are Θ-independent and proportional to the powers of spatial volume. The interaction with the monopole background reveals itself at zero temperature, yielding a Θ-dependence of a specific type, which is due to a possibility of appearance of a bound state with zero energy in the one-particle electron spectrum, see Eq. (113). This fact has immediate consequences when we turn to a question: whether the values of charge and squared total angular momentum at zero temperature are observed in a single quantum measurement, or whether they are to be regarded as expected averages of many such measurements.

We recall that *CP* invariance is violated, unless

$$\Theta = n\pi. \qquad (116)$$

Induced vacuum quantum numbers, as functions of the vacuum angle, are discontinuous at points $\Theta = \pi \bmod 2\pi$ (i.e. when the bound state with zero energy appears in the one-particle electron spectrum), otherwise they are continuous, vanishing at points $\Theta = 0 \bmod 2\pi$. Since the electric charge of a dyon in the case of *CP* conservation can be either integer or half-integer in units of e, this dictates that the induced charge and all other

quantum numbers in the case of Eq. (116) have to take the same, i.e. equal to zero, values. In other reasoning, it is sufficient to choose range $|\Theta| \leq \pi$, where end points $\Theta = \pi$ and $\Theta = -\pi$ have to be equivalent, and the equivalence obliges to choose the mean between the right and left limiting values, i.e. zero value for the induced quantum number. Also, if we start from nonzero temperatures, when the induced quantum numbers are continuous in Θ everywhere, see Eq. (93) or (94), and tend temperature to zero, then we get the induced vacuum quantum numbers which are vanishing at $\Theta = \pi \bmod 2\pi$. However, as it follows from the expressions for quadratic fluctuations at zero temperature, Eqs. (114) and (115), charge and squared total angular momentum are sharp observables (quantum-mechanical eigenvalues), unless $\Theta = \pi \bmod 2\pi$. Thus, *CP* conserving values of the vacuum angle, Eq. (116), differ significantly: in contrast to the case of $\Theta = 0 \bmod 2\pi$, charge and squared total angular momentum in the case of $\Theta = \pi \bmod 2\pi$ are expected average values, not eigenvalues.

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[1] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[2] R. Jackiw and J. R. Schrieffer, Nucl. Phys. **B190**, 253 (1981).
[3] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[4] A. J. Niemi and G. W. Semenoff, Phys. Rep. **135**, 99 (1986).
[5] M. B. Paranjape, Phys. Rev. Lett. **55**, 2390 (1985); Phys. Rev. D **36**, 3766 (1987).
[6] E. Farhi, N. Graham, R. L. Jaffe, and H. Weigel, Nucl. Phys. **B595**, 536 (2001).
[7] E. Witten, Phys. Lett. **86B**, 283 (1979).
[8] B. Grossman, Phys. Rev. Lett. **50**, 464 (1983).
[9] H. Yamagishi, Phys. Rev. D **27**, 2383 (1983).
[10] C. Coriano and R. R. Parwani, Phys. Lett. B **363**, 71 (1995); A. S. Goldhaber, R. Parwani, and H. Singh, Phys. Lett. B **386**, 207 (1996).
[11] G. Dunne and J. Feinberg, Phys. Lett. B **477**, 474 (2000).
[12] P. A. M. Dirac, Proc. R. Soc. A **133**, 60 (1931).
[13] R. Jackiw, Phys. Rev. Lett. **54**, 159 (1985); Int. J. Mod. Phys. A **19S1**, 137 (2004).
[14] T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).
[15] J. Schwinger, Phys. Rev. **144**, 1087 (1966); **173**, 1536 (1968).
[16] D. Zwanziger, Phys. Rev. **176**, 1480 (1968); **176**, 1489 (1968).
[17] Yu. A. Sitenko, Phys. Lett. B **387**, 334 (1996); Phys. At. Nucl. **60**, 2102 (1997); Yad. Fiz. **60**, 2285 (1997); **62**, 1898 (1999); [Phys. At. Nucl. **62**, 1767 (1999)].
[18] Yu. A. Sitenko and V. M. Gorkavenko, Nucl. Phys. **B679**, 597 (2004); **B714**, 217 (2005); Yu. A. Sitenko and N. D. Vlasii, hep-th/0608216 [Ann. Phys. (to be published)].
[19] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).
[20] A. Das, *Finite Temperature Field Theory* (World Scientific, Singapore, 1997).
[21] A. J. Niemi, Nucl. Phys. **B251**, 155 (1985).
[22] T. T. Wu and C. N. Yang, Nucl. Phys. **B107**, 365 (1976).
[23] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[24] Y. Kazama, C. N. Yang, and A. S. Goldhaber, Phys. Rev. D **15**, 2287 (1977).
[25] A. S. Goldhaber, Phys. Rev. D **16**, 1815 (1977); C. J. Callias, Phys. Rev. D **16**, 3068 (1977).