Second-order gravitational self-force

Eran Rosenthal

Department of Physics, University of Guelph, Guelph, Ontario N1G 2W1, Canada

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We derive an expression for the second-order gravitational self-force that acts on a self-gravitating compact object moving in a curved background spacetime. First we develop a new method of derivation and apply it to the derivation of the first-order gravitational self-force. Here we find that our result conforms with the previously derived expression. Next we generalize our method and derive a new expression for the second-order gravitational self-force. This study also has a practical motivation: The data analysis for the planned gravitational wave detector LISA requires construction of waveform templates for the expected gravitational waves. Calculation of the two leading orders of the gravitational self-force will enable one to construct highly accurate waveform templates, which are needed for the data analysis of gravitational waves that are emitted from extreme mass-ratio binaries.

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I. INTRODUCTION

When a self-gravitating compact object moves in curved spacetime it interacts with its own gravitational field. This interaction induces a (gravitational) self-force that the object exerts on itself, and thereby affects its own motion. The analysis of this phenomenon is much simplified by considering a point particle limit, in which the spatial dimensions of the object approach zero. In this limit, the motion of the object is described by a timelike worldline, and the dynamics of the object is governed by an equation of motion.

To derive an expression for the self-force, it is useful to employ perturbations methods. In this approximation, one assumes that the object self-gravity can be represented by small metric perturbations on a given curved background metric. At the leading order of approximation, the worldline of the object traces a geodesic in the background spacetime (see e.g. [1]). At higher orders, the metric perturbations produced by the object, give rise to a self-force that induces a deviation from the geodesic trajectory. The linear metric perturbations that are proportional to the mass of the object μ induce a first-order self-force proportional to μ^2 [2,3]. At the next order, the second-order metric perturbations that are proportional to μ^2, give rise to a second-order self-force proportional to μ^3.

The main difficulty in deriving an expression for the self-force originates from the fact that we consider a point particle limit. In this limit, the metric perturbations become singular at the location of the particle, and one is required to introduce a regularization method to be able to derive the correct (and finite) expression for self-force. Using different methods Mino Sasaki and Tanaka [2], and independently Quinn and Wald [3] have derived the general expression for the first-order gravitational self-force in a vacuum background spacetime. To be able to go beyond the first-order self-force and calculate the trajectory of an object to a higher accuracy, one is required to calculate the interaction of the object with its own second-order metric perturbations. These perturbations satisfy perturbations equations with highly singular source terms. Recent studies have introduced a regularization method that enables one to solve these equations and construct the second-order metric perturbations [4,5]. In this article we follow Refs. [4,5] and use the second-order metric perturbations to construct an expression for the second-order gravitational self-force.

The study of self-force has attracted much attention in recent years (see e.g. [6] and references therein) mostly because a calculation of the self-force is needed for future detection of gravitational waves with the planned laser interferometer space antenna (LISA) [7]. One of the most interesting types of sources for LISA is an extreme mass-ratio binary, in which a compact object (e.g. a neutron star or a stellar mass black hole) gradually inspirals towards a supermassive black hole with a mass M (e.g., $M/\mu = 10^5$). Calculating the self-force that the compact object exerts on itself enables one to calculate the inspiral trajectory, and thereby prepare gravitational waveform templates for the expected gravitational waves. These templates are needed for matched-filtering data-analysis techniques. To determine the binary parameters using matched-filtering techniques one is often required to prepare gravitational waveform templates with a phase error of less than one cycle over a year of inspiral [8]. Calculating waveform templates to this accuracy is a challenging task and requires one to take into account the interaction of the compact object with its own second-order metric perturbations [4,5]. This provides a practical motivation for the study of the second-order gravitational self-force.

Our analysis is based on perturbation theory. We therefore decompose the full spacetime metric $g^{\rm full}_{\mu\nu}$ as follows

$$g^{\rm full}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}. \qquad (1)$$

Here $g_{\mu\nu}$ denotes the background metric satisfying the vacuum Einstein's field equations in the absence of the compact object, and $\delta g_{\mu\nu}$ denotes the metric perturbations produced by the compact object. Throughout we shall use $g_{\mu\nu}$ to raise and lower tensorial indices and to evaluate covariant derivatives. Recall that the gravitational self-force originates from the interaction of the object with its own metric perturbations. Therefore, the dynamics of the object should be described by an equation of motion of the following form

$$\mu \frac{D^2 z^\mu}{D\tau^2} = f^\mu[g_{\mu\nu}, \delta g_{\mu\nu}, z(\tau)], \tag{2}$$

where f^μ denotes the desired gravitational self-force,[1] $z(\tau)$ denotes the worldline of the object in the background geometry (at the point particle limit), and τ denotes the proper time with respect to the background geometry. Recall that the perturbations $\delta g_{\mu\nu}$ are singular at the worldline. This singularity suggests that $f^\mu[g_{\mu\nu}, \delta g_{\mu\nu}, z(\tau)]$ should be interpreted as a regularization operator, which takes singular quantities in its input and produces a regular self-force as an output. To make this statement meaningful (and useful), we assume that f^μ depends on the values of $\delta g_{\mu\nu}$ in the vicinity of the worldline. More precisely, we consider the hypersurface of constant time $\Sigma(\tau)$ generated by spacelike geodesics normal to the worldline at $z(\tau)$, and assume that the dependence of $f^\mu(\tau)$ on the metric perturbations is in fact a dependence on $\delta g_{\mu\nu}[\Sigma(\tau)]$ in the vicinity of $z(\tau)$. Throughout this article we shall ignore corrections to the self-force which originate from finite size and asphericity of the object. These corrections could be analyzed separately using other methods (see e.g. [9,10]). For sufficiently spherical objects these terms do not affect the first two leading orders which concern us here.[2]

We formally expand the metric perturbations in powers of μ which gives

$$\delta g_{\mu\nu} = \mu h_{\mu\nu} + \mu^2 l_{\mu\nu} + O(\mu^3). \tag{3}$$

Throughout this paper we keep the dependence on μ explicit. In the context of perturbations theory we can separate the calculation of the self-force into several consecutive steps. In the first step, we completely ignore the self-force and consider a geodesic worldline $z_G(\tau)$. Using $z_G(\tau)$ as a leading order worldline, we can calculate the first-order metric perturbations $h_{\mu\nu}$. In this context the first-order self-force is calculated using $h_{\mu\nu}$ and $z_G(\tau)$. The corrections to $z_G(\tau)$ due to the first-order self-force are required as an input for the higher orders terms of the self-force, for example, they are required for the calculation of the second-order self-force. Formally expanding f^μ in powers of μ we obtain

$$f^\mu = \mu^2 f^\mu_{(1)}[g_{\mu\nu}, h_{\mu\nu}, z_G(\tau)] + \mu^3 f^\mu_{(2)}[g_{\mu\nu}, h_{\mu\nu}, l_{\mu\nu}, z(\tau)] + O(\mu^4). \tag{4}$$

Here $f^\mu_{(1)}$ and $f^\mu_{(2)}$ denote the first-order self-force and the second-order self-force, respectively.

To calculate the self-force we shall use a decomposition technique as a part of our derivation method (several authors have previously used a similar technique together with other methods of derivation in calculating various types of self-forces, see for example [13–15]). Roughly speaking, in this technique one first decomposes the metric perturbations into a certain regular piece and a certain singular piece. Next, one calculates the self-force and shows (using a certain method) that the self-force is completely determined by the regular piece and does not depend on the singular piece. To apply such a technique to the derivation of $f^\mu_{(1)}$ one might attempt to introduce the following decomposition $h_{\mu\nu} = h^{\rm sing}_{\mu\nu} + h^{\rm reg}_{\mu\nu}$, where $h^{\rm sing}_{\mu\nu}$ is a certain singular piece and $h^{\rm reg}_{\mu\nu}$ is a certain regular piece. Here there is a difficulty, since this decomposition is nonunique. We can easily generate a family of such decompositions by invoking a transformation of the form $h^{\rm sing}_{\mu\nu} \to h^{\rm sing}_{\mu\nu} + q_{\mu\nu}$, $h^{\rm reg}_{\mu\nu} \to h^{\rm reg}_{\mu\nu} - q_{\mu\nu}$, where $q_{\mu\nu}$ is an arbitrary regular field. Therefore, there is a concern that our final expression for $f^\mu_{(1)}$ will depend on the specific decomposition in use. In this article we employ a specific decomposition (see below) that suffers from a similar nonuniqueness problem. Nevertheless, we show that this nonuniqueness does not affect the formulas that we derive for $f^\mu_{(1)}$ and $f^\mu_{(2)}$.

This article is organized as follows: In Sec. II we derive an expression for the first-order gravitational self-force in a vacuum background spacetime. Tackling this simpler problem first allows us to introduce the main principles of our derivation method. In addition, this analysis also serves as a check of our method, since we are able to compare our formula for $f^\mu_{(1)}$ with the previously derived expression for the first-order self-force [2,3]. Next, in Sec. III we generalize the principles which have been introduced in Sec. II, and derive an expression for the second-order gravitational self-force in a vacuum background spacetime.

II. FIRST-ORDER SELF-FORCE

We consider the first-order metric perturbations in the Lorenz gauge

$$\bar{h}_{\mu\nu}{}^{;\nu} = 0,$$

where an overbar denotes the trace-reversal operator de-

[1]Here it is understood that the dependence on $z(\tau)$ is other than terms of the form $C\frac{D^2 z^\mu}{D\tau^2}$ where C is a constant. No observation can distinguish between these terms and the term on the left-hand side. Therefore, these terms should be absorbed within the term on left-hand side in a mass redefinition procedure.

[2]In a simpler case of an electromagnetic self-force in flat spacetime it was recently demonstrated that the self-force is universal—i.e. at the point particle limit it is independent of the object charge distribution [11,12].

fined by $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - (1/2)g_{\mu\nu}h_\alpha{}^\alpha$. In the Lorenz gauge the first-order metric perturbations satisfy the following wave equation

$$\Box\bar{h}_{\mu\nu} + 2R^\eta{}_\mu{}^\rho{}_\nu\bar{h}_{\eta\rho} = -16\pi T_{\mu\nu}, \tag{5}$$

where $\Box \equiv g^{\rho\sigma}\nabla_\rho\nabla_\sigma$, $R_{\eta\mu\rho\nu}$ denotes the Riemann tensor of the background geometry with the sign convention of Ref. [16]. Throughout this article we use the signature $(-, +, +, +)$ and geometrized units $G = c = 1$. In a local neighborhood of the worldline the energy-momentum tensor (at the point particle limit) takes the form of

$$T_{\mu\nu}(x) = \mu \int_{-\infty}^{\infty} \bar{g}_\mu{}^\alpha(x|z_G)u_\alpha\bar{g}_\nu{}^\beta(x|z_G)u_\beta\delta^4(x - z_G) \times [-g(z_G)]^{-1/2}d\tau,$$

where the four velocity is denoted $u^\alpha = \frac{dz_G^\alpha}{d\tau}$; for an arbitrary point x' the notation $\bar{g}_\mu{}^\alpha = \bar{g}_\mu{}^\alpha(x|x')$ denotes the geodetic parallel propagator from x' to x. This bivector transports an arbitrary vector $A_\alpha(x')$ to a vector $A_\mu(x) = \bar{g}_\mu{}^\alpha(x|x')A_\alpha(x')$ by a parallel propagation of the vector A_α along the geodesic connecting x' and x (see Appendix A); g denotes the determinant of the background metric, and $\delta^4(x - x')$ denotes the four-dimensional (coordinate) Dirac delta-function.

Equation (4) implies that $f^\mu_{(1)}$ depends on the metric perturbations $h_{\mu\nu}$, where hereafter $h_{\mu\nu}$ shall denote the retarded solution of Eq. (5). In a local neighborhood of the worldline it is useful to expand the perturbations $h_{\mu\nu}$ using the distance from the worldline as a small parameter. For definiteness let us consider an arbitrary point on the worldline $\hat{z} = z_G(\hat{\tau})$, and expand $h_{\mu\nu}$ on $\Sigma(\hat{\tau})$ as follows

$$h_{\alpha\beta}(x) = \bar{g}_\alpha{}^{\hat\alpha}\bar{g}_\beta{}^{\hat\beta}[c^{(-1)}_{\hat\alpha\hat\beta}\varepsilon^{-1} + c^{(0)}_{\hat\alpha\hat\beta} + c^{(1)}_{\hat\alpha\hat\beta}\varepsilon + O(\varepsilon^2)]. \tag{6}$$

Here the overhat indices refer to $\hat{z}$, ε denotes the length of the geodesic connecting $\hat{z}$ and x. The coefficients $\{c^{(n)}_{\hat\alpha\hat\beta}\}$ transform as tensors under a coordinate transformation at $\hat{z}$, and transform as scalars under a coordinate transformation at x. By definition these coefficients are independent of ε. The explicit expressions for the coefficients $\{c^{(n)}_{\hat\alpha\hat\beta}\}$ are not required in this paper, but we should point out that some of the coefficients are nonlocal (in particular $c^{(1)}_{\hat\alpha\hat\beta}$) i.e., they depend on the entire past history of the worldline (see e.g. [6]). Notice that the perturbations $h_{\mu\nu}$ diverge as ε^{-1} and therefore the calculation of the self-force $f^\mu_{(1)}$ requires one to introduce a regularization method.

Our regularization method is based on a decomposition of $h_{\mu\nu}$ into the following tensor fields in a local neighborhood of $z_G(\tau)$:

$$h_{\mu\nu} = h^I_{\mu\nu} + h^{SR}_{\mu\nu}. \tag{7}$$

Here $h^I_{\mu\nu}$ denotes an *instantaneous* piece and $h^{SR}_{\mu\nu}$ denotes a *sufficiently regular* piece. We define an instantaneous piece $h^I_{\mu\nu}$ as a piece that admits a local expansion on $\Sigma(\hat{\tau})$, reading

$$h^I_{\alpha\beta}(x) = \bar{g}_\alpha{}^{\hat\alpha}\bar{g}_\beta{}^{\hat\beta}[d^{(-1)}_{\hat\alpha\hat\beta}\varepsilon^{-1} + d^{(0)}_{\hat\alpha\hat\beta} + d^{(1)}_{\hat\alpha\hat\beta}\varepsilon], \tag{8}$$

where the expansion coefficients $\{d^{(-1)}_{\hat\alpha\hat\beta}, d^{(0)}_{\hat\alpha\hat\beta}, d^{(1)}_{\hat\alpha\hat\beta}\}$ are constructed only from combinations of quantities in the following list: the metric tensor $g_{\hat\mu\hat\nu}$ and tensors constructed from the metric and its derivatives (e.g. Riemann tensor and its covariant derivatives), hereafter we shall refer to these tensors as background tensors; the four velocity $u^{\hat\mu}$; the unit tangent vector $\nabla_{\hat\alpha}\varepsilon$ which is tangent to the geodesic connecting $\hat{z}$ and x (see Eq. (A3) in Appendix A), this quantity transforms as a vector at $\hat{z}$ and as a scalar at x; and numerical coefficients. We define a sufficiently regular piece $h^{SR}_{\mu\nu}$ as a piece whose first-order derivative $\nabla_\alpha h^{SR}_{\mu\nu}$ is continuous in a local neighborhood of z_G, and its higher-order derivatives are not too singular (more precisely we demand that $\varepsilon^{n-1}\nabla_{\delta_n}\ldots\nabla_{\delta_1}h^{SR}_{\mu\nu;\gamma}$, where $n \geq 1$, remains bounded as $x \rightarrow \hat{z}$.).

Roughly speaking, the instantaneous piece captures the singularity content of $h_{\mu\nu}$, and therefore the piece $h^{SR}_{\mu\nu}$ is sufficiently regular. Another important property of $h^I_{\alpha\beta}(x)$ is the "instantaneous" property: unlike $h_{\mu\nu}$, the instantaneous piece $h^I_{\alpha\beta}(x)$ on $\Sigma(\hat{\tau})$ does not depend on the past history of z_G, it depends only on instantaneous quantities defined on $\Sigma(\hat{\tau})$.

A priori it is not clear that $h_{\mu\nu}$ can be decomposed into an instantaneous piece and a sufficiently regular piece. For now let us suppose without proof that this decomposition exists. Later (in Sec. II E) we shall provide a specific prescription for its construction. Notice that decomposition (7) is nonunique, and we are at liberty to introduce an alternative decomposition that satisfies the same conditions, reading

$$h_{\mu\nu} = (h^I_{\mu\nu} + k_{\mu\nu}) + (h^{SR}_{\mu\nu} - k_{\mu\nu}), \tag{9}$$

where $k_{\mu\nu}$ is both sufficiently regular and instantaneous (i.e. it satisfies all the conditions that these fields satisfy), but is otherwise completely arbitrary. At this stage we do not provide a specific prescription for the construction of $h^I_{\mu\nu}$ and $h^{SR}_{\mu\nu}$, and our derivation will be based only on the general properties that define this decomposition.

Substituting Eq. (4) into Eq. (2) and using Eq. (7) we find that at the first-order [ignoring $O(\mu^3)$ corrections] the equation of motion reads

$$\mu\frac{D^2z^\mu}{D\tau^2} = \mu^2 f^\mu_{(1)}[g_{\mu\nu}, h^I_{\mu\nu} + h^{SR}_{\mu\nu}, z_G(\tau)]. \tag{10}$$

The left-hand side of this equation is dimensionless, and therefore the vector $f^\mu_{(1)}$ must have dimensions of $(\text{Length})^{-2}$. Notice that the first-order self-force can have a complicated nonlinear dependence on the background metric. However, it must be linear in $h_{\mu\nu}$, since terms which are nonlinear in $h_{\mu\nu}$ must also come with higher powers of μ and therefore belong to higher-order corrections to f^μ. Because of this linearity in $h_{\mu\nu}$ we may group the terms in the expression for $f^\mu_{(1)}$ into a piece originating from $\{g_{\mu\nu}, h^I_{\mu\nu}, z_G(\tau)\}$ denoted $f^\mu_{(1)I}$, and a piece originating from $\{g_{\mu\nu}, h^{SR}_{\mu\nu}, z_G(\tau)\}$ denoted $f^\mu_{(1)SR}$. We formally write this decomposition as

$$f^\mu_{(1)}[g_{\mu\nu}, h^I_{\mu\nu} + h^{SR}_{\mu\nu}, z_G(\tau)] = f^\mu_{(1)I}[g_{\mu\nu}, h^I_{\mu\nu}, z_G(\tau)] + f^\mu_{(1)SR}[g_{\mu\nu}, h^{SR}_{\mu\nu}, z_G(\tau)]. \tag{11}$$

Here both $f^\mu_{(1)SR}$ and $f^\mu_{(1)I}$ are well-defined vector fields with dimensions of $(\text{Length})^{-2}$.

A. Instantaneous piece

We now introduce the principles of our derivation method that are used extensively in this article. Let us focus on the instantaneous piece $f^\mu_{(1)I}[g_{\mu\nu}, h^I_{\mu\nu}, z_G(\tau)]$. For definiteness, and without loss of generality, we calculate $f^{\hat\mu}_{(1)I}$ (i.e. $f^\mu_{(1)I}$ at $\hat z$). To derive an expression for $f^{\hat\mu}_{(1)I}$ we first make a list of all the possible tensors (at $\hat z$) that may be included in the expression for $f^{\hat\mu}_{(1)I}$. We refer to these tensors as the *tensorial constituents* of $f^{\hat\mu}_{(1)I}$. In practice, our list of tensorial constituents is constructed by studying the possible tensors that can be constructed from the quantities $\{g_{\mu\nu}, h^I_{\mu\nu}, z_G(\tau)\}$. Next, we combine these tensorial constituents to find all the possible vector expressions for $f^{\hat\mu}_{(1)I}$. In this method we first make a list of individual tensors (without their combinations), and consider the combinations only at the next step, when we construct the possible vector expressions. Notice that these vector expressions must be well defined and must have dimensions of $(\text{Length})^{-2}$.

The explicit dependence of $f^{\hat\mu}_{(1)I}$ on $z_G(\tau)$ and $g_{\mu\nu}$ implies that the four velocity and the background tensors must be included in the list of the tensorial constituents of $f^{\hat\mu}_{(1)I}$. Since $z_G(\tau)$ is a geodesic, higher-order covariant time derivatives of the four velocity vanish. The dependence of $f^{\hat\mu}_{(1)I}$ on $h^I_{\mu\nu}$ implies that tensors that can be constructed from $h^I_{\mu\nu}$ should also be included in our list of tensorial constituents. All these tensors can be expressed using the quantities that are listed in Eq. (8). We now show that these tensors do not extend our list of tensorial constituents. From its definition $h^I_{\mu\nu}$ depends on both the four velocity and the background tensors. These tensors have already been included in our list. In addition, $h^I_{\mu\nu}$ also depends on $\nabla_{\hat\alpha}\varepsilon$. However, the tangent vector $\nabla_{\hat\alpha}\varepsilon$ is well defined only for a specific value of x but it becomes multivalued as we change x. Since the self-force at $\hat z$ is independent of x, it cannot depend on $\nabla_{\hat\alpha}\varepsilon$. Moreover, we cannot generate a new well-defined vector by taking the limit $\varepsilon \to 0$ of $\nabla_{\hat\alpha}\varepsilon$. This limit is ill defined since its value depends on the direction from which the limit is taken. We therefore conclude that $\nabla_{\hat\alpha}\varepsilon$ must be excluded from our list. A similar argument shows that ε must also be excluded from our list. In addition, $h^I_{\mu\nu}$ also depends on the parallel propagator $\bar g_\alpha{}^{\hat\alpha}$. Since $\bar g_\alpha{}^{\hat\alpha}$ is a bivector rather then a tensor, it cannot be included in our list of tensorial constituents. However, one can construct a well-defined tensor from $\bar g_\alpha{}^{\hat\alpha}$, for example, by taking the limit $\varepsilon \to 0$ of $\bar g_\alpha{}^{\hat\alpha}$ (or by taking the limit $\varepsilon \to 0$ of its covariant derivatives) and thereby construct a tensor at $\hat z$. Notice, however, that $\bar g_\alpha{}^{\hat\alpha}$ is completely determined from the background metric and the geodesic connecting $\hat z$ and x. Therefore, any well-defined tensor (independent of x) that is constructed from $\bar g_\alpha{}^{\hat\alpha}$ must be a background tensor. We therefore conclude that the tensors that are constructed from $h^I_{\mu\nu}$ do not extend our list of tensorial constituents. Finally, we need to consider the possibility that the Levi-Civita tensor might appear in our list. Notice that a term with an even number of Levi-Civita tensors does not introduce any new tensor to our list, since these combinations can be expressed using combinations of Kronecker deltas.[3] It is therefore sufficient to consider a single Levi-Civita tensor in our final expression for $f^{\hat\mu}_{(1)I}$.

Summarizing the above analysis, we have found that the expression for $f^{\hat\mu}_{(1)I}$ must be composed from the following tensorial constituents: the four velocity, the background tensors, and a single Levi-Civita tensor. The fact that $f^{\hat\mu}_{(1)I}$ has a dimension of $(\text{Length})^{-2}$ further constrains our list of tensorial constituents. A background tensor with a dimension of $(\text{Length})^{-2}$ may contain only the following quantities: $g_{\hat\mu\hat\nu}$, $g_{\hat\mu\hat\nu,\hat\alpha}$, and $g_{\hat\mu\hat\nu,\hat\alpha\hat\beta}$. Clearly a combination of $g_{\hat\mu\hat\nu}$ together with $g_{\hat\mu\hat\nu,\hat\alpha}$ cannot produce such a tensor, since we may always employ a locally flat coordinate system (at $\hat z$) in which $g_{\hat\mu\hat\nu,\hat\alpha} = 0$. Consequently, a background tensor that is constructed from $g_{\hat\mu\hat\nu}$, $g_{\hat\mu\hat\nu,\hat\alpha}$ and has a dimensionality of $(\text{Length})^{-2}$ must vanish. Therefore, the desired tensor must be linear in $g_{\hat\mu\hat\nu,\hat\alpha\hat\beta}$ in addition to a combination of $g_{\hat\mu\hat\nu}$ and $g_{\hat\mu\hat\nu,\hat\alpha}$. Reference [18] provides a uniqueness theorem that ensures us that the only background tensor that satisfies all these requirements is the Riemann tensor $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ (up to a combination with $g_{\hat\mu\hat\nu}$). Tensors that include higher-order derivatives of the metric, or tensors which are nonlinear in the second derivative of the metric [e.g. tensors of the form $\nabla_{\hat\epsilon}R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$, or tensors

[3] Any two Levi-Civita tensors $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\eta}$ can be expressed as a sum over terms of the form $\pm\delta^\alpha_\mu\delta^\beta_\nu\delta^\gamma_\rho\delta^\delta_\eta$ with appropriate index permutations. This sum is often expressed as a 4×4 determinant of Kronecker deltas, see e.g. [17].

that are quadratic in the Riemann tensor] have a "wrong" dimensionality of $(\text{Length})^{-(2+n)}$ where $n > 0$, and are therefore excluded from our list.[4] We conclude that each term in $f^{\hat{\mu}}_{(1)I}$ must be constructed from a single Riemann tensor $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ and may also include the following dimensionless tensors: $u^{\hat{\mu}}$, $g_{\hat{\mu}\hat{\nu}}$, and a single Levi-Civita tensor $\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$.

We now show that in vacuum it is impossible to construct a vector from our list of tensorial constituents. First we consider vectors that are constructed from: $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$, $g_{\hat{\mu}\hat{\nu}}$, and $u^{\hat{\mu}}$. We find that all these vectors vanish by virtue of the symmetries $R_{\hat{\alpha}\hat{\beta}(\hat{\gamma}\hat{\delta})} = 0$, $R_{(\hat{\alpha}\hat{\beta})\hat{\gamma}\hat{\delta}} = 0$, and the fact that $R_{\hat{\alpha}\hat{\beta}} = 0$. Considering the next vector expressions involving the tensor $\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\eta}}$ leads to the same vanishing result: An attempt to contract three or four indices of $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ with $\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\eta}}$ vanishes due to the identity $R^{\hat{\alpha}}_{[\hat{\beta}\hat{\gamma}\hat{\delta}]} = 0$, contracting a pair of indices of $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ with $\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\eta}}$ produces a tensor with two pairs of indices (a pair from each tensor). Each of these pairs is antisymmetric, and therefore it is impossible to combine the resulting tensor with the four velocity and construct a vector. The remaining possibilities vanish trivially.

Since one cannot construct a nonvanishing vector from the above list of tensorial constituents we are led to the conclusion that

$$f^{\mu}_{(1)I} = 0. \tag{12}$$

Ori has used a similar argument to show that the expression for $f^{\mu}_{(1)}$ does not consist of any local terms [19]. In this article we shall extend this argument in the analysis of $f^{\mu}_{(2)}$ (see below). After this analysis was completed we have learned that Anderson, Flanagan, and Ottewill have further extended and simplified Ori's argument in their calculation of a quasilocal expansion of the first-order self-force [20].

B. Sufficiently regular piece

Following the method of Sec. II A we begin our analysis by studying the tensorial constituents of the sufficiently regular piece of the self-force $f^{\hat{\mu}}_{(1)SR}[g_{\mu\nu}, h^{SR}_{\mu\nu}, z_G(\tau)]$. Recall that $f^{\hat{\mu}}_{(1)SR}$ must be a well-defined vector with a dimension of $(\text{Length})^{-2}$. The dependence of $f^{\hat{\mu}}_{(1)SR}$ on $g_{\mu\nu}$ and $z_G(\tau)$ implies that the background tensors and the four velocity must be included in our list of the tensorial constituents of $f^{\hat{\mu}}_{(1)SR}$. However, in Sec. II A we found that in vacuum these tensorial constituents (together with the Levi-Civita tensor) cannot be combined to give a nonvanishing vector with a dimension of $(\text{Length})^{-2}$. Therefore, all the terms in the expression for $f^{\hat{\mu}}_{(1)SR}$ must contain an explicit (linear) dependence on $h^{SR}_{\mu\nu}$.

Since $h^{SR}_{\mu\nu}$ is sufficiently regular it admits the following local expansion on $\Sigma(\hat{\tau})$ (see Appendix A)

$$h^{SR}_{\alpha\beta}(x) = \bar{g}_{\alpha}{}^{\hat{\alpha}}\bar{g}_{\beta}{}^{\hat{\beta}}[h^{SR}_{\hat{\alpha}\hat{\beta}} - h^{SR}_{\hat{\alpha}\hat{\beta};\hat{\gamma}}\varepsilon\varepsilon^{;\hat{\gamma}} + O(\varepsilon^2)]. \tag{13}$$

For illustration consider the simplest case of static particle in flat spacetime. In this case Eq. (13) is reduced to a Taylor expansion: using Cartesian coordinates $x^a = (x^1, x^2, x^3)$ we find that in this case we have $\bar{g}_{\alpha}{}^{\hat{\alpha}} = \delta_{\alpha}{}^{\hat{\alpha}}$, $\varepsilon = \sqrt{(x^a - \hat{z}^a)(x^b - \hat{z}^b)\delta_{ab}}$, and $\varepsilon\varepsilon^{;\hat{c}} = -(x^c - \hat{z}^c)$.

We now examine the terms in Eq. (13) that may appear in our list of the tensorial constituents of $f^{\hat{\mu}}_{(1)SR}$. Following Sec. II A we find that the quantities $\bar{g}_{\beta}{}^{\hat{\beta}}$, ε, and $\varepsilon^{;\hat{\gamma}}$ that appear in Eq. (13) are excluded from our list. We are therefore left with the terms $h^{SR}_{\hat{\alpha}\hat{\beta}}$, $h^{SR}_{\hat{\alpha}\hat{\beta};\hat{\gamma}}$, and possibly higher-order derivatives (if they exist). Recall that in our notation $h^{SR}_{\hat{\alpha}\hat{\beta}}$ has a dimension of $(\text{Length})^{-1}$ [see Eq. (3)]. Therefore, higher-order derivatives of the form $h^{SR}_{\hat{\mu}\hat{\nu};\hat{\gamma}\hat{\delta}_1\ldots\hat{\delta}_n}$, $n \geq 1$, must have dimensions of $(\text{Length})^{-2-n}$. By dimensionality, these terms must be excluded from our list [21]. Consequently, from the various terms that are listed in Eq. (13) only the terms $h^{SR}_{\hat{\alpha}\hat{\beta}}$ and $h^{SR}_{\hat{\alpha}\hat{\beta};\hat{\gamma}}$ can appear in the expression for $f^{\hat{\mu}}_{(1)SR}$. For example we can construct the following vectors $(g^{\hat{\alpha}\hat{\nu}} + u^{\hat{\alpha}}u^{\hat{\nu}})u^{\hat{\mu}}u^{\hat{\rho}}h^{SR}_{\hat{\mu}\hat{\nu};\hat{\rho}}$ and $(g^{\hat{\alpha}\hat{\rho}} + u^{\hat{\alpha}}u^{\hat{\rho}})u^{\hat{\nu}}u^{\hat{\mu}}h^{SR}_{\hat{\mu}\hat{\nu};\hat{\rho}}$. Since the set of all possible vector expressions that may contribute to $f^{\hat{\mu}}_{(1)SR}$ is not an empty set, we cannot immediately implement the method of Sec. II A. To get around this difficulty we employ a gauge transformation. Previous studies have shown that the self-force is a gauge dependent quantity (see [22] and also discussion in Sec. II C below). Therefore, we may employ a gauge transformation to simplify our analysis. Consider a regular gauge transformation of the form $x^\mu \to x^\mu - \mu\xi^\mu$. We denote the first-order perturbations in the new gauge with $h^F_{\mu\nu}$. In this gauge we have $h^F_{\mu\nu} = h^I_{\mu\nu} + h^{SR(F)}_{\mu\nu}$, where $h^{SR(F)}_{\mu\nu} \equiv h^{SR}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$. Here we have included the entire gauge transformation in the definition of the new sufficiently regular piece $h^{SR(F)}_{\mu\nu}$, thus leaving $h^I_{\mu\nu}$ gauge invariant. We demand that $\xi_{\mu;\nu\rho}$ will be continuous in a local neighborhood of the worldline, and thereby we guarantee that $h^{SR(F)}_{\mu\nu}$ will satisfy the conditions of a sufficiently regular piece. We shall refer to the new gauge as Fermi gauge. In Fermi gauge Eq. (11) reads

$$f^{\mu}_{F(1)}[g_{\mu\nu}, h^I_{\mu\nu} + h^{SR(F)}_{\mu\nu}, z_G(\tau)] = f^{\mu}_{(1)I}[g_{\mu\nu}, h^I_{\mu\nu}, z_G(\tau)] + f^{\mu}_{(1)SR(F)}[g_{\mu\nu}, h^{SR(F)}_{\mu\nu}, z_G(\tau)], \tag{14}$$

where $f^{\mu}_{F(1)}$ is the first-order self-force in Fermi gauge and

[4] Notice that tensors with dimensions of $(\text{Length})^n$, where $n > 0$ [e.g. $(R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}R^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}})^{-2}R_{\hat{\mu}\hat{\nu}\hat{\eta}\hat{\rho}}$] must be excluded since they give rise to terms that are ill defined in flat spacetime.

$f^{\mu}_{(1)SR(F)}$ is its corresponding sufficiently regular piece. We impose the following gauge conditions along the worldline $z_G(\tau)$[5]

$$[h^{SR(F)}_{\mu\nu}]_{z_G(\tau)} \equiv [h^{SR}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}]_{z_G(\tau)} = 0, \tag{15}$$

$$[h^{SR(F)}_{\mu\nu;\rho}]_{z_G(\tau)} \equiv [(h^{SR}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu})_{;\rho}]_{z_G(\tau)} = 0. \tag{16}$$

In complete analogy with the Lorenz gauge, each term in $f^{\hat{\mu}}_{(1)SR(F)}$ (in Fermi gauge) must explicitly depend on one of the first two coefficients in the expansion of $h^{SR(F)}_{\mu\nu}$ [similar to Eq. (13)]. Since these coefficients vanish all these terms must be equal to zero. We therefore conclude that

$$f^{\mu}_{(1)SR(F)} = 0. \tag{17}$$

We now briefly explain how to construct Fermi gauge. By contracting Eq. (16) with u^{ρ}, we find that $\frac{D}{D\tau} h^{SR(F)}_{\mu\nu} = 0$. This result is consistent with Eq. (15). We can now solve Eq. (15) at an initial point $z_G(\tau_0)$. Satisfying Eq. (16) guarantees the validity of Eq. (15) everywhere along $z_G(\tau)$. We construct an arbitrary gauge vector $\xi_{(0)\mu}$ at $z_G(\tau_0)$, and demand that

$$\xi_{(0)\mu;\nu} = \xi_{(0)\nu;\mu} = -\tfrac{1}{2} h^{SR}_{\mu\nu}(\tau_0), \tag{18}$$

thereby satisfying Eq. (15). We now turn to Eq. (16). Viewing these equations as algebraic equations, we find that we have a set of 40 equations for the 64 variables $\xi_{\mu;\nu\rho}$. By introducing the commutation relation

$$2\xi_{\mu;[\nu\alpha]} = R^{\epsilon}{}_{\mu\nu\alpha}\xi_{\epsilon}, \tag{19}$$

we include another 24 equations which brings us to 64 equations, as desired. Using the identities of the Riemann tensor we obtain from Eqs. (16) and (19) the following relation

$$\xi_{\nu;\mu\alpha} = R^{\epsilon}{}_{\alpha\mu\nu}\xi_{\epsilon} - \delta\Gamma^{SR}_{\nu\mu\alpha}, \tag{20}$$

where

$$\delta\Gamma^{SR}_{\nu\mu\alpha} = \tfrac{1}{2}(h^{SR}_{\mu\nu;\alpha} + h^{SR}_{\nu\alpha;\mu} - h^{SR}_{\mu\alpha;\nu}).$$

Here all quantities are evaluated on the worldline. By contracting Eq. (20) with $u^{\alpha}u^{\mu}$ we obtain

$$\ddot{\xi}_{\nu} = R^{\epsilon}{}_{\alpha\mu\nu}\xi_{\epsilon}u^{\alpha}u^{\mu} - u^{\alpha}u^{\mu}\delta\Gamma^{SR}_{\nu\mu\alpha}, \tag{21}$$

where $\ddot{\xi}_{\nu} \equiv \frac{D^2\xi_{\nu}}{D\tau^2}$. One can solve this second-order transport equation by using $\xi_{(0)\mu}$, and $\dot{\xi}_{(0)\mu}$ as initial conditions, and thereby obtain a gauge vector $\xi_{\mu}(\tau)$ along the worldline. Similarly we can construct a first-order transport equation for $\xi_{\mu;\nu}$ by contracting Eq. (20) with u^{α}. Using $\xi_{\mu}(\tau)$ together with Eq. (18) this first-order transport equation can be integrated to give $\xi_{\mu;\nu}(\tau)$. $\xi_{\mu;\nu\alpha}(\tau)$ is then obtained by substituting $\xi_{\mu}(\tau)$ into Eq. (20). This completes the formal solution of Eqs. (15) and (16), and allows one to calculate the leading terms in a local expansion of ξ_{μ} in the vicinity of the worldline. This local expansion could be arbitrarily continued, thereby obtaining a global definition for Fermi gauge.

[5]Fermi gauge was first introduced in Ref. [4]. In this reference the gauge vector ξ^{μ} was defined using Eqs. (15) and (16) with the replacement $h^{SR}_{\mu\nu} \to h^{R}_{\mu\nu}$, where $h^{R}_{\mu\nu}$ is the Detweiler and Whiting regular piece [15]. These different definitions coincide once we express $h^{SR}_{\mu\nu}$ in terms of $h^{R}_{\mu\nu}$ in Eq. (31) below.

C. First-order self-force in Fermi gauge

Combining Eqs. (12) and (17) together with Eq. (14) we find that in Fermi gauge the first-order self-force vanishes, namely

$$f^{\mu}_{F(1)} = 0. \tag{22}$$

This result might appear surprising at first. The surprise is removed once one recalls that within the framework of general relativity the complete information about the motion of a point like object includes two pieces of information: the object's worldline together with the spacetime metric. It is therefore not surprising that one may transfer information about the motion of the object from the description of the worldline to the description of the gauge. This also implies that an expression for the self-force is physically meaningful only if the gauge is specified.

D. First-order self-force in the Lorenz gauge

We now derive the expression for the first-order self-force in the Lorenz gauge. For this purpose, we consider the inverse gauge transformation i.e. a transformation from Fermi gauge to the original Lorenz gauge, generated by $x^{\alpha} \to x^{\alpha} + \mu\xi^{\alpha}$, where ξ^{α} satisfies Eqs. (15) and (16). Barack and Ori have derived the gauge-transformation formula for the first-order self-force [22]. Their analysis shows that under a regular gauge transformation of the form $x^{\alpha} \to x^{\alpha} - \mu\xi^{\alpha}$ the first-order self-force transforms as

$$f^{\alpha}_{(1)} \to f^{\alpha}_{(1)} + \delta f^{\alpha}_{(1)}, \tag{23}$$

where

$$\delta f^{\alpha}_{(1)} = -(g^{\alpha\lambda} + u^{\alpha}u^{\lambda})\ddot{\xi}_{\lambda} - R^{\alpha}{}_{\mu\lambda\nu}u^{\mu}\xi^{\lambda}u^{\nu}. \tag{24}$$

Using Eqs. (22)–(24) we find that in the Lorenz gauge the self-force is given by

$$f^{\alpha}_{(1)} = f^{\alpha}_{F(1)} - \delta f^{\alpha}_{(1)} = -\delta f^{\alpha}_{(1)}, \tag{25}$$

where the minus sign originates from the fact that we are considering the inverse gauge transformation. We now calculate $f^{\alpha}_{(1)}$ by first substituting Eq. (21) into Eq. (24), and then substituting $\delta f^{\alpha}_{(1)}$ into Eq. (25). In this way we obtain the following expression for the first-order self-force in the Lorenz gauge

$$\mu^2 f^\mu_{(1)} = \mu^2 K^{\mu\alpha\beta\gamma} h^{SR}_{\alpha\beta;\gamma}. \tag{26}$$

where

$$K^{\mu\alpha\beta\gamma} = -g^{\mu\beta}u^\gamma u^\alpha - (1/2)u^\mu u^\alpha u^\beta u^\gamma + (1/2)g^{\mu\gamma}u^\alpha u^\beta.$$

We have already mentioned that decomposition (7) is nonunique, and one can generate a family of alternative decompositions (9), where $k_{\mu\nu}$ is both sufficiently regular and instantaneous, but is otherwise completely arbitrary. Clearly our result in Eq. (26) is self-consistent only if it is invariant under the transformation $h^{SR}_{\alpha\beta} \to h^{SR}_{\alpha\beta} - k_{\alpha\beta}$. This invariance follows from properties of $k_{\alpha\beta}$. Since $k_{\alpha\beta}$ is sufficiently regular the vector $K^{\mu\alpha\beta\gamma}k_{\alpha\beta;\gamma}$ is well defined on the worldline, and since $k_{\alpha\beta}$ is instantaneous it follows from subsection II A that $K^{\mu\alpha\beta\gamma}k_{\alpha\beta;\gamma} = 0$, meaning that Eq. (26) is invariant under the transformation $h^{SR}_{\alpha\beta} \to h^{SR}_{\alpha\beta} - k_{\alpha\beta}$.

E. Specific construction of the metric decomposition

So far we did not specify how to practically construct decomposition (7), nor did we show that this decomposition exists. In this subsection we complete our derivation by providing a specific prescription for the construction of decomposition (7). Decomposition (7) can be constructed in more than one method. For simplicity we choose to follow a previous analysis by Detweiler and Whiting [15] and relate decomposition (7) to their decomposition, reading

$$h_{\alpha\beta} = h^S_{\alpha\beta} + h^R_{\alpha\beta}. \tag{27}$$

Here the (trace reversed) singular piece $\bar{h}^S_{\alpha\beta}$ satisfies the inhomogeneous wave equation (5). In a local neighborhood of the worldline it is defined by

$$\bar{h}^{\,S}_{\mu\nu}(x) = 4\int_{-\infty}^{\infty} G^S_{\mu\nu\alpha\beta}[x|z(\tau)]u^\alpha(\tau)u^\beta(\tau)d\tau, \tag{28}$$

where the singular Green's function is given by

$$G^S_{\mu\nu\alpha'\beta'}[x|x'] = \tfrac{1}{2}[U_{\mu\nu\alpha'\beta'}[x|x']\delta(\sigma) + V_{\mu\nu\alpha'\beta'}[x|x']\theta(-\sigma)]. \tag{29}$$

Here $\sigma = \sigma(x|x')$ denotes half the square of the invariant distance measured along a geodesic connecting x and x'; $U_{\mu\nu\alpha'\beta'}[x|x']$ and $V_{\mu\nu\alpha'\beta'}[x|x']$ are certain regular bitensors (for their definitions and properties see e.g. [6,23]), θ denotes a step function. Equation (29) implies that for a fixed x only a finite interval contributes to the integral in Eq. (28). This interval approaches zero as the evaluation point x approaches point $\hat{z}$ on the worldline. The regular field $\bar{h}^R_{\alpha\beta}$ satisfies a homogeneous wave equation [Eq. (5) with $T_{\mu\nu} = 0$], and is a smooth field in the vicinity of the worldline.

Expanding $\bar{h}^S_{\mu\nu}$ on the hypersurface of constant time $\Sigma(\hat{\tau})$ gives (see Appendix B)

$$\bar{h}^{\,S}_{\alpha\beta}(x) = \bar{g}_{(\alpha}{}^{\hat{\alpha}}\bar{g}_{\beta)}{}^{\hat{\beta}}[4u_{\hat{\alpha}}u_{\hat{\beta}}\varepsilon^{-1} - (\tfrac{2}{3}R_{\hat{\mu}\hat{\gamma}\hat{\nu}\hat{\delta}}u^{\hat{\mu}}u^{\hat{\nu}}\varepsilon^{;\hat{\gamma}}\varepsilon^{;\hat{\delta}}u_{\hat{\alpha}}u_{\hat{\beta}} + 4R_{\hat{\mu}\hat{\alpha}\hat{\nu}\hat{\beta}}u^{\hat{\mu}}u^{\hat{\nu}})\varepsilon + O(\varepsilon^2)]. \tag{30}$$

Comparing Eq. (30) with Eq. (8) we find that the leading terms in the expansion $\bar{h}^S_{\alpha\beta}(x)$ [and $h^S_{\alpha\beta}(x)$] satisfy the conditions of an instantaneous piece. We therefore identify $\bar{h}^I_{\alpha\beta}(x)$ with the expression in Eq. (30) up to $O(\varepsilon)$ inclusive. This identification implies that in the vicinity of the worldline we have

$$h^{SR}_{\alpha\beta} = h^R_{\alpha\beta} + O(\varepsilon^2), \tag{31}$$

where the $O(\varepsilon^2)$ discrepancy between $\bar{h}^{SR}_{\alpha\beta}$ and $h^R_{\alpha\beta}$ originate from terms of order ε^2 which appear in $\bar{h}^S_{\alpha\beta}$ but are absent from $\bar{h}^I_{\alpha\beta}$. Equation (31) implies that the expression that has been identified as $h^{SR}_{\alpha\beta}$ satisfies the required condition of a sufficiently regular piece.[6] We therefore conclude that we have constructed a decomposition which satisfies the conditions of decomposition (7). Substituting Eq. (31) into Eq. (26) yields the standard expression for the first-order self-force [15]

$$\mu^2 f^\mu_{(1)} = \mu^2 K^{\mu\alpha\beta\gamma} h^R_{\alpha\beta;\gamma}. \tag{32}$$

F. Summary

Before we generalize our method and tackle the second-order problem it is instructive to summarize the derivation method that we have used. First, in Eq. (2) we introduced the general form of the equation of motion. Next, in Eq. (7) we introduced a decomposition of the first-order metric perturbations into instantaneous and sufficiently regular pieces, and defined their properties. Using these properties we showed in Sec. II A that one cannot construct a well-defined vector (with the desired dimensionality and the correct scaling in μ) from the tensorial constituents of the instantaneous piece. Next, in Sec. II B, we invoked a gauge transformation from Lorenz gauge to Fermi gauge, and used the same method (as in Sec. II A) to show that the sufficiently regular piece of the first-order self-force vanishes. At this stage we concluded that in Fermi gauge the first-order self-force must vanish. Next, in Sec. II D, we used the inverse gauge transformation together with Barack and Ori transformation formula to derive the expression for the first-order self-force in the original Lorenz gauge. Finally, in Sec. II E, we employed Detweiler and Whiting decomposition and provided a prescription for the

[6]Notice that the singularities of the derivatives of the $O(\varepsilon^2)$ term in Eq. (31) are constrained by the form of the $O(\varepsilon^2)$ terms in Eq. (B4) in Appendix B, as required.

construction of decomposition (7). Using this prescription we have obtained the standard expression for the first-order self-force in the Lorenz gauge, given by Eq. (32) above. The fact that the entire derivation followed from the general properties of decomposition (7) led to a simple analysis. This simplicity enables us now to tackle the more complicated second-order problem.

III. SECOND-ORDER SELF-FORCE

From its definition the second-order self-force depends on the second-order metric perturbations. It is therefore necessary to begin our discussion by introducing a construction method for these second-order perturbations. Already this preliminary stage introduces difficulties. The difficulties and their resolution were studied in Refs. [4,5]. We now briefly summarize the main results of these references.

A. Construction of the second-order metric perturbations

Following Refs. [4,5] we specialize to a compact object which is a Schwarzschild black hole with a mass μ. We focus on the region far from the black hole i.e. at distances larger than r_E, where $r_E(\mu) \gg \mu$, and refer to this region as the external zone. In the external zone the geometry is dominated by the background geometry, and the full metric is represented by Eq. (1), where the metric perturbations are given by Eq. (3). We let $r_E(\mu) \to 0$ as $\mu \to 0$. In this limit we write the perturbations equations as

$$D_{\mu\nu}[\bar{h}] = 0, \qquad x \notin \gamma, \tag{33}$$

$$D_{\mu\nu}[\bar{l}] = S_{\mu\nu}[\bar{h}], \qquad x \notin \gamma. \tag{34}$$

Here γ is a timelike worldline, the operators $D_{\mu\nu}$ and $S_{\mu\nu}$ are obtained from an expansion of the full Ricci tensor, where for brevity we have omitted tensorial indices inside the squared brackets. This expansion is obtained by substituting Eq. (1) into the Ricci tensor, which gives $R^{\rm full}_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu}[\delta g] + R^{(2)}_{\mu\nu}[\delta g] + O(\delta g^3)$ (for explicit expressions of the terms in this expansion see e.g. [16]). To simplify the notation we denote $\bar{R}^{(1)}_{\mu\nu}[h]$ with $D_{\mu\nu}[\bar{h}]$, and denote $-\bar{R}^{(2)}_{\mu\nu}[h]$ with $S_{\mu\nu}[\bar{h}]$, where $h_{\mu\nu}$ is expressed in terms of $\bar{h}_{\mu\nu}$. D'Eath has shown that in the limit, the perturbations $h_{\mu\nu}$ are identical to the retarded first-order metric perturbations produced by a unit-mass point particle tracing the worldline γ [1]. At the leading order of approximation, γ is a geodesic of the background spacetime [1], which we continue to denote with $z_G(\tau)$. Therefore, in the Lorenz gauge the perturbations $h_{\mu\nu}$ are given by the retarded solution of Eq. (5). At the next order of approximation, the worldline γ has an $O(\mu)$ acceleration, induced by the first-order self-force. Hereafter we shall adopt Fermi gauge for the first-order perturbations. Recall that in this gauge the first-order self-force vanishes and the geodesic worldline $z_G(\tau)$ becomes sufficiently accurate up to acceleration of order μ, inclusive. In this gauge we may replace γ with $z_G(\tau)$ when we construct the second-order metric perturbations [4,5].

References [4,5] show that the second-order metric perturbations can be decomposed as follows

$$\bar{l}_{\mu\nu} = \bar{\psi}_{\mu\nu} + \delta\bar{l}_{\mu\nu}, \tag{35}$$

where the first piece $\bar{\psi}_{\mu\nu}$ is given by the following linear combination of terms that are quadratic in $\bar{h}^F_{\mu\nu}$

$$\bar{\psi}_{\mu\nu} = \tfrac{1}{64}[2(c_A\varphi^A_{\mu\nu} + c_B\varphi^B_{\mu\nu}) - 7(c_C\varphi^C_{\mu\nu} + c_D\varphi^D_{\mu\nu})]. \tag{36}$$

Here $\varphi^A_{\mu\nu} = \bar{h}^{F\rho}{}_{\mu}\bar{h}^F_{\rho\nu}$, $\varphi^B_{\mu\nu} = \bar{h}^{F\rho}{}_{\rho}\bar{h}^F_{\mu\nu}$, $\varphi^C_{\mu\nu} = (\bar{h}^{F\eta\rho}\bar{h}^F{}_{\eta\rho})g_{\mu\nu}$, $\varphi^D_{\mu\nu} = (\bar{h}^{F\rho}{}_{\rho})^2 g_{\mu\nu}$; and the constants c_A, c_B, c_C, c_D must satisfy $c_A + c_B = 1$, $c_C + c_D = 1$, but are otherwise arbitrary. The second piece $\delta\bar{l}_{\mu\nu}$ is given by the retarded solution of the following wave equation

$$\Box\delta\bar{l}_{\mu\nu} + 2R^{\eta}{}_{\mu}{}^{\rho}{}_{\nu}\delta\bar{l}_{\eta\rho} = -2\delta S_{\mu\nu}. \tag{37}$$

Here $\delta S_{\mu\nu} \equiv S_{\mu\nu} - D_{\mu\nu}[\bar{\psi}]$.

B. General considerations

The second term in Eq. (4), namely

$$\mu^3 f^{\mu}_{(2)}[g_{\mu\nu}, h^F_{\mu\nu}, l_{\mu\nu}, z_G(\tau)] \tag{38}$$

provides a complete list of the quantities that the second-order self-force may depend upon. Notice that working in Fermi gauge justifies the substitution $z(\tau) \to z_G(\tau)$ which was made in Eq. (38). This equation also states that the second-order self-force is proportional to μ^3, meaning that this self-force induces an acceleration which scales as μ^2. To obtain the desired μ^2 scaling of the acceleration, the second-order self-force must be linear in $\mu^2 l_{\mu\nu}$. Higher powers of $\mu^2 l_{\mu\nu}$ are therefore excluded from $f^{\mu}_{(2)}$. Similarly, terms in $f^{\mu}_{(2)}$ that depend explicitly on $h^F_{\mu\nu}$ must be quadratic in $h^F_{\mu\nu}$. Combining all the quantities that $f^{\mu}_{(2)}$ depends upon together with Eq. (35) we obtain

$$f^{\mu}_{(2)} = f^{\mu}_{(2)A}[g_{\alpha\beta}, z_G(\tau), h^F_{\gamma\delta}] + f^{\mu}_{(2)B}[g_{\alpha\beta}, z_G(\tau), \delta l_{\gamma\delta}]. \tag{39}$$

Here the first term is quadratic in $h^F_{\mu\nu}$ and the second term is linear in $\delta l_{\gamma\delta}$.

From Eqs. (2) and (4) we find that $f^{\mu}_{(2)}$ has a dimension of $(\text{Length})^{-3}$. Following Sec. II, we shall list all the tensorial constituents of $f^{\mu}_{(2)}$. The dimensionality of $f^{\mu}_{(2)}$ restricts our list of tensorial constituents to tensors with dimensionality of $(\text{Length})^{-n}$, $0 \leq n \leq 3$. This follows from the fact that we do not have tensors with dimensionality of $(\text{Length})^j$, $j > 0$ at our disposal.

C. First term $f^{\mu}_{(2)A}$

First we focus on the term $f^{\mu}_{(2)A}[g_{\alpha\beta}, z_G(\tau), h^F_{\gamma\delta}]$ in Eq. (39). Here it proves simpler to consider a general dependence (not necessarily quadratic) of $f^{\mu}_{(2)A}$ on $h^F_{\mu\nu}$. In the following analysis we shall often use decomposition (27). We write this decomposition in Fermi gauge, and obtain

$$h^F_{\alpha\beta} = h^S_{\alpha\beta} + h^{R(F)}_{\alpha\beta}, \qquad h^{R(F)}_{\alpha\beta} \equiv h^R_{\alpha\beta} + \xi_{\alpha;\beta} + \xi_{\beta;\alpha}, \tag{40}$$

where ξ_α satisfies Eqs. (15) and (16). Using this decomposition we schematically write the first piece as

$$f^{\mu}_{(2)A}[g_{\alpha\beta}, z_G(\tau), h^F_{\gamma\delta}] = f^{\mu}_{(2)A}[g_{\alpha\beta}, z_G(\tau), h^S_{\gamma\delta} + h^{R(F)}_{\gamma\delta}]. \tag{41}$$

We shall now list all the tensorial constituents of $f^{\mu}_{(2)A}$. First, we shall consider the quantities $\{g_{\alpha\beta}, z_G(\tau), h^S_{\gamma\delta}\}$, list the relevant tensors, and discuss the possible expressions for $f^{\mu}_{(2)A}$ that can be constructed from these quantities. Next, we shall consider all the remaining possibilities.

1. h^S terms

We now list all the tensorial constituents that can be constructed from the quantities $\{g_{\alpha\beta}, z_G(\tau), h^S_{\gamma\delta}\}$ and have dimensions of $(\text{Length})^{-n}$, $0 \le n \le 3$. From the quantities $\{g_{\alpha\beta}, z_G(\tau)\}$ we obtain the following list: $u^{\hat\mu}$, $g_{\hat\mu\hat\nu}$, $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$, and $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta;\hat\epsilon}$. In addition, our list includes the Levi-Civita tensor. Recall that $\bar h^S_{\gamma\delta}$ admits the local expansion (30). The coefficients of this expansion up to $O(\varepsilon)$ (inclusive) satisfy the conditions of an instantaneous piece. More generally, the higher-order coefficients in this expansion are composed from the same kind of instantaneous quantities, namely: background tensors, $u^{\hat\mu}$, $\nabla_{\hat\alpha}\varepsilon$, numerical coefficients (see Appendix B). Following the arguments of Sec. II A we now find that $h^S_{\gamma\delta}$ does not add any new tensor to the above list of tensorial constituents. Examining this list we find that a vector with the desired dimensionality of $(\text{Length})^{-3}$ must be linear in $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta;\hat\epsilon}$ and may also include combinations of the following dimensionless tensors: $u^{\hat\mu}$, $g_{\hat\mu\hat\nu}$, and $\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}$.

Examining our list of tensorial constituents shows that they cannot be combined to give a vector with a dimension of $(\text{Length})^{-3}$: Consider first vectors that are constructed from the following tensors $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta;\hat\epsilon}$, $u^{\hat\mu}$, and $g_{\hat\mu\hat\nu}$. By virtue of the symmetries $R_{\hat\alpha\hat\beta(\hat\gamma\hat\delta)} = 0$, $R_{(\hat\alpha\hat\beta)\hat\gamma\hat\delta} = 0$, and the fact that $R_{\hat\alpha\hat\beta} = 0$, we find that up to a sign the only possibility is $R^{\hat\epsilon}{}_{\hat\beta\hat\gamma\hat\delta;\hat\epsilon} u^{\hat\beta} u^{\hat\delta}$. This possibility vanishes by virtue of the Bianchi identities which imply that in vacuum we have $R^{\hat\epsilon}{}_{\hat\beta\hat\gamma\hat\delta;\hat\epsilon} = 0$. Similarly, using the properties of the Riemann tensor together with properties of the Levi-Civita tensor one finds that all the vector expressions composed from the tensor $\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ together with $R^{\hat\epsilon}{}_{\hat\beta\hat\gamma\hat\delta;\hat\epsilon}$, $g_{\hat\mu\hat\nu}$, and $u^{\hat\mu}$, vanish as well. We conclude that by combining tensorial constituents that originate from the list $\{g_{\alpha\beta}, u^{\mu}, h^S_{\gamma\delta}\}$ one cannot construct a vector with dimensionality of $(\text{Length})^{-3}$.

2. Remaining terms

Since the tensors $\{g_{\alpha\beta}, u^{\mu}, h^S_{\gamma\delta}\}$ cannot produce a vector with the desired dimensionality we find that each term in $f^{\mu}_{(2)A}$ must explicitly depend on $h^{R(F)}_{\alpha\beta}$. Expanding $h^{R(F)}_{\alpha\beta}$ in a local neighborhood of the worldline gives

$$h^{R(F)}_{\alpha\beta}(x) = \bar g_{\alpha}{}^{\hat\alpha} \bar g_{\beta}{}^{\hat\beta}[\tfrac{1}{2}\varepsilon^2 h^{R(F)}_{\hat\alpha\hat\beta;\hat\gamma\hat\delta}\varepsilon^{;\hat\gamma}\varepsilon^{;\hat\delta} + O(\varepsilon^3)]. \tag{42}$$

Notice that the leading terms that scale as ε^0 and ε^1 vanished by virtue of Fermi gauge. In this expansion only the coefficient of the ε^2 term has a dimension which fits our dimensionality condition of $(\text{Length})^{-n}$, $0 \le n \le 3$. Therefore, only this coefficient can contribute to our list of tensorial constituents. To construct a vector with a dimension of $(\text{Length})^{-3}$ we must combine $h^{R(F)}_{\hat\alpha\hat\beta;\hat\gamma\hat\delta}$ with the other dimensionless tensors from our list, namely $u^{\hat\mu}$, $g_{\hat\mu\hat\nu}$, and $\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}$. One can easily construct such a vector, for example $(g^{\hat\alpha\hat\nu} + u^{\hat\alpha}u^{\hat\nu})u^{\hat\mu}u^{\hat\rho}u^{\hat\epsilon}h^{R(F)}_{\hat\nu\hat\mu;\hat\rho\hat\epsilon}$. Following Sec. II B we use the gauge freedom to eliminate all the possibilities of constructing such a vector. We extend the definition of Fermi gauge by restricting the values of $\xi_{\mu;\nu\alpha\beta}(\tau)$. From Eq. (40) we obtain

$$h^{R(F)}_{\mu\nu;\alpha\beta} = h^R_{\mu\nu;\alpha\beta} + \xi_{\mu;\nu\alpha\beta} + \xi_{\nu;\mu\alpha\beta}. \tag{43}$$

Following Eq. (20) we introduce the ansatz

$$\xi_{\nu;\mu\alpha\beta}(\tau) = [R^{\epsilon}{}_{\alpha\mu\nu}\xi_{\epsilon} - \delta\Gamma^R{}_{\nu\mu\alpha}]_{;\beta} + D_{\nu\mu\alpha\beta}. \tag{44}$$

Here all the quantities are evaluated on the worldline, $\delta\Gamma^R_{\nu\mu\alpha} = \frac{1}{2}(h^R_{\mu\nu;\alpha} + h^R_{\nu\alpha;\mu} - h^R_{\mu\alpha;\nu})$, and $D_{\nu\mu\alpha\beta}$ is a tensor field which is defined on the worldline (see Appendix C). Substituting Eq. (44) into Eq. (43) gives

$$h^{R(F)}_{\mu\nu;\alpha\beta} = 2D_{(\mu\nu)\alpha\beta}. \tag{45}$$

Unfortunately, $D_{\nu\mu\alpha\beta}$ is not completely arbitrary and its values are constrained by various identities [e.g. identity (19)]. These constrains prevent us from annihilating $h^{R(F)}_{\mu\nu;\alpha\beta}$ by invoking a gauge transformation. In Appendix C, we choose $D_{\nu\mu\alpha\beta}$ such that all the required constraints are satisfied. We shall refer to this specific choice of gauge as extended Fermi gauge. In this gauge we have

$$h^{R(F)}_{\mu\nu;\alpha\beta} = P_{\mu}{}^{\lambda}{}_{\nu}{}^{\eta}{}_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma}\delta R^{R(F)}_{\lambda\eta\rho\sigma}, \tag{46}$$

where $P_{\mu}{}^{\lambda}{}_{\nu}{}^{\eta}{}_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma}$ is a certain combination of u^{μ} and δ^{ν}_{μ} (see Appendix C); and $\mu\delta R^{R(F)}_{\alpha\beta\gamma\delta}$ is the linear term in the following expansion in powers of μ

$$(g_{\mu\alpha} + \mu h^{R(F)}_{\mu\alpha})R^{\mu}{}_{\beta\gamma\delta}[g_{\mu\nu} + \mu h^{R(F)}_{\mu\nu}] - g_{\mu\alpha}R^{\mu}{}_{\beta\gamma\delta}[g_{\mu\nu}] = \mu\delta R^{R(F)}_{\alpha\beta\gamma\delta} + O(\mu^2). \quad (47)$$

Here the Riemann tensors are evaluated using the metric inside the squared brackets. Meaning that $\mu\delta R^{R(F)}_{\alpha\beta\gamma\delta}$ is the linear perturbation of the Riemann tensor, evaluated with the metric $g_{\mu\nu} + \mu h^{R(F)}_{\mu\nu}$. Notice that $\delta R^{R(F)}_{\alpha\beta\gamma\delta}$ has the same algebraic symmetries as the Riemann tensor. Furthermore, in Appendix C we show that in vacuum $\delta R^{R(F)}_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ is traceless. We now follow the arguments in Sec. II A. There we showed that in vacuum no vector can be constructed from the tensors $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$, $u^{\hat\mu}$, $\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}$, and $g_{\hat\mu\hat\nu}$. Similarly, the above mentioned properties of $\delta R^{R(F)}_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ imply that in vacuum no vector can be constructed from $\delta R^{R(F)}_{\hat\lambda\hat\eta\hat\rho\hat\sigma}$, $u^{\hat\alpha}$, $\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}$, and $g_{\hat\alpha\hat\beta}$. Employing Eq. (46) we now find that in the extended Fermi gauge we have

$$f^{\mu}_{(2)A} = 0. \quad (48)$$

D. Second term $f^{\mu}_{(2)B}$

Here we focus on the second term $f^{\mu}_{(2)B}[g_{\alpha\beta}, z_G(\tau), \delta l_{\gamma\delta}]$ in Eq. (39). Following Sec. II, we decompose $\delta l_{\gamma\delta}$, in a local neighborhood of $z_G(\tau)$, into a generalized instantaneous piece $\delta l^{I}_{\mu\nu}$ and a sufficiently regular piece $\delta l^{SR}_{\mu\nu}$, reading

$$\delta l_{\mu\nu} = \delta l^{I}_{\mu\nu} + \delta l^{SR}_{\mu\nu}. \quad (49)$$

Here $\delta l^{I}_{\mu\nu}$ is defined as a piece that admits the following a local expansion on $\Sigma(\hat\tau)$

$$\delta l^{I}_{\alpha\beta}(x) = \bar g_{\alpha}{}^{\hat\alpha}\bar g_{\beta}{}^{\hat\beta}[e^{(0)}_{\hat\alpha\hat\beta} + e^{(1)}_{\hat\alpha\hat\beta}\varepsilon + e^{(2)}_{\hat\alpha\hat\beta}\varepsilon\log\varepsilon], \quad (50)$$

where the expansion coefficients $\{e^{(0)}_{\hat\alpha\hat\beta}, e^{(1)}_{\hat\alpha\hat\beta}, e^{(2)}_{\hat\alpha\hat\beta}\}$ are composed only from the quantities in the following list: background tensors (at $\hat z$), $u^{\hat\mu}$, $\nabla_{\hat\alpha}\varepsilon$, $\delta R^{R(F)}_{\hat\lambda\hat\eta\hat\rho\hat\sigma}$, and numerical coefficients. We define a sufficiently regular piece $\delta l^{SR}_{\mu\nu}$ such that its first-order covariant derivative $\nabla_\alpha\delta l^{SR}_{\mu\nu}$ is continuous in a local neighborhood of z_G and its higher-order derivatives are not too singular (More precisely, we demand that $\varepsilon^{-1+n}\nabla_{\delta_n}\ldots\nabla_{\delta_1}\delta l^{SR}_{\mu\nu;\gamma}$, where $n \geq 1$, remains bounded as $x \to \hat z$.).

We initially assume that decomposition (49) exists. Later, in Sec. III F, we shall provide a specific prescription for its construction. As in Sec. II D, the properties of decomposition (49) imply that this decomposition in nonunique. Following Eq. (9), we can generate a family of decompositions by invoking the transformation $\delta l^{I}_{\mu\nu} \to \delta l^{I}_{\mu\nu} + j_{\mu\nu}$, $\delta l^{SR}_{\mu\nu} \to \delta l^{SR}_{\mu\nu} - j_{\mu\nu}$, where $j_{\mu\nu}$ satisfies the definitions of both $\delta l^{I}_{\mu\nu}$ and $\delta l^{SR}_{\mu\nu}$. Following the discussion in Sec. II D reveals that the nonuniqueness of decomposition (49) does not affect our final expression for the second-order self-force [see Eq. (58) below].

Using Eqs. (39), (48), and (49) together with the fact that $f^{\mu}_{(2)}$ must be linear in the second-order perturbations we find that we can decompose $f^{\mu}_{(2)}$ as follows

$$f^{\mu}_{(2)} = f^{\mu}_{(2)B}[g_{\alpha\beta}, z_G(\tau), \delta l_{\gamma\delta}] = f^{\mu}_{(2)I}[g_{\alpha\beta}, z_G(\tau), \delta l^{I}_{\gamma\delta}] + f^{\mu}_{(2)SR}[g_{\alpha\beta}, z_G(\tau), \delta l^{SR}_{\gamma\delta}]. \quad (51)$$

1. Instantaneous piece

First we consider the instantaneous piece $f^{\mu}_{(2)I}[g_{\alpha\beta}, z_G(\tau), \delta l^{I}_{\gamma\delta}]$ in Eq. (51). The quantities $g_{\alpha\beta}$, $z_G(\tau)$, $\delta l^{I}_{\gamma\delta}$ in this expression together with the definition of $\delta l^{I}_{\mu\nu}$ reveal that the tensorial constituents of $f^{\mu}_{(2)I}$ are: the background tensors, the four velocity, the Levi-Civita tensor, and the tensor $\delta R^{R(F)}_{\hat\lambda\hat\eta\hat\rho\hat\sigma}$. Recall that in Sec. III C we showed that in vacuum these tensors cannot be combined to give a vector with the desired dimensionality of $(\text{Length})^{-3}$. We therefore conclude that

$$f^{\mu}_{(2)I} = 0. \quad (52)$$

2. Sufficiently regular piece

Next we consider the sufficiently regular piece $f^{\hat\mu}_{(2)SR}$. Following the analysis in Sec. II B, we list the tensorial constituents of $f^{\hat\mu}_{(2)SR}[g_{\alpha\beta}, z_G(\tau), \delta l^{SR}_{\gamma\delta}]$. The explicit dependence of $f^{\hat\mu}_{(2)SR}$ on the quantities $\{g_{\alpha\beta}, z_G(\tau)\}$ implies that the background tensors and the four velocity must be included in the our list of the tensorial constituents of $f^{\hat\mu}_{(2)SR}$. In addition, this list must also include the Levi-Civita tensor. Recall that in Sec. III C we showed that in vacuum these tensors cannot be combined to give a nonvanishing vector with a dimension of $(\text{Length})^{-3}$. Therefore, each term in the expression for $f^{\hat\mu}_{(2)SR}$ must depend (linearly) on $\delta l^{SR}_{\mu\nu}$.

Since $\delta l^{SR}_{\mu\nu}$ is sufficiently regular it admits the following expansion on $\Sigma(\hat\tau)$

$$\delta l^{SR}_{\alpha\beta}(x) = \bar g_{\alpha}{}^{\hat\alpha}\bar g_{\beta}{}^{\hat\beta}[\delta l^{SR}_{\hat\alpha\hat\beta} - \delta l^{SR}_{\hat\alpha\hat\beta;\hat\gamma}\varepsilon\varepsilon^{;\hat\gamma} + O(\varepsilon^2)]. \quad (53)$$

Recall that $\delta l^{SR}_{\alpha\beta}$ has a dimension of $(\text{Length})^{-2}$. Thus $\delta l^{SR}_{\hat\alpha\hat\beta;\hat\gamma\hat\delta_1\ldots\hat\delta_n}$ $n \geq 1$ must have dimensions of $(\text{Length})^{-3-n}$. By dimensionality, these higher-order derivatives must be excluded from our list [21]. Consequently, each term in the expression of $f^{\hat\mu}_{(2)SR}$ must depend on $\delta l^{SR}_{\hat\alpha\hat\beta}$ or $\delta l^{SR}_{\hat\alpha\hat\beta;\hat\gamma}$. To eliminate all the possibilities of constructing a vector with a dimension of $(\text{Length})^{-3}$ we employ a purely second-order gauge transformation of the form $x^\mu \to x^\mu - \mu^2 \xi^\mu_{(2)}$. The second-order perturbations in this second-order Fermi gauge are given by $l^F_{\mu\nu} = l_{\mu\nu} + \xi_{(2)\mu;\nu} + \xi_{(2)\nu;\mu}$. Similar to Sec. II B, we include the entire gauge transformation in the definition of the new sufficiently regular piece $\delta l^{SR(F)}_{\mu\nu} \equiv \delta l^{SR}_{\mu\nu} + \xi_{(2)\mu;\nu} + \xi_{(2)\nu;\mu}$ and demand that $\xi_{(2)\mu;\nu\rho}$ will be continuous in a local neighborhood of the worldline and thereby ensure that $\delta l^{SR(F)}_{\mu\nu}$ will satisfy the conditions of a sufficiently regular piece. In the second-order Fermi gauge Eq. (51) reads

$$f^\mu_{F(2)} = f^\mu_{(2)I}[g_{\alpha\beta}, z_G(\tau), \delta l^I_{\gamma\delta}] + f^\mu_{(2)SR(F)}[g_{\alpha\beta}, z_G(\tau), \delta l^{SR(F)}_{\gamma\delta}], \quad (54)$$

where $f^\mu_{F(2)}$ is the second-order self-force in the second-order Fermi gauge, and $f^\mu_{(2)SR(F)}$ is its corresponding sufficiently regular piece. Following Sec. II B we impose the following gauge conditions along the worldline $z_G(\tau)$

$$[\delta l^{SR(F)}_{\mu\nu}]_{z_G(\tau)} = 0, \qquad [\delta l^{SR(F)}_{\mu\nu;\rho}]_{z_G(\tau)} = 0. \quad (55)$$

Recall that each term in $f^{\hat\mu}_{(2)SR}$ must depend on $\delta l^{SR}_{\hat\alpha\hat\beta}$ or $\delta l^{SR}_{\hat\alpha\hat\beta;\hat\gamma}$. In complete analogy, each term in $f^{\hat\mu}_{(2)SR(F)}$ must depend on $\delta l^{SR(F)}_{\hat\alpha\hat\beta}$ or $\delta l^{SR(F)}_{\hat\alpha\hat\beta;\hat\gamma}$. Notice that these coefficients vanish by virtue of Eq. (55). We therefore conclude that

$$f^\mu_{(2)SR(F)} = 0. \quad (56)$$

E. Second-order self-force

Combining Eqs. (52) and (56) together with Eq. (54) we find that in the second-order Fermi gauge the second-order self-force vanishes, namely

$$f^\mu_{F(2)} = 0. \quad (57)$$

We now invoke an inverse gauge transformation of the form $x^\mu \to x^\mu + \mu^2 \xi^\mu_{(2)}$ and calculate the second-order self-force in our original second-order gauge. For this purpose we need to generalize the Barack and Ori gauge-transformation formula of Ref. [22] [see Eqs. (23) and (24)] to second order. The generalization turns out to be trivial. The first-order expression (24) is obtained by calculating the leading order acceleration of the worldline which is induced by a first-order gauge transformation. Here we consider a purely second-order gauge transformation of the form $x^\mu \to x^\mu - \mu^2 \xi^\mu_{(2)}$, but similar to Ref. [22], we are still calculating the leading order acceleration which this transformation induces. Therefore, the final expression has the same form as Eqs. (23) and (24), with the following substitutions $\xi^\mu \to \xi^\mu_{(2)}$, $f^\alpha_{(1)} \to f^\alpha_{(2)}$, $\delta f^\alpha_{(1)} \to \delta f^\alpha_{(2)}$.

The final piece of the calculation directly follows from the analysis in Sec. II D. We use the gauge conditions (55) to derive an expression for $\ddot\xi^\mu_{(2)}$, and use this expression to find $\delta f^\alpha_{(2)}$ and $f^\mu_{(2)}$. This calculation produces our main result which is the following expression for the second-order self-force, reading

$$\mu^3 f^\mu_{(2)} = \mu^3 K^{\mu\alpha\beta\gamma} \delta l^{SR}_{\alpha\beta;\gamma}. \quad (58)$$

Here the first-order metric perturbations satisfy the conditions of the extended Fermi gauge, the second-order metric perturbations are constructed using the prescription of Sec. III A, and the piece $\delta \bar l_{\mu\nu}$ is decomposed according to Eq. (49). To complete the derivation we now provide a prescription for the construction of this decomposition.

F. Specific decomposition of $\delta l_{\mu\nu}$

To construct decomposition (49) we need to study the singular behavior of $\delta l_{\mu\nu}$ in the vicinity of the worldline. For this purpose, we first study the singular behavior of the source term $-2\delta S_{\alpha\beta}$ of Eq. (37). From its definition $\delta S_{\alpha\beta}$ has the schematic form $\delta S = \nabla h^F \nabla h^F \& h^F \nabla\nabla h^F$, where h^F schematically denotes the first-order metric perturbations in the extended Fermi gauge, and "&" denotes "and terms of the form ..." Recall that h^F diverges as ε^{-1} as $\varepsilon \to 0$. Therefore, the schematic form of $\delta S_{\alpha\beta}$ suggests that $\delta S_{\alpha\beta}$ should diverge as ε^{-4}. However, Refs. [4,5] show that the leading singular terms in $\delta S_{\alpha\beta}$ cancel out and $\delta S_{\alpha\beta}$ diverges only as ε^{-2}. Formally expanding $-2\delta S_{\alpha\beta}$ on $\Sigma(\hat\tau)$ gives

$$-2\delta S_{\alpha\beta} = \bar g_\alpha{}^{\hat\alpha} \bar g_\beta{}^{\hat\beta} [\varepsilon^{-2} A_{\hat\alpha\hat\beta} + \varepsilon^{-1} B_{\hat\alpha\hat\beta} + O(\varepsilon^0)], \quad (59)$$

where the coefficients $A_{\hat\alpha\hat\beta}$, $B_{\hat\alpha\hat\beta}$ are independent of ε. Notice that this equation uniquely defines $A_{\hat\alpha\hat\beta}$ and $B_{\hat\alpha\hat\beta}$. We now list the tensorial constituents of the coefficients $A_{\hat\alpha\hat\beta}$, $B_{\hat\alpha\hat\beta}$. To construct this list, we substitute decomposition (40) into the schematic expression for $\delta S_{\alpha\beta}$ and use Eqs. (30), (42), and (B4) in Appendix B. We find that each term in the expression for $A_{\hat\alpha\hat\beta}$ must be linear in $R_{\hat\lambda\hat\eta\hat\rho\hat\sigma}$, and each term in the expression for $B_{\hat\alpha\hat\beta}$ must be either linear in $\delta R^{R(F)}_{\hat\lambda\hat\eta\hat\rho\hat\sigma}$ or linear in $\nabla_{\hat\alpha} R_{\hat\lambda\hat\eta\hat\rho\hat\sigma}$. In addition to these tensors, the coefficients $A_{\hat\alpha\hat\beta}$ and $B_{\hat\alpha\hat\beta}$ include only combinations of the following dimensionless quantities: $u^{\hat\mu}$, $g_{\hat\mu\hat\nu}$, $\nabla_{\hat\alpha}\varepsilon$, and numerical coefficients.

To construct $\delta \bar l^I_{\alpha\beta}$ we first construct a solution to the following equation, defined in a local neighborhood of the worldline

$$\Box \bar\varphi_{\alpha\beta} + 2R^\eta{}_\alpha{}^\rho{}_\beta \bar\varphi_{\eta\rho} = \bar g_\alpha{}^{\hat\alpha} \bar g_\beta{}^{\hat\beta} [\varepsilon^{-2} A_{\hat\alpha\hat\beta} + \varepsilon^{-1} B_{\hat\alpha\hat\beta} + O_a(\varepsilon^0)]. \quad (60)$$

Here $O_a(\varepsilon^0)$ denotes an arbitrary quantity of order ε^0, meaning that we do not restrict the values of the $O(\varepsilon^0)$ terms on the right-hand side. To construct a solution in the desired form we substitute

$$\bar{\varphi}_{\alpha\beta} = \bar{g}_{\alpha}{}^{\hat{\alpha}}\bar{g}_{\beta}{}^{\hat{\beta}}\bar{\varphi}_{\hat{\alpha}\hat{\beta}} \tag{61}$$

into Eq. (60), and solve directly for $\bar{\varphi}_{\hat{\alpha}\hat{\beta}}$. Notice that $A_{\hat{\alpha}\hat{\beta}}$, $B_{\hat{\alpha}\hat{\beta}}$, and $\bar{\varphi}_{\hat{\alpha}\hat{\beta}}$ transform as scalars under a coordinate transformation at x. It is useful to decompose these quantities into scalar spherical harmonics

$$\begin{aligned}
\bar{\varphi}_{\hat{\alpha}\hat{\beta}} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)\bar{\phi}^{lm}_{\hat{\alpha}\hat{\beta}}(\varepsilon,t),\\
A_{\hat{\alpha}\hat{\beta}} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)a^{lm}_{\hat{\alpha}\hat{\beta}}(t),\\
B_{\hat{\alpha}\hat{\beta}} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)b^{lm}_{\hat{\alpha}\hat{\beta}}(t).
\end{aligned} \tag{62}$$

To define the variables (t, θ, φ), consider Fermi-normal coordinates based on the worldline, where x^a ($a = 1, 2, 3$) denote the spatial coordinates and t denotes the time coordinate. The angular variables are defined through the canonical angular parametrization, namely $x^a = \varepsilon(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. In Appendix D we construct the following solution to Eq. (60), reading

$$\begin{aligned}
\bar{\phi}^{00}_{\hat{\alpha}\hat{\beta}} &= \tfrac{1}{2}b^{00}_{\hat{\alpha}\hat{\beta}}\varepsilon, \qquad l=0\\
\bar{\phi}^{1m}_{\hat{\alpha}\hat{\beta}} &= -\tfrac{1}{2}a^{1m}_{\hat{\alpha}\hat{\beta}} + \tfrac{1}{3}b^{1m}_{\hat{\alpha}\hat{\beta}}\varepsilon\log\varepsilon, \qquad l=1,\\
\bar{\phi}^{lm}_{\hat{\alpha}\hat{\beta}} &= -[l(l+1)]^{-1}a^{lm}_{\hat{\alpha}\hat{\beta}} + [2-l(l+1)]^{-1}b^{lm}_{\hat{\alpha}\hat{\beta}}\varepsilon, \qquad l>1.
\end{aligned} \tag{63}$$

Notice that the angular dependence of $A_{\hat{\alpha}\hat{\beta}}$ and $B_{\hat{\alpha}\hat{\beta}}$ is completely described by combinations of the quantity $\nabla_{\hat{\alpha}}\varepsilon$. Therefore, when we calculate the spherical-harmonics coefficients $a^{lm}_{\hat{\alpha}\hat{\beta}}$ and $b^{lm}_{\hat{\alpha}\hat{\beta}}$, we end up integrating only over combinations of $\nabla_{\hat{\alpha}}\varepsilon$. Therefore, $\bar{\varphi}_{\hat{\alpha}\hat{\beta}}$ is composed form the same well-defined tensorial constituents that appear in the expressions of $A_{\hat{\alpha}\hat{\beta}}$ and $B_{\hat{\alpha}\hat{\beta}}$. In addition, $\bar{\varphi}_{\hat{\alpha}\hat{\beta}}$ also has a nontrivial angular dependence which can be expressed using combinations of $\nabla_{\hat{\alpha}}\varepsilon$. Comparing Eqs. (61)–(63) with Eq. (50) we find that $\bar{\varphi}_{\alpha\beta}$ satisfies the conditions of a generalized instantaneous piece. We therefore identify $\bar{\varphi}_{\alpha\beta}$ with the desired generalized instantaneous piece, namely

$$\delta\bar{l}^{I}_{\alpha\beta} = \bar{\varphi}_{\alpha\beta}. \tag{64}$$

By subtracting Eq. (60) (for the above solution $\bar{\varphi}_{\alpha\beta}$) from Eq. (37) we obtain

$$\begin{aligned}
&\Box(\delta\bar{l}_{\alpha\beta} - \bar{\varphi}_{\alpha\beta}) + 2R^{\eta}{}_{\alpha}{}^{\rho}{}_{\beta}(\delta\bar{l}_{\eta\rho} - \bar{\varphi}_{\eta\rho})\\
&\quad = \bar{g}_{\alpha}{}^{\hat{\alpha}}\bar{g}_{\beta}{}^{\hat{\beta}}[O(\varepsilon^0)].
\end{aligned} \tag{65}$$

In Appendix D we use the fact that the source term of Eq. (65) is nondivergent to show that $\delta\bar{l}_{\alpha\beta} - \bar{\varphi}_{\alpha\beta}$ satisfies the conditions of a sufficiently regular piece. We therefore make the following identification

$$\delta\bar{l}^{SR}_{\alpha\beta} = \delta\bar{l}_{\alpha\beta} - \bar{\varphi}_{\alpha\beta},$$

which completes the construction of decomposition (49).

G. Summary

Equation (58) provides a formula for calculating the second-order gravitational self-force in a vacuum background spacetime, given the sufficiently regular piece $\delta l^{SR}_{\alpha\beta}$. To use this formula one has to be able to calculate both first-order and second-order metric perturbations, in specified first and second-order gauges. We now briefly summarize a prescription for the construction of $\delta l^{SR}_{\alpha\beta}$.

The first step in this construction is to calculate the retarded solution of Eq. (5) which provides us with $\bar{h}_{\mu\nu}$—the (traced reversed) first-order metric perturbations in the Lorenz gauge. The next step is to calculate the gauge vector ξ_{μ} that generates the gauge-transformation $h_{\mu\nu} \to h_{\mu\nu} + \xi_{\nu;\mu} + \xi_{\mu;\nu}$ from the Lorenz gauge to the extended Fermi gauge. To be able to calculate ξ_{μ}, one first has to calculate the regular piece $h^{R}_{\mu\nu}(z_G)$ together with its first and second-order derivatives at the worldline. This preliminary calculation follows from Eqs. (27)–(29). Next, one follows the prescription in Sec. II B [immediately after Eq. (17)] and calculates $\xi^{\mu}(z_G)$ together with its first-order and second-order derivatives. To construct the third- order derivative of ξ^{μ} one should use Eq. (44) together with Eqs. (C6) and (C7) in Appendix C. Once $\xi^{\mu}(z_G)$ together with its first, second, and third-order derivatives on the worldline are obtained, one can use these quantities to construct a local expansion for ξ^{μ} (this expansion can be continued in an arbitrary manner away from the worldline).

The next step in our construction is to calculate the (traced reversed) second-order metric perturbations $\bar{l}_{\mu\nu}$. For this purpose one can use the decomposition $\bar{l}_{\mu\nu} = \bar{\psi}_{\mu\nu} + \delta\bar{l}_{\mu\nu}$. The first piece of this decomposition—$\bar{\psi}_{\mu\nu}$ is given by Eq. (36), and the second piece—$\delta\bar{l}_{\mu\nu}$ is equal to the retarded solution of Eq. (37). The piece $\delta\bar{l}_{\mu\nu}$ is further decomposed according to $\delta\bar{l}_{\mu\nu} = \delta\bar{l}^{I}_{\mu\nu} + \delta\bar{l}^{SR}_{\mu\nu}$. Equations (59) and (61)–(64) provide a prescription for the calculation of the desired sufficiently regular piece $\delta l^{SR}_{\alpha\beta}$ that should be substituted in Eq. (58).

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APPENDIX A: LOCAL EXPANSION OF $h^{SR}_{\alpha\beta}$

In this appendix we derive Eq. (13), which provides a local expansion of a sufficiently regular tensor field on a spacelike hypersurface $\Sigma(\hat{\tau})$. Most of our derivation follows a similar derivation in Ref. [6]. We begin our analysis by studying some of the properties of the quantities $\varepsilon\varepsilon^{;\hat{\mu}}$ and $\bar{g}_{\alpha}{}^{\hat{\alpha}}$ that appear in Eq. (13).

First we derive an expression for $\varepsilon\varepsilon^{;\hat{\mu}}$. Consider two points x' and x, and suppose that x' lies in a local neighborhood of x, such that within this local neighborhood there is a unique geodesic $y(\lambda)$ that connects x' with x. Here λ denotes an affine parameter ranging from λ_0 to λ_1, where $x' = y(\lambda_0)$ and $x = y(\lambda_1)$. We denote the square of the invariant length of $y(\lambda)$ with $2\sigma(x|x')$, and express $\sigma(x|x')$ as

$$\sigma(x|x') = \frac{1}{2}(\lambda_1 - \lambda_0)\int_{\lambda_0}^{\lambda_1} g_{\mu\nu}(y)\dot{y}^{\mu}\dot{y}^{\nu}d\lambda, \tag{A1}$$

where $\dot{y}^{\mu} = \frac{dy}{d\lambda}$. We now calculate $\sigma_{,\mu'} = \frac{\partial\sigma}{\partial x^{\mu'}}$. For this purpose we consider an infinitesimal displacement of the point x' to a point $x' + \delta x'$, and let $y(\lambda) + \delta y(\lambda)$ denote the unique geodesic connecting $x' + \delta x'$ with x. On this geodesic we scale the affine parameter λ to run from λ_0 to λ_1, such that $x' + \delta x' = y(\lambda_0) + \delta y(\lambda_0)$ and $\delta y(\lambda_1) = 0$. Expanding $\delta\sigma = \sigma(x|x' + \delta x') - \sigma(x|x')$ to the first order in the variation, and using integration by parts we obtain

$$\delta\sigma = (\lambda_1 - \lambda_0)[g_{\mu\nu}(y)\dot{y}^{\nu}\delta y^{\mu}]_{\lambda_0}^{\lambda_1} - (\lambda_1 - \lambda_0) \times \int_{\lambda_0}^{\lambda_1}\frac{D\dot{y}_{\mu}}{D\lambda}d\lambda\delta y^{\mu} + O(\delta y^2).$$

Recalling that $y(\lambda)$ is a geodesic and noting that $\delta y(\lambda_1) = 0$ we obtain

$$\sigma_{,\mu'} = -(\lambda_1 - \lambda_0)g_{\mu'\nu'}\dot{y}^{\nu'}. \tag{A2}$$

Hereafter we shall specialize to the spacelike geodesic that connects the points $\hat{z}$ and x on $\Sigma(\hat{\tau})$. Here we have $2\sigma(y|\hat{z}) = \varepsilon^2(y|\hat{z})$. Employing Eq. (A2) and using $\varepsilon(y)$ as the affine parameter of the geodesic, we find that

$$\varepsilon\nabla^{\hat{\mu}}\varepsilon = -\varepsilon\frac{dy^{\hat{\mu}}}{d\varepsilon}, \tag{A3}$$

which implies that $\nabla^{\hat{\mu}}\varepsilon$ is a unit vector, tangent to the geodesic $y(\varepsilon)$.

Next we study the properties of the parallel propagator $\bar{g}_{\alpha}{}^{\hat{\alpha}}$. From its definition $\bar{g}_{\alpha}{}^{\hat{\alpha}}$ transports an arbitrary vector $A_{\hat{\alpha}}$ to a vector

$$A_{\mu}(y) = \bar{g}_{\mu}{}^{\hat{\alpha}}(y|\hat{z})A_{\hat{\alpha}}(\hat{z}), \tag{A4}$$

by a parallel propagation of the vector $A_{\hat{\alpha}}$ on the geodesic $y(\varepsilon)$, implying that

$$\lim_{\varepsilon\to 0}\bar{g}_{\mu}{}^{\hat{\alpha}}[y(\varepsilon)|\hat{z}] = \delta_{\hat{\mu}}{}^{\hat{\alpha}}. \tag{A5}$$

Since $\frac{D}{D\varepsilon}A_{\mu} = 0$ we find from Eq. (A4) that

$$(\bar{g}_{\mu}{}^{\hat{\alpha}})_{;\rho}\frac{dy^{\rho}}{d\varepsilon} = 0. \tag{A6}$$

The geometric content of $\bar{g}_{\alpha}{}^{\hat{\alpha}}$ implies that $\bar{g}_{\alpha}{}^{\hat{\alpha}}\bar{g}^{\beta}{}_{\hat{\alpha}} = \delta^{\beta}_{\alpha}$, and $\bar{g}_{\alpha}{}^{\hat{\alpha}}\bar{g}^{\alpha}{}_{\hat{\beta}} = \delta^{\hat{\alpha}}_{\hat{\beta}}$.

We now construct a local expansion for $h^{SR}_{\alpha\beta}(x)$ on $\Sigma(\hat{\tau})$, this construction follows directly from the expansion of $h^{SR}_{\alpha\beta}(y)$ on the geodesics $y(\varepsilon)$. First we introduce the following bitensor field on $y(\varepsilon)$

$$B_{\hat{\alpha}\hat{\beta}}[\hat{z}, y] = \bar{g}^{\alpha}{}_{\hat{\alpha}}(y|\hat{z})\bar{g}^{\beta}{}_{\hat{\beta}}(y|\hat{z})h^{SR}_{\alpha\beta}(y). \tag{A7}$$

Notice that $B_{\hat{\alpha}\hat{\beta}}$ transforms as a scalar under a coordinate transformation at y. Expanding $B_{\hat{\alpha}\hat{\beta}}(\varepsilon) = B_{\hat{\alpha}\hat{\beta}}[\hat{z}, y(\varepsilon)]$ in a Taylor series on the geodesic $y(\varepsilon)$, with $\hat{z}$ fixed, gives

$$B_{\hat{\alpha}\hat{\beta}}(\varepsilon) = B_{\hat{\alpha}\hat{\beta}}(0) + \varepsilon\left[\frac{dB_{\hat{\alpha}\hat{\beta}}}{d\varepsilon}\right]_{\varepsilon=0} + O(\varepsilon^2). \tag{A8}$$

Equations (A5) and (A7) imply that

$$B_{\hat{\alpha}\hat{\beta}}(0) = h^{SR}_{\hat{\alpha}\hat{\beta}}(\hat{z}). \tag{A9}$$

Using Eqs. (A6) and (A7) we find that

$$\frac{dB_{\hat{\alpha}\hat{\beta}}}{d\varepsilon} = B_{\hat{\alpha}\hat{\beta};\gamma}\frac{dy^{\gamma}}{d\varepsilon} = \bar{g}^{\alpha}{}_{\hat{\alpha}}(y|\hat{z})\bar{g}^{\beta}{}_{\hat{\beta}}(y|\hat{z})h^{SR}_{\alpha\beta;\gamma}(y)\frac{dy^{\gamma}}{d\varepsilon}.$$

Employing Eqs. (A3) and (A5) we obtain

$$\left[\frac{dB_{\hat{\alpha}\hat{\beta}}}{d\varepsilon}\right]_{\varepsilon=0} = -h^{SR}_{\hat{\alpha}\hat{\beta};\hat{\gamma}}(\hat{z})\varepsilon^{;\hat{\gamma}}. \tag{A10}$$

We can now rewrite $B_{\hat{\alpha}\hat{\beta}}[\varepsilon(y)]$ for $y = x$ by substituting Eqs. (A9) and (A10) into Eq. (A8). Using this expression for $B_{\hat{\alpha}\hat{\beta}}$ together with Eq. (A7) we finally obtain Eq. (13), reading

$$h^{SR}_{\alpha\beta}(x) = \bar{g}_{\alpha}{}^{\hat{\alpha}}(x|\hat{z})\bar{g}_{\beta}{}^{\hat{\beta}}(x|\hat{z})[h^{SR}_{\hat{\alpha}\hat{\beta}} - h^{SR}_{\hat{\alpha}\hat{\beta};\hat{\gamma}}\varepsilon\varepsilon^{;\hat{\gamma}} + O(\varepsilon^2)].$$

This expansion takes a very simple form in Fermi-normal coordinates based on z_G (for the definition and properties of these coordinates see e.g. [6]). In these coordinates we have $\bar{g}_{\alpha}{}^{\hat{\alpha}} = \delta_{\alpha}{}^{\hat{\alpha}} + O(\varepsilon^2)$, and Eq. (13) is reduced to

$$h^{SR}_{\alpha\beta}(x) \stackrel{*}{=} h^{SR}_{\hat{\alpha}\hat{\beta}}(t) + h^{SR}_{\hat{\alpha}\hat{\beta},a}(x^a = 0, t)x^a + O(\varepsilon^2). \tag{A11}$$

Here $\stackrel{*}{=}$ denotes equality in Fermi-normal coordinates, t denotes the time coordinate, x^a denote the spatial coordinates, where Latin indices run from 1 to 3. Equation (A11) clearly shows that $h^{SR}_{\alpha\beta}$ is a regular tensor field in a local neighborhood of the worldline. In Fermi-normal coordinates it is also easy to evaluate the first-order covariant derivatives of Eq. (A11) since $[\nabla_{\alpha}]_{\hat{z}} \stackrel{*}{=} [\partial_{\alpha}]_{\hat{z}}$.

APPENDIX B: LOCAL EXPANSION OF $\bar{h}^S_{\mu\nu}$

In this appendix we study the general form of a local expansion of $\bar{h}^S_{\mu\nu}$ (for more details on the construction method of such local expansions see e.g. Ref. [6].). First, we substitute Eq. (29) into Eq. (28) and obtain

$$\bar{h}^S_{\mu\nu}(x) = 2U_{\mu\nu}[x|z_G(\tau^-)](\dot{\sigma}_{\tau^-})^{-1} - 2U_{\mu\nu}[x|z_G(\tau^+)](\dot{\sigma}_{\tau^+})^{-1} + 2\int_{\tau^-}^{\tau^+} V_{\mu\nu}[x|z_G(\tau)]d\tau. \quad \text{(B1)}$$

Here we introduced the following notation: $\sigma_\tau = \sigma(z_G(\tau)|x)$, $\dot{\sigma}_{\tau^\pm} = (\frac{d\sigma}{d\tau})_{\tau^\pm}$, $U_{\mu\nu}[x|z_G(\tau)] = U_{\mu\nu\alpha\beta}[x|z_G(\tau)]u^\alpha(\tau)u^\beta(\tau)$, $V_{\mu\nu}[x|z_G(\tau)] = V_{\mu\nu\alpha\beta}[x|z_G(\tau)]u^\alpha(\tau)u^\beta(\tau)$; the retarded and advanced times are denoted τ^- and τ^+, respectively. These times satisfy $\sigma[x|z_G(\tau^\mp)] = 0$, where $\tau^+ > \hat{\tau}$ and $\tau^- < \hat{\tau}$. Introducing the notation $\Delta\tau^\mp = |\tau^\mp - \hat{\tau}|$, we expand the quantities in Eq. (B1) in Taylor series around $\hat{\tau}$ and obtain

$$\begin{aligned}\dot{\sigma}_{\tau^\mp} &= \dot{\sigma}_{\hat{\tau}} \mp \ddot{\sigma}_{\hat{\tau}}\Delta\tau^\mp + (1/2)\dddot{\sigma}_{\hat{\tau}}(\Delta\tau^\mp)^2 + O[(\Delta\tau^\mp)^3],\\ U_{\mu\nu}[x|z_G(\tau^\mp)] &= U_{\mu\nu}[x|\hat{z}] \mp \Delta\tau^\mp \dot{U}_{\mu\nu}[x|\hat{z}] + (1/2)(\Delta\tau^\mp)^2\ddot{U}_{\mu\nu}[x|\hat{z}] + O[(\Delta\tau^\mp)^3],\\ V_{\mu\nu}[x|z_G(\tau)] &= V_{\mu\nu}[x|\hat{z}] + (\tau - \hat{\tau})\dot{V}_{\mu\nu}[x|\hat{z}] + (1/2)(\tau - \hat{\tau})^2\ddot{V}_{\mu\nu}[x|\hat{z}] + O[(\tau - \hat{\tau})^3].\end{aligned} \quad \text{(B2)}$$

From the relation $\sigma[x|z_G(\tau^\mp)] = 0$ we find that

$$\Delta\tau^\mp = \varepsilon(1 - \tfrac{1}{6}\varepsilon^2 R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}}\varepsilon^{;\hat{\gamma}}\varepsilon^{;\hat{\delta}}u^{\hat{\alpha}}u^{\hat{\beta}}) + O(\varepsilon^4). \quad \text{(B3)}$$

Notice that expansion (B3) was derived using only geometrical consideration. Therefore, the higher-order terms in this expansion must also have a geometric content. Substituting Eq. (B2) together with Eq. (B3) into Eq. (B1) one obtains an expansion in powers of ε involving only quantities defined on $\Sigma(\hat{\tau})$. Each of these quantities can be further expanded in powers of ε, for example

$$U_{\alpha\beta}[x|\hat{z}] = \bar{g}_\alpha{}^{\hat{\alpha}}\bar{g}_\beta{}^{\hat{\beta}}[u_{\hat{\alpha}\hat{\beta}} - \varepsilon\varepsilon^{;\hat{\gamma}}u_{\hat{\alpha}\hat{\beta};\hat{\gamma}} + \tfrac{1}{2}\varepsilon^2\varepsilon^{;\hat{\gamma}}\varepsilon^{;\hat{\delta}}u_{\hat{\alpha}\hat{\beta};(\hat{\gamma}\hat{\delta})} + O(\varepsilon^3)].$$

In general the coefficients in these expansions are obtained from coincidence limits of the expanded quantity and its covariant derivatives. In this example the tensors $u_{\hat{\alpha}\hat{\beta}}$, $u_{\hat{\alpha}\hat{\beta};\hat{\gamma}}$, and $u_{\hat{\alpha}\hat{\beta};(\hat{\gamma}\hat{\delta})}$ are equal to the coincidence limits (as $x \to \hat{z}$) of $U_{\alpha\beta}$, $U_{\alpha\beta;\gamma}$, and $U_{\alpha\beta;(\gamma\delta)}$, respectively. These coincidence limits have a geometrical content and they are obtained from the definitions of σ, $U_{\mu\nu\alpha\beta}$, and $V_{\mu\nu\alpha\beta}$, see e.g. [6]. The form of the above-mentioned expansions reveals that the expansion of $\bar{h}^S_{\mu\nu}(x)$ has the following form

$$\bar{h}^S{}_{\mu\nu}(x) = \bar{g}_\alpha{}^{\hat{\alpha}}\bar{g}_\beta{}^{\hat{\beta}}\Sigma_{n=-1}^N f^{(n)}_{\hat{\alpha}\hat{\beta}}\varepsilon^n. \quad \text{(B4)}$$

Here the expansion coefficients $f^{(n)}_{\hat{\alpha}\hat{\beta}}$ are composed only from the following geometrical quantities: background tensors, $u^{\hat{\alpha}}$, $\varepsilon^{;\hat{\alpha}}$; together with numerical coefficients. The first two terms in expansion (B4) are provided by Eq. (30).

APPENDIX C: EXTENDED FERMI GAUGE

In this appendix, we provide a prescription for the construction of the extended Fermi gauge. In addition, we derive Eq. (46) which provides an expression for $h^{R(F)}_{\mu\nu;\alpha\beta}$ in this gauge. Our derivation is based on ansatz (44), reading

$$\xi_{\nu;\mu\alpha\beta}(\tau) = [R^\epsilon{}_{\alpha\mu\nu}\xi_\epsilon - \delta\Gamma^R_{\nu\mu\alpha}]_{;\beta} + D_{\nu\mu\alpha\beta}.$$

First, we derive an expression for $D_{\nu\mu\alpha\beta}$ on $z_G(\tau)$. Contracting the ansatz with u^β and comparing with Eq. (20) we obtain

$$D_{\nu\mu\alpha\beta}u^\beta = 0. \quad \text{(C1)}$$

Taking the covariant derivative ∇_β of identity (19) and using ansatz (44) we obtain

$$D_{\nu[\mu\alpha]\beta} = 0. \quad \text{(C2)}$$

Using the identity

$$2\xi_{\nu;\mu[\alpha\beta]} = R^\epsilon{}_{\nu\alpha\beta}\xi_{\epsilon;\mu} + R^\epsilon{}_{\mu\alpha\beta}\xi_{\nu;\epsilon}, \quad \text{(C3)}$$

together with ansatz (44) we obtain

$$D_{\nu\mu[\alpha\beta]} = \tfrac{1}{2}\delta R^{R(F)}_{\mu\nu\alpha\beta}. \quad \text{(C4)}$$

To define $\delta R^{R(F)}_{\mu\nu\alpha\beta}$ we first introduce $\mu\delta R^R_{\alpha\beta\gamma\delta}$ which denotes the linear term in the following expansion of the Riemann tensor evaluated with the metric $g_{\mu\nu} + \mu h^R_{\mu\nu}$ (in the Lorenz gauge)

$$(g_{\mu\alpha} + \mu h^R_{\mu\alpha})R^\mu{}_{\beta\gamma\delta}[g_{\mu\nu} + \mu h^R_{\mu\nu}] - g_{\mu\alpha}R^\mu{}_{\beta\gamma\delta}[g_{\mu\nu}] = \mu\delta R^R_{\alpha\beta\gamma\delta} + O(\mu^2).$$

From which we find that

$$\delta R^R_{\nu\mu\alpha\beta} = \delta\Gamma^R_{\nu\mu\beta;\alpha} - \delta\Gamma^R_{\nu\mu\alpha;\beta} + h^R_{\epsilon\nu}R^\epsilon{}_{\mu\alpha\beta}.$$

Now $\delta R^{R(F)}_{\mu\nu\alpha\beta}$ is defined by transforming $\delta R^R_{\mu\nu\alpha\beta}$ to Fermi gauge, which gives

$$\begin{aligned}\mu\delta R^{R(F)}_{\alpha\beta\gamma\delta} &= \mu(\delta R^R_{\alpha\beta\gamma\delta} + \mathcal{L}_\xi R_{\alpha\beta\gamma\delta})\\ &= (g_{\mu\alpha} + \mu h^{R(F)}_{\mu\alpha})R^\mu{}_{\beta\gamma\delta}[g_{\mu\nu} + \mu h^{R(F)}_{\mu\nu}] - g_{\mu\alpha}R^\mu{}_{\beta\gamma\delta}[g_{\mu\nu}] + O(\mu^2),\end{aligned} \quad \text{(C5)}$$

where $\mathcal{L}_\xi$ denotes a Lie derivative, with respect to ξ^μ. Combining the fact that $h^{R(F)}_{\hat{\mu}\hat{\alpha}} = 0$, together with the fact that $h^{R(F)}_{\mu\alpha}$ satisfies a homogeneous perturbation equation in vacuum, reading $\delta R^F_{\beta\delta} = 0$, where

$$R_{\beta\delta}[g_{\mu\nu} + \mu h^{R(F)}_{\mu\nu}] - R_{\beta\delta}[g_{\mu\nu}] = \mu\delta R^{F}_{\beta\delta} + O(\mu^2),$$

we find that $\delta R^{R(F)}_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ is traceless.

Using Eqs. (C1), (C2), and (C4) we calculate some of the components of $D_{\nu\mu\alpha\beta}$. For simplicity we express these components in Fermi-normal coordinates, and obtain

$$D_{\nu\mu\alpha t} \overset{*}{=} 0, \qquad D_{\nu t\mu\beta} \overset{*}{=} D_{\nu\mu t\beta} \overset{*}{=} \delta R^{R(F)}_{\nu\mu\beta t}. \tag{C6}$$

Here $\overset{*}{=}$ denotes equality in Fermi-normal coordinates. The remaining components are not uniquely determined from Eqs. (C1), (C2), and (C4). We make the following choice which satisfies these equations

$$D_{\mu abc} \overset{*}{=} -\tfrac{1}{3}(\delta R^{R(F)}_{\mu abc} + \delta R^{R(F)}_{\mu bac}). \tag{C7}$$

Here spatial components are denoted with Latin indices that run from 1 to 3. To construct the extended Fermi gauge one first follows the prescription laid out in Sec. II B. Next, one uses ansatz (44) together with Eqs. (C6) and (C7) to construct $\xi_{\nu;\mu\alpha\beta}$.

Using Eqs. (C6) and (C7) together with Eq. (45) we obtain Eq. (46) reading

$$h^{R(F)}_{\mu\nu;\alpha\beta} = P_{\mu}{}^{\lambda}{}_{\nu}{}^{\eta}{}_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma}\delta R^{R(F)}_{\lambda\eta\rho\sigma},$$

where

$$\begin{aligned} P_{\mu}{}^{\lambda}{}_{\nu}{}^{\eta}{}_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} = &-2u_\mu u^\eta u_\nu u^\sigma P_\alpha{}^\lambda P_\beta{}^\rho \\ &- \tfrac{8}{3}u^\lambda u_{(\mu}P_{\nu)}{}^\rho P^{(\eta}_\alpha P_\beta{}^{\sigma)} \\ &- \tfrac{2}{3}P^{(\lambda}_\mu P_\nu{}^{\rho)}P_\alpha{}^\eta P_\beta{}^\sigma. \end{aligned}$$

Here $P^\beta_\alpha = \delta^\beta_\alpha + u_\alpha u^\beta$.

APPENDIX D: CONSTRUCTION OF $\bar\varphi_{\alpha\beta}$

In this appendix we discuss the solutions to Eqs. (60) and (65). First, we consider Eq. (60), reading

$$\Box\bar\varphi_{\alpha\beta} + 2R^\eta{}_\alpha{}^\rho{}_\beta\bar\varphi_{\eta\rho} = \bar g_\alpha{}^{\hat\alpha}\bar g_\beta{}^{\hat\beta}[\varepsilon^{-2}A_{\hat\alpha\hat\beta} + \varepsilon^{-1}B_{\hat\alpha\hat\beta} + O_a(\varepsilon^0)].$$

We substitute Eq. (61) into Eq. (60), and expand $\bar\varphi_{\hat\alpha\hat\beta}$ in powers of $\varepsilon\mathcal{R}^{-1}$, where $\mathcal{R}$ is the smallest of the length scales that characterize Riemann curvature tensor of the background geometry. Thus we obtain

$$\bar\varphi_{\hat\alpha\hat\beta} = \bar\varphi^{(0)}_{\hat\alpha\hat\beta} + \varepsilon\mathcal{R}^{-1}\bar\varphi^{(1)}_{\hat\alpha\hat\beta} + \varepsilon^2\mathcal{R}^{-2}\bar\varphi^{(2)}_{\hat\alpha\hat\beta} + O(\varepsilon^3\mathcal{R}^{-3}).$$

We use Fermi-normal coordinates based on the worldline, and expand $g_{\alpha\beta}$ and $\bar g^{\hat\alpha}{}_\alpha$ in powers of $\varepsilon\mathcal{R}^{-1}$ as follows

$$g_{\alpha\beta} \overset{*}{=} \eta_{\alpha\beta} + O(\varepsilon^2\mathcal{R}^{-2}), \qquad \bar g_\alpha{}^{\hat\alpha} \overset{*}{=} \delta^{\hat\alpha}_\alpha + O(\varepsilon^2\mathcal{R}^{-2}).$$

Notice that the source of Eq. (60) changes on a time scale of $O(\mathcal{R})$. In Fermi-normal coordinates we have $\Box \overset{*}{=} (\delta^{ab}\partial_a\partial_b) + O(\mathcal{R}^{-2})$. We therefore find that the leading term $\bar\varphi^{(0)}_{\hat\alpha\hat\beta}$ satisfies the following Poisson equation

$$(\delta^{ab}\partial_a\partial_b)\bar\varphi^{(0)}_{\hat\alpha\hat\beta} \overset{*}{=} \varepsilon^{-2}A_{\hat\alpha\hat\beta} + \varepsilon^{-1}B_{\hat\alpha\hat\beta} + O_a(\varepsilon^0). \tag{D1}$$

We decompose $\bar\varphi^{(0)}_{\hat\alpha\hat\beta}$ into spherical harmonics

$$\bar\varphi^{(0)}_{\hat\alpha\hat\beta} \overset{*}{=} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)^{(0)}\bar\phi^{lm}_{\hat\alpha\hat\beta}(\varepsilon,t), \tag{D2}$$

and use the spherical-harmonics decompositions (62). We construct the following solution for the spherical-harmonics coefficients

$$\begin{aligned} {}^{(0)}\bar\phi^{00}_{\hat\alpha\hat\beta} &\overset{*}{=} \tfrac{1}{2}b^{00}_{\hat\alpha\hat\beta}\varepsilon, \quad l=0, \\ {}^{(0)}\bar\phi^{1m}_{\hat\alpha\hat\beta} &\overset{*}{=} -\tfrac{1}{2}a^{1m}_{\hat\alpha\hat\beta} + \tfrac{1}{3}b^{1m}_{\hat\alpha\hat\beta}\varepsilon\log\varepsilon, \quad l=1, \\ {}^{(0)}\bar\phi^{lm}_{\hat\alpha\hat\beta} &\overset{*}{=} -[l(l+1)]^{-1}a^{lm}_{\hat\alpha\hat\beta} + [2-l(l+1)]^{-1}b^{lm}_{\hat\alpha\hat\beta}\varepsilon, \quad l>1. \end{aligned} \tag{D3}$$

Calculation of $a^{00}_{\hat\alpha\hat\beta}$ (using Mathematica software) reveals that it vanishes, and therefore it is absent from Eq. (D3). As both sides of Eq. (D3) are composed from bitensors we may replace the notation "$\overset{*}{=}$" with "$=$". Equation (D3) states that $\bar\phi^{lm}_{\hat\alpha\hat\beta}$ is bounded in a local neighborhood of the worldline. Therefore, $\bar g_\alpha{}^{\hat\alpha}\bar g_\beta{}^{\hat\beta}\bar\varphi^{(0)}_{\hat\alpha\hat\beta}$ satisfies the original Eq. (60) up to an arbitrary $O(\varepsilon^0)$ term. We therefore set $\bar\varphi_{\hat\alpha\hat\beta} = \bar\varphi^{(0)}_{\hat\alpha\hat\beta}$.

We now turn to Eq. (65), which reads

$$\begin{aligned} \Box(\delta\bar l_{\alpha\beta} - \bar\varphi_{\alpha\beta}) + 2R^\eta{}_\alpha{}^\rho{}_\beta(\delta\bar l_{\eta\rho} - \bar\varphi_{\eta\rho}) \\ = \bar g_\alpha{}^{\hat\alpha}\bar g_\beta{}^{\hat\beta}[O(\varepsilon^0)]. \end{aligned}$$

Here again we expand $(\delta\bar l_{\alpha\beta} - \bar\varphi_{\alpha\beta})$ and the corresponding source term in powers of $\varepsilon\mathcal{R}^{-1}$. Similar to Eq. (D1), the dominant terms as $\varepsilon \to 0$ are obtained by solving the following Poisson equation

$$(\delta^{ab}\partial_a\partial_b)(\delta\bar l_{\hat\alpha\hat\beta} - \bar\varphi_{\hat\alpha\hat\beta})^{(0)} \overset{*}{=} \rho_{\hat\alpha\hat\beta}. \tag{D4}$$

Here $\bar g_\alpha{}^{\hat\alpha}\bar g_\beta{}^{\hat\beta}(\delta\bar l_{\hat\alpha\hat\beta} - \bar\varphi_{\hat\alpha\hat\beta})^{(0)}$ is the leading order term in the expansion of $(\delta\bar l_{\alpha\beta} - \bar\varphi_{\alpha\beta})$, and $\bar g_\alpha{}^{\hat\alpha}\bar g_\beta{}^{\hat\beta}\rho_{\hat\alpha\hat\beta}$ is the corresponding leading term in the expansion of the source term, this term scales like ε^0. We decompose $\rho_{\hat\alpha\hat\beta}(\theta,\varphi,t)$ and $(\delta\bar l_{\hat\alpha\hat\beta} - \bar\varphi_{\hat\alpha\hat\beta})^{(0)}$ into spherical harmonics, and denote the corresponding spherical-harmonics coefficients with lm indices. Solving for the spherical-harmonics coefficients we obtain

$$\begin{aligned} (\delta\bar l_{\hat\alpha\hat\beta} - \bar\varphi_{\hat\alpha\hat\beta})^{(0)}_{lm} &= [(l+3)(2-l)]^{-1}\rho^{lm}_{\hat\alpha\hat\beta}\varepsilon^2, \qquad l\neq 2, \\ (\delta\bar l_{\hat\alpha\hat\beta} - \bar\varphi_{\hat\alpha\hat\beta})^{(0)}_{2m} &= -(1/5)\rho^{2m}_{\hat\alpha\hat\beta}\varepsilon^2(1/5 - \log\varepsilon), \qquad l=2. \end{aligned} \tag{D5}$$

This solution satisfies the criteria of a sufficiently regular piece. The most general solution to Eq. (D4) is obtained by adding to Eq. (D5) a general homogeneous solution. A homogeneous solution which is regular at the worldline has the form of $\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c^{lm}(t)Y_{lm}(\theta,\varphi)\varepsilon^{l}$, and therefore satisfies the criteria of a sufficiently regular piece. Notice that solutions which are homogeneous off the worldline but diverge at the worldline are excluded, since they correspond to a source in the form of a distribution, and are therefore nonhomogeneous. The higher-order corrections to $(\delta\bar{l}_{\alpha\beta}-\bar{\varphi}_{\hat{\alpha}\hat{\beta}})^{(0)}$ are also sufficiently regular. We therefore conclude that $(\delta\bar{l}_{\alpha\beta}-\bar{\varphi}_{\alpha\beta})$ satisfies the criteria of a sufficiently regular piece.

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[21] Notice that the dimensionality argument applies even if the derivatives are not well defined at $\hat{z}$. Suppose that $h^{SR}_{\hat{\mu}\hat{\nu};\hat{\gamma}\hat{\delta}_1\ldots\hat{\delta}_n}$, $n\geq 1$ is ill-defined (a similar argument also applies to $\delta l^{SR}_{\hat{\mu}\hat{\nu};\hat{\gamma}\hat{\delta}_1\ldots\hat{\delta}_n}$). Recall that $\varepsilon^{-1+n}h^{SR}_{\mu\nu;\gamma\delta_1\ldots\delta_n}$ is bounded if x is sufficiently close to $\hat{z}$. However, this quantity (at $\hat{z}$) cannot be included in our list of tensorial constituents since it has a dimensionality of $(\text{Length})^{-3}$. A similar argument shows that if the singularity of $h^{SR}_{\mu\nu;\gamma\delta_1\ldots\delta_n}$ is less severe (than ε^{1-n}), then the bounded quantities that can be constructed from $h^{SR}_{\mu\nu;\gamma\delta_1\ldots\delta_n}$ must also have the wrong dimensions. For example, suppose that $\varepsilon^{-1+n-k}h^{SR}_{\mu\nu;\gamma\delta_1\ldots\delta_n}$, where $k>0$, is bounded as $x\rightarrow\hat{z}$. This quantity has a dimension of $(\text{Length})^{-(3+k)}$ and therefore, by dimensionality, it must be excluded from our list.
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