Lagrangian for doubly special relativity particle and the role of noncommutativity

Subir Ghosh

Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Calcutta 700108, India

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In this paper we have constructed a coordinate space (or geometric) Lagrangian for a point particle that satisfies the exact doubly special relativity (DSR) dispersion relation in the Magueijo-Smolin framework. Next we demonstrate how a noncommutative phase space is needed to maintain Lorentz invariance for the DSR dispersion relation. Lastly we address the very important issue of velocity of this DSR particle. Exploiting the above noncommutative phase space algebra in a Hamiltonian framework, we show that the speed of massless particles is c and for massive particles the speed saturates at c when the particle energy reaches the maximum value κ, the Planck mass.

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Motivated by ideas from quantum gravity [1], an extension of special theory of relativity known as doubly (or deformed) special relativity (DSR) [2] has been proposed. Indeed, it should be emphasized that, (as in special theory of relativity), DSR is also based on the sacred (Einsteinian) relativity principle of inertial observers that automatically removes the idea of a preferred reference frame (see for example [3]). However, unlike special theory of relativity that has a single observer-independent scale—the velocity of light c—in DSR there are *two* observer-independent scales: a length scale, ($\sim$ Planck length?) and the velocity of light c. This construction generalizes the conventional energy-momentum dispersion relation to $p^2 = m^2 + F(\kappa, ..)$ where m is the rest mass and the extension function F depends on a new parameter κ (besides the existing variables and parameter). κ is related (maybe) to the Planck mass. However, in the limit $\kappa \to \infty$ one recovers the relation $p^2 = m^2$.

This additional mass scale κ plays crucial roles in two very different contexts: The new dispersion relation is useful in explaining observations of ultrahigh energy cosmic ray particles and photons that violate the Greisen-Zatsepin-Kuzmin bound [4]. Note that one can "solve" these threshold anomaly problems by introducing explicitly Lorentz symmetry violating schemes but this is possible only at the cost of abandoning the relativity principle. At the same time, existence of a length scale is directly linked to the breakdown of spacetime continuum and the emergence of a noncommutative (NC) spacetime (below e.g. Planck length) [5–7]. Once again, this is in accord with the relativity principle since all inertial observers should agree to the (energy) scale that signals the advent of new physics, (maybe) in the form of an underlying NC spacetime structure and its related consequences. In this paper we will focus on the second aspect, after constructing a dynamical model for a DSR particle.

The theoretical development in this area so far has been mainly kinematical in the sense that various forms of the generalized dispersion relation (i.e. forms of F) have been suggested that are consistent with κ-dependent extensions of Poincaré algebras [8–10]. However, a satisfactory geometrical picture of the model, in terms of a *coordinate space Lagrangian*, so far has not appeared. In the present work we have provided such a Lagrangian that can describe a particular form of DSR dispersion relation, known as the Magueijo-Smolin (MS) relation [9],

$$p^2 = m^2\left[1 - \frac{(\eta^\mu p_\mu)}{\kappa}\right]^2, \tag{1}$$

where $\eta^0 = 1$, $\vec{\eta} = 0$. In (1), $p_0 = \kappa$ provides the particle energy upper bound, which can be identified with the Planck mass. A first step in model building was taken in [11] where the system described the MS particle only for $m = \kappa$. The model presented here is valid for the exact MS relation (1).

The other important issue is the connection between this particle model with a NC spacetime (or more generally phase space) [7–10]. Exploiting the notion of duality in the context of quantum group ideas, it has been demonstrated [8–10] that each DSR relation is *uniquely* associated with a particular form of NC phase space. More precisely, a DSR relation is Casimir of a particular κ-deformed Poincaré algebra and the latter is connected to a NC phase space in a unique way. In particular, the MS relation is related to a specific representation of κ-Minkowski NC phase space [8–10].

From a different perspective, one can directly obtain the phase space algebra of a point particle model simply by studying its symplectic structure (in a first order phase space Lagrangian formulation [12]) or by performing a constraint analysis in a Hamiltonian framework [13]. The most popular example in this connection is the canonical NC Moyal plane that one gets in studying the planar motion of a charge in a large perpendicular magnetic field [7]. Generation of NC phase space with Lie algebraic forms of noncommutativity have appeared in [14]. The motivation of our work is also to see in an explicit way how the connection between a DSR relation and a specific NC phase space is uniquely established, in a dynamical framework, as an alternative to the quantum group duality approach [8–10]. The conclusion drawn from our present analysis of an explicit model is quite interesting. We find that for a modified dispersion relation (such as DSR mass-energy law) appearance of a modified phase space algebra (or an NC phase space) is *necessary*, but the association between a DSR relation and a NC phase space is *not* unique. The first assertion is originated from a subtle interplay between Lorentz invariance and the DSR dispersion relation in question. The nonuniqueness in the choice of NC phase space is due to the gauge choice and relative strengths of the parameters. In the present work we demonstrate that the MS law is consistent with a NC phase space algebra that is more general than the κ-Minkowski. This is a new and mixed form of NC phase space algebra that interpolates between *two* Lie algebraic structures: Snyder [6] and κ-Minkowski (in MS base) [8,10].

The important question of three velocity of the particle is answered very clearly in our dynamical framework. Our results show that the massless particles move with c and the maximum speed of massive particles is also c, when their energy reaches the upper bound κ and there are subtle κ effects for the general case. These conclusions agree with [15].

Let us start with the Lagrangian of our proposed model of a MS particle,

$$L = \frac{m\kappa}{\sqrt{\kappa^2 - m^2}}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{m^2}{\kappa^2 - m^2}(g_{\mu\nu}\dot{x}^\mu\eta^\nu)^2\right]^{1/2} - \frac{m^2\kappa}{\kappa^2 - m^2}g_{\mu\nu}\dot{x}^\mu\eta^\nu \equiv \frac{m\kappa}{\sqrt{\kappa^2 - m^2}}\Lambda - \frac{m^2\kappa}{\kappa^2 - m^2}(\dot{x}\eta). \quad (2)$$

Here $g_{\mu\nu}$ represents the flat Minkowski metric $g_{00} = -g_{ii} = 1$. We have adopted the shorthand notation $(AB) = g_{\mu\nu}A^\mu B^\nu$ and $c = 1$.

First we derive the DSR dispersion relation. The conjugate momentum is

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\kappa}{\sqrt{\kappa^2 - m^2}}\frac{(\dot{x}_\mu + \frac{m^2}{\kappa^2 - m^2}(\dot{x}\eta)\eta_\mu)}{\Lambda} - \frac{m^2\kappa}{\kappa^2 - m^2}\eta^\nu. \quad (3)$$

It is straightforward to check that (3) satisfies the MS dispersion law (1). The structure of a point particle model of the kind (2) is new and is one of our main results. We note that the last term (although being a total derivative) and the specific overall scale factor in (2) is required to yield the MS relation (1).

Let us now discuss why the NC phase space is necessary. To begin with, one can construct the above Lagrangian from the first order form,

$$L = (\dot{x}p) - \frac{\lambda}{2}\left[p^2 - m^2\left(1 - \frac{(\eta p)}{\kappa}\right)^2\right], \quad (4)$$

by eliminating λ and p_μ. In (4) λ plays the role of a multiplier that enforces the MS mass-shell condition. The symplectic structure in (4) clearly suggests a canonical phase space with the only nontrivial Poisson bracket $\{x_\mu, p_\nu\} = -g_{\mu\nu}$. But notice that the MS law is *not* compatible with Lorentz invariance if one employs a canonical phase space. Quite obviously the Lorentz generator $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ transforms x_σ and p_σ properly,

$$\{J_{\mu\nu}, x_\sigma\} = g_{\nu\sigma}x_\mu - g_{\mu\sigma}x_\nu; \quad \{J_{\mu\nu}, p_\sigma\} = g_{\nu\sigma}p_\mu - g_{\mu\sigma}p_\nu, \quad (5)$$

but it fails to keep the MS dispersion law invariant,

$$\left\{J_{\mu\nu}, \left(p^2 - m^2\left(1 - \frac{(\eta p)}{\kappa}\right)^2\right)\right\} = -\frac{2}{\kappa}\left(1 - \frac{(\eta p)}{\kappa}\right)(\eta_\mu p_\nu - \eta_\nu p_\mu). \quad (6)$$

The remedy is to introduce a modified or NC phase space algebra that is consistent with the present Lagrangian structure (2) and keeps the MS relation invariant.

This is possible thanks to the τ-reparameterization invariance of the Lagrangian (2), which is evident from the vanishing Hamiltonian,

$$H = (p\dot{x}) - L = 0. \quad (7)$$

This local gauge invariance allows us to choose appropriate gauge fixing conditions such that specific forms of NC phase space structures are induced via Dirac brackets [13]. In the terminology of Dirac, the noncommuting constraints are termed as second class constraints (SCC) and the commuting constraints, that induce local gauge invariance are first class (FCC). In the presence of SCCs (ψ_1, ψ_2) that do not commute, $\{\psi_1, \psi_2\} \neq 0$, the Dirac brackets are defined in the following way:

$$\{A, B\}^* = \{A, B\} - \{A, \psi_i\}\{\psi_i, \psi_j\}^{-1}\{\psi_j, B\}, \quad (8)$$

where $\{\psi_i, \psi_j\}$ refers to the constraint matrix. From now on we will always use Dirac brackets and refer to them simply as $\{A, B\}$. In the present instance, the MS mass-shell condition (1) is the only FCC present and there are no SCC. We choose the gauge

$$\psi_1 \equiv (xp) = 0, \quad (9)$$

that has been considered before [14,16] in similar circumstances. Together with the mass-shell condition (1), $\psi_2 \equiv p^2 - m^2(1 - (\eta p)/\kappa)^2 = 0$ they constitute a SCC [13] pair with the only nonvanishing constraint matrix element $\{\psi_1, \psi_2\} = -m^2(1 - (\eta p)/\kappa)$. Hence the Dirac brackets follow:

$$\{x_\mu, x_\nu\} = \frac{1}{\kappa}(x_\mu \eta_\nu - x_\nu \eta_\mu) + \frac{1}{m^2(1-(\eta p)/\kappa)}(x_\mu p_\nu - x_\nu p_\mu),$$
$$\{x_\mu, p_\nu\} = -g_{\mu\nu} + \frac{1}{\kappa}\eta_\mu p_\nu + \frac{p_\mu p_\nu}{m^2(1-(\eta p)/\kappa)},$$
$$\{p_\mu, p_\nu\} = 0. \tag{10}$$

Performing an (invertible) transformation on the variables,

$$\tilde{x}_\mu = x_\mu - \frac{1}{\kappa}(x\eta)p_\mu, \tag{11}$$

we find an interesting form of algebra that interpolates between Snyder [6] and κ-Minkowski [2,8,10]:

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = \frac{1}{\kappa}(\tilde{x}_\mu \eta_\nu - \tilde{x}_\nu \eta_\mu) + \frac{\kappa^2 - m^2}{\kappa^2 m^2}(\tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu),$$
$$\{\tilde{x}_\mu, p_\nu\} = -g_{\mu\nu} + \frac{1}{\kappa}(p_\mu \eta_\nu + p_\nu \eta_\mu) + \frac{\kappa^2 - m^2}{\kappa^2 m^2}p_\mu p_\nu,$$
$$\{p_\mu, p_\nu\} = 0. \tag{12}$$

In absence of the $1/\kappa$ term or the $(\kappa^2 - m^2)/(\kappa^2 m^2)$ term, one obtains the Snyder [6] or the κ-Minkowski algebra [8,10], respectively.

We now show that the novel phase space algebra (12) is indeed consistent with Lorentz invariance. With $J_{\mu\nu} = \tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu$ and using (12), one can easily compute,

$$\left\{J_{\mu\nu}, \left(p^2 - m^2\left(1 - \frac{(\eta p)}{\kappa}\right)^2\right)\right\} = 2\left(p^2 - m^2\left(1 - \frac{(\eta p)}{\kappa}\right)^2\right) \approx \psi_2(\eta_\mu p_\nu - \eta_\nu p_\mu), \tag{13}$$

so that the MS relation is Lorentz invariant on shell. Next, using (12), we check that the Lorentz algebra is intact,

$$\{J^{\mu\nu}, J^{\alpha\beta}\} = g^{\mu\beta}J^{\nu\alpha} + g^{\mu\alpha}J^{\beta\nu} + g^{\nu\beta}J^{\alpha\mu} + g^{\nu\alpha}J^{\mu\beta}. \tag{14}$$

This, in itself, is expected since individually both Snyder [6] and κ-Minkowski [8] algebras do not modify the Lorentz sector, but all the same it is reassuring to note that the mixed form (12) also has this property. However, Lorentz transformations of x_μ and p_μ are indeed affected,

$$\{J^{\mu\nu}, \tilde{x}^\rho\} = g^{\nu\rho}\tilde{x}^\mu - g^{\mu\rho}\tilde{x}^\nu - \frac{1}{\kappa}(\eta^\mu J^{\rho\nu} - \eta^\nu J^{\rho\mu});$$
$$\{J^{\mu\nu}, p^\rho\} = g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu - \frac{1}{\kappa}(\eta^\nu p^\mu - \eta^\mu p^\nu)p^\rho. \tag{15}$$

Notice that only the κ-Minkowski part of the algebra (12) is responsible for the above modified forms and also that the extra terms appear only for J^{0i} and not for J^{ij} so that only boost transformations are changed. This concludes our discussion on the construction of MS particle Lagrangian and its associated NC phase space.

We now address the very important issue of speed of the κ particle [15]. We stress that since we have an explicit Lagrangian construction, the definition of velocity is very natural and unambiguous in this scheme. We extract the Hamiltonian p_0 from the MS mass-shell condition (1),

$$p_0 = \frac{\kappa}{(\kappa^2 - m^2)}(-m^2 + (\kappa^2 m^2 + (\kappa^2 - m^2)\vec{p}^2)^{1/2}) \tag{16}$$

with $p_0 \sim \sqrt{m^2 + \vec{p}^2}$ as $\kappa \to \infty$. Next, exploiting the NC algebra (12), we derive the particle dynamics:

$$\dot{\tilde{x}}_0 \equiv \{\tilde{x}_0, p_0\} = \frac{\vec{p}^2}{m^2};$$
$$\dot{\tilde{x}}_i \equiv \{\tilde{x}_i, p_0\} = \sqrt{\left(1 + \frac{(\kappa^2 - m^2)}{\kappa^2 m^2}\vec{p}^2\right)}\frac{p_i}{m}. \tag{17}$$

As a consistency check, note that (17) can be directly read off from the $\{\tilde{x}_\mu, p_\nu\}$ bracket given in (12). The natural definition for three velocity [15], $v_i \equiv \dot{\tilde{x}}_i/\dot{\tilde{x}}_0$, is not naïvely applicable in the present case as it does not lead to normal particle velocity in the $\kappa \to \infty$ limit. However, this is not surprising since we have used a nonstandard gauge choice (9) and further redefinitions (11). Let us insist that all the physical quantities in the limit κ, $\kappa m \to \infty$ should reduce to normal particle properties since then the algebra (12) becomes completely canonical. Keeping this in mind, we define a new variable,

$$X \equiv \left(\frac{(\kappa^2 - m^2)}{\kappa^2} + \frac{m^2}{\vec{p}^2}\right)\tilde{x}_0, \tag{18}$$

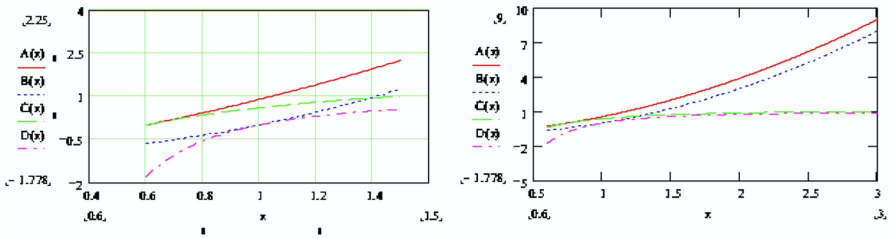
and hence obtain,

$$v_i \equiv \dot{\tilde{x}}_i/\dot{X} = p_i/\sqrt{\left(m^2 + \frac{(\kappa^2 - m^2)}{\kappa^2}\vec{p}^2\right.},$$
$$\vec{v}^2 = \vec{p}^2/\left(m^2 + \frac{(\kappa^2 - m^2)}{\kappa^2}\vec{p}^2\right). \tag{19}$$

First of all, we justify our choice of X by noting that

$$\{X, p_0\} = 1 + \frac{(\kappa^2 - m^2)}{\kappa^2 m^2}\vec{p}^2 \approx 1 + O\left(\frac{1}{\kappa^2 m^2}\right), \tag{20}$$

so that in the canonical limit X behaves as a conjugate variable to p_0, the Hamiltonian, as it should. The velocity in (19) has the correct $\kappa \to \infty$ canonical limit. Moreover, $m^2 = 0 \Rightarrow \vec{v}^2 = 1$ showing that massless DSR particles move with c irrespective of their energy. On the other hand, for massive particles, the MS relation (1) saturates at $p_0 = \kappa$ for which $\vec{p}^2 = \kappa^2$. Putting this back in (19) we find once again $\mid v \mid = 1$. Lastly, $m = \kappa \Rightarrow \mid v \mid = \mid p \mid /m$ so that Planck mass particles appear to be nonrelativistic, which agrees with their dispersion relation (1). All these



(a) (b)

FIG. 1 (color online). Plot of (velocity)^2 vs energy for DSR and normal particle.

conclusions are in accord with [15]. In the two Figs. 1(a) and 1(b) for $m = 1$, $\kappa = 1.5$ and $m = 1$, $\kappa = 3$, respectively, we plot $\vec{p}^2 \equiv A(x)$, $\vec{v}^2 \equiv C(x)$ against energy $p_0 \equiv x$ for the MS particle and compare them with the normal particle $\vec{p}^2 \equiv B(x)$, $\vec{v}^2 \equiv D(x)$. The MS energy upper bound $p_0 \equiv x = \kappa$ is used in the graphs. They indicate that MS particles can survive for smaller energy than normal particle (for the same mass) and are always faster than normal particles of same energy. However, the velocity of massive MS particles is also bounded by c that occurs at $p_0 = \kappa$. Figure 1(b) shows that, for larger κ, the MS particle tends towards its normal cousin very quickly. We emphasize that although we have worked in a particular gauge, the above limiting results are general since they involve only κ-relativistic invariants.

It is interesting to consider generalization of the invariant "length" l^2 in our geometry,

$$\{J^{\mu\nu}, l^2\} = \left\{J^{\mu\nu}, \left(\tilde{x}^2\left(1 - \frac{(\eta p)}{\kappa}\right)^2\right) - \frac{(\tilde{x}p)^2}{m^2}\right\} = \frac{2(\tilde{x}p)}{m^2\kappa}\left(p^2 - m^2\left(1 - \frac{(\eta p)}{\kappa}\right)^2\right)(\tilde{x}_\mu \eta_\nu - \tilde{x}_\nu \eta_\mu). \tag{21}$$

We find that in this type of phase space geometry the notion of an absolute coordinate space length is replaced by a combination of both $\tilde{x}_\mu$ and p_μ that is invariant only on shell (for MS law), and for $\kappa \rightarrow \infty$ one recovers the length for Snyder geometry. For Snyder algebra, this is also consistent with the interpretation,

$$\{x^S_\mu, p_\nu\} = -g_{\mu\nu} + \frac{1}{m^2} p_\mu p_\nu \equiv -G^S_{\mu\nu}, \quad (l^2)^s = G^S_{\mu\nu}(x^S)^\mu (x^S)^\nu, \tag{22}$$

where, as noted before, the Snyder algebra (and metric) is obtained from (12) in the limit $\kappa \rightarrow \infty$. However, this interpretation does not work if one includes the κ-Minkowski component of the algebra.

We conclude by noting that more dramatic changes in our perception are awaiting us, as and when we are able to construct a quantum field theory with the underlying κ-Minkowski NC spacetime structure and with fundamental excitations obeying DSR kinematics. To that end, it is essential that one has a clear understanding of the physics involved in the classical and quantum mechanical scenario. We hope that the present work is a first step in this direction.

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