Geometrodynamics in a spherically symmetric, static crossflow of null dust

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The spherically symmetric, static space-time generated by a crossflow of noninteracting radiation streams, treated in the geometrical optics limit (null dust), is equivalent to an anisotropic fluid forming a radiation atmosphere of a star. This reference fluid provides a preferred/internal time, which is employed as a canonical coordinate. Among the advantages we encounter a new Hamiltonian constraint, which becomes linear in the momentum conjugate to the internal time (therefore yielding a functional Schrödinger equation after quantization), and a strongly commuting algebra of the new constraints.

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I. INTRODUCTION

Beside the covariant description of general relativity, a Hamiltonian formalism of gravity based on the existence of foliations for globally hyperbolic space-times was developed by Arnowitt, Deser, and Misner (hereafter ADM) [1]. In this approach the canonical coordinates are the components of the induced metric on the 3-leaves of the foliation, while the canonical momenta are related in a simple way to the extrinsic curvature of these spatial hypersurfaces. Gravitational evolution is therefore quoted as geometrodynamics. The freedom to perform coordinate transformations on the leaves of the foliation leads to the diffeomorphism (or momentum) constraints. The true dynamics is encompassed in the so-called Hamiltonian constraint. These four constraints (per point) form a Dirac algebra [2], which is not a true algebra in the mathematical sense, as its closure is obstructed by the appearance of the induced metric in the Poisson brackets of the Hamiltonian constraints.

The problem of time in canonical gravity was reviewed by Isham in Ref. [3]. Approaches for introducing the concept of time are of three types: time is either identified before or after quantization, or in certain approaches time plays no fundamental role at all. In what follows, we are interested in identifying time at the classical level. Time is not preselected by any Hamiltonian description of gravity, there are infinitely many ways to choose the time (many-fingered time formalism) [4]. Despite this ambiguity, in certain cases it is possible to select a preferred time function, by either imposing coordinate conditions [5] or by filling space-time with an adequate reference fluid [6,7] and letting gravity to evolve in the (proper) time of the chosen reference fluid.

In certain cases new canonical variables can be introduced, providing new constraints for gravity [8–10]. Among the advantages we count that the Dirac algebra transforms into a true algebra and the quantization of the Hamiltonian constraint, usually leading to the Wheeler-de Witt equation (which has no linear space of solutions), rather gives a Schrödinger equation. This program has been particularly successful for incoherent dust, as presented by Brown and Kuchař in Ref. [6].

A similar formalism [11] was applied by Bičák and Kuchař for null dust, the geometrical optics approximation to nongravitational radiation. Null dust however provides no natural time function, basically because, unlike the congruence of the incoherent dust particles, null world lines have no natural parametrization. While for ordinary dust the Hamiltonian and supermomentum constraints depend on four pairs of canonical variables associated with the proper time and the comoving coordinate frame of the dust, the constraint equations for null dust contain only three pairs of comoving coordinates.

The quantum theory of gravitational collapse can be modeled in the most simple spherically symmetric case by a collapsing thin shell of null dust [12,13]. A second null-dust shell can be introduced in the model in order to test the quantum behavior of the geometry induced by the first shell. Motivated by certain problems in the above scenario, the canonical formalism in the presence of a null dust has been recently extended to the case of two cross-flowing, noninteracting null-dust streams in a spherically symmetric space-time by Bičák and Hájíček [14]. This formalism combines ingredients of the canonical formalisms developed for null dust [11] with elements of the geometrodynamics of the Schwarzschild space-time [15], developed by Kuchař. The lack of a time standard for a single null dust however deprived the canonical formalism of the cross-streaming null dust from a time standard as well. This is because the starting point of the canonical description [14] is simply the sum of the spherically reduced Einstein-Hilbert action for gravity and two pieces of the null-dust action, also reduced by spherically symmetry. The null-dust variables are therefore doubled, without any of them becoming an internal time. The basic assumption of Ref. [14] is that the cross-flowing null-dust streams interact only gravitationally, therefore the energy-momentum tensors of the components are conserved separately. The analysis of the equations of motion provides two pairs of integrals of motion (per point), one pair for each null-dust component. Unfortunately, the Hamiltonian density could not be explicitly expressed in terms of these quantities, except in the case when one of the null-dust components is switched off. In this case the action can be transformed such that the matter part of the Liouville form contains the integrals of motion associated with the null-dust component in question.

The formalism derived in Ref. [14] is valid for certain known spherically symmetric space-times, for example, the Vaidya space-time, describing the one-component null dust [16], and the *static* space-time found in Ref. [17] by one of the present authors. The latter space-time represents the geometry in the presence of a static crossflow of noninteracting null-dust streams. Although it is asymptotically nonflat and it has a central naked singularity, it can be conveniently interpreted as the radiation atmosphere of a star. A second interpretation presented in Ref. [17] is of a 2-dimensional dilatonic model, in the presence of a pair of 2-dimensional scalar fields. While the dilaton is the square of the radial coordinate, the scalar fields are related to the energy densities of the null-dust streams. The third interpretation, based on previous work of Letelier [18], is of an anisotropic fluid, with radial pressure equal to its energy density and no tangential pressures. The static solution [17] has a homogenous counterpart [19], which can be interpreted as a Kantowski-Sachs–type cosmology. These two space-times obey a unicity theorem, as they are the only spherically symmetric solutions of the Einstein equation in the presence of a crossflow of null-dust streams with an additional (fourth) Killing vector [19]. Interestingly, for null-dust streams with negative energy density, wormhole space-times emerge [20,21].

The anisotropic fluid interpretation of the crossflow of null-dust streams with positive energy densities is particularly important for our purposes. The physical model of the anisotropic fluid *has* a preferred time, which is the time elapsed in the rest frame of the fluid. This suggests that, in contrast with the single null-dust model, for the two-component null dust an internal time formalism can be constructed. In this paper we will explicitly construct the matter action for the static configuration of noninteracting null-dust streams in terms of suitable variables, containing the internal time singled out uniquely by the crossflow of null dust.

In Sec. II we summarize the basic ingredients necessary for the purposes of the present work. We present:

(I) the canonical formalism of ordinary incoherent dust [6], with special emphasis on how the proper choice of the internal time allows us to introduce a set of new constraints for gravity, such that the new super-Hamiltonian constraint becomes *linear* in the canonical momentum conjugate to the internal time,

(II) the geometrodynamics of the spherically symmetric static vacuum [15], with special emphasis on the introduction of geometrically motivated canonical variables (including the Schwarzschild mass) in the gravitational sector,

(III) the spherically symmetric, static space-time with cross-flowing null-dust streams [17], and

(IV) the anisotropic fluid interpretation of the crossflow of noninteracting null-dust streams [18], which provides the internal time for the two-component null dust.

In Sec. III we introduce an action functional of three scalar fields characterizing the static crossflow of null dust minimally coupled to gravity. We show that variation with respect to the metric together with the equations of motion reproduces the energy-momentum tensor of two noninteracting radiation streams. Two pairs of conservation equations for the rest mass currents and the momentum currents also emerge.

In Sec. IV we derive the contribution of the two null-dust streams to the super-Hamiltonian and diffeomorphism constraints. Then we fulfill the program of replacing the total super-Hamiltonian and diffeomorphism constraints by an equivalent set, in which both momenta conjugate to the temporal and radial canonical variables appear *linearly*. We also prove that the new constraints form an Abelian algebra.

Section V contains a discussion of the falloff conditions the gravitational variables, the lapse, and the shift should obey.

In Sec. VI we compare our findings with the results presented in Ref. [14] and we show that similar techniques can be employed in the more generic context of Ref. [14] as well. We also underline the connections between our canonical variables and those employed in Ref. [14], specified for the static case.

Finally in Sec. VII we summarize our results.

II. PRELIMINARIES

In this section, we present a more technical summary of the results of Refs. [6,15,17,18] needed later in the paper.

A. Geometrodynamics of space-times with ordinary dust

The space-time action of ordinary dust was constructed by Brown and Kuchař [6] from eight scalar fields Z^k, W_k, T, M $(k = 1, 2, 3)$ minimally coupled to the space-time metric ${}^{(4)}g_{ab}$:

$$S^D[T, Z^k, M, W_k; {}^{(4)}g_{ab}] = -\frac{1}{2}\int d^4x\sqrt{-{}^{(4)}g}M(U_aU^a + 1), \tag{1}$$

The four-velocity U_a is expressed as the Pfaff form of seven scalar fields:

$$U_a = -T_{,a} + W_kZ^k{}_{,a}. \tag{2}$$

The equations of motion are

$$0 = \frac{\delta S^D}{\delta M} = -\frac{1}{2}\sqrt{-{}^{(4)}g}(U_a U^a + 1), \tag{3}$$

$$0 = \frac{\delta S^D}{\delta W_k} = -\sqrt{-{}^{(4)}g} M Z^k{}_{,a} U^a, \tag{4}$$

$$0 = \frac{\delta S^D}{\delta T} = -(\sqrt{-{}^{(4)}g} M U^a)_{,a}, \tag{5}$$

$$0 = \frac{\delta S^D}{\delta Z^k} = -(\sqrt{-{}^{(4)}g} M W_k U^a)_{,a}. \tag{6}$$

According to Eq. (4) the three vector fields Z^k are constant along the flow lines of U^a (they can be interpreted as comoving coordinates for the dust). Equation (3) shows that the four-velocity U^a is a unit timelike vector field. Equation (5) allows us to interpret M as the rest mass density of the dust and it represents mass conservation. Equation (6) can be interpreted as the momentum conservation law. From Eqs. (2)–(4) it is straightforward to deduce that T is the proper time along the dust world lines, measured between a fiducial hypersurface $T = 0$ and an arbitrary hypersurface with constant T. The dust energy-momentum tensor T_{ab} can be found from the variation of the action (1) with respect to ${}^{(4)}g_{ab}$. From the conservation of T_{ab} and M, it follows that the dust particles evolve along geodesics.

The Legendre transformed action is

$$S^D[T, Z^k, P, P_k, g_{ab}, N, N^a] = \int dt \int d^3x (P\dot{T} + P_k \dot{Z}^k - NH^D - N^a H^D_a), \tag{7}$$

where g_{ab} denotes the induced metric on the leaves, N and N^a are the lapse function and shift vectors, respectively, and the momenta P and P_k are conjugate to T and Z^k. (The original variables W_k were expressed in terms of P and P_k.) The constraints are

$$H^D_\perp = \frac{1}{2}\frac{P^2}{Mg^{1/2}} + \frac{1}{2}\frac{Mg^{1/2}}{P^2}(P^2 + g^{ab}H^D_a H^D_b), \tag{8}$$

$$H^D_a = PT_{,a} + P_k Z^k_{,a}. \tag{9}$$

The dependence of the Hamiltonian constraint on the variable M is spurious. This can be shown as follows. By varying the action with respect to M, we obtain an algebraic expression from which M can be given in terms of the other variables. Substituting this into the Hamiltonian constraint gives

$$H^D_\perp = \sqrt{P^2 + g^{ab}H^D_a H^D_b}, \tag{10}$$

so the mass multiplier M is eliminated from the action.

By employing that the *total* (gravitational + dust) constraints have to vanish, e.g. $H^D_\perp = -H^G_\perp$ and $H^D_k = -H^G_k$ on the constraint hypersurface, and solving the constraints (9) and (10) with respect to the momenta, we can replace the old constraints by an equivalent set. The new super-Hamiltonian constraint can be cast into the form

$$H_\uparrow = P + h[g_{ab,} p^{ab}] = 0, \tag{11}$$

$$h = -\sqrt{(H^G_\perp)^2 - g^{ij}H^G_i H^G_j}, \tag{12}$$

where p^{ab} are the momenta conjugate to g_{ab}. Similarly the new supermomentum constraint is

$$H_{\uparrow k} = P_k + h_k[T, Z^k, g_{ab,} p^{ab}] = 0, \tag{13}$$

$$h_k = Z^a_k H^G_a - hT_{,a}Z^a_k. \tag{14}$$

The quantization of the linearized constraint (11) gives a Schrödinger equation [6].

B. Geometrodynamics of spherically symmetric static vacuum

After the preliminary studies on the canonical formalism of the spherically symmetric space-times [22], a comprehensive analysis of Hamiltonian dynamics for Schwarzschild black holes was given by Kuchař [15]. In this section we summarize those results of his work which are relevant for our purposes.

The space-time was foliated by spherically symmetric leaves Σ_t which were labeled by the parameter time t. The induced metric on these 3-leaves can be characterized by two metric functions Λ and R,

$$g_{ab}dx^a dx^b = \Lambda^2(t, r)dr^2 + R^2(t, r)d\Omega^2, \tag{15}$$

where r is a spacelike coordinate and $d\Omega$ is the line element on the unit sphere. Under coordinate transformations R behaves as a scalar and Λ as a scalar density. In the ADM decomposition of the spherically symmetric geometry, the shift vector has a nonvanishing component only in the radial direction, denoted with N^r, which together with the lapse function N depend solely on the variables t and r.

The metric functions R and Λ are chosen as canonical coordinates and their momenta, as derived in [15], are

$$\begin{aligned} P_\Lambda &= -N^{-1}R(\dot{R} - R'N^r), \\ P_R &= -N^{-1}[\Lambda(\dot{R} - R'N^r) + R(\dot{\Lambda} - (\Lambda N^r)')]. \end{aligned} \tag{16}$$

The vacuum action for the spherically symmetric geometry can be written as

$$S_\Sigma[g_{ab}, N, N^a] = \int dt \int_{\Sigma_t} d^3x(\dot{\Lambda}P_\Lambda + \dot{R}P_R - NH^G_\perp - N^r H^G_r), \tag{17}$$

with super-Hamiltonian and supermomentum constraints

$$H_{\perp}^{G}[R, \Lambda, P_{\Lambda}, P_{R}] = \frac{1}{R} P_{R} P_{\Lambda} + \frac{1}{2R^2} \Lambda P_{\Lambda}^{2} + \frac{1}{\Lambda} R R'' - \frac{R}{\Lambda^2} R' \Lambda' + \frac{1}{2\Lambda} R'^2 - \frac{1}{2} \Lambda, \quad (18)$$

$$H_{r}^{G}[R, \Lambda, P_{\Lambda}, P_{R}] = P_{R} R' - P_{\Lambda}' \Lambda. \quad (19)$$

There exists a canonical transformation, through which the only dynamical characteristics of the Schwarzschild space-time, the Schwarzschild mass M turns into a canonical variable. The new set of variables is $(M, \mathrm{R}; P_M, P_{\mathrm{R}})$, where $M(t, r)$ is expressed in terms of the old variables $(\Lambda, R; P_{\Lambda}, P_R)$ through the formula of the Schwarzschild mass derived by Kuchař:

$$M = \tfrac{1}{2} R^{-1} P_{\Lambda}^{2} - \tfrac{1}{2} \Lambda^{-2} R R' + \tfrac{1}{2} R. \quad (20)$$

The remaining part of the canonical transformation is

$$P_M = \Lambda P_{\Lambda} \left(1 - \frac{2M}{R}\right)^{-1} R^{-1},$$
$$\mathrm{R} = R,$$
$$P_{\mathrm{R}} = P_R - \frac{1}{2} R^{-1} \Lambda P_{\Lambda} - \frac{1}{2}\left(1 - \frac{2M}{R}\right)^{-1} R^{-1} \Lambda P_{\Lambda} - R^{-1} \Lambda^{-1} \left(1 - \frac{2M}{R}\right)^{-1} \times [(\Lambda P_{\Lambda})' R R' - (\Lambda P_{\Lambda})(R R')']. \quad (21)$$

The second advantage of the new set of canonical variables is that the momentum P_M is the gradient T' of the Schwarzschild time [cf. Eq. (80) in Ref. [15]]. The gravitational constraints (18) and (19), written in terms of the new canonical variables, become

$$H_{\perp}^{G}[M, \mathrm{R}, P_M, P_{\mathrm{R}}] = -\left(1 - \frac{2M}{\mathrm{R}}\right)^{-1} \frac{M' \mathrm{R}'}{\Lambda} + \left(1 - \frac{2M}{\mathrm{R}}\right) \frac{P_M P_{\mathrm{R}}}{\Lambda}, \quad (22)$$

$$H_{r}^{G}[M, \mathrm{R}, P_M, P_{\mathrm{R}}] = P_{\mathrm{R}} \mathrm{R}' + P_M' M, \quad (23)$$

where Λ, rather than being a canonical variable, is only a shorthand notation for the following expression of the new canonical variables:

$$\Lambda = \left(1 - \frac{2M}{\mathrm{R}}\right)^{-1} M'^2 - \left(1 - \frac{2M}{\mathrm{R}}\right) P_M^2. \quad (24)$$

We will also introduce the canonical variable M in the description of the gravitational sector of the cross-streaming null-dust space-time.

We mention here the related result of Varadarajan [23], who has derived a transformation from the usually employed canonical variables (induced metric + extrinsic curvature), to a set of new canonical variables, which have the interpretation of Kruskal coordinates. This transformation is regular on the whole space-time, including the horizon. The constraints simplify to such an extent that those are equivalent to the vanishing of the canonical momenta.

C. The spherically symmetric, static space-time with cross-flowing null-dust streams

The static superposition of two noninteracting null-dust streams propagating along the null congruences u^a and v^a is characterized by the energy-momentum tensor

$$T_{ab}^{\mathrm{2ND}} = \rho(u_a u_b + v_a v_b), \quad (25)$$

with

$$u_a u^a = v_a v^a = 0, \qquad u_a v^a \neq 0. \quad (26)$$

The same time-independent energy density ρ was chosen for both null-dust components in order to assure no net energy flow (static configuration).

The spherically symmetric, static space-time containing such a crossflow of two noninteracting null-dust streams has been presented in Ref. [17]:

$$ds^2 = -2a \frac{e^{L^2}}{R(L)} [dZ^2 - R^2(L) dL^2] + R^2(L) d\Omega^2, \quad (27)$$

where Z and L are the time and radial coordinates adapted to the symmetry and R is the following expression of the radial coordinate:

$$-R(L) = a(e^{L^2} - 2L\Phi_B), \qquad \Phi_B = B + \int^L e^{x^2} dx. \quad (28)$$

Here a is a positive constant and B is a parameter.

The four-velocity null vectors of the null-dust streams are then

$$u_a = W Z_{,a} + R W L_{,a}, \qquad v_a = W Z_{,a} - R W L_{,a}, \quad (29)$$

with

$$W = \sqrt{\frac{a e^{L^2}}{R}}. \quad (30)$$

The energy density becomes

$$\rho = (8\pi R^2 W^2)^{-1}. \quad (31)$$

The superposition of the in- and outgoing null-dust streams can be interpreted as an anisotropic fluid. This indicates that there may be a possibility to use the same procedure as in the case of the incoherent dust to obtain an internal time for the canonical dynamics of cross-flowing (but otherwise noninteracting) null-dust streams, minimally coupled to gravity.

D. The anisotropic fluid interpretation of the crossflow of noninteracting null-dust streams

Letelier has shown that the energy-momentum tensor of two null-dust streams is equivalent to the energy-momentum tensor of a specific anisotropic fluid [18]. In consequence, the source of the static, spherically symmetric space-time (27) can be interpreted as an anisotropic fluid with radial pressure equaling its energy density and no tangential pressures:

$$T_{ab} = \rho(U_a U_b + \chi_a \chi_b). \tag{32}$$

Here χ^α is the (normalized) radial direction and U^α is the unit four-velocity of the fluid particles, obeying

$$-U_a U^a = \chi_a \chi^a = 1, \qquad U_a \chi^a = 0.$$

They are related to the null vectors by

$$U^\alpha = \frac{1}{\sqrt{2}}(u^a + v^a), \qquad \chi^\alpha = \frac{1}{\sqrt{2}}(u^a - v^a). \tag{33}$$

By employing Eqs. (29), we can also express the vector fields U^a and χ^a in the coordinate basis defined by Z and L:

$$U_a = \sqrt{2}WZ_{,a}, \qquad \chi_a = \sqrt{2}RWL_{,a}. \tag{34}$$

In the anisotropic fluid picture ρ represents both the energy density and the pressure, while no tangential pressure components to the spheres of constant L are present. The fluid is isotropic only about a single point, the origin.

III. ACTION PRINCIPLE FOR THE STATIC, SPHERICALLY SYMMETRIC CROSSFLOW OF TWO NONINTERACTING RADIATION STREAMS

A generic spherically symmetric space-time in coordinates (T, R, θ, φ) is characterized by two metric functions h and f as

$$ds^2 = -h(T,R)dT^2 + f^{-1}(T,R)dR^2 + R^2 d\Omega^2. \tag{35}$$

Let us introduce two scalar fields $Z(T)$ and $L(R)$, and the following advanced-type and retarded-type combinations of the 1-forms dZ and dL, which span the (T, R) sector:

$$u_a = WZ_{,a} + RWL_{,a}, \qquad v_a = WZ_{,a} - RWL_{,a}, \tag{36}$$

with W given by Eq. (30). Thus in this cobasis the 1-forms u_a and v_a do not have time-dependent components. They are entirely expressed in terms of the two scalar fields Z and L (as the coefficient functions W and R can be given in terms of L). Note that the expressions (36) are identical with (and in fact motivated by) Eqs. (29), but this time the scalars Z and L are *not* related to any exact solution, and in consequence the 1-forms u_a and v_a are not necessarily null for the generic spherically symmetric metric (35). They do have instead the same length:

$$u^a u_a = v^a v_a = -W^2\left[h^{-1}\left(\frac{dZ}{dT}\right)^2 - fR^2\left(\frac{dL}{dR}\right)^2\right]. \tag{37}$$

We also note that

$$u^a v_a = -W^2\left[h^{-1}\left(\frac{dZ}{dT}\right)^2 + fR^2\left(\frac{dL}{dR}\right)^2\right]. \tag{38}$$

Let us define a dynamical system by the action

$$S^{2ND}[{}^{(4)}g_{ab}, \rho, Z, L] = -\frac{1}{2}\int d^4x\sqrt{-{}^{(4)}g}\,\rho(u_a u^a + v_a v^a), \tag{39}$$

where $\rho(L)$ is a third scalar field. We do not know at this stage the dynamical system described by the action (39).

Variation of the action with respect to the metric gives the energy-momentum tensor:

$$T^{ab} = \frac{-2}{\sqrt{-{}^{(4)}g}}\frac{\delta S^{2ND}}{\delta^{(4)}g_{ab}} = \frac{1}{2}{}^{(4)}g^{ab}\rho(u^c u_c + v^c v_c) + \rho(u^a u^b + v^a v^b), \tag{40}$$

while the variation with respect to the coordinates Z, L, and the parameter ρ give the Euler-Lagrange equations:

$$0 = \frac{\delta S^{2ND}}{\delta Z} = -2[\sqrt{-{}^{(4)}g}\,\rho W(u^a + v^a)]_{,a}, \tag{41}$$

$$0 = \frac{\delta S^{2ND}}{\delta L} = 2\sqrt{-{}^{(4)}g}\,\rho L(u^a u_a + v^a v_a) - 2R[\sqrt{-{}^{(4)}g}\,\rho W(u^a - v^a)]_{,a}, \tag{42}$$

$$0 = \frac{\delta S^{2ND}}{\delta \rho} = -\frac{1}{2}\sqrt{-{}^{(4)}g}(u_a u^a + v_a v^a). \tag{43}$$

In Eq. (42) we have employed the relation $dW/dL = W(2RL - dR/dL)/2R$.

Equation (43) together with Eq. (37) implies that both u^a and v^a are null vectors. Then the energy-momentum tensor (40) reduces to

$$T^{ab} = \rho(u^a u^b + v^a v^b), \tag{44}$$

characterizing a noninteracting crossflow (in the null directions u^a and v^a) of null-dust streams with energy density ρ.

One can define rest mass currents as in [6]

$$\mathcal{J}^a := \sqrt{-{}^{(4)}g}\,\rho u^a, \qquad \mathcal{K}^a := \sqrt{-{}^{(4)}g}\,\rho v^a, \tag{45}$$

and momentum currents as

$$\mathcal{J}_L^a := W\mathcal{J}^a, \qquad \mathcal{K}_L^a := W\mathcal{K}^a. \tag{46}$$

In terms of these, Eq. (41) is a continuity equation for the net flow of radiation:

$$\nabla_a(\mathcal{J}_L^a + \mathcal{K}_L^a) = 0. \tag{47}$$

As the vectors u^a and v^a are null, Eq. (42) simplifies to

$$\nabla_a(\mathcal{J}^a_L - \mathcal{K}^a_L) = 0, \tag{48}$$

implying that both momentum currents are conserved individually:

$$\nabla_a \mathcal{J}^a_L = 0, \qquad \nabla_a \mathcal{K}^a_L = 0, \tag{49}$$

as expected for noninteracting radiation fields.

We have shown that the action defined by Eqs. (36) and (39) describes a crossflow of noninteracting null-dust streams in a static configuration with energy density ρ. As the vectors u^a and v^a are null, we can partially normalize them as $u^a v_a = -1$. Also, from Eq. (37) we get $dZ/dT = (fh)^{1/2} R dL/dR$. Then Eq. (38) allows one to express both metric functions as

$$f^{-1} = 2ae^{L^2} R \left(\frac{dL}{dR}\right)^2, \tag{50}$$

$$h = \frac{2ae^{L^2}}{R}\left(\frac{dZ}{dT}\right)^2. \tag{51}$$

By inserting these into the generic spherically symmetric metric (35), we obtain the metric form (27), however without the additional information (28) and (31). In order to recover these, we need the Einstein equations, derived from the sum of the Einstein-Hilbert action and the cross-flowing null-dust action (39). These are identical to those presented in Ref. [17], thus lead to the solution summarized in Sec. II C.

At the end of this section we note that the equivalent action in the anisotropic fluid picture is

$$S^F[{}^{(4)}g^{ab}, \rho, Z, L] = -\frac{1}{2}\int d^4x \sqrt{-{}^{(4)}g}\,\rho(U_a U^a + \chi_a \chi^a), \tag{52}$$

with U_a and χ_a given by Eq. (34). Because of the equivalence of the two interpretations, all equations are the same, irrespective of being derived from the cross-streaming null-dust action (39) or from the anisotropic fluid action (52).

IV. CANONICAL FORMALISM

In this section we present the calculations yielding linearized constraints for the two-component null dust, similar to Eqs. (11) and (13) derived for ordinary dust.

A. 3 + 1 decomposition of the two null-dust Lagrangian

The ADM decomposition of any spherically symmetric metric yields [15]

$$ds^2 = -(N^2 - \Lambda^2 N^{r2})dt^2 + 2\Lambda^2 N^r dr dt + \Lambda^2 dr^2 + R^2 d\Omega^2, \tag{53}$$

where Λ and R are the metric functions from the induced line element (15) and (t, r) are generic coordinates orthogonal to the (θ, φ) sector. The variables ρ, Z, L characterizing the radiation crossflow thus depend on both coordinates: $\rho = \rho(t, r)$, $Z = Z(t, r)$, and $L = L(t, r)$. From Eq. (53) $\sqrt{{}^{(4)}g} = N\sqrt{g}$.

The (3 + 1)-split form of the Lagrangian density taken from the action (39) is

$$L^{2\mathrm{ND}} = \frac{a\sqrt{g}\rho W^2}{N}\left(\dot{Z}^2 - 2N^r \dot{Z} Z' - \frac{N^2 - \Lambda^2 N^{r2}}{\Lambda^2} Z'^2\right) + \frac{a\sqrt{g}\rho R^2 W^2}{N}\left(\dot{L}^2 - 2N^r \dot{L} L' - \frac{N^2 - \Lambda^2 N^{r2}}{\Lambda^2} L'^2\right). \tag{54}$$

The canonical momenta conjugate to the radiation variables Z and L become

$$P_Z := \frac{\partial L^{2\mathrm{ND}}}{\partial \dot{Z}} = \frac{2a\sqrt{g}\rho W^2}{N}(\dot{Z} - N^r Z'),$$
$$P_L := \frac{\partial L^{2\mathrm{ND}}}{\partial \dot{L}} = \frac{2a\sqrt{g}\rho R^2 W^2}{N}(\dot{L} - N^r L'), \tag{55}$$

or inverted with respect to the velocities we obtain:

$$\dot{Z} = \frac{N}{2a\sqrt{g}\rho W^2} P_Z + N^r Z',$$
$$\dot{L} = \frac{N}{2a\sqrt{g}\rho R^2 W^2} P_L + N^r L'. \tag{56}$$

By inserting the velocities only in one factor of the velocity-squared terms of (54), we obtain the Lagrangian in the "already Hamiltonian" form

$$L^{2\mathrm{ND}} = \dot{Z} P_Z + \dot{L} P_L - N H_\perp^{2\mathrm{ND}} - N^r H_r^{2\mathrm{ND}}, \tag{57}$$

where the Hamiltonian and momentum constraints associated with the crossflow of null-dust streams are found to be

$$H_\perp^{2\mathrm{ND}} = \frac{1}{2a\sqrt{g}\rho W^2}\left(P_Z^2 + \frac{P_L^2}{R^2}\right) + \frac{2a\sqrt{g}\rho W^2}{\Lambda^2}(Z'^2 + R^2 L'^2), \tag{58}$$

$$H_r^{2\mathrm{ND}} = Z' P_Z + L' P_L. \tag{59}$$

Remarkably, the momentum constraint has the same form as the dust constraint (9).

B. Introduction of new dust constraints

If we vary the dust action (54) with respect to the comoving density ρ of the dust, we obtain

$$\frac{\delta S^{2\mathrm{ND}}}{\delta \rho} = -N \frac{\partial H_\perp^{2\mathrm{ND}}}{\partial \rho} = 0, \tag{60}$$

from which ρ can be expressed as

$$2a\sqrt{g}\rho W^2 = \Lambda \sqrt{\frac{P_Z^2 + P_L^2/R^2}{Z'^2 + R^2 L'^2}}. \tag{61}$$

By substituting this result into the Hamiltonian constraint (58), we get

$$H_\perp^{2\mathrm{ND}} = \frac{2}{\Lambda}\sqrt{R^2(P_Z L')^2 + \frac{(P_L Z')^2}{R^2} + (P_Z Z')^2 + (P_L L')^2}. \tag{62}$$

Since (59) implicates that the last two terms below the root appear in $(H_r^{2\mathrm{ND}})^2$, we eliminate them from (62). The final form of the Hamiltonian constraint is

$$H_\perp^{2\mathrm{ND}} = 2\sqrt{\left(\frac{P_Z L' R}{\Lambda} - \frac{P_L Z'}{\Lambda R}\right)^2 + g^{rr} H_r^{2\mathrm{ND}} H_r^{2\mathrm{ND}}}. \tag{63}$$

We note that in the spherically symmetric case the momentum constraint $H_r^{2\mathrm{ND}}$ can also be brought to a square-root form. From Eq. (15) and (63) we have

$$H_r^{2\mathrm{ND}} = \sqrt{-\left(P_Z L' R - \frac{P_L Z'}{R}\right)^2 + \frac{1}{4}(\Lambda H_\perp^{2\mathrm{ND}})^2}. \tag{64}$$

Equation (63) is of similar form to the Hamiltonian constraint of the incoherent dust derived in [6]. There is one difference, namely, that the Hamiltonian constraint (11) of the incoherent dust depends on the momenta conjugate to the 3-dimensional coordinate frame variables only through the momentum constraint, while in (63) P_L appears both explicitly and through $H_r^{2\mathrm{ND}}$. In spite of this, we can still follow the algorithm of [6], as will become transparent in the following.

The ADM decomposition of the total action leads to the super-Hamiltonian and supermomentum constraints

$$H_\perp := H_\perp^G + H_\perp^{2\mathrm{ND}} = 0, \tag{65}$$

$$H_r := H_r^G + H_r^{2\mathrm{ND}} = 0, \tag{66}$$

where the vacuum constraints $H_\perp^G$ and H_r^G are expressed in terms of the canonical variables $(M, \mathrm{R}; P_M, P_\mathrm{R})$ in Eqs. (22) and (23). By using Eqs. (59), (65), and (66) we can eliminate P_L, $H_\perp^{2\mathrm{ND}}$, and $H_r^{2\mathrm{ND}}$ from Eq. (63) to obtain

$$-H_\perp^G = 2\sqrt{\left(\frac{(Z'^2 + L'^2R^2)P_Z + Z'H_r^G}{L'\Lambda R}\right)^2 + g^{rr} H_r^G H_r^G}. \tag{67}$$

Then P_Z can be separated from the other variables in Eq. (67):

$$H_{\uparrow Z} = P_Z + h_Z[M, \mathrm{R}, Z, L, P_M, P_\mathrm{R}] = 0,$$
$$h_Z = \frac{L'\Lambda \mathrm{R} h + Z' H_r^G}{Z'^2 + L'^2\mathrm{R}^2}. \tag{68}$$

From (68) we know

$$P_Z = -h_Z = -\frac{L'\Lambda \mathrm{R} h + Z' H_r^G}{Z'^2 + L'^2\mathrm{R}^2}. \tag{69}$$

Here we used the notation (12).

By using (59) and (69), the constraint (66) can be written as

$$0 = H_r := H_r^G + H_r^{2\mathrm{ND}} = H_r^G + Z'P_Z + L'P_L$$
$$= H_r^G - Z'\frac{L'\Lambda R h + Z' H_r^G}{Z'^2 + L'^2R^2} + L'P_L, \tag{70}$$

which gives

$$0 = P_L + \frac{-Z'\Lambda R h + H_r^G L' R^2}{Z'^2 + L'^2R^2}. \tag{71}$$

We will denote the constraint (71) by $H_{\uparrow L}$:

$$H_{\uparrow L} = P_L + \pi_L[M, \mathrm{R}, Z, L, P_M, P_\mathrm{R}] = 0,$$
$$\pi_L = \frac{-Z'\Lambda \mathrm{R} h + L'\mathrm{R}^2 H_r^G}{Z'^2 + L'^2\mathrm{R}^2}. \tag{72}$$

Thus we have obtained a new, more convenient set of super-Hamiltonian constraint $H_{\uparrow Z}$ and supermomentum constraint $H_{\uparrow L}$. Both linearized constraints contain exactly one null-dust momentum. The Dirac algebra of the old constraints turns into an Abelian algebra of the new constraints:

$$\{H_{\uparrow J}(r), H_{\uparrow J'}(r')\} = 0, \tag{73}$$

where $H_{\uparrow J} = (H_{\uparrow Z}, H_{\uparrow L})$. This feature is similar to the case of the one-component ordinary dust [6], and in fact the proof proceeds exactly in the same way. Following [6], first we note that the Poisson brackets of the new constraints must vanish, at least weakly (on the constraint hypersurface). However, due to the linearity of the constraints (68) and (72) in the momenta P_Z, P_L, the brackets do not depend on any of P_Z, P_L. But then there is no way the constraints (68) and (72) would help in turning into zero the Poisson brackets. Therefore they have to strongly vanish.

V. FALLOFF OF THE CANONICAL VARIABLES

A. Falloff conditions for the eternal Schwarzschild black hole

The proof of Kuchař in Ref. [15] that the mapping $(\Lambda, R, P_\Lambda, P_R) \to (M, \mathrm{R}, P_M, P_\mathrm{R})$ is a canonical transformation in the gravitational sector relies on the check that the difference of the Liouville forms is an exact form. This translates to show that the expression

$$\mathcal{B}(r) = \frac{1}{2} R\delta R \ln\left|\frac{RR' + \Lambda P_\Lambda}{RR' - \Lambda P_\Lambda}\right| \tag{74}$$

vanishes on the boundaries of the domain of integration. For the eternal Schwarzschild black hole discussed there, the desired behavior was assured at $r \to \pm\infty$ by imposing suitable falloff conditions for the canonical variables, based on the treatment of Beig and O'Murchadha [24]. The proper falloff of the variables Λ, R, P_Λ, P_R, Killing time T, lapse function N, and shift N^r, given by Eqs. (49)–

(55) of Ref. [15], assure that the Kuchař mapping is a canonical transformation.

B. Falloff conditions for $r \to 0$ in flat space-time

Hájíček and Kiefer have studied the evolution of a spherically symmetric null-dust shell in the space-time generated by an other spherically symmetric null-dust shell [12]. The (innermost) region surrounded by the incoming null shell is Minkowski. In order to avoid the occurrence of a conical singularity at $r = 0$, following the method developed for cylindrical gravitational waves [25], they have imposed boundary conditions on both the coordinates and their spatial derivatives at the regular center. Based on these, Bičák and Hájíček [14] have shown that the boundary term (74) also vanishes at $r \to 0$.

Louko, Whiting, and Friedman have discussed the Hamiltonian dynamics of a thin (distributional) null-dust shell under both sets of boundary conditions: first at the two spatial infinities $r \to \pm\infty$ of the Kruskal-like manifold and second at $r \to 0$ and $r \to \infty$ [26]. In the latter case, the falloff conditions at $r \to 0$ for the canonical variables, lapse, and shift in the flat geometry within the null shell are given by their system of Eqs. (7.1):

$$\begin{aligned} \Lambda(t,r) &= \Lambda_0 + \mathcal{O}(r^2), & R(t,r) &= R_1 r + \mathcal{O}(r^3), \\ P_\Lambda(t,r) &= P_{\Lambda_2} r^2 + \mathcal{O}(r^4), & P_R(t,r) &= P_{R_1} r + \mathcal{O}(r^3), \\ N(t,r) &= N_0 + \mathcal{O}(r^2), & N^r(t,r) &= N_1^r r + \mathcal{O}(r^3), \end{aligned} \tag{75}$$

where Λ_0, R_1, P_{Λ_2}, P_{R_1}, N_0, and N_1^r are functions of time. With these falloffs, the expression $\mathcal{B}(0)$ vanishes, in accordance with the conclusion of Ref. [14].

Given the falloff behaviors (75), all terms in the vacuum gravitational super-Hamiltonian constraint (18) are $\mathcal{O}(r^2)$, with two exceptions: $R'^2/2\Lambda = R_1^2/2\Lambda_0 + \mathcal{O}(r^2)$ and $-\Lambda/2 = -\Lambda_0/2 + \mathcal{O}(r^2)$. Therefore

$$H_\perp^G = \frac{R_1^2 - \Lambda_0^2}{2\Lambda_0} + \mathcal{O}(r^2). \tag{76}$$

The leading term vanishes for

$$R_1 = \Lambda_0. \tag{77}$$

For this choice, the falloff conditions obey the vacuum gravitational super-Hamiltonian constraint. The gravitational supermomentum constraint (19) in turn behaves as

$$H_r^G = -2P_{\Lambda_2}\Lambda_0 r + \mathcal{O}(r^2). \tag{78}$$

Thus, the falloff conditions are consistent with the vacuum constraints. They are also preserved in time, as noted in [26], but we will show that only for

$$P_{\Lambda_2} = 0. \tag{79}$$

This can be seen from the following argument. The time evolution of the super-Hamiltonian and supermomentum constraints are linear combinations of the constraints and their covariant derivatives on the leaves:

$$\begin{aligned} \dot H_\perp^G &= 2H_r^G D^r N - 2NKH_\perp^G + N^r D_r H_\perp^G + N D_r H^{Gr}, \\ \dot H_r^G &= 2H_\perp^G D_r N - NKH_r^G + H_r^G D_r N^r + N D_r H_\perp^G \\ &\quad + N^r D_r H_r^G. \end{aligned} \tag{80}$$

Here $K = \Lambda^{-1}R^{-2}(RP_R - \Lambda P_\Lambda) + 2R^{-2}P_\Lambda$ is the trace of the extrinsic curvature of the leaves Σ_t, given in Ref. [15]. From the falloff conditions (75) we obtain

$$K = \Lambda_0^{-1}R_1^{-2}(R_1 P_{R_1} + \Lambda_0 P_{\Lambda_2}) + \mathcal{O}(r). \tag{81}$$

Thus, the terms proportional to the gravitational constraints, whether they contain K or not, will decay at least as $\mathcal{O}(r)$ and $\mathcal{O}(r^2)$, respectively (provided $R_1 = \Lambda_0$ was chosen). Problems could arise only from the terms containing derivatives of the constraints. The falloff conditions (75) and the covariant derivatives of the scalar and vector densities $H_\perp^G$ and H_r^G give at $r \to 0$

$$\begin{aligned} N^r D_r H_\perp^G &= -N_1^r \frac{R_1^2 - \Lambda_0^2}{\Lambda_0^2} + \mathcal{O}(r), \\ N D_r H^{Gr} &= -2N_0 \frac{P_{\Lambda_2}}{\Lambda_0} + \mathcal{O}(r), \\ N D_r H_\perp^G &= -N_0 \frac{R_1^2 - \Lambda_0^2}{2\Lambda_0^3}(2r^{-1} + \Lambda_0^{-1}) + \mathcal{O}(r), \\ N^r D_r H_r^G &= -2N_1^r \Lambda_0 P_{\Lambda_2} r + \mathcal{O}(r^2). \end{aligned} \tag{82}$$

Therefore

$$\begin{aligned} \dot H_\perp^G &= -N_1^r \frac{R_1^2 - \Lambda_0^2}{\Lambda_0^2} - 2N_0 \frac{P_{\Lambda_2}}{\Lambda_0} + \mathcal{O}(r), \\ \dot H_r^G &= -N_0 \frac{R_1^2 - \Lambda_0^2}{2\Lambda_0^3}(2r^{-1} + \Lambda_0^{-1}) + \mathcal{O}(r). \end{aligned} \tag{83}$$

By choosing the condition (77), the expression for $\dot H_r^G$ will decay as $\mathcal{O}(r)$. As we exclude the possibility $N_0 = 0$ (which would freeze time evolution at $r = 0$), the only possibility remaining for a proper decay of $\dot H_\perp^G$ is to set $P_{\Lambda_2} = 0$, which completes our proof.

C. Falloff conditions for the radiative atmosphere of a star

Now we study the question, whether the Kuchař mapping $(\Lambda, R, P_\Lambda, P_R) \to (M, \mathrm{R}, P_M, P_\mathrm{R})$ of the gravitational variables remains a canonical transformation in the configuration discussed in this paper. In order to answer this question, first we remark that the range of the cross-streaming null-dust metric parameter B is restricted by $R > 0$. This determines a lower boundary $L_{\min}$ of L, corresponding to $R = 0$. The quasilocal mass function $m(L)$ (for a definition see [17]) vanishes at some $L_{m=0}(a, B)$ and takes negative values below, in the interval $L_{\min} < L < L_{m=0}$. Besides, for $L \to \infty$ the solution (27) is not asymptotically flat. One can escape these unpleasant features by cutting off the space-time between certain $L_1 > L_{m=0}$ and an appropriate high value $L_2 > L_1$ and matching with appropriate metrics across these boundaries (see Fig. 1). The cross-streaming null-dust region (34) is matched then from the interior with the interior Schwarzschild solution, representing a static star with mass M_1, whereas from the exterior it is bounded by incoming and outgoing Vaidya regions, and it touches three exterior Schwarzschild regions in three points (these are 2-spheres, if we take into account the angles θ, φ). Therefore the solution (27) is interpreted as a thick shell of 2-component radiation, created from the intersection of incoming and outgoing thick radiation shells.

The intersection of the last incoming ray with the first outgoing ray is the point (2-sphere) where the junction to the outermost Schwarzschild region (characterized by mass M_2) is done. This region extends towards the spatial infinity i^0. As the fluid region is bounded, only the proper falloff at i^0 of the gravitational variables M, R, P_M, P_R has to hold, as summarized in the first subsection of this section.

The situation is not so trivial on the other boundary, at $r \to 0$. There, in contrast with the previous treatments of Refs. [12,14,26], we do not have vacuum, but rather the

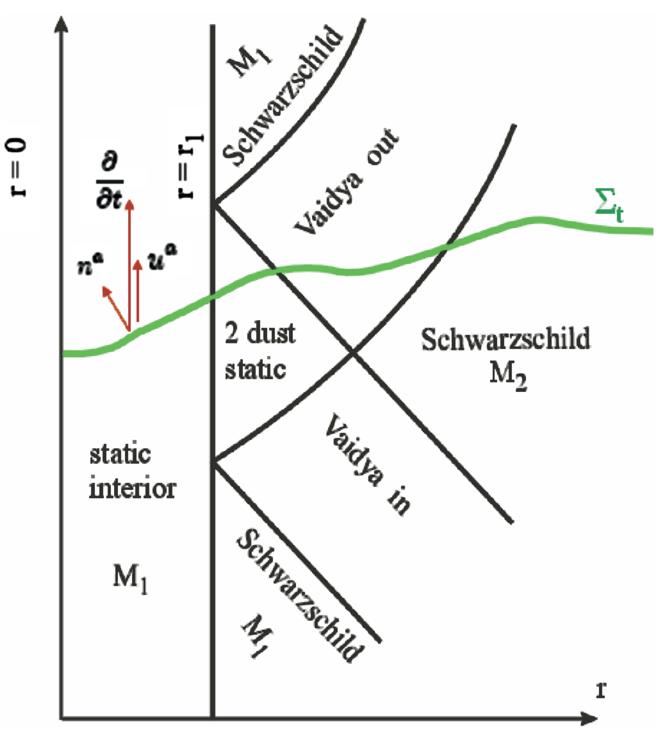

FIG. 1 (color online). The static star, incoming and outgoing radiation zones, cross-flowing null-dust region, and three exterior Schwarzschild domains. The normal vector n^a of the foliation leaf Σ_t is different from the fluid 4-velocity u^a. The foliation is chosen so that time evolution in the static star region proceeds in the direction of the fluid world lines.

center of a static star represented by the interior Schwarzschild solution, where the falloff conditions are not yet known. The line element representing the gravitational field in the interior Schwarzschild solution,

$$ds^2 = -(a - bF^{1/2})^2 dt^2 + F^{-1}dr^2 + r^2 d\Omega^2, \tag{84}$$

$$F(r) = 1 - \frac{\kappa^2 \rho r^2}{3}, \tag{85}$$

is generated by a perfect fluid with energy-momentum tensor

$$T_{ab} = (\rho + p)u_a u_b + p^{(4)}g_{ab}, \tag{86}$$

where the energy density ρ and pressure p (with respect to the 4-velocity u^a of the fluid particles) are given as

$$\rho = \text{const}, \qquad p = \rho \frac{bF^{1/2} - a/3}{a - bF^{1/2}}. \tag{87}$$

Here $\kappa^2 = 8\pi G$ and a, b are constants, chosen such that $p \geq 0$.

As the canonical treatment of the interior Schwarzschild solution has not been developed yet (and it is beyond the scope of the present paper), we will impose the simplifying condition that the world lines of the fluid particles of the stellar material are along the time-evolution vector $\partial/\partial t$:

$$\alpha u^a = \left(\frac{\partial}{\partial t}\right)^a = N n^a + N^r \Lambda^{-1}\left(\frac{\partial}{\partial r}\right)^a,$$

where $\alpha(t, r) > 0$ is a scaling function. From the condition of normalization of the 4-velocity $u^a u_a = -1$ we obtain $\alpha^2 = N^2 - N^{r2}$. This choice of the allowable foliations is in accordance with the generic expectation, that whenever a reference fluid is present in the system, it is advantageous to introduce the parameter associated with the world lines of the reference fluid as a time variable. Outside the interior Schwarzschild region, the leaves Σ_t are still allowed to be arbitrary spacelike hypersurfaces.

The energy density and energy current density of the fluid *with respect to the chosen foliation* become

$$\mu = T_{ab}n^a n^b = \left(\frac{N}{\alpha}\right)^2 \rho + \left(\frac{N^r}{\alpha}\right)^2 p,$$
$$j_r = T_{ab}n^a g^b_r = -\frac{NN^r}{\alpha^2}\Lambda(\rho + p). \tag{88}$$

With the falloff conditions (75) at $r \to 0$ the condition $\mathcal{B} \to 0$ will continue to hold, thus the Kuchař transformation is canonical. But are these falloff conditions consistent with the constraints? In order to respond affirmatively, first we note that for the fluid variables we have the following falloff conditions:

$$\begin{aligned} p &= \frac{3b-a}{3(a-b)}\rho_S + \mathcal{O}(r^2), \\ \alpha &= N_0 + \mathcal{O}(r^2), \\ \mu &= \rho_S + \mathcal{O}(r^2), \\ j_r &= -\frac{2a\rho_S}{3(a-b)} N_0^{-1} N_1^r \Lambda_0 r + \mathcal{O}(r^2). \end{aligned} \tag{89}$$

These, together with $\sqrt{g} = \Lambda R^2 \sin\theta$, imply that

$$H_\perp^{\rm star} = 2\kappa^2 \sqrt{g}\mu = \mathcal{O}(r^2), \tag{90}$$

$$H_r^{\rm star} = 2\kappa^2 \sqrt{g} j_r = \mathcal{O}(r^3), \tag{91}$$

which shows that the total constraints of gravity and fluid are obeyed for the chosen falloffs on the boundary, provided condition (77) holds.

The last question to address is whether time evolution conserves these falloffs. In order to see this, we note that both $\dot{H}_\perp^{\rm star}$ and $\dot{H}_r^{\rm star}$ vanish for the interior Schwarzschild solution and for the chosen class of foliations as $\mathcal{O}(r^2)$ and $\mathcal{O}(r^3)$, respectively. Thus, the falloff of the matter part of the constraints is faster than the falloff of the gravitational part, given by Eqs. (83). Fulfilling the conditions (77) and (79) is sufficient for the consistency with the constraints in the interior Schwarzschild solution.

Alternatively, if we do not insist on the interpretation of the cross-streaming null-dust space-time region as a radiation atmosphere of a star, we can let the outgoing radiation emerge from the origin and the incoming component be absorbed by the boundary at $r = 0$. In this case Cauchy surfaces can be chosen in such a way that their boundary at $r \to 0$ is in a flat space-time, as in Fig. 3 of Ref. [14]. In this setup, the expression $\mathcal{B}$ again vanishes, and the Kuchař transformation is proved to be canonical.

VI. COMPARISON WITH PREVIOUS RESULTS

In this section we will establish the connections between the sets of canonical variables employed in this paper and in Ref. [14]. In order to do this, first we illustrate in Sec. VI A that an internal time can be introduced for a generic spherically symmetric crossflow of radiation streams. We start from the variables employed in Ref. [14]. In Sec. VI B we show that the connection of those variables with our variables can be written up explicitly.

A. Constraints of null-dust crossflow

Bičák and Hájíček [14] generalized the canonical formulation of the one-component null dust, presented in Ref. [11] for a two-component null dust, with the specification of spherical symmetry. The gravitational part of their action was given by (17), whereas the energy-momentum tensor has been chosen as

$$T^{ab} = \frac{1}{4\pi}(l_+^a l_+^b + l_-^a l_-^b), \tag{92}$$

with

$$l_\pm^a = \frac{\sqrt{|\Pi_\pm \Phi'_\pm|}}{\Lambda R}\left[n^a \pm \Lambda^{-1}\left(\frac{\partial}{\partial r}\right)^a\right] \tag{93}$$

being the four-velocity null vectors of the ingoing and outgoing null-dust streams. The latter were characterized by the canonical coordinates Φ_+, Φ_- and their conjugate momenta Π_+, Π_-. The unit normal to the leaves was denoted by n^a. The canonical action of the system became

$$\begin{aligned} &S^T[N, N^r, \Lambda, R, \Phi_+, \Phi_-, P_\Lambda, P_R, \Pi_+, \Pi_-] \\ &= \int dt \int dr (P_\Lambda \dot{\Lambda} + P_R \dot{R} + \Pi_+ \dot{\Phi}_+ + \Pi_- \dot{\Phi}_- \\ &\quad - N H_\perp^T - N^r H_r^T), \end{aligned} \tag{94}$$

with the super-Hamiltonian constraint

$$\begin{aligned} H_\perp^T &:= H_\perp^G + H_\perp^{BH} = 0, \\ H_\perp^{BH} &= \frac{|\Pi_+ \Phi'_+| + |\Pi_- \Phi'_-|}{\Lambda}, \end{aligned} \tag{95}$$

and supermomentum constraint

$$H_r^T := H_r^G + H_r^{BH} = 0, \qquad H_r^{BH} = \Pi_+ \Phi'_+ + \Pi_- \Phi'_-. \tag{96}$$

Following the convention of Ref. [14], we assume that $\Pi_+ \Phi'_+ < 0 < \Pi_- \Phi'_-$. Thus we will use

$$H_\perp^{BH} = \frac{-\Pi_+ \Phi'_+ + \Pi_- \Phi'_-}{\Lambda}. \tag{97}$$

The constraints (95) and (96) can be conveniently combined as follows:

$$0 = \frac{\Lambda H_\perp^T - H_r^T}{-2\Phi'_+} = \Pi_+ + \frac{-\Lambda H_\perp^G + H_r^G}{2\Phi'_+}. \tag{98}$$

Similarly

$$0 = \frac{\Lambda H_\perp^T + H_r^T}{2\Phi'_-} = \Pi_- + \frac{\Lambda H_\perp^G + H_r^G}{2\Phi'_-}. \tag{99}$$

The new constraints (98) and (99) are analogous to the previously introduced constraints (68) and (72) in containing the canonical momenta only linearly.

There is nothing to prevent us in introducing square-root-type new constraints by properly transforming the constraints (98) and (99) as well. The product of the null-dust momenta is

$$\Pi_+ \Pi_- = \frac{-(\Lambda H_\perp^G)^2 + (H_r^G)^2}{4\Phi'^2_+ \Phi'^2_-} = \frac{-(\Lambda H_\perp^{BH})^2 + (H_r^{BH})^2}{4\Phi'^2_+ \Phi'^2_-}, \tag{100}$$

where we have employed the constraints (98) and (99) in the first equality and the constraints (95) and (96) in the second equality. Now we can express either $H_\perp^{BH}$ or H_r^{BH} as a square root:

$$\begin{aligned} H_\perp^{BH} &= \sqrt{-\frac{4}{\Lambda^2}\Phi'_+\Phi'_-\Pi_+\Pi_- + g^{rr}H_r^{BH}H_r^{BH}}, \\ H_r^{BH} &= \sqrt{\Phi'_+\Phi'_-\Pi_+\Pi_- + (\Lambda H_\perp^{BH})^2}. \end{aligned} \tag{101}$$

These formulas are analogous to the result (63).

The simple linear transformation

$$\begin{aligned} T &= \frac{1}{\sqrt{2}}(\Phi_+ + \Phi_-), \qquad \sigma = \frac{1}{\sqrt{2}}(\Phi_+ - \Phi_-), \\ P_T &= \frac{1}{\sqrt{2}}(\Pi_+ + \Pi_-), \qquad P_\sigma = \frac{1}{\sqrt{2}}(\Pi_+ - \Pi_-) \end{aligned} \tag{102}$$

is a canonical transformation as can be checked by calculating the Poisson brackets of the new canonical variables T, σ, P_T, P_σ. The sum of the constraints (98) and (99), divided by $\sqrt{2}$, in the new canonical chart becomes

$$0 = P_T + \frac{\sigma'\Lambda H_\perp^G + T'H_r^G}{T'^2 - \sigma'^2}. \tag{103}$$

Similarly the difference of (98) and (99), divided by $\sqrt{2}$, reads

$$0 = P_\sigma + \frac{-T'\Lambda H_\perp^G - \sigma'H_r^G}{T'^2 - \sigma'^2}. \tag{104}$$

The canonical momenta of the cross-flowing null dust are then completely separated from the rest of the variables in the new constraints (103) and (104).

If the null vectors $l_{\pm a} \propto \Phi_{\pm,a}$ are both future oriented, then T is timelike coordinate. Therefore the quantization of the constraint (103) will give a functional Schrödinger equation.

B. Static case

In this subsection we establish the connection between the canonical variables $(\Phi_\pm, \Pi_\pm)$ employed in Ref. [14] and our canonical coordinates (Z, L) and momenta (P_Z, P_L). In order to do this, first we introduce the "tortoise-type" radial coordinate R^* defined as $dR^* = R(L)dL$. Next we define null coordinates $X_\pm = Z \pm R^*$ (see [17]).

As in the static scenario the metric is uniquely given by Eq. (27), we would like to identify the energy-momentum tensors (25) and (92), which yields $l_+^a = \kappa^2 u^a/R^2W$ and $l_-^a = \kappa^2 v^a/R^2W$, with a proportionality constant κ^2. According to Ref. [14] the null forms are $l_\pm = \lambda^2(|\Pi_\pm|/\sqrt{g})d\Phi_\pm$ (λ^2 a possible second proportionality constant). By employing Eq. (29), we conclude that

$$\Pi_\pm d\Phi_\pm = \mp\sqrt{g}\left(\frac{\kappa}{\lambda}\right)^2\frac{dX_\pm}{R^2}, \tag{105}$$

with R regarded as a function of $X_\pm$.

Equivalently, the derived canonical coordinates (T, σ) are related in a simple way to (Z, L):

$$\begin{aligned} P_\sigma dT + P_T d\sigma &= -\sqrt{g}\left(\frac{2\kappa}{\lambda}\right)^2\frac{dZ}{R^2}, \\ P_T dT + P_\sigma d\sigma &= \sqrt{g}\left(\frac{2\kappa}{\lambda}\right)^2\frac{dL}{R}, \end{aligned} \tag{106}$$

with R representing the function (28) of L.

The transformations (105) and (106) establish the relation between our results and the results derived in Ref. [14].

VII. CONCLUDING REMARKS

The geometrical optics approximation of radiation fields is represented by null dust. This approximation is a very good one whenever the wavelength of the radiation is short as compared to the typical local curvature scale of the space-time. The crossflow of two such radiation streams describes the interesting situation of a two-component radiation atmosphere of a star. The assumptions of spherical symmetry and staticity lead to the space-time (27). The two null-dust components interact only gravitationally (through the curvature of the space-time they jointly produce) and formally they are equivalent to an anisotropic fluid.

A previous canonical treatment of such a system [14], besides its many achievements, still suffers from the lack of an internal time (a difficulty first encountered in the case of one-component null dust). The existence of such a preferred time function would simplify the quantization of the gravitational field in question. The absence of the internal time from the formalism of Ref. [14] is not a major inconvenience for the analysis presented there, where the one-component null-dust limit (Vaidya space-time) is discussed in detail.

If one does not aim to have this limit in the formalism, the situation is different. We have shown on the example of the static crossflow of radiation streams how to construct an internal time for the two-component system. By suitable canonical transformations we have introduced the time function Z as canonical coordinate and we have constructed the new super-Hamiltonian and supermomentum constraints, Eqs. (68) and (72), which have strongly vanishing Poisson brackets. With this, we have turned the Dirac algebra of the original constraints into an Abelian algebra.

The new constraints contain the momenta conjugate to the cross-flowing null-dust variables linearly. This convenient feature can be further exploited in the process of quantization, which will turn the new super-Hamiltonian constraint into a functional Schrödinger equation. The latter has the obvious advantage over the Wheeler-deWitt equation obtained by the quantization of the original super-Hamiltonian constraint, that its space of solutions is linear. Further properties of the resulting functional Schrödinger equation are under investigation and we propose to discuss this topic in detail elsewhere.

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