

Integrable models and confinement in (2 + 1)-dimensional weakly-coupled Yang-Mills theory

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We generalize the (2 + 1)-dimensional Yang-Mills theory to an anisotropic form with two gauge coupling constants e and e' . In axial gauge, a regularized version of the Hamiltonian of this gauge theory is $H_0 + e'^2 H_1$, where H_0 is the Hamiltonian of a set of (1 + 1)-dimensional principal chiral sigma nonlinear models and H_1 couples charge densities of these sigma models. We treat H_1 as the interaction Hamiltonian. For gauge group SU(2), we use form factors of the currents of the principal chiral sigma models to compute the string tension for small e' , after reviewing exact S-matrix and form-factor methods. In the anisotropic regime, the dependence of the string tension on the coupling constants is not in accord with generally-accepted dimensional arguments.

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I. INTRODUCTION

In this paper we calculate the string tension of pure (2 + 1)-dimensional Yang-Mills theory at weak coupling. This is done in an anisotropic version of the theory, with two different, but small, coupling constants. The result clearly establishes that confinement is not restricted to the strong-coupling region.

The method used is a development of the proposal of Ref. [1], which in turn was inspired by Mandelstam's arguments for confinement in QCD [2]. The starting point is a regularized Hamiltonian in an axial gauge, $A_1 = 0$, where A_j , $j = 1, 2$ is the Lie-algebra valued gauge field (the component A_0 is also set to zero). The Hamiltonian may be written as a sum of two terms, namely

$$H_0 = \int d^2x \left(\frac{e^2}{2} \text{Tr} \mathcal{E}_2^2 + \frac{1}{2e^2} \text{Tr} \mathcal{B}^2 \right),$$

and

$$H_1 = \frac{e'^2}{2} \int d^2x \text{Tr} \mathcal{E}_1^2,$$

where \mathcal{E}_j , $j = 1, 2$ are the components of the electric field and \mathcal{B} is the magnetic field $\mathcal{B} = i[\partial_1 - iA_1, \partial_2 - iA_2] = \partial_1 A_2$. Though we may write $\mathcal{E}_2 = -i\delta/\delta A_2$, in the Schrödinger representation, the formula for \mathcal{E}_1 is a nonlocal expression, obtained from Gauss's law [2]

$$\begin{aligned} \mathcal{E}_1(x) &= - \int^{x^1} dy^1 [\partial_2 - iA_2(y^1, x^2), \mathcal{E}_2(y^1, x^2)] \\ &= - \int^{x^1} dy^1 \mathcal{D}_2(y^1, x^\perp) \cdot \mathcal{E}_2(y^1, x^2), \end{aligned} \quad (1.1)$$

where \mathcal{D}_2 is the adjoint covariant derivative in the two-

direction. The local form of Gauss's law is explicitly satisfied with (1.1), provided a residual gauge invariance

$$\int dx^1 \mathcal{D}_2 \mathcal{E}_2 \Psi = 0, \quad (1.2)$$

is satisfied by physical states Ψ (this condition must be modified slightly if quarks are present).

If the theory is regularized on a lattice, H_0 is the Hamiltonian for a set of decoupled $SU(N) \times SU(N)$ principal chiral nonlinear sigma models. These sigma models are coupled together by the interaction Hamiltonian H_1 . We shall treat the coefficient of H_1 as small, meaning that we consider an anisotropic modification,

$$H_1 = \frac{(e')^2}{2} \int d^2x \text{Tr} \mathcal{E}_1^2.$$

By assuming $e' \ll e$, simple arguments were made that the theory is in a confining phase [1]. It was also suggested that Rayleigh-Schrödinger perturbation theory in e' was infrared finite—but it is now clear to the author that this is not the case. A better understanding is needed to obtain quantitative results.

In this paper, we shall work with gauge group SU(2) and exploit exact knowledge of the form factors of the (1 + 1)-dimensional $O(4) \simeq SU(2) \times SU(2)$ nonlinear sigma model [3]. The expressions for the form factors are valid for small e , as we shall show, so both gauge couplings must be small. After reintroducing the component A_0 of the gauge field, we integrate out A_2 using these form factors. We determine the resulting effective action of A_0 to leading order. This effective action is infrared finite.

We shall find a result for the string tension in the x^1 -direction which is different from that in Ref. [1] and will discuss how the difference comes about. We do not calculate the string tension in the x^2 -direction here, which is still under investigation.

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In the course of these investigations, we discovered a paper by Griffin which suggests a program very similar to ours [4]. Evidently, his ideas were taken no further. Griffin's proposal was for the light-cone gauge, instead of the axial gauge, but the two are very similar in certain respects.

We should say a few words about other analytic approaches to Yang-Mills theories in $(2 + 1)$ -dimensions. Probably the best known paper is that of Feynman [5]. Though some of the mathematical details of Feynman's paper are not correct, evidence for some of his conjectures were found in Ref. [6], which utilized some methods introduced in Ref. [7]. In particular, the low-magnetic-energy region of configuration space was argued to be bounded in $(2 + 1)$ dimensions (it is unbounded in $(3 + 1)$ dimensions [7]), which is indicative of a mass gap.

An approach, which at first appears rather different, was developed by Karabali and Nair [8] in which new coordinates for configuration space were devised. A rederivation of one form of the Hamiltonian discussed in Ref. [8] was done in [9] using a kind of non-Abelian exterior differentiation; in this way a connection was made with the ideas of Refs. [6,7].

The major achievement of Ref. [8] was the finding of a mass gap at strong coupling and a *resummation* of the strong-coupling expansion. This resummation was used to obtain a string tension which is independent of the ultraviolet cut-off. We should mention, however, that very similar results can be obtained by analytic lattice methods. The strong-coupling spectrum in the lattice Hamiltonian formalism has the same dependence on the continuum coupling as in Karabali and Nair's formalism. A different resummation method on the lattice, due to Greensite [10] also yields the strong-coupling vacuum wave functional. The form of Greensite's vacuum wave functional in the continuum limit in $2 + 1$ dimensions is

$$\Psi_0 = \exp - \frac{1}{4(N - 1/N)e^4} \int d^2x \text{Tr}[F_{ij}(x)]^2.$$

The wave functional that Karabali and Nair obtained has the form, in the infrared limit,

$$\Psi_0 = \exp - \frac{\pi}{C_A e^4} \int d^2x \text{Tr}[F_{ij}(x)]^2,$$

where C_A is the quadratic Casimir for the adjoint representation. Both vacua yield a string tension proportional to e^4 . It does not seem to have been noticed before that the essential features of Karabali and Nair's strong-coupling expansion and those of the lattice strong-coupling expansion worked out by Greensite are the same.

To obtain a genuine proof of confinement through strong-coupling methods, it must be shown that the strong-coupling expansion has a finite radius of convergence in the $1/e$ or an infinite radius of convergence in the dimen-

sionless coupling $1/g_0$. This has not been accomplished yet.

Recently a new approach has appeared [11], which uses a scheme to improve a Gaussian *Ansatz* for the vacuum and excited-state wave functionals in the variables of reference [8].

What distinguishes the approach of [1] and this paper from other work is that does not exploit strong-coupling approximations or any *Ansatz* for wave functionals. We do make assumptions, which are quite different from the assumption of reference [11]. We assume that an anisotropic weak-coupling expansion makes sense and that the result for the $SU(2) \times SU(2)$ principal-chiral-sigma-model exact form factor [3] (which has been checked in the $1/N$ expansion for the $O(4)$ formulation) is correct. We think that our approach leaves no doubt that confinement occurs at weak coupling, granting that the anisotropy is something we would like to get beyond. Or perhaps not—the anisotropic theory is asymptotically free and not finite like the isotropic theory. In this respect it is more like real QCD in $3 + 1$ dimensions.

By treating the coefficient of $\text{Tr}\mathcal{E}_1^2$ (instead of $\text{Tr}\mathcal{B}^2$) as small, our method is inherently a weak-coupling approach. We think much insight can be gained by working systematically at small coupling. Furthermore, to solve the much harder problem of QCD in $(3 + 1)$ dimensions, a weak-coupling understanding is essential.

There is a well-accepted argument concerning how the mass gap M and string tension σ depend on the coupling constant (see, for example, Refs. [8,12]). We find that such an argument fails in the anisotropic weak-coupling regime. The argument goes as follows: Yang-Mills theory is a perturbatively-ultraviolet-finite field theory (there are severe infrared divergences, but let us ignore these). Thus, after being suitably regularized, the coupling has a nonzero finite value, as the ultraviolet regulator is removed. Since the coupling squared e^2 , has units of cm^{-1} , the mass gap must behave as $M \sim e^2$ and the string tension must behave as $\sigma \sim e^4$. In our anisotropic case, the same dimensional reasoning implies

$$M \sim \sum_p C_p e^{2-p}(e')^p, \quad \sigma \sim \sum_P K_P e^{4-P}(e')^P, \quad (1.3)$$

for some set of numbers p and P and dimensionless constants C_p and K_P . Our final answer for the string tension is quite different from (1.3). We show in Sec. VI that for two quarks separated in the x^1 -direction,

$$\sigma \sim \frac{e^2}{a} \exp - \frac{4\pi}{e^2 a},$$

where a is a short-distance cutoff. We believe that for $e = e'$ the dimensional argument does yield the right answer, but that there is a crossover phenomenon between (1.3) to the behavior we find in the anisotropic regime.

Another application of exact form factors to the (2 + 1)-dimensional SU(2) gauge theory has just appeared [13]. In this work, the form factors of the two-dimensional Ising model are used to find the profile of the electric string, near the high-temperature deconfining transition, assuming the Svetitsky-Yaffe conjecture.

An interesting question is the value of k -string tensions for gauge group SU(N) (see reference [14] for detailed review of this matter). This is not discussed in this paper, since the gauge group is SU(2). Thus the value of k is always one. Another question is whether adjoint sources are confined. Both of these issues are addressed in a new paper [15]. The sine law is clearly seen for the vertical string tensions. The situation is less clear for horizontal string tensions; at zeroth order in g_0' there is a Casimir law, but there are corrections. We are unable to calculate these corrections, because we do not know the form factors for SU(N) \times SU(N) principal chiral sigma models. Adjoint sources are shown not to be confined.

In the next section, we discuss the regularized Hamiltonian. In Section III, we go to the axial gauge and split this Hamiltonian into the Hamiltonians of (1 + 1)-dimensional O(4) \simeq SU(2) \times SU(2) nonlinear sigma models H_0 and a nonlocal term H_1 . We discuss how to find the effective action of the temporal gauge field in terms of correlators of the nonlinear sigma model in Sec. IV. In Sec. V, we determine the leading-order effective action using the exact form factors of the O(N) nonlinear sigma model in (1 + 1)-dimensions. The static potential is then found between two quarks separated in the x^1 -direction in Sec. VI. The physical picture of confinement of glueball excitations and quarks separated in the x^2 -direction is presented in Sec. VII. We discuss some future endeavors in Sec. VIII. We give a review for non-experts on the exact S-matrix [16] and form factors [3] of the (1 + 1)-dimensional O(N) nonlinear sigma model in the appendix.

II. THE REGULARIZED HAMILTONIAN

We will quickly review the Kogut-Susskind Hamiltonian formulation of lattice gauge theory. If the reader finds this discussion incomplete, we refer him or her to the book by Creutz [17].

Consider a lattice of sites x of size $L^1 \times L^2$, with sites x whose coordinates are x^1 and x^2 . We require that x^1/a and x^2/a are integers, where a is the lattice spacing. There are 2 space directions, labeled $j = 1, 2$. Each link is a pair x, j , and joins the site x to $x + \hat{j}a$, where \hat{j} is a unit vector in the j th direction.

We use generators t_b , $b = 1, 2, 3$, of the Lie algebra of SU(2), which are related to the Pauli matrices by $t^b = \sigma^b/\sqrt{2}$. The identity matrix will be denoted by 1.

For now, the Hamiltonian lattice gauge theory will be in the temporal gauge $A_0 = 0$. The basic degrees of freedom, before any further gauge fixing, are elements of the group

SU(2) in the fundamental (2 \times 2)-dimensional matrix representation $U_j(x) \in$ SU(2) at each link x, j . The relation between these variables and the continuum gauge field is $U_j(x) = e^{-iaA_j(x)}$. There are three self-adjoint electric-field operators at each link $l_j(x)_b$, $b = 1, 2, 3$. The commutation relations on the lattice are

$$\begin{aligned} [l_j(x)_b, l_k(y)_c] &= i\sqrt{2}\delta_{xy}\delta_{jk}\epsilon^{bcd}l_j(x)_d, \\ [l_j(x)_b, U_k(y)] &= -\delta_{xy}\delta_{jk}t_b U_j(x), \end{aligned} \quad (2.1)$$

all others zero.

The Hamiltonian is

$$\begin{aligned} H &= \sum_x \sum_{j=1}^2 \sum_{b=1}^3 \frac{g_0^2}{2a} [l_j(x)_b]^2 \\ &\quad - \sum_x \frac{1}{2g_0^2 a} [\text{Tr}U_{12}(x) + \text{Tr}U_{21}(x)], \end{aligned} \quad (2.2)$$

where

$$U_{jk}(x) = U_j(x)U_k(x + \hat{j}a)U_j(x + \hat{k}a)^\dagger U_k(x)^\dagger,$$

where \hat{j} and \hat{k} are the unit vectors in the j - and k -directions, respectively, and the bare coupling constant g_0 is dimensionless. The coefficient of the kinetic term is just half the square of the continuum coupling constant e , namely $g_0^2/(2a) = e^2/2$. This is why the mass gap in (2 + 1) dimensions is proportional to e^2 in strong-coupling expansions.

We denote the adjoint representation of the SU(2) gauge field by $\mathcal{R}_j(x)$. The precise definition is $\mathcal{R}_j(x)_b{}^c t_c = U_j(x)t_b U_j(x)^\dagger$. The matrix \mathcal{R}_j lies in the group SO(3).

Color charge operators $q(x)_b$, may be placed at lattice sites. These obey

$$[q(x)_b, q(y)_c] = i\sqrt{2}\epsilon^{bca}\delta_{xy}q(x)_a. \quad (2.3)$$

In the presence of static charges, Gauss's law is the condition on physical wave functions

$$[(\mathcal{D} \cdot l)(x)_b - q(x)_b]\Psi(\{U\}) = 0. \quad (2.4)$$

where

$$[\mathcal{D}_j l_j(x)]_b = l_j(x)_b - \mathcal{R}_j(x - \hat{j}a)_b{}^c l_j(x - \hat{j}a)_c. \quad (2.5)$$

III. THE AXIAL GAUGE; SPLITTING THE HAMILTONIAN

Next we discuss the axial-gauge-fixing procedure. This is most easily done on a cylinder of dimensions $L^1 \times L^2$, with open boundary conditions in the x^1 -direction and periodic boundary conditions in the x^2 -direction [1]. For any function $f(x^1, x^2)$, we require that $f(x^1, x^2 + L^2) = f(x^1, x^2)$. The range of coordinates is $x^1 = 0, a, 2a, \dots, L^1$, $x^2 = 0, a, 2a, \dots, L^2 - a$. For any physical state Ψ , Gauss's law implies that

$$l_1(x^1, x^2)\Psi = [\mathcal{R}_1(x^1 - a, x^2)l_1(x^1 - a, x^2) - (\mathcal{D}_2 l_2)(x^1 - a, x^2) + q(x)]\Psi,$$

so the operators on each side of this expression may be identified. Taking the gauge condition $U_1(x^1, x^2) = \mathbb{1}$, which is possible everywhere on a cylinder, we sum over the 1-coordinate to obtain

$$l_1(x^1, x^2) = \sum_{y^1=0}^{x^1} q(y^1, x^2) - \sum_{y^1=0}^{x^1} (\mathcal{D}_2 l_2)(y^1, x^2), \quad (3.1)$$

which is the lattice analogue of (1.1). There is a remnant of Gauss's law not determined by (3.1), which is the condition

$$\left[\sum_{x^1=0}^{L^1} (\mathcal{D}_2 l_2)(x^1, x^2) - \sum_{x^1=0}^{L^1} q(x^1, x^2) \right] \Psi = 0 \quad (3.2)$$

on physical states Ψ . The condition (3.2) is the lattice analogue of (1.2).

In the axial gauge, using the nonlocal expression (3.1) for the electric field in the x^1 -direction (which henceforth will be called the horizontal direction), the Hamiltonian (2.2) becomes $H = H_0 + H_1$, where

$$H_0 = \sum_{x^2=0}^{L^2-a} H_0(x^2), \quad (3.3)$$

with

$$H_0(x^2) = \sum_{x^1=0}^{L^1} \frac{g_0^2}{2a} [l_2(x^1, x^2)]^2 - \sum_{x^1=0}^{L^1-a} \frac{1}{2g_0^2 a} \times [\text{Tr} U_2(x)^{\dagger} U_2(x^1 + a, x^2) + \text{c.c.}], \quad (3.4)$$

and

$$H_1 = -\frac{(g'_0)^2}{2a} \sum_{x^2=0}^{L^2-a} \sum_{x^1, y^1=0}^{L^1} |x^1 - y^1| [l_2(x^1, x^2) - \mathcal{R}_2(x^1, x^2 - a)l_2(x^1, x^2 - a) - q(x^1, x^2)] \times [l_2(y^1, x^2) - \mathcal{R}_2(y^1, x^2 - a)l_2(y^1, x^2 - a) - q(y^1, x^2)], \quad (3.5)$$

and where we have now introduced the second dimensionless coupling constant g'_0 , defined by $(e')^2 = (g'_0)^2/a$. The only interaction in the x^2 -direction (which from now on will be called the vertical direction) is due to H_1 .

If g'_0 vanishes, the cylindrical lattice splits apart as shown in Fig. 1. The dashed line at the top of the unsplit lattice on the left in this figure indicates that this line is identified with the bottom line. No such identification is made in the split lattice on the right.

The operators $H_0(x^2)$ are Hamiltonians of $SU(2) \times SU(2)$ principal chiral nonlinear sigma models, as noted in Ref. [1] (see also Ref. [4]). Each sigma model is represented by a horizontal ladder of plaquettes on the right-

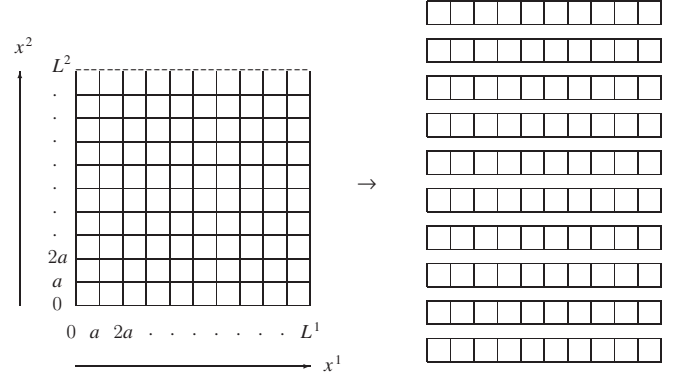


FIG. 1. The splitting of the lattice at $g'_0 = 0$.

hand side of Fig. 1. We see that there is a ladder for each value of x^2 . Setting g'_0 to zero results in decoupled layers of $(1 + 1)$ -dimensional sigma models. Increasing g'_0 leads to an interaction between the vertically-separated layers. This fact was used to give a set of simple arguments for confinement for $g'_0 \ll g_0$. The expression for H_1 , acting on physical states, with $q = 0$, is identical to the expression on the right-hand side of Eq. (6) in Ref. [1], by virtue of (3.2).

For readers who are not familiar with Hamiltonian strong-coupling expansions, we remark that they start by neglecting the second term of (2.2) or (3.4). This term is reintroduced by Rayleigh-Schrödinger perturbation theory, yielding the strong-coupling expansion in $1/g_0^2$.

In a strong-coupling expansion, the magnetic flux is allowed to flow through space unconstrained. By including the “magnetic” or “plaquette” term of order $1/g_0^2$, perturbatively, this unphysical assumption is corrected for. In contrast, we start by allowing the 1-component of the electric field to flow through space unconstrained. By including H_1 (we shall discuss how, at the end of the next section) we correct for this assumption.

At this point, the reader has noticed that our aim is to exploit a type of dimensional reduction from $(2 + 1)$ to $(1 + 1)$ dimensions. This reduction is very different from “compactification,” that is, making L^2 small. It is in some sense the opposite of Fu and Nielsen’s idea of a “layer phase” [18], in which the coupling between lattice layers is made strong instead of weak. The reduction is similar to the “deconstruction” of Arkani-Hamed, Cohen, and Georgi [19] in that the difference between the $(1 + 1)$ -dimensional Hamiltonians and the $(2 + 1)$ -dimensional Hamiltonians is that the sigma models can be regarded as gauged and coupled together through the external gauge field.

IV. THE EFFECTIVE ACTION FOR THE ELECTROSTATIC POTENTIAL

Before we can make much use of the axial-gauge formulation, we need to examine H_0 and H_1 in more detail.

First we consider H_0 . If we adopt the interaction representation, time derivatives of operators \mathfrak{A} are given by $\partial_0 \mathfrak{A} = i[H_0, \mathfrak{A}]$. By working out the time derivative of $U_2(x^1, x^2)$, we find

$$\begin{aligned} l(x^1, x^2)_b &= \frac{ia}{g_0^2} \text{Tr} t_b \partial_0 U(x^1, x^2) U(x^1, x^2)^\dagger, \\ \mathcal{R}(x^1, x^2)_b {}^c l(x^1, x^2)_c &= \frac{ia}{g_0^2} \text{Tr} t_b U(x^1, x^2)^\dagger \partial_0 U(x^1, x^2), \end{aligned} \quad (4.1)$$

where we have dropped the subscript 2, since there is only one spatial component of the gauge field. The time dependence of operators is implicit in these expressions. The (1 + 1)-dimensional $SU(2) \times SU(2)$ principal chiral sigma model of the field $U \in SU(2)$ has the Lagrangian

$$\mathcal{L}_{\text{PCSM}} = \frac{1}{2g_0^2} \eta^{\mu\nu} \text{Tr} \partial_\mu U^\dagger \partial_\nu U, \quad \mu, \nu = 0, 1. \quad (4.2)$$

The left-handed and right-handed currents are, respectively,

$$\begin{aligned} j_\mu^L(x)_b &= i \text{Tr} t_b \partial_\mu U(x) U(x)^\dagger, \\ j_\mu^R(x)_b &= i \text{Tr} t_b U(x)^\dagger \partial_\mu U(x). \end{aligned} \quad (4.3)$$

The Hamiltonian obtained from (4.2) is

$$\begin{aligned} H_{\text{PCSM}} &= \int dx^1 \frac{1}{2g_0^2} \{ [j_0^L(x)_b]^2 + [j_1^L(x)_b]^2 \} \\ &= \int dx^1 \frac{1}{2g_0^2} \{ [j_0^R(x)_b]^2 + [j_1^R(x)_b]^2 \}. \end{aligned} \quad (4.4)$$

By comparing (4.1) with (4.3), we can see that H_{PCSM} in (4.4) is the continuum limit of $H_0(x^2)$ in (3.4).

Next we turn to H_1 . This interaction Hamiltonian is nonlocal, but can be made local by reintroducing the temporal component of the gauge field. In one continuous infinite dimension \mathbb{R} , the function $g(x^1 - y^1) = |x^1 - y^1|/2$ is the Green's function of the ‘‘Laplacian’’; in other words

$$-\partial_1^2 g(x^1 - y^1) = \delta(x^1 - y^1).$$

On our lattice, with x^1 and y^1 taking values $0, a, 2a, \dots, L^1$, the same function $g(x^1 - y^1) = |x^1 - y^1|/2$ is the Green's function of an $(L^1/a + 1)$ -dimensional operator $\Delta_{L^1, a}$, by which it is meant

$$\Delta_{L^1, a} g(x^1 - y^1) = \sum_{z^1=0}^{L^1} (\Delta_{L^1, a})_{x^1 z^1} g(z^1 - y^1) = \frac{1}{a} \delta_{x^1 y^1}.$$

In the continuum limit $a \rightarrow 0$ and thermodynamic limit $L^1 \rightarrow \infty$, $\Delta_{L^1, a} \rightarrow -\partial_1^2$. We use this operator to introduce

an auxiliary field $\Phi(x^1, x^2)_b$ to replace (3.5) by

$$\begin{aligned} H_1 &= \sum_{x^2=0}^{L^2-a} \sum_{x^1=0}^{L^2} \left\{ \frac{(g'_0)^2 a}{4} \Phi(x^1, x^2) \Delta_{L^1, a} \Phi(x^1, x^2) \right. \\ &\quad - (g'_0)^2 [l_2(x^1, x^2) - \mathcal{R}_2(x^1, x^2 - a) l_2(x^1, x^2 - a) \\ &\quad \left. - q(x^1, x^2)] \Phi(x^1, x^2) \right\}. \end{aligned} \quad (4.5)$$

Let us assume that there are only two color charges—a quark with charge q at site u and another quark with charge q' at site v (note: the gauge group is $SU(2)$, so it makes no difference if we have a pair of heavy quarks or a heavy quark and antiquark). For small lattice spacing, we approximate the sum over x^1 *only* as an integral to obtain

$$\begin{aligned} H_1 &= \sum_{x^2=0}^{L^2-a} \int dx^1 \frac{(g'_0)^2 a^2}{4} \partial_1 \Phi(x^1, x^2) \partial_1 \Phi(x^1, x^2) \\ &\quad - \left(\frac{g'_0}{g_0} \right)^2 \sum_{x^2=0}^{L^2-a} \int dx^1 [j_0^L(x^1, x^2) \Phi(x^1, x^2) \\ &\quad - j_0^R(x^1, x^2) \Phi(x^1, x^2 + a)] + (g'_0)^2 q_b \Phi(u^1, u^2)_b \\ &\quad - (g'_0)^2 q'_b \Phi(v^1, v^2)_b. \end{aligned} \quad (4.6)$$

We wish to stress that we are not really taking $a \rightarrow 0$, but only assuming that a is small. Though the expression (4.5) makes the physical meaning of the interaction clearer than (4.6), we shall keep the regulator, at least implicitly. From the coupling to charges, we see that Φ_b is proportional to the temporal component of the gauge field A_{0b} . We will call Φ_b the electrostatic potential for this reason.

From (4.6) we see that the left-handed charge of the sigma model at x^2 is coupled to the electrostatic potential at x^2 . The right-handed charge of the sigma model is coupled to the electrostatic potential at $x^2 + a$.

We now state the mathematical problem we wish to solve. In the presence of a quark at u and an antiquark at v , what is the effective action $\mathfrak{E}(\Phi)$, after integrating out U ? If $u^2 = v^2$, that is, the quarks are only separated horizontally, the effective action is given by

$$e^{i\mathfrak{E}(\Phi)} = \langle 0 | \mathcal{T} e^{-i \int dx^0 H_1} | 0 \rangle, \quad (4.7)$$

where the state $|0\rangle$ is the tensor product of sigma-model vacua (in other words, it is the vacuum of H_0) and where \mathcal{T} denotes time (x^0) ordering. If $u^2 \neq v^2$, the expression (4.7) is no longer correct. In that case, the expectation value needs to be taken with respect to different eigenstates of H_0 , as we shall discuss in Sec. VII.

The effective action may be expanded in terms of vacuum expectation values of products of currents:

$$\begin{aligned}
 i\mathfrak{S}(\Phi) = & -i \sum_{x^2=0}^{L^2-a} \int d^2x (g'_0)^2 a^2 \partial_1 \Phi(x^0, x^1, x^2) \partial_1 \Phi(x^0, x^1, x^2) \\
 & - \frac{1}{2} \left(\frac{g'_0}{g_0}\right)^4 \sum_{x^2=0}^{L^2-a} \int d^2x \int d^2y [\langle 0 | \mathcal{T} j_0^L(x^0, x^1, x^2)_b j_0^L(y^0, y^1, x^2)_c | 0 \rangle + \langle 0 | \mathcal{T} j_0^R(x^0, x^1, x^2)_b j_0^R(y^0, y^1, x^2)_c | 0 \rangle] \\
 & \times \Phi(x^0, x^1, x^2)_b \Phi(y^0, y^1, x^2)_c + O(\Phi^4) - i \int dx^0 [(g'_0)^2 q(x^0)_b \Phi(x^0, u^1, u^2)_b - (g'_0)^2 q'(x^0)_b \Phi(x^0, v^1, u^2)_b] \\
 & + iS_{\text{WZWN}}(q) + iS_{\text{WZWN}}(q'), \tag{4.8}
 \end{aligned}$$

where $d^2x = dx^0 dx^1$, $d^2y = dy^0 dy^1$, and $S_{\text{WZWN}}(q)$ is the Wess-Zumino-Witten-Novikov action of a single SU(2) quark charge (see, for example, Refs. [20]) the details of which are not important for our purposes. We will determine the two-point correlators of currents in (4.8) at large separations.

If we ignore the quantum corrections in (4.8), the potential between the quark and the antiquark is

$$\begin{aligned}
 V(u^1 - v^1) &= \sigma |u^1 - v^1|, \\
 \sigma &= q^2 \frac{(g'_0)^4}{(g'_0)^2 a^2} = \frac{3}{2} \frac{(g'_0)^2}{a^2}.
 \end{aligned}$$

This is the result of Section 6 of Ref. [1]. In the next section, we show the quantum corrections from the current-current correlators will drastically change this result. Physically, these correlators correspond to transverse (that is, vertical) fluctuations of the electric string, as we discuss in Sec. VII.

V. THE LEADING-ORDER CORRECTIONS TO THE EFFECTIVE ACTION

From the exact result for the two-point form factor (A35), discussed in the appendix, we will determine the correlation functions

$$\begin{aligned}
 D(x, y)_{bc} &= \langle 0 | \mathcal{T} j_0^L(x^0, x^1, x^2)_b j_0^L(y^0, y^1, x^2)_c | 0 \rangle \\
 &= \langle 0 | \mathcal{T} j_0^R(x^0, x^1, x^2)_b j_0^R(y^0, y^1, x^2)_c | 0 \rangle, \tag{5.1}
 \end{aligned}$$

in (4.8) for the $SU(2) \times SU(2) \simeq O(4)$ sigma model. We will then use this result to find the string tension for horizontally-separated color charges.

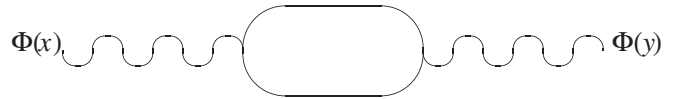
The *exact* correlation functions will also have contributions from two- as well as all higher-point form factors. The complete formula for the Wightman (non-time-ordered) expectation value of two operators in terms of form factors is

$$\begin{aligned}
 D(x, y)_{bc} = & \frac{4m^2 \delta_{bc}}{2!(2\pi)^2} \int d\theta_1 d\theta_2 (\cosh\theta_1 - \cosh\theta_2)^2 |F(\theta_2 - \theta_1)|^2 \exp\{-im \operatorname{sgn}(x^0 - y^0)[(x^0 - y^0)(\cosh\theta_1 + \cosh\theta_2) \\
 & - (x^1 - y^1)(\sinh\theta_1 + \sinh\theta_2)]\}, \tag{5.3}
 \end{aligned}$$

where $\operatorname{sgn}(x^0 - y^0)$ is defined as 1 if $x^0 > y^0$ and -1 if $x^0 < y^0$.

$$\begin{aligned}
 \langle 0 | \mathfrak{B}(x) \mathfrak{C}(y) | 0 \rangle &= \langle 0 | \mathfrak{B}(x) | 0 \rangle \langle 0 | \mathfrak{C}(y) | 0 \rangle \\
 &+ \sum_{M=1}^{\infty} \int \frac{d\theta_1 \cdots d\theta_M}{(2\pi)^M M!} \\
 &\times \langle 0 | \mathfrak{B}(x) | \theta_M, j_M, \dots, \theta_1, j_1 \rangle \\
 &\times \langle \theta_1, j_1, \dots, \theta_M, j_M | \mathfrak{C}(y) | 0 \rangle. \tag{5.2}
 \end{aligned}$$

To obtain this result, we used the resolution of the identity (A3). Time-ordered expectation values can be written in terms of Wightman functions. In a field theory with a mass gap m , the largest contribution to (5.2) for large $m|x - y|$ comes from the terms with the smallest number of particle exchanges M . For our problem, with two-sigma-model charge densities, the first term, *i.e.* the vacuum channel, gives no contribution; therefore we can consider just the case of $M = 2$. We will evaluate (5.1) in this way. Viewing the two-point form factor as a vertex between the electrostatic potential Φ and two excitations of the sigma model, called Faddeev-Zamolodchikov (FZ) particles, we consider the one-loop diagram:



This diagram is infrared and ultraviolet finite. A scale is set by the sigma-model mass gap m . We will expand this diagram in derivatives of Φ . Lowest order in derivatives means lowest order in momentum, in the Fourier transform of this amplitude. The two-particle form factors should therefore be sufficient. If we wanted many higher-derivative terms in the effective action of the electrostatic potential $\mathfrak{S}(\Phi)$, this would no longer be the case.

Taking the complex conjugate of the expression for the form factor (A36), in the appendix, with x replaced by y , applying (5.2), truncating $M > 2$, and finally ordering in the time coordinates x^0 and y^0 , yields the following for the current-current correlation functions (5.1)

The integration in (5.3) can be made somewhat easier by introducing new parameters $\Omega = (\theta_1 + \theta_2)/2$ and $\omega = (\theta_1 - \theta_2)/2$:

$$D(x, y)_{bc} = \frac{4m^2 \delta_{bc}}{\pi^2} \int d\Omega d\omega \sinh^2 \Omega \sinh^2 \omega \exp\{-2im \cosh \Omega \operatorname{sgn}(x^0 - y^0)[(x^0 - y^0) \cosh \omega - (x^1 - y^1) \sinh \omega]\} \\ \times \exp - \int_0^\infty \frac{d\xi}{\xi} \frac{e^{-\xi}}{\cosh^2 \frac{\xi}{2}} \left(1 - \cosh \xi \cos \frac{2\xi \omega}{\pi}\right). \quad (5.4)$$

The action $\mathfrak{S}(\Phi)$ is nonlocal, but the excitations of the sigma model are massive particles. Hence we expect that $\mathfrak{S}(\Phi)$ is dominated by local terms, obtained from the derivative expansion. Let us introduce the new coordinates X^μ and r^μ by $x^\mu = X^\mu + r^\mu/2$ and $y^\mu = X^\mu - r^\mu/2$. We expand $\Phi(x)$ and $\Phi(y)$ in powers of r^μ , in the standard way as

$$\Phi(x) = \Phi(X) + \frac{r^\mu}{2} \partial_\mu \Phi(X) + \frac{r^\mu r^\nu}{8} \partial_\mu \partial_\nu \Phi(X) + \dots, \\ \Phi(y) = \Phi(X) - \frac{r^\mu}{2} \partial_\mu \Phi(X) + \frac{r^\mu r^\nu}{8} \partial_\mu \partial_\nu \Phi(X) \pm \dots, \quad (5.5)$$

where ∂_μ now denotes $\partial/\partial X^\mu$.

Taking care to sum over L and R , the term we want to evaluate in the effective action, which is quadratic in the fields, is given by

$$i\mathfrak{S}^{(2)}(\Phi) = -\left(\frac{g'_0}{g_0}\right)^4 \sum_{x^2=0}^{L^2-a} \int d^2 X d^2 r D\left(X + \frac{r}{2}, X - \frac{r}{2}\right)_{bc} \\ \times \Phi\left(X + \frac{r}{2}, x^2\right)_b \Phi\left(X - \frac{r}{2}, x^2\right)_c. \quad (5.6)$$

At the risk of overemphasizing a point, we remark that the truncation of (5.2) to $M = 2$ and the use of the derivative expansion (5.5) have the same justification. Both are valid approximations in a massive theory. Though we expand in r in (5.5), it is not a short-distance expansion in the usual sense. We can integrate over r precisely because large $m|r|$ contributions are suppressed. The result of substituting (5.5) into (5.6) is really a small-momentum expansion.

After substituting (5.4) and (5.5) into (5.6) we integrate over r^μ . The integrals over r^μ are of the form

$$I(Q_0, Q_1, A) = \int d^2 r e^{-iQ_0|r^0| + iQ_1 r^1 \operatorname{sgn}(r^0)} A(r^0, r^1),$$

for some polynomial $A(r^0, r^1)$, where

$$Q_0 = 2m \cosh \Omega \cosh \omega, \quad Q_1 = 2m \cosh \Omega \sinh \omega.$$

We therefore need to evaluate $I(Q_0, Q_1, A)$, for a few choices of $A(r^0, r^1)$.

Suppose that $A(r^0, r^1)$ has no dependence on r^1 . Then the r^1 -integration will produce a term proportional to $\delta(Q_1)$, which, in turn, is proportional to $\delta(\sinh \omega)$. From the factor $\sinh^2 \omega$ in the integral expression (5.4), this will give no contribution to $i\mathfrak{S}^{(1)}(\Phi)$ in (5.6). Since we are working to quadratic order in r^0 and r^1 , we therefore need only consider $A(r^0, r^1) = r^1, r^0 r^1, (r^1)^2$.

It is elementary to show that $I(Q_0, Q_1, r^1) = 0$.

The integral $I(Q_0, Q_1, r^0 r^1)$ is proportional to $\delta'(\sinh \omega)$. Again, the factor $\sinh^2 \omega$ in the integral expression (5.4) insures that this will give no contribution to $i\mathfrak{S}^{(2)}(\Phi)$ in (5.6).

The only integral remaining is

$$I[Q_0, Q_1, (r^1)^2] = -\frac{2i\delta''(Q_1)}{Q_0 - i\varepsilon} \\ = -\text{P.V.} \frac{2i\delta''(Q_1)}{Q_0} + 2\pi\delta(Q_0)\delta''(Q_1).$$

Only the principal value contributes to the effective action, which is

$$i\mathfrak{S}^{(2)}(\Phi) = -\frac{2mi}{\pi^2} \left(\frac{g'_0}{g_0}\right)^4 \sum_{x^2=0}^{L^2-a} \int d^2 X \int d\Omega d\omega \frac{\sinh^2 \Omega \sinh^2 \omega}{\cosh \Omega \cosh \omega} \delta''(2m \cosh \Omega \sinh \omega) \\ \times \exp - \int_0^\infty \frac{d\xi}{\xi} \frac{e^{-\xi}}{\cosh^2 \frac{\xi}{2}} \left(1 - \cosh \xi \cos \frac{2\xi \omega}{\pi}\right) [\partial_1 \Phi(X, x^2)_b]^2 + \text{higher derivative terms.}$$

The integration over Ω and ω can now be done. After some work, we find

$$i\mathfrak{S}^{(2)}(\Phi) = -\frac{i}{3m^2 \pi^2} \left(\frac{g'_0}{g_0}\right)^4 \exp\left[-2 \int_0^\infty \frac{d\xi}{\xi} e^{-\xi} \tanh^2 \frac{\xi}{2}\right] \sum_{x^2=0}^{L^2-a} \int d^2 X [\partial_1 \Phi(X, x^2)_b]^2 + \text{h.d.t.}, \quad (5.7)$$

which is the central result of this paper.

The correction to $\mathfrak{S}(\Phi)$ which is cubic in Φ can be shown to vanish by symmetry considerations, $\mathfrak{S}^{(3)}(\Phi) = 0$. The quartic correction $\mathfrak{S}^{(4)}(\Phi)$, does not vanish. This can also be obtained using form-factor methods, though the calculation will be longer than that above. An interesting aspect of $\mathfrak{S}^{(4)}(\Phi)$ is that it couples fields together at neighboring values of the vertical coordinate, e.g. x^2 and $x^2 + a$. It may lead to interesting nonlinear dynamics of electric strings. At this order in g'_0 the form factors and the mass spectrum will be altered [21], which will also need to be implemented.

VI. THE HORIZONTAL STRING TENSION

With our result (5.7) serving as the second term of the expression (4.8) for the effective action $\mathfrak{S}(\Phi)$, we find

$$\begin{aligned} \mathfrak{S}(\Phi) = & -K \sum_{x^2=0}^{L^2-a} \int d^2x [\partial_1 \Phi(x, x^2)_b]^2 + \text{h.d.t.} + \text{O}(\Phi^4) \\ & - \int dx^0 [(g'_0)^2 q(x^0)_b \Phi(x^0, u^1, u^2)_b \\ & - (g'_0)^2 q'(x^0)_b \Phi(x^0, v^1, u^2)_b] + S_{\text{WZWN}}(q) \\ & + S_{\text{WZWN}}(q'), \end{aligned} \quad (6.1)$$

Where the factor K is given by

$$\begin{aligned} K = & \frac{(g'_0)^2 a^2}{4} + \frac{1}{3m^2 \pi^2} \left(\frac{g'_0}{g_0} \right)^4 \\ & \times \exp \left[-2 \int_0^\infty \frac{d\xi}{\xi} e^{-\xi} \tanh^2 \frac{\xi}{2} \right]. \end{aligned} \quad (6.2)$$

Notice that there is no induced mass term in Φ . This is not hard to understand; it is due to the fact that the FZ particles of the principal chiral sigma model are adjoint charges, hence do not screen quarks.

Notice that at leading order, there are no time derivatives of Φ in $\mathfrak{S}(\Phi)$. The effective Hamiltonian is

$$\begin{aligned} E = & K \sum_{x^2=0}^{L^2-a} \int dx^1 [\partial_1 \Phi(x, x^2)_b]^2 + \text{h.d.t.} + \text{O}(\Phi^4) \\ & + (g'_0)^2 q_b \Phi(u^1, u^2)_b - (g'_0)^2 q'_b \Phi(v^1, u^2)_b. \end{aligned} \quad (6.3)$$

Thus the horizontal string tension is

$$\sigma_{\text{H}} = \frac{1}{4} (q_b)^2 \frac{(g'_0)^4}{K} = \frac{3(g'_0)^4}{8K},$$

where we used, as before, $q^2 = (q')^2 = \frac{3}{2}$ for our normalization of $\text{SU}(2)$ charges.

Let us look a bit more closely at the factor K . In the asymptotically-free $\text{SU}(2) \times \text{SU}(2)$ nonlinear sigma model, the mass of the FZ particles depends on the coupling g_0 as

$$m = \frac{C}{a} (g_0^{-1} e^{-2\pi/g_0^2} + \dots),$$

where C is a nonuniversal constant, which depends on the cut-off method. Thus the second term in K according to (6.2) is significant. Our result for the horizontal string tension is therefore

$$\sigma_{\text{H}} = \frac{3}{2} \left(\frac{g'_0}{a} \right)^2 \left[1 + \frac{4}{3} \frac{0.7296}{C^2 \pi^2} \frac{(g'_0)^2}{g_0^2} e^{4\pi/g_0^2} \right]^{-1}. \quad (6.4)$$

Notice that for very small g_0 , the exponential dominates the denominator, even if $g'_0 \ll g_0$. Thus (6.4) depends on the couplings as

$$\sigma_{\text{H}} \approx \frac{9C^2 \pi^2}{(0.7296)8} \frac{g_0^2}{a^2} e^{-4\pi/g_0^2}. \quad (6.5)$$

This expression is unusual in that all g'_0 -dependence has disappeared.

If we rewrite our expression for the string tension (6.4) in terms of the continuum couplings $e = g_0 \sqrt{a}$ and $e' = g'_0 \sqrt{a}$, we see an expression which is different than (1.3). In terms of these constants and the lattice spacing, that

$$\begin{aligned} \sigma_{\text{H}} = & \frac{3}{2} \frac{(e')^2}{a} \left[1 + \frac{4}{3} \frac{0.7296}{C^2 \pi^2} \frac{(e')^2}{e^2} e^{4\pi/(e^2 a)} \right]^{-1} \\ \approx & \frac{9C^2 \pi^2}{(0.7296)8} \frac{e^2}{a} e^{-4\pi/(e^2 a)}, \end{aligned} \quad (6.6)$$

where the approximation is valid for small a . We cannot take a continuum limit of our string tension, holding e and e' fixed. The reason the naive argument leading to (1.3) fails is in the assumption of no ultraviolet divergences. What our method shows is that, at least for the anisotropic range of couplings we consider, there are such divergences. The resulting dependence of the string tension on the couplings is not analytic.

We believe that (6.4) cannot be extended to the isotropic regime $g'_0 \sim g_0$. To see why, imagine that we generalize the regularized theory to one with three couplings. In the Euclidean Wilson lattice gauge theory, with lattice spacing a and link fields $U_\mu(x) \in \text{SU}(2)$, where x lies on a three-dimensional lattice, this means an action of the form

$$S = \sum_{\mu \neq \nu} \frac{1}{4g_{\mu\nu}^2} \text{Tr} U_\mu(x) U_\nu(x + \hat{\mu}a) U_\mu(x + \hat{\nu}a) U_\nu(x), \quad (6.7)$$

where $\hat{\mu}$ denotes the unit vector in the μ -direction and $g_{\mu\nu} = g_{\nu\mu}$. There are three distinct couplings in this model. The regime analogous to that we consider is $g_{01} \ll g_{02} = g_{12}$. There are other regimes, we could apply our result (6.4) to, namely $g_{02} \ll g_{01} = g_{12}$ and $g_{12} \ll g_{01} = g_{02}$. Clearly there must be a crossover phenomenon between different behaviors of the string tension. It seems plausible that there is other crossover behavior between these regimes to $g_{01} \sim g_{02} \sim g_{12}$ in which (1.3) takes

place. Since the string tension does not depend on g'_0 to our order of approximation, it may be that the crossover occurs at a value of g'_0/g_0 which is not extremely small.

VII. PHYSICAL ASPECTS OF EXCITATIONS

In this section, we complete the picture of the confining phase by discussing the vertical string and the nature of the pure-gluon excitations.

We already presented the basic mechanisms of linear potential between vertically-separated quarks and the area decay of spacelike Wilson loops in Ref. [1]. We will show how these mechanisms fit into a general picture of the excitations.

Let us begin by asking what the excitations are if $g_0 > 0$ and $g'_0 = 0$. As we discussed in Sec. III, the system splits up into decoupled layers of nonlinear sigma models. The possible excitations are massive FZ particles which can travel horizontally, but not vertically. There is, however a restriction on these excitations, which is that the residual gauge-invariance condition (3.2) must be satisfied. If we approximate the sum over x^1 as an integral, this condition states that for each x^2

$$\left\{ \int dx^1 [j_0^L(x^1, x^2)_b - j_0^R(x^1, x^2 - a)_b] - g_0^2 Q(x^2)_b \right\} \Psi = 0, \quad (7.1)$$

where $Q(x^2)_b$ is the total color charge from quarks at x^2 and Ψ is any physical state. If there are no quarks, the total right-handed charge of FZ particles in the sigma model at $x^2 - a$ is equal to the total left-handed charge of FZ particles in the sigma model at x^2 .

Let us picture two-dimensional space partitioned into a set of parallel horizontal layers. Each layer contains a sigma model. The FZ particles move within a layer, as in the left-hand side of Fig. 2. Now suppose we increase g'_0 from zero to a small value. Since an FZ particles at x^2 has left-SU(2) charge at x^2 and right-SU(2) charge at $x^2 + a$, horizontal electric strings must join the FZ particles together. The strings lie between the layers. Because the constraint (7.1) is satisfied, these strings can be consistently introduced. We now have a similar picture of excitations, but now with strings with the tension calculated in the last

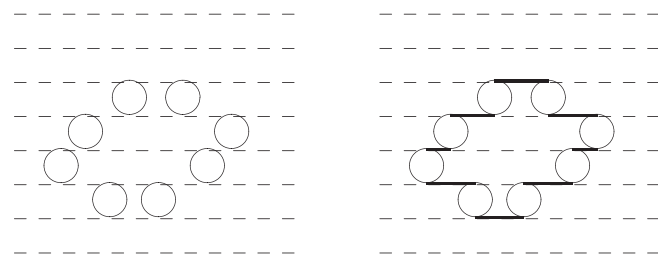


FIG. 2. The circles represent FZ particles in the layers between the dashed lines. On the right, electric strings connect the FZ particles.

section. This is shown in the right-hand side of Fig. 2. We see now that the vertical electric flux is carried by the FZ particles themselves—they are short segments of vertical electric flux. In fact, if we introduce a quark and antiquark with a vertical separation, there will simply be a line of FZ particles and horizontal strings joining them together.

The term in the effective action of the electrostatic potential we calculated in Sec. V is due to charge fluctuations in the sigma model. According to the picture we have just outlined, this means it is due to transverse (that is, vertical) fluctuations of the string joining a quark-antiquark pair.

The reader should not be misled by the right side of Fig. 2 into thinking that there is no 1-component of the electric field if $g'_0 = 0$. There is electric field produced by the FZ particles in this case, but this field carries no energy.

Naively, the vertical string tension is simply [1]

$$\sigma_V = \frac{m}{a} \approx \frac{C}{g_0 a^2} e^{-2\pi/g_0^2}. \quad (7.2)$$

This result follows from assuming that the energy in each layer is m . Strictly speaking, the right-hand side of (7.2) is a lower bound to the vertical string tension, $\sigma_V \geq m/a$. We expect that there will be corrections to this formula. These can be found, in principle, just as we did as for the horizontal string tension. There is a subtle difference however. In the expression for the effective action. Instead of (4.7), we must calculate

$$e^{i\Xi(\Phi)} = \lim_{T \rightarrow \infty} \langle \Psi | \mathcal{T} e^{-i \int_0^T dx^0 H_1} | \Psi \rangle. \quad (7.3)$$

The state Ψ is not the vacuum. We must now have an FZ particle in every layer between the quark and antiquark. This problem is under study.

VIII. CONCLUSIONS

By spitting $(2 + 1)$ -dimensional SU(2) Yang-Mills theory anisotropically into integrable models, with an interaction between these models, we obtained an expression for the string tension (in one direction only) at small values of the couplings.

Two problems yet to be solved for this gauge theory are finding a more precise result for the vertical string tension (discussed in the last section) and the mass gap of the gauge theory. The latter problem is not straightforward, as the glueball excitations are collections of many FZ particles. Furthermore, we do not expect the number of these FZ particles is fixed. The right approach may be to obtain a better understanding of the vacuum state first, then examine gauge-invariant correlators in this state.

The effective action we used to find the horizontal string tension contains further nonlinear terms, as we mentioned at the end of Sec. V. The quartic term contains mixed correlators of left-handed and right-handed charges of the sigma model. It is necessary to use (A38), as well as (A36)

There is no reason why this term cannot be determined; all that is needed is sufficient effort. Perhaps this term could shed light on the crossover phenomenon discussed at the end of Sec. VI. It will also be important to include corrections to the mass spectrum and the form factors at this order [21].

We could extend our methods to $SU(N)$ gauge theories, if we knew the form factors for the $SU(N) \times SU(N)$ sigma model with $N > 2$. Our basic idea can be formulated for any value of N [1]. The form-factor problem for $N > 2$ is tough, in part because of the presence of bound states of the fundamental FZ particles. The S-matrix was worked out some time ago [22], and agrees with the Bethe *Ansatz* approach for an equivalent Fermionic model [23]. The S-matrix becomes unity in the 't Hooft limit $N \rightarrow \infty$, $g_0^2 N$ fixed, and the form factors should simplify in this limit (note: there is a large- N limit of the $SU(N) \times SU(N)$ sigma model, whose S-matrix is nontrivial [24]. This is a different model, because the limit is *not* the standard 't Hooft limit).

The picture of confinement and excitations described in Sec. VII suggests that the non-Abelian gauge theory may be dual to another field theory. The weak-coupling diagrams would correspond to the strong-coupling terms of the dual theory.

A splitting similar to the one we have used can be done in $(3 + 1)$ dimensions. There is an important difference, however. As in $(2 + 1)$ dimensions, some electric-field components squared are included in the interaction Hamiltonian. The new feature is that the interaction Hamiltonian also contains a magnetic-field component squared. Our methods would therefore not yield, strictly speaking, a weak-coupling result. Investigations in this direction may be of some value, nonetheless.

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APPENDIX A: EXACT S-MATRICES AND FORM FACTORS

We now present an introduction to how S-matrices and form factors are exactly determined for the $O(N)$ nonlinear sigma model in $(1 + 1)$ dimensions. Though not a comprehensive treatment of $(1 + 1)$ -dimensional S-matrices and form factors, this review is self-contained. Since most people who might find this paper of interest are not

experts on these subjects, we felt it was necessary to provide complete explanations of their less obvious aspects, which are not abundantly available. We hope this appendix is sufficiently complete for the reader to understand the results used in the remainder of the paper.

Integrability of the $(1 + 1)$ -dimensional quantized asymptotically-free $O(N)$ nonlinear classical sigma model [25] became of interest, after it was first established by Pohlmeyer for the classical case [26]. Then general arguments were made that the infinite set of dynamical charges can be generalized to the quantized theory [27]. The generation of a mass gap in the $1/N$ -expansion was discovered considerably earlier [28].

The Lagrangian of the $O(N)$ sigma model depends on an N -component vector field $n = (n^1, \dots, n^N)$, of unit length $n \cdot n = 1$:

$$\mathcal{L}_{\text{NLS}} = \frac{1}{2g_0^2} \eta^{\mu\nu} \partial_\mu n \cdot \partial_\nu n. \quad (\text{A1})$$

For $N = 4$, the identification between (A1) and the $SU(2) \times SU(2)$ Lagrangian (4.2) is made by $U = n^4 \mathbb{1} - i n^b \sigma_b$.

From the $1/N$ -expansion, we know that there are N basic species of massive particle, which we label by letters j, k, l, m, \dots taking the values $1, 2, \dots, N$. The particle states are eigenstates of momentum q, p, \dots as well as species j, k, \dots , created on the vacuum by Faddeev-Zamolodchikov operators or FZ operators $\mathfrak{A}(q)_j^\dagger$:

$$|q, j, p, k, \dots\rangle = \mathfrak{A}(q)_j^\dagger \mathfrak{A}(p)_k^\dagger \dots |0\rangle.$$

We are using the Heisenberg representation, so we work with in-states and in-operators or out-states and out-operators.

From either the the $1/N$ -expansion or from the assumption of integrability, we find that particles are neither created nor destroyed when scattered (that is, there is no particle production). Hence multiparticle scattering may be decomposed as a sequence of two-particle scatterings. The requirement that this decomposition is consistent is the Yang-Baxter equation or factorization equation. For non-integrable theories, there is no such decomposition.

We can find expressions for Green's functions which are valid at large distances using exact form factors, for models with a mass gap. Green's functions are not exactly known, except those for the spins of the Ising model, where *all* the form factors are known [29]. Nonetheless, an expression can be written down for the vacuum expectation value of a product of operators which approaches the correct answer at large separations.

1. The exact S-matrix of the $O(N)$ nonlinear sigma model

The two-particle S-matrix should have, on fairly general intuitive grounds, the following form:

$$\begin{aligned} \langle \text{out} | q', j, p', l | q, m, p, k \rangle_{\text{in}} &= \delta(q'_1 - p_1) \delta(p'_1 - q_1) \mathcal{S}_{mk}^{jl}(s) \\ &\pm \delta(q'_1 - q_1) \delta(p'_1 - p_1) \mathcal{S}_{km}^{jl}(s), \end{aligned} \quad (\text{A2})$$

where the four-tensor in species indices $\mathcal{S}(s)$, depends on the center-of-mass energy squared $s = (q + p)^2$, and \pm refers to Bose or Fermi statistics (we are going to consider the former only). We introduce the other Mandelstam variable $t = (p' - p)^2$ and the rapidities θ_1, θ_2 , related to the momenta by the standard relation $p_0 = m \cosh \theta_1$, $p_1 = m \sinh \theta_1$, $q_0 = m \cosh \theta_2$, $q_1 = m \sinh \theta_2$. The relative rapidity is $\theta_{12} = \theta_2 - \theta_1$.

Bound states of two particles can form only when the center-of-mass energy is less than the total rest-mass energy. Above the the total rest-mass energy, the energy spectrum is continuous. Thus there may be poles in $\mathcal{S}(s)$ in the complex s plane for $s < 4m^2$, and there is a cut in $\mathcal{S}(s)$ for $s > 4m^2$. Assuming crossing $s \rightleftharpoons t$, there is a cut for $t > 4m^2$. Kinematically, $s = 2m^2(1 + \cosh \theta_{12})$ and $t = 2m^2(1 - \cosh \theta_{12})$. Hence there are really two cuts in the complex s plane; one for $s > 4m^2$ and another for $s < 0$. A minimal analyticity assumption is that any poles present lie on the real s axis on the interval $0 < s < 4m^2$.

If we make the change of variable from s to $\theta \equiv \theta_{12}$, we find the analyticity structure is as follows. The s plane is

mapped to the region bounded by $\text{Im} \theta = 0$ and $\text{Im} \theta = \pi$, called the physical strip (the physical strip is not the same as the physical region, which is the on-shell kinematically allowed region of θ). Each of the boundaries $\text{Im} \theta = 0, \pi, -\infty < \text{Re} \theta < \infty$ is a cut. Poles lie on the $\text{Re} \theta = 0$ axis only. There may be poles on this axis outside the physical strip, but these do not correspond to physical bound states (there are no bound states for the model we are studying). Note that crossing corresponds to $\theta \rightarrow i\pi - \theta$. The resolution of the identity or overcompleteness relation in rapidity space, which is straightforwardly obtained from that in momentum space, is

$$\begin{aligned} \mathbb{1} &= |0\rangle\langle 0| + \sum_{M=1}^{\infty} \int \frac{d\theta_1 \cdots d\theta_M}{(2\pi)^M M!} |\theta_M, j_M \cdots, \theta_1, j_1\rangle \\ &\times \langle \theta_1, j_1, \dots, \theta_M, j_M|. \end{aligned} \quad (\text{A3})$$

The tensor $\mathcal{S}(\theta)$ can be decomposed in the following way, following Zamolodchikov and Zamolodchikov [16,30]:

$$\mathcal{S}_{mk}^{jl}(\theta) = \delta^{jl} \delta_{mk} S_1(\theta) + \delta_m^l \delta_k^j S_2(\theta) + \delta_m^j \delta_k^l S_3(\theta). \quad (\text{A4})$$

We represent this decomposition pictorially as

$$S_{mk}^{jl}(\theta) = S_1(\theta) \begin{array}{c} l \quad j \\ \diagdown \quad \diagup \\ k \quad m \end{array} + S_2(\theta) \begin{array}{c} l \quad j \\ \diagup \quad \diagdown \\ k \quad m \end{array} + S_3(\theta) \begin{array}{c} l \quad j \\ \diagup \quad \diagdown \\ k \quad m \end{array} = \theta \begin{array}{c} l \quad j \\ \diagdown \quad \diagup \\ k \quad m \end{array}$$

Each line connecting two indices in this picture is a Kronecker delta, in accordance with diagrammatic lore. The rapidity θ is drawn as an angle between incoming and outgoing lines (though its range is, of course, not restricted from 0 to π).

The functions $S_1(\theta)$, $S_2(\theta)$ and $S_3(\theta)$ are assumed to be real on the real θ axis. Therefore, by the Schwartz reflection principle, $S_1(-\theta) = S_1(\theta)^*$, $S_2(-\theta) = S_2(\theta)^*$ and $S_3(-\theta) = S_3(\theta)^*$.

If the reader stares at the diagrammatic form for $\mathcal{S}(\theta)$, he or she should be able to see that under crossing, $S_2(i\pi - \theta) = S_2(\theta)$ and $S_3(i\pi - \theta) = S_1(\theta)$.

There are two nontrivial conditions satisfied by the S-matrix elements which, together with maximal analyticity, will give its complete determination. The first of these is unitarity within the two-particle sector. This is satisfied because there is no particle production. Applying unitarity to (A2) yields

$$S_{mk}^{gh}(-\theta) S_{jl}^{mk}(\theta) = \delta_j^g \delta_l^h.$$

This expression can be worked out by multiplying Kronecker deltas together and contracting indices, or by

diagrammatic methods. In either case, we obtain the three relations

$$S_2(\theta) S_2(-\theta) + S_3(\theta) S_3(-\theta) = 1, \quad (\text{A5})$$

$$S_2(\theta) S_3(-\theta) + S_3(\theta) S_2(-\theta) = 0, \quad (\text{A6})$$

$$\begin{aligned} NS_1(\theta) S_1(-\theta) + S_1(\theta) S_2(-\theta) + S_1(\theta) S_3(-\theta) \\ + S_2(\theta) S_1(-\theta) + S_3(\theta) S_1(-\theta) = 0. \end{aligned} \quad (\text{A7})$$

The second nontrivial condition is scattering factorization or the Yang-Baxter relation. This condition makes the three- and higher-particle scattering amplitudes unique. If we imagine three classical particles scattering, there are two possible orderings in which this scattering can occur. Each of these orderings corresponds to a certain decomposition of three-particle S-matrix elements in terms of two-particle S-matrix elements in the quantum theory. If either decomposition is correct, they both are; we therefore *identify* them.

Let us imagine three particles scattering with particle rapidities θ_1, θ_2 and θ_3 . The relative rapidities are

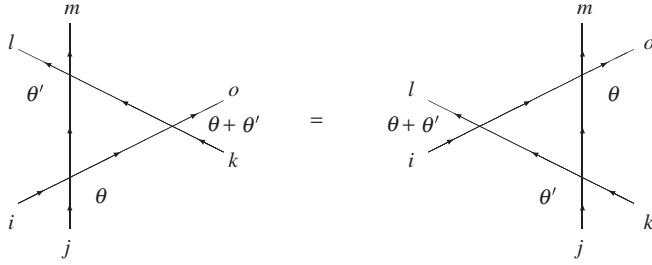


FIG. 3. The factorization equation for three-particle S-matrix elements.

$\theta = \theta_2 - \theta_1$, $\theta' = \theta_3 - \theta_2$, and $\theta + \theta' = \theta_3 - \theta_1$. The Yang-Baxter equation is shown pictorially in Fig. 3. We have labeled each of the incoming particles with a species index i, j, k and each of the outgoing particles by another species index l, m, o .

Now comes the real work—turning the Yang-Baxter equation into something manageable. Each side of the Yang-Baxter equation can be decomposed into 27 terms, each proportional to a six-index tensor, which is a product of three Kronecker deltas. The left-hand and right-hand sides of the Yang-Baxter equation are decomposed

pictorially in Figs. 4 and 5, respectively. Each of the terms is equal to a product of three functions $S_\alpha(\theta)$ times a product of three Kronecker deltas. We abbreviate products of three such functions, e.g. $S_2(\theta)S_3(\theta + \theta')S_2(\theta')$ as $S_2S_3S_2$, keeping the arguments in the order $\theta, \theta + \theta', \theta'$ always. Though there are 27 terms on each side, each term must be one of only 15 distinct tensors, up to the factor of 3 S's. That is because there are exactly 15 ways to make a six-index tensor from a product of three Kronecker deltas.

For $N = 2$, there is an additional subtlety, which is that the 15 tensors constructed from products of three Kronecker deltas are not linearly independent. This is important for understanding the sine-Gordon/massive-Thirring model, but is of no relevance to this paper. These tensors are linearly independent for $N \geq 3$.

With the labeling of species as in Fig. 2, we can see the form of each term in Figs. 4 and 5. For example, the first term on the left-hand side of the Yang-Baxter equation is seen from Fig. 4 to be $S_1(\theta)S_1(\theta + \theta')S_3(\theta')\delta_{ij}\delta_k^l\delta^{mo}$. In this way, the Yang-Baxter equation reduces to 15 algebraic equations. Of these 15 equations, the seven which are proportional to one of the following:

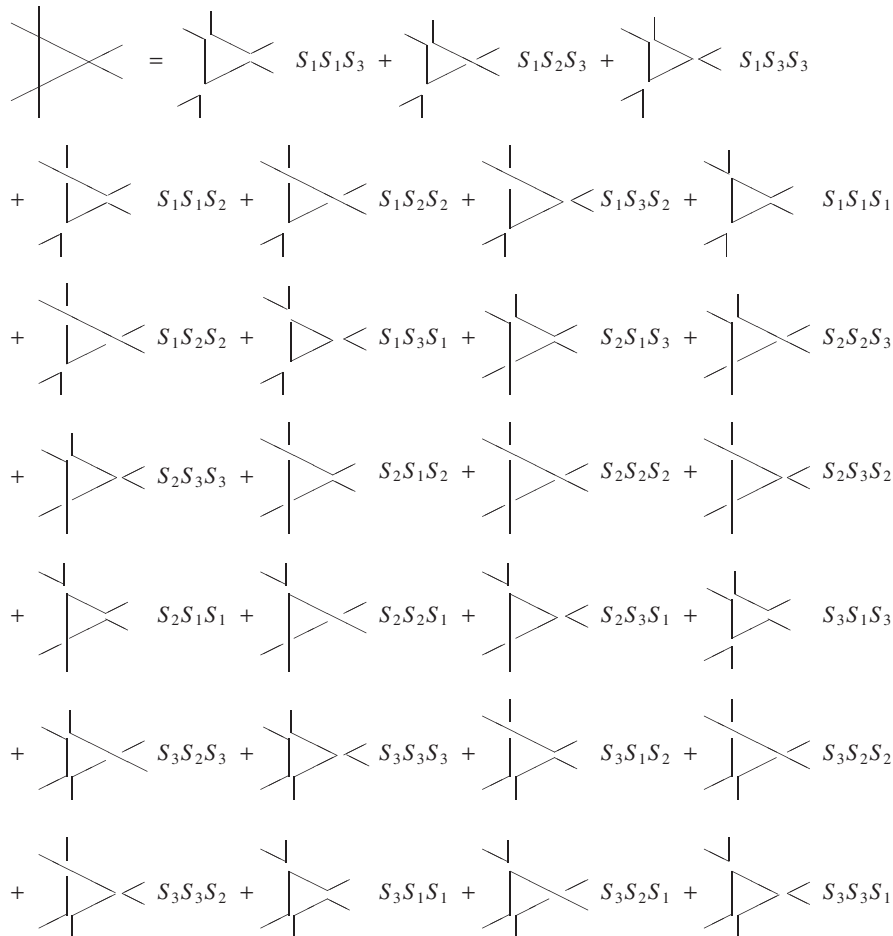


FIG. 4. Expansion of the left-hand side of the factorization equation.

FIG. 5. Expansion of the right-hand side of the factorization equation.

$$\delta_i^l \delta_j^m \delta_k^o, \quad \delta^{lo} \delta_j^m \delta_{ik}, \quad \delta_k^l \delta_j^m \delta_i^o, \quad \delta_k^l \delta^{mo} \delta_{ij},$$

$$\delta^{lm} \delta_i^o \delta_{jk}, \quad \delta_j^l \delta_k^m \delta_i^o, \quad \delta_k^l \delta_i^m \delta_j^o,$$

are trivially satisfied. Of those eight remaining, five are redundant, leaving three nontrivial equations [16,30]. The terms proportional to $\delta_i^l \delta_k^m \delta_j^o$ and $\delta_j^l \delta_i^m \delta_k^o$ each give

$$S_2 S_3 S_3 + S_3 S_3 S_2 = S_3 S_2 S_3. \quad (\text{A8})$$

The terms proportional to $\delta^{lo} \delta_i^m \delta_{jk}$ and $\delta^{lm} \delta_{ik} \delta_j^o$ each give

$$S_2 S_1 S_1 + S_3 S_2 S_1 = S_3 S_1 S_2. \quad (\text{A9})$$

The terms proportional to $\delta_j^l \delta^{mo} \delta_{ik}$ and $\delta^{lo} \delta_k^m \delta_{ij}$, also give (A9), but with the arguments θ and θ' reversed. Finally, the terms proportional to $\delta_i^l \delta^{mo} \delta_{jk}$ and $\delta^{lo} \delta_k^m \delta_{ij}$ each give

$$NS_1 S_3 S_1 + S_1 S_3 S_2 + S_1 S_3 S_3 + S_1 S_2 S_1 + S_2 S_3 S_1$$

$$+ S_3 S_3 S_1 + S_1 S_1 S_1 = S_3 S_1 S_3, \quad (\text{A10})$$

In each of (A8)–(A10) the arguments are θ , $\theta + \theta'$ and θ' , respectively.

Next, let us solve the unitarity and factorization equations. If the function $h(\theta)$ is defined as $h(\theta) = S_2(\theta)/S_3(\theta)$, (A5) becomes

$$h(\theta) + h(\theta') = h(\theta + \theta').$$

Unless $h(\theta)$ vanishes, the only possible solution is

$$h(\theta) = \frac{i}{\lambda} \theta, \quad (\text{A11})$$

for some constant λ . Note that (A6) yields $h(-\theta) + h(\theta) = 0$, which is automatically satisfied. Defining another function $\rho(\theta) = S_1(\theta)/S_3(\theta)$, Eqs. (A9) and (A11) imply that

$$\rho(\theta + \theta')^{-1} + \rho(\theta')^{-1} = -\frac{i\theta}{\lambda}.$$

The solution for $\rho(\theta)$ is

$$\rho(\theta) = -\frac{i\lambda}{i\kappa - \theta}, \quad (\text{A12})$$

where κ is another constant. In terms of the original functions,

$$S_3(\theta) = \frac{-i\lambda}{\theta} S_2(\theta), \quad S_1(\theta) = -\frac{i\lambda}{i\kappa - \theta} S_2(\theta). \quad (\text{A13})$$

Next we substitute (A13) into (A10) and multiply both sides by $\theta(i\kappa - \theta)(\theta + \theta')(i\kappa - \theta - \theta')\theta'(i\kappa - \theta')$. The result is that a fourth-order polynomial in rapidities θ and θ' is zero. The zeroth-, first-, and fourth-order terms are identically zero. Both the second- and third-order terms are zero provided

$$\kappa = \frac{\lambda(N-2)}{2}.$$

By the crossing property and (A13), we have that $\kappa = \pi$ and thus $\lambda = 2\pi/(N-2)$. By (A5) and (A13), we obtain

$$\begin{aligned} S_2(\theta)S_2(-\theta) &= \frac{\theta^2}{\theta^2 + \frac{4\pi^2}{(N-2)^2}}, \\ S_3(\theta) &= -\frac{2\pi i}{(N-2)\theta} S_2(\theta), \\ S_1(\theta) &= -\frac{2\pi i}{(N-2)(i\pi - \theta)} S_2(\theta). \end{aligned} \quad (\text{A14})$$

Using crossing, we can write the first of these equations as

$$S_2(\theta)S_2(\pi i + \theta) = \frac{\theta^2}{\theta^2 + \frac{4\pi^2}{(N-2)^2}}. \quad (\text{A15})$$

We will solve (A15) for the solution of the two-particle S-matrix inside the physical strip, assuming maximal analyticity. Rather than following Refs. [16,30] at this stage, we will instead use an elegant prescription invented by Karowski, Thun, Truong, and Weisz for the sine-Gordon/massive-Thirring model [31]. We use this prescription not only because it is straightforward, but because it gives the solution in precisely the form we need for obtaining form factors of current operators. We are not aware of this method being used for models other than sine-Gordon in the literature (we suspect it lies buried in the notes of the serious practitioners of the subject), but the result is certainly well known, and can be obtained by other methods.

To use the prescription of Ref. [31], it is convenient to decompose $S(\theta)$ differently. Instead of (A4), we write

$$S(\theta) = P_0 S_0(\theta) + P_S S_S(\theta) + P_A S_A(\theta), \quad (\text{A16})$$

where P_0 , P_S , and P_A project to the singlet, symmetric-traceless and antisymmetric irreducible representations, respectively, on $N \otimes N$:

$$\begin{aligned} (P_0)_{mk}^{jl} &= \frac{1}{N} \delta^{jl} \delta_{mk}, & (P_S)_{mk}^{jl} &= \frac{1}{2} (\delta_k^j \delta_m^l - \delta_m^j \delta_k^l), \\ (P_A)_{mk}^{jl} &= \frac{1}{2} (\delta_k^j \delta_m^l + \delta_m^j \delta_k^l) - \frac{1}{N} \delta^{jl} \delta_{mk}. \end{aligned} \quad (\text{A17})$$

The Eqs. (A14) and (A15) become

$$\begin{aligned} S_A(\theta)S_A(\pi i + \theta) &= \left(1 + \frac{i\lambda}{\theta}\right) \left(1 + \frac{i\lambda}{\theta + \pi i}\right) \frac{\theta^2}{\theta^2 + \lambda^2} \\ &= \frac{1 + \frac{i\lambda}{\theta + i\pi}}{1 - \frac{i\lambda}{\theta}}, \\ S_0(\theta) &= \frac{(\theta - i\lambda)(\theta - i\pi) + iN\lambda\theta}{(\theta + i\lambda)(\theta - i\pi)} S_A(\theta), \\ S_S(\theta) &= \frac{\theta - i\lambda}{\theta + i\lambda} S_A(\theta), \end{aligned} \quad (\text{A18})$$

where, as before, $\lambda = 2\pi/(N-2)$. We will use the first of (A18) to solve for $S_A(\theta)$.

Following Ref. [31] we assume

- (1) The function $S_A(\theta)$ is analytic¹ and nonzero in the interior of the physical strip (it has, of course, cuts on the boundaries).
- (2) $|\ln S_A(\theta)/\sinh(z - \theta)| \rightarrow 0$ as $|\text{Re}z| \rightarrow \infty$, for any fixed choice of θ in the physical strip.

Now $\sinh(z - \theta)$ has only one zero of z in the physical strip, provided θ is in the physical strip. Let C be the counter-clockwise contour enclosing the physical strip; so C extends from $-\infty$ to ∞ and from $\pi i + \infty$ to $\pi i - \infty$. Then

$$\begin{aligned} \ln S_A(\theta) &= \frac{1}{2\pi i} \int_C \frac{dz}{\sinh(z - \theta)} \ln S_A(z) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dz}{\sinh(z - \theta)} \ln [S_A(z)S_A(z + \pi i)] \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dz}{\sinh(z - \theta)} \ln \frac{1 + \frac{i\lambda}{z + i\pi}}{1 - \frac{i\lambda}{z}} \end{aligned} \quad (\text{A19})$$

where in the last step, we used (A18).

Though (A19) is a succinct expression for the S-matrix, it is not yet the most useful form. To find a better form, we need the following Fourier transforms

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{i\xi z}}{\sinh(z - \theta)} &= \frac{ie^{i(\theta - \pi i/2)}}{2 \cosh \frac{\pi\xi}{2}}, \\ \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{i\xi z} \ln \left(1 - \frac{i\lambda}{z}\right) &= \frac{1}{\xi} (1 - e^{-\xi\lambda}), \end{aligned}$$

which can be worked out using basic complex-integration methods. Substituting these Fourier transforms into (A19), we obtain

$$S_A(\theta) = \exp \int_0^{\infty} \frac{d\xi}{\xi} \frac{e^{-2\xi/(N-2)} - 1}{e^\xi + 1} \sinh \left(\frac{\xi\theta}{\pi i} \right). \quad (\text{A20})$$

This is the form of the S-matrix we shall use to discuss

¹In the case of the sine-Gordon model, there are bound-state poles on the imaginary axis in the analogous S-matrix element, for some choices of the coupling [30,32–34]. Hence this analyticity assumption is only valid for the case where the solitons have repulsive interaction. After the answer is found for this case, it can be generally applied by analytic continuation [31].

form factors. We note that this expression (A20) can be converted to the Zamolodchikovs' rational expression of gamma functions [16,30] using the integral formula [35]

$$\Gamma(z) = \exp \int_0^\infty \frac{d\xi}{\xi} \left[\frac{e^{-\xi z} - e^{-\xi}}{1 - e^{-\xi}} + (z-1)e^{-\xi} \right], \quad \text{Re} z > 0,$$

which can be checked by differentiation and comparison with the integral formula for the psi function and verification that the right-hand side is equal to one at $z = 1$. This integral formula was first used by Weisz [36] to write the infinite-gamma-function-product-ratio expressions of Zamolodchikov [30,32] for the sine-Gordon S-matrix in a more compact form.

We have obtained the *minimal* $O(N)$ -symmetric S-matrix, i.e. that with as much analyticity as possible. When expanded in powers of $1/N$, this agrees with the S-matrix of the nonlinear sigma model obtained by standard $1/N$ -expansion methods [16,37]. If the two-particle S-matrix elements are multiplied by CDD factors [38]

$$\prod_k \frac{\sinh \theta + i \sin \alpha_k}{\sinh \theta - i \sin \alpha_k},$$

unitarity and factorization are unaffected. One such S-matrix obtained in this way is that of the $O(N)$ Gross-Neveu model. See Refs. [16,30] for more discussion. The supersymmetric sigma model is a nonlinear sigma model and a Gross-Neveu model coupled together; its S-matrix was obtained in Ref. [39].

2. Exact form factors

The determination of exact two-particle form factors was initiated by Vergeles and Gryanik [40] for the fundamental particles of the sinh-Gordon model and by Weisz [36] for the solitons of the sine-Gordon model. Subsequently, Karowski and Wiesz [3] obtained generalizations of this result for the sine-Gordon model and extended the method to other models. Smirnov and later Kirillov and Smirnov [41] found extensions to higher-point form factors. Smirnov formulated a set of axioms underlying the entire subject [42] and which was useful in studying specific field theories. In the meantime, Smirnov's axioms have actually been proved as theorems, assuming maximal analyticity and the LSZ axioms [43]. Though establishing their validity from deeper principles is certainly worthwhile, the validity of Smirnov's axioms can be argued from symmetries, crossing, integrability and a little physical intuition.

Our interest in form factors is that they contain off-shell information about integrable quantum field theories. This

means that they can be used to study deformations of such theories which are no longer integrable. The form-factor program has led to nonperturbative calculations in statistical mechanics [29,44] and condensed-matter physics [45,46] which agree well with experimental measurements.

A form factor is a matrix element of an operator $\mathfrak{B}(x)$ between multiparticle states. We can obtain all of these from

$$f^{\mathfrak{B}}(\theta_1, \dots, \theta_M)_{j_1 \dots j_M} = \langle 0 | \mathfrak{B}(0) | \theta_M, j_M, \dots, \theta_1, j_1 \rangle_{\text{in}} \quad (\text{A21})$$

by crossing and the Lorentz-transformation properties of the operator $\mathfrak{B}(x)$. For example

$$\begin{aligned} f^{\mathfrak{B}}(\theta_1, \dots, \theta_M)_{j_1 \dots j_M} \exp -i \sum_{l=1}^M p_{j_l} \cdot x_l \\ = \langle 0 | \mathfrak{B}(x) | \theta_M, j_M, \dots, \theta_1, j_1 \rangle_{\text{in}}. \end{aligned} \quad (\text{A22})$$

Let us state some basic commutation relations of the FZ particle creation in-operators and annihilation in-operators $\mathfrak{A}_j(\theta)^\dagger$ and $\mathfrak{A}_j(\theta)$, respectively, (the arguments of these operators are now rapidities instead of momenta):

$$\begin{aligned} \mathfrak{A}_i(\theta_2)^\dagger \mathfrak{A}_j(\theta_1)^\dagger &= S_{ij}^{lm}(\theta_2 - \theta_1) \mathfrak{A}_m(\theta_1)^\dagger \mathfrak{A}_l(\theta_2)^\dagger, \\ \mathfrak{A}_i(\theta_2) \mathfrak{A}_j(\theta_1) &= S_{ij}^{lm}(\theta_2 - \theta_1) \mathfrak{A}_m(\theta_1) \mathfrak{A}_l(\theta_2), \\ \mathfrak{A}_i(\theta_2) \mathfrak{A}_j(\theta_1)^\dagger &= 2\pi \delta_{ij} \delta(\theta_2 - \theta_1) \\ &\quad + S_{jl}^{mi}(\theta_2 - \theta_1) \mathfrak{A}_m(\theta_1)^\dagger \mathfrak{A}_l(\theta_2). \end{aligned} \quad (\text{A23})$$

The third equation is obeyed for a free field theory with the S-matrix element replaced by the identity on two-particle space. The 2π in the normalization of the first term on the right-hand side of this equation follows from the over-completeness relation (A3) in rapidity space. The form of the second term on the right-hand side of this equation follows from crossing.

For pedagogical reasons, we shall list the form-factor axioms for general multiparticle form factors and attempt to convince the reader that they are reasonable. Our list follows Essler and Konik [46], though we attempt to provide more justification for some of the axioms. We use only four of these five axioms. We will only consider the case of operators with mutually-local commutation relations. For discussion of how to deal with other commutation relations, see Chapter 6 of Smirnov's book [42] or the article by Essler and Konik [46].

(1) *Scattering Axiom*. This follows from the properties (A23) of the FZ operators. It is

$$\begin{aligned} f^{\mathfrak{B}}(\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \theta_i, \theta_{i+2}, \dots, \theta_M)_{j_1 \dots j_{i-1} j_{i+1} j_i j_{i+2} \dots j_M} \\ = S_{j_i j_{i+1}}^{k_i k_{i+1}}(\theta_i - \theta_{i+1}) f^{\mathfrak{B}}(\theta_1, \dots, \theta_{i-1}, \theta_i, \theta_{i+1}, \theta_{i+2}, \dots, \theta_M)_{j_1 \dots j_{i-1} k_i k_{i+1} j_{i+2} \dots j_M}. \end{aligned} \quad (\text{A24})$$

This axiom is sometimes called Watson's theorem [47].

(2) *Periodicity axiom.*—This axiom is subtle. It is motivated by crossing. The axiom states

$$f^{\mathfrak{B}}(\theta_1, \dots, \theta_M)_{j_1 \dots j_M} = f^{\mathfrak{B}}(\theta_M - 2\pi i, \theta_1, \dots, \theta_{M-1})_{j_M j_1 \dots j_{M-1}}. \quad (\text{A25})$$

To see where this axiom comes from, let us see what happens when a creation FZ operator in front of a ket (state vector) is replaced by an annihilation operator behind a bra (dual state vector) by crossing. Consider the Green's function of FZ operators and \mathfrak{B}

$$\begin{aligned} & \langle 0 | \mathfrak{A}_{j_1}(\theta_1) \mathfrak{B}(0) \mathfrak{A}_{j_M}(\theta_M)^\dagger \mathfrak{A}_{j_{M-1}}(\theta_{M-1})^\dagger \cdots \mathfrak{A}_{j_2}(\theta_2)^\dagger | 0 \rangle_{\text{connected}} \\ &= \langle 0 | \mathfrak{A}_{j_1}(\theta_1) \mathfrak{B}(0) \mathfrak{A}_{j_M}(\theta_M)^\dagger \mathfrak{A}_{j_{M-1}}(\theta_{M-1})^\dagger \cdots \mathfrak{A}_{j_2}(\theta_2)^\dagger | 0 \rangle \\ & \quad \times \langle 0 | \mathfrak{A}_{j_M}(\theta_M)^\dagger \mathfrak{A}_{j_{M-1}}(\theta_{M-1})^\dagger \cdots \mathfrak{A}_{j_2}(\theta_2)^\dagger | 0 \rangle, \end{aligned}$$

which is “connected” in the sense that the vacuum intermediate channel is subtracted [43]. This Green's function can be thought of as $M - 1$ incoming particles being absorbed by a vertex corresponding to the operator $\mathfrak{B}(0)$ which then emits a single particle. Consider the pair of particles, with labels 1 (the outgoing particle) and M . Under crossing, these both become incoming particles, but with θ_1 replaced by $\theta_1 + \pi i$. To see this, notice that this change preserves all the relativistic invariants $(p_j \pm p_{j+1})^2$, $j = 2, \dots, M - 1$, but it switches the two invariants $s_{1M} = (p_1 + p_M)^2$ and $t_{1M} = (p_1 - p_M)^2$. Thus

$$\langle 0 | \mathfrak{A}_{j_1}(\theta_1) \mathfrak{B}(0) \mathfrak{A}_{j_M}(\theta_M)^\dagger \mathfrak{A}_{j_{M-1}}(\theta_{M-1})^\dagger \cdots \mathfrak{A}_{j_2}(\theta_2)^\dagger | 0 \rangle_{\text{connected}} = f^{\mathfrak{B}}(\theta_1 + \pi i, \theta_1, \dots, \theta_M)_{j_1 \dots j_M}. \quad (\text{A26})$$

Suppose that instead of switching the invariants s_{1M} and t_{1M} , we switch the invariants $s_{12} = (p_1 + p_2)^2$ and $t_{12} = (p_1 - p_2)^2$. Then we find

$$\langle 0 | \mathfrak{A}_{j_1}(\theta_1) \mathfrak{B}(0) \mathfrak{A}_{j_M}(\theta_M)^\dagger \mathfrak{A}_{j_{M-1}}(\theta_{M-1})^\dagger \cdots \mathfrak{A}_{j_2}(\theta_2)^\dagger | 0 \rangle_{\text{connected}} = f^{\mathfrak{B}}(\theta_2, \theta_1, \dots, \theta_M, \theta_1 - \pi i)_{j_2 \dots j_M j_1}. \quad (\text{A27})$$

Comparing (A26) and (A27), the axiom follows.

(3) *Annihilation pole axiom.*—This axiom is important for relating form factors of M particles to form factors of $M - 2$ particles. Though we will not apply $M > 2$ form factors in this paper, we mention the axiom anyway, since higher-point form factors should be eventually be of interest in our problem. The idea is the following: let us again consider the form factor $f^{\mathfrak{B}}(\theta_1, \dots, \theta_M)_{j_1 \dots j_M}$. Now there is the possibility of the $(M - 1)$ st and M th particles annihilating (they can be antiparticles) at $\theta_M = \theta_{M-1} + \pi i$ (so that $s_{M-1M} = (p_{M-1} + p_M)^2$ will vanish). Thus there must be a pole in the form factor which corresponds to this annihilation at $\theta_M = \theta_{M-1} + \pi i$. There are two terms in the residue of this pole: i) The $(M - 1)$ st particle may scatter with the $(M - 2)$ nd, $(M - 3)$ rd, \dots , 1st particles before annihilating the M th particle, or ii) the $(M - 1)$ st particle may not scatter with any particles before annihilating the M th particle. The axiom is

$$\begin{aligned} i \text{Res} f^{\mathfrak{B}}(\theta_1, \dots, \theta_M)_{j_1 \dots j_M} |_{\theta_M = \theta_{M-1} + \pi i} &= f^{\mathfrak{B}}(\theta_1, \dots, \theta_{M-2})_{j_1 \dots j_{M-2}} C_{j_{M-1} j_M} - S_{t_1 j_1}^{k_{M-1} k_1}(\theta_1 - \theta_{M-1}) \\ & \quad \times S_{t_2 j_2}^{t_1 k_2}(\theta_2 - \theta_{M-1}) \cdots S_{t_{M-3} j_{M-3}}^{t_{M-4} k_{M-3}}(\theta_{M-3} - \theta_{M-1}) S_{j_{M-1} j_{M-2}}^{k_{M-3} k_{M-2}}(\theta_{M-2} - \theta_{M-1}) \\ & \quad \times f^{\mathfrak{B}}(\theta_1, \dots, \theta_{M-2})_{k_1 \dots k_{M-2}} C_{k_{M-1} j_M}, \end{aligned} \quad (\text{A28})$$

where C is the charge-conjugation matrix. The normalization of the left-hand side follows from the standard state normalization, e.g. $\langle \theta | \theta' \rangle = 2\pi \delta(\theta - \theta')$.

(4) *Lorentz-invariance axiom.*—If an operator $\mathfrak{B}(x)$ carries Lorentz spin s , the form factors must transform under a boost $\theta_j \rightarrow \theta_j + \alpha$ for all $j = 1, \dots, M$ as

$$f^{\mathfrak{B}}(\theta_1 + \alpha, \dots, \theta_M + \alpha) = e^{s\alpha} f^{\mathfrak{B}}(\theta_1, \dots, \theta_M), \quad (\text{A29})$$

We hope the reader will have no trouble distinguishing spin s from center-of-mass energy squared s .

(5) *Minimality axiom.*—Just as we assume S-matrix elements have as much analyticity as possible, so we make a similar assumption of form factors. In order to check the validity of this principle, all that can be done is to compare with some perturbative method, which means either standard covariant perturbation theory or the $1/N$ -expansion. In cases where we can find form factors, minimality stands up very well. If a first guess for the form factor

$f^{\mathfrak{S}}(\theta_1, \dots, \theta_M)$ satisfies the first four axioms, then so does

$$f_{\text{minimal}}^{\mathfrak{S}}(\theta_1, \dots, \theta_M) = f^{\mathfrak{S}}(\theta_1, \dots, \theta_M) \times \frac{P_M(\{\cosh(\theta_j - \theta_k)\})}{Q_M(\{\cosh(\theta_j - \theta_k)\})}, \quad (\text{A30})$$

where P_M and Q_M are symmetric polynomials. In this way, we can eliminate all the poles in form factors, except those corresponding to bound states. In order for Axiom 3 to be satisfied:

$$\begin{aligned} P_M|_{\theta_M=\theta_{M-1}+\pi i} &= P_{M-2}, \\ Q_M|_{\theta_M=\theta_{M-1}+\pi i} &= Q_{M-2}. \end{aligned}$$

The overall normalization of the form factors is not determined by these axioms. The normalization can be found for the case of current operators, which is what we shall apply to the (2 + 1)-dimensional Yang-Mills theory.

3. Currents of the $O(N)$ nonlinear sigma model

After this review of form-factor concepts, we next apply these ideas to the case of two-particle form factors, which we shall find explicitly for currents of the $O(N)$ sigma model. The two-particle form factor may be written as

$$f^{\mathfrak{S}}(\theta_1, \theta_2)_{j_1 j_2} = e^{-s\theta_1} F^{\mathfrak{S}}(\theta_2 - \theta_1)_{j_1 j_2},$$

by axiom 4. In terms of the function $F^{\mathfrak{S}}(\theta_1, \theta_2)_{j_1 j_2}$, Axioms 1 and 2 are

$$\begin{aligned} F^{\mathfrak{S}}(\theta)_{k_1 k_2} &= e^{-s\theta} S_{k_1 k_2}^{j_1 j_2}(\theta) F^{\mathfrak{S}}(-\theta)_{j_1 j_2}, \\ F^{\mathfrak{S}}(2\pi i - \theta)_{j_2 j_1} &= e^{s\theta} F^{\mathfrak{S}}(\theta)_{j_1 j_2}, \end{aligned} \quad (\text{A31})$$

respectively.

Let us now recall the projectors P_0 , P_S , and P_A defined in (A17). We have already used the fact that these diagonalize the S-matrix acting on species tensors transforming as $N \otimes N$. We define

$$F_{0,S,A}^{\mathfrak{S}}(\theta)_{k_1 k_2} = (P_{0,S,A})_{k_1 k_2}^{j_1 j_2} F^{\mathfrak{S}}(\theta)_{j_1 j_2},$$

which allows us to rewrite (A31) as

$$\begin{aligned} F_{0,S,A}^{\mathfrak{S}}(\theta) &= e^{-s\theta} S_{0,S,A}(\theta) F_{0,S,A}^{\mathfrak{S}}(-\theta), \\ F_{0,S}^{\mathfrak{S}}(\theta) &= e^{s\theta} F_{0,S}^{\mathfrak{S}}(2\pi i - \theta), \\ F_A^{\mathfrak{S}}(\theta) &= -e^{s\theta} F_A^{\mathfrak{S}}(2\pi i - \theta). \end{aligned} \quad (\text{A32})$$

We can solve the Eqs. (A32) for the form factors, up to an overall normalization by a contour-integration method [3].

Consider a function $F(\theta)$ satisfying

$$F(2\pi i - \theta) = -e^{s\theta} F(\theta), \quad F(\theta) = e^{s\theta} S_A(\theta) F(-\theta).$$

Define the contour C to be that from $-\infty$ to ∞ and from $\infty + 2\pi i$ to $-\infty + 2\pi i$. Then

$$\begin{aligned} \ln F(\theta) &= \int_C \frac{dz}{4\pi i} \coth \frac{z - \theta}{2} \ln F(z) \\ &= \int_{-\infty}^{\infty} \frac{dz}{4\pi i} \coth \frac{z - \theta}{2} \ln \frac{F(z)}{F(z + 2\pi i)} \end{aligned}$$

Differentiating this formula with respect to θ (we will explain why in a moment)

$$\begin{aligned} \frac{dF(\theta)}{d\theta} &= \int_{-\infty}^{\infty} \frac{dz}{8\pi i} \frac{1}{\sinh^2 \frac{z - \theta}{2}} \ln \frac{F(z)}{-e^{s\theta} F(-z)} \\ &= \int_{-\infty}^{\infty} \frac{dz}{8\pi i} \frac{1}{\sinh^2 \frac{z - \theta}{2}} \ln S_A(z). \end{aligned}$$

From our expression for S_A in (A20) and differentiating the integral formula

$$\int_{-\infty}^{\infty} \frac{dz}{4\pi i} \coth \frac{z - \theta}{2} \sinh \frac{\xi z}{\pi i} = \frac{\sin^2 \frac{\xi}{\pi} (\pi i - \theta)}{\sinh \xi} - \frac{1}{2 \sinh \xi} \quad (\text{A33})$$

(which can be done using basic complex-integration methods) with respect to θ , and finally integrating $dF(\theta)/d\theta$ with respect to θ ,

$$F(\theta) = G \exp 2 \int_0^{\infty} \frac{d\xi}{\xi} \frac{e^{-2\xi/(N-2)} - 1}{e^\xi + 1} \frac{\sin^2 \frac{\xi}{2\pi} (\pi i - \theta)}{\sinh \xi}, \quad (\text{A34})$$

where G is a constant. The reason we had to differentiate and integrate with respect to θ is that otherwise, the second term of (A33) will not lead to a convergent answer.

Now we consider the form factors of the $O(N)$ current operator of the sigma model. This operator is $J_\mu^{jk} = n^j \partial_\mu n^k - n^k \partial_\mu n^j$. By Hermiticity, translation invariance, antisymmetry and Lorentz invariance

$$\langle 0 | J_\mu^{jk}(0) | \theta_2, m, \theta_1, l \rangle = iG (\delta_l^j \delta_m^k - \delta_m^j \delta_l^k) (p_1 - p_2)_\mu \exp 2 \int_0^{\infty} \frac{d\xi}{\xi} \frac{e^{-2\xi/(N-2)} - 1}{e^\xi + 1} \frac{\sin^2 \frac{\xi}{2\pi} (\pi i - \theta_{12})}{\sinh \xi}. \quad (\text{A35})$$

This is the expression for the current-operator form factor with as much analyticity as possible. The normalization of the right-hand side of (A35) is obtained by the crossing relation (A26) for the two-point case. We will fix this normalization for the case of $N = 4$, which is the case we wish to study further. It is not difficult, however, to obtain the normalization for any N using slightly different methods.

The normalization is fixed by examining the matrix elements of a charge operator. The eigenvalues of this operator are fixed by symmetry considerations. In this way the value of the form factor can be specified for particular rapidities. We will find that $G = 1$.

As we already mentioned, the connection between the $SU(2)$ -valued field U and the unit four-vector n is $U = n^4 - in^b \sigma_b$. The relation between the currents defined in (4.3) for the principal chiral sigma model and those for the vector sigma model is therefore

$$j_{\mu b}^L = \sqrt{2} \left(J_{\mu}^{Ab} + \frac{1}{2} \epsilon_{bcd} J_{\mu}^{cd} \right),$$

$$j_{\mu b}^R = \sqrt{2} \left(J_{\mu}^{Ab} - \frac{1}{2} \epsilon_{bcd} J_{\mu}^{cd} \right).$$

The left-handed charge density obeys the algebra

$$[j_0^L(x^1)_b, j_0^L(y^1)_c] = i\sqrt{2} \epsilon^{bcd} \delta(x^1 - y^1) j_{\mu}^L(x^1)_d,$$

and the left-handed charge is $Q_b = \int dx^1 j_0^L(x^1)_b$. The charge therefore obeys the commutation relations

$$[Q_b^L, Q_c^L] = i\sqrt{2} \epsilon^{bcd} Q_d^L,$$

Since the charge is in an orbital representation, the possible eigenvalues of Q_3^L are 0 (isospin zero), $0, \pm\sqrt{2}$ (isospin one), etc. We find the same relations for the right-handed charge.

From (A22) and (A35), we have

$$\langle 0 | j_0^{L,R}(x)_b | \theta_2, j_2, \theta_1, j_1 \rangle = i\sqrt{2} G (\delta_{j_1 4} \delta_{j_2 b} - \delta_{j_2 4} \delta_{j_1 b} \pm \epsilon_{b j_1 j_2}) m (\cosh \theta_1 - \cosh \theta_2) \\ \times \exp\{-im[x^0(\cosh \theta_1 + \cosh \theta_2) - x^1(\sinh \theta_1 + \sinh \theta_2)]\} F(\theta_2 - \theta_1), \quad (\text{A36})$$

where the plus or minus sign corresponds to the left-handed (L) or right-handed (R) current, respectively, and

$$F(\theta) = \exp 2 \int_0^{\infty} \frac{d\xi}{\xi} \frac{e^{-\xi} - 1}{e^{\xi} + 1} \frac{\sin^2 \frac{\xi(\pi i - \theta)}{2\pi}}{\sinh \xi} = \exp - \int_0^{\infty} \frac{d\xi}{\xi} \frac{e^{-\xi}}{\cosh^2 \frac{\xi}{2}} \sin^2 \frac{\xi(\pi i - \theta)}{2\pi}. \quad (\text{A37})$$

Now under crossing, the relation (A26) for our simple two-point case yields

$$\langle \theta_1, j_1 | j_0^{L,R}(x)_b | \theta_2, j_2 \rangle = iG\sqrt{2} (\delta_{j_1 4} \delta_{j_2 b} - \delta_{j_2 4} \delta_{j_1 b} \pm \epsilon_{b j_1 j_2}) m (\cosh \theta_1 + \cosh \theta_2) \\ \times \exp\{-im[x^0(\cosh \theta_1 - \cosh \theta_2) - x^1(\sinh \theta_1 - \sinh \theta_2)]\} F(\theta_2 - \pi i + \theta_1), \quad (\text{A38})$$

Integrating each side of (A38) over x^1 yields

$$\langle \theta_1, j_1 | Q_b^{L,R} | \theta_2, j_2 \rangle = iG\sqrt{2} (\delta_{j_1 4} \delta_{j_2 b} - \delta_{j_2 4} \delta_{j_1 b} \pm \epsilon_{b j_1 j_2}) \langle \theta_1 | \theta_2 \rangle.$$

This result yields the eigenvalues of $Q_b^{L,R}$ for isospin one, provided $G = 1$.

Equation (A35) with $G = 1$ is a result of Karowski and Weisz [3], who also checked its validity in the $1/N$ -expansion. We shall use it to obtain an expression for the effective action of the electrostatic potential Φ .

When an interaction is added to the action of an integrable model, the form factors will receive corrections [21]. The mass spectrum will be altered as well. We shall not work to high enough order in g'_0 for these effects to be taken into account. If higher-order calculations can be done, however, they will need to be included.

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