New derivation of the variational principle for the dynamics of a gravitating spherical shell

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The dynamics of a self-gravitating matter shell in general relativity is discussed in general. The case of a spherical shell composed of an arbitrary ideal fluid is then considered, and its Lagrangian function is derived from first principles. For this purpose, the standard Hilbert action is modified by an appropriate surface term at spatial infinity. The total Hamiltonian of the composed ''shell + gravity'' system is then calculated. Known results for the dust matter are recovered as particular cases. The above ''surface renormalization'' of the Hilbert action may be used for any spatially flat spacetime.

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I. INTRODUCTION

In his seminal paper [1], Werner Israel considered the dynamics of a self-gravitating thin matter shell as the simplest toy model describing gravitational collapse. Since field equations for realistic collapse are very difficult to handle, these models prove useful to analyze many aspects of this phenomenon (see e.g. [2,3]). The dynamics of such a system indeed reduces to a proper ''tailoring'' of the two different spacetimes, describing the two sides of the shell.

Both the variational and the canonical (Hamiltonian) formulations of the dynamics of the composed ''shell + gravity'' system were constructed in [4,5], starting from first principles, without assuming any symmetry of the system. Recently, a considerable simplification of this theory was obtained *via* a consistent use of the theory of distributions (see [6] for lightlike shells and [7] for massive shells).

In the case of a spherically symmetric shell in vacuum, this formulation leads to a simple Hamiltonian system with 1 degree of freedom. The configuration variable is the area of the shell, whereas the canonical momentum equals the hyperbolic angle between the surfaces $\{t_{\mathrm{Schwarzschild}} = \mathrm{const}\}$ on one side and the surfaces $\{t_{\mathrm{Minkowski}} = \mathrm{const}\}$ on the other side of the shell (cf. [5] for a shell composed of the dust matter; the general result, valid for an arbitrary ideal fluid, is proved in the present paper). Transformations of this Hamiltonian structure due to changes of a time variable were thoroughly discussed in [8].

The spherically symmetric shell is, therefore, well understood as a special case of an arbitrary shell. Nevertheless, an intriguing puzzle persists: ''imposing spherical symmetry of the model'' and ''performing variation of the action'' seem to be noncommuting procedures, at least in the simplest version, when the variation is performed within the family of configurations in which an internal Minkowski geometry is tailored to an external Schwarzschild geometry of a fixed but arbitrary mass parameter. Indeed, the standard Hilbert action (with appropriate modifications, necessary to handle singular—Dirac deltalike—objects), restricted to those configurations only, *does not* imply the correct dynamics of the ''shell + gravity'' system. As will be seen in the sequel, this is due to the fact that the Arnowit-Deser-Misner (ADM) mass at infinity, which later plays the role of the Hamiltonian of the system, is fixed *a priori* within the above family of configurations instead of being a function of the configuration variables.

To obtain the missing relation between the energy (i.e. the ADM mass) and the configuration of the system, one has to extend the family of admissible configurations in such a way that the ADM mass is no longer fixed *a priori* but is also subject to variation. This leads to an even more difficult problem: the total action diverges at infinity.

In the present paper we propose a simple remedy for this disease. It consists in improving the Hilbert action by an appropriate boundary term at spatial infinity. This solution is suggested by the analysis of boundary terms in canonical relativity, which was given in [9].

The above boundary term may also be used as a remedy for the standard difficulty in general relativity, consisting in the fact that the integral of the Hilbert action over an asymptotically flat spacelike surface is infinite when calculated for a generic metric $g = \eta + h$ (where η is the flat Minkowski background), if standard falloff conditions ($h \sim \frac{1}{r}$ and $\partial h \sim \frac{1}{r^2}$) are imposed. The divergence of this integral at spatial infinity does not lead to any difficulty in classical theory, where we only need a local version of the action to derive the field equations. Global problems arise however on the canonical level, when we want to define the global Hamiltonian of the gravitational field. Here, appropriate ''boundary Legendre transformations'' are necessary. But, as shown in [9], all of these boundary manipulations may always be performed on finite two-surfaces S. As a result we obtain quasilocal Hamiltonians H_S (see also [10]). For this purpose no boundary improve-

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ments of the action are necessary, because the boundary manipulations are always performed on the Hamiltonian level and are related to an appropriate choice of the control of the boundary data for the Hamiltonian field dynamics. Finally, the global Hamiltonian H_∞ can be directly obtained by simply shifting the surface S to infinity. In this procedure no ‘‘global action’’ is necessary. However, if we think about quantization of gravitational field and want to implement the idea of Feynman path integrals, the (spatially) global action is necessary as a building block for the field propagator. The techniques proposed by us in the present paper may be used for this purpose.

The plan of the paper is as follows. In Sec. II we briefly summarize the framework of the singular Riemann tensor related to the tailoring of the spacetime and show how the use of the tensor-densities simplifies the entire formalism. In Sec. III we restrict ourselves to the spherically symmetric case and calculate the (hyperbolic) angle μ between the Schwarzschild and the Minkowski foliations on both sides of the shell. This angle later plays the role of the momentum canonically conjugate to the volume of the shell. In Sec. IV we show that the simplest Hilbert action principle does not lead to the correct description of the dynamics of the ‘‘shell + gravity’’ system: the dependence of the total mass of the system upon the configuration variables is still missing. In Sec. V we show how to improve the above variational principle by taking into account boundary terms at spatial infinity. Finally, in Sec. VI we show that the angle μ is indeed the canonical momentum of our theory and prove that the Hamiltonian function of the system is numerically equal to the ADM mass at spatial infinity. We stress that the Legendre transformation between the velocity and the momentum of the physical system in question is not univocal. Indeed, to one velocity there correspond in general two different values of the momentum. Hence, only the canonical variables, position and momentum (and not position and velocity), provide the global coordinate chart on the phase space of the system. Detailed calculations are found in the Appendices.

II. SINGULAR CURVATURE TENSOR

We consider a spacetime $\mathcal{M}$ consisting of two parts: $\mathcal{M} = \mathcal{M}_- \cup \mathcal{M}_+$, tailored together along their common boundary $\Sigma = \partial\mathcal{M}_- = -\partial\mathcal{M}_+$. The 3-dimensional hypersurface Σ describes the history of a moving 2-dimensional matter shell. Tailoring means that the induced metric g_{ab} on Σ is continuous, whereas its derivatives (i.e. also the four-dimensional connection coefficients $\Gamma^\lambda_{\mu\nu}$) may be discontinuous across Σ. We assume that the matter distributed on Σ is massive. This means that the intrinsic metric g_{ab} of Σ is one-timelike and two-spacelike, i.e. has signature $(-, +, +)$.

As shown in [7], a considerable simplification of the theory of thin shells is obtained if we use only such coordinate systems (x^μ), $\mu = 0, 1, 2, 3$; on $\mathcal{M}$, for which not only the induced metric g_{ab} on Σ but also all the 10 components $g_{\mu\nu}$ of the spacetime metric are continuous. This is not a physical restriction imposed on possible configurations of the system, but merely a gauge condition, which can be always satisfied. Indeed, it is easy to see that starting from any field configuration which does not fulfill our gauge condition (i.e. only 6 components of the metric along Σ are continuous), there exists a transformation of variables which also makes the remaining 4 components (transversal to Σ) continuous. We stress, however, that our formulation of the dynamics of the system contains only intrinsic geometric quantities, which do not depend upon any choice of coordinates. Hence, our results may be translated to any coordinate system, also those not fulfilling our gauge condition; however, the mathematical formalism which leads to Lagrangian and Hamiltonian formulations of the dynamics becomes much simpler in this gauge. Indeed, for a metric $g_{\mu\nu}$ satisfying this condition, we may use Einstein tensor density: $\mathcal{G}^\mu{}_\nu = \sqrt{\det g}(\mathbf{R}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu \mathbf{R})$, where the Riemann tensor $\mathbf{R}$ is defined *via* the standard geometric formulas, with derivatives of the connection coefficients $\Gamma^\lambda_{\mu\nu}$ understood in the sense of distributions. Because of discontinuities of Γ's across Σ it contains a singular part

$$^{\rm sing}\mathcal{G}^\mu{}_\nu = \mathbf{G}^\mu{}_\nu \boldsymbol{\delta}_\Sigma \tag{2.1}$$

proportional to the Dirac delta $\boldsymbol{\delta}_\Sigma$ on Σ. It may be easily proved (see [7]) that $\mathbf{G}$ is the 3-dimensional tensor-density living on Σ, whose components transversal with respect to Σ vanish identically. Moreover, its components tangent to Σ are given by the following identity:

$$\mathbf{G}^a{}_b = [Q^a{}_b], \tag{2.2}$$

where $Q^a{}_b$ denotes the extrinsic curvature of Σ (written in the ADM representation) and square brackets denote its jump between both sides of Σ.

Formula (2.1) shows the reason why the theory simplifies considerably if we use consequently tensor densities instead of tensors: all the singular four-dimensional densities which arise in the theory, factorize in a natural way into products of: (i) three-dimensional densities on Σ and (ii) the Dirac delta $\boldsymbol{\delta}_\Sigma$ which, in fact, is a one-dimensional density in the direction transversal to Σ. No such factorization exists for tensors.

The entire dynamics of the gravitational field interacting with the shell is described by the Einstein equations: $^{\rm sing}\mathcal{G}^\mu{}_\nu = 8\pi\mathcal{T}^\mu{}_\nu$, where $\mathcal{T}$ denotes the energy momentum tensor density of the matter. In case of a matter shell, $\mathcal{T}$ vanishes outside of Σ and, again, factorizes into the product of the Dirac delta and a three-dimensional tensor density $\mathbf{T}$ on Σ:

$$\mathcal{T} = \mathbf{T}\boldsymbol{\delta}_\Sigma. \tag{2.3}$$

Of course, components of $\mathbf{T}$ transversal with respect to Σ

vanish identically. Consequently, the singular part of Einstein equations may be expressed in terms of 3-dimensional intrinsic objects living on Σ: $\mathbf{G}^a{}_b = 8\pi\mathbf{T}^a{}_b$. The remaining dynamical equations of the theory are the vacuum Einstein equations outside of Σ and the mechanical equations of motion for the matter, implied by its constitutive equation.

III. SPHERICAL SYMMETRY

In the present paper we consider the simplest, spherically symmetric case. Assuming the trivial $\mathbb{R}^3$ topology of each Cauchy surface $\{t = \text{const}\}$ and the $S^2 \times \mathbb{R}^1$ topology of the world tube Σ, we conclude that the interior of the shell (i.e. the component $\mathcal{M}_-$) must be a portion of the flat Minkowski space

$$ds_-^2 = -d\tau^2 + d\rho^2 + \rho^2 d\Omega^2 \tag{3.1}$$

corresponding to $\rho \le \phi(\tau)$, with the shell located at $\rho = \phi(\tau)$, whereas the exterior (i.e. the component $\mathcal{M}_+$) must carry a Schwarzschild geometry

$$ds_+^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.2}$$

for $r \ge \psi(t)$ and some value $M \ge 0$, with the shell located at $r = \psi(t)$. As the time variable we choose the external Schwarzschild time t, measured at spatial infinity by an external observer.[1] Consequently, we treat the value of the Minkowski time τ on the shell as a (monotonic) function of t:

$$\tau = f(t). \tag{3.3}$$

Matching conditions for the metric on the shell imply that the function ϕ is fully determined by f and ψ:

$$\phi(f(t)) = \psi(t). \tag{3.4}$$

The remaining matching conditions are equivalent to the fact that the four-velocity $\frac{\partial}{\partial t}$ of the particles constituting the shell has the same length with respect to both metric tensors. This velocity is given by the following vectors, which must coincide:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_+ := \frac{\partial}{\partial t} + \dot\psi\frac{\partial}{\partial r},$$
$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_- := \frac{\mathrm{d}\tau}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau} = \dot f\left(\frac{\partial}{\partial\tau} + \dot\phi\frac{\partial}{\partial\rho}\right) = \dot f\frac{\partial}{\partial\tau} + \dot\psi\frac{\partial}{\partial\rho}.$$

Equating the length of this vector on shell, calculated with respect to (3.2)

$$-\left\|\frac{\partial}{\partial t} + \dot\psi\frac{\partial}{\partial r}\right\|^2 = -\left(1 - \frac{2M}{\psi}\right) + \dot\psi^2\left(1 - \frac{2M}{\psi}\right)^{-1}, \tag{3.5}$$

and with respect to (3.1), respectively,

$$-\left\|\dot f\frac{\partial}{\partial\tau} + \dot\psi\frac{\partial}{\partial\rho}\right\|^2 = \dot f^2 - \dot\psi^2, \tag{3.6}$$

we obtain the remaining matching condition for the metric:

$$\dot f^2 = 1 - \frac{2M}{\psi}\cdot\frac{1 - \frac{2M}{\psi} + \dot\psi^2}{1 - \frac{2M}{\psi}}. \tag{3.7}$$

An important role in further calculations is played by the hyperbolic angle μ between surfaces $\{t = \text{const}\}$ on the Schwarzschild side and surfaces $\{\tau = \text{const}\}$ on the Minkowski side of the shell. More precisely, we take the four-vector

$$\vec{\mathbf{n}} := \left(1 - \frac{2M}{r}\right)^{-(1/2)}\frac{\partial}{\partial t},$$

(ortho)normal with respect to the first (Schwarzschild) foliation and the four-vector

$$\vec{\mathbf{m}} := \frac{\partial}{\partial\tau},$$

(ortho)normal to the latter (Minkowski) foliation. Consequently, we define the angle (see also [5]): $|\mu| := \text{arcosh}|(\vec{\mathbf{n}}|\vec{\mathbf{m}})|$. Similarly as in Euclidean geometry, the sign of the angle may be chosen arbitrarily. A convenient choice is the sign of the velocity $\dot\psi$. So, we define

$$\mu := (\text{sign}\,\dot\psi)\text{arcosh}|(\vec{\mathbf{n}}|\vec{\mathbf{m}})| = (\text{sign}\,\dot\psi)\text{arcosh}\{-(\vec{\mathbf{n}}|\vec{\mathbf{m}})\} \tag{3.8}$$

[remember that both vectors are timelike and belong to the future light cone; their scalar product $(\vec{\mathbf{n}}|\vec{\mathbf{m}}) := g_{\mu\nu}\mathbf{n}^\mu\mathbf{m}^\nu$ is, therefore, negative]. To use the above formula we need, in principle, a system of coordinates in a neighborhood of Σ, such that the metric g is continuous across Σ. Before we construct such coordinates (see Appendix A), we may calculate μ using the following method. Take the normalized velocity vector written in Schwarzschild coordinates:

$$\vec{\mathbf{v}} := \frac{\frac{\partial}{\partial t} + \dot\psi\frac{\partial}{\partial r}}{\|\frac{\partial}{\partial t} + \dot\psi\frac{\partial}{\partial r}\|},$$

or, equivalently, written in Minkowski coordinates:

$$\vec{\mathbf{v}} := \frac{\dot f\frac{\partial}{\partial\tau} + \dot\psi\frac{\partial}{\partial\rho}}{\|\dot f\frac{\partial}{\partial\tau} + \dot\psi\frac{\partial}{\partial\rho}\|}.$$

Now, we may use (3.2) [together with (3.5)] to calculate

[1] This way we are able to cover only this portion of the evolution of the shell which lies outside of the event horizon. To cover e.g. the collapse, we have to pass to a different time variable. The transition between different times in the Hamiltonian description of the shell dynamics was thoroughly discussed in [8].

$$\cosh\mu_+ = |(\vec{\mathbf{v}}|\vec{\mathbf{n}})| = \frac{(1-\frac{2M}{\psi})}{\sqrt{\dot{\psi}^2-(1-\frac{2M}{\psi})^2}} \tag{3.9}$$

and (3.1) to calculate

$$\cosh\mu_- = |(\vec{\mathbf{v}}|\vec{\mathbf{m}})| = \frac{\dot{f}(1-\frac{2M}{\psi})^{3/2}}{\sqrt{\dot{\psi}^2-(1-\frac{2M}{\psi})^2}}. \tag{3.10}$$

Finally, we calculate $\mu = \mu_+ \pm \mu_-$ according to formula

$$\cosh(\mu_+ \pm \mu_-) = \cosh\mu_+\cosh\mu_- \pm \sinh\mu_+\sinh\mu_-, \tag{3.11}$$

equivalent to the following identity:

$$(\vec{\mathbf{n}}|\vec{\mathbf{m}})^2 + 2(\vec{\mathbf{n}}|\vec{\mathbf{m}})(\vec{\mathbf{m}}|\vec{\mathbf{v}})(\vec{\mathbf{n}}|\vec{\mathbf{v}}) + (\vec{\mathbf{m}}|\vec{\mathbf{v}})^2 + (\vec{\mathbf{n}}|\vec{\mathbf{v}})^2 - 1 = 0, \tag{3.12}$$

valid for any triple $(\vec{\mathbf{m}}, \vec{\mathbf{n}}, \vec{\mathbf{v}})$ of normalized, timelike four-vectors.

For a given value of configuration variables $(\psi, \dot{\psi})$ of the shell, there are two possible values of μ (see Fig. 1), corresponding to the two possible solutions (3.11) of identity (3.12) treated as a quadratic equation for $(\vec{\mathbf{n}}|\vec{\mathbf{m}})$. Using (3.9) and (3.10) for $(\vec{\mathbf{m}}|\vec{\mathbf{v}})$ and $(\vec{\mathbf{n}}|\vec{\mathbf{v}})$, one may derive the following version of (3.12):

$$\cosh^2\mu - 1 = \frac{\dot{\psi}^2}{(1-\frac{2M}{\psi})^2}\left(\cosh\mu - \sqrt{1-\frac{2M}{\psi}}\right)^2, \tag{3.13}$$

which, according to the convention (3.8) concerning the sign of μ, may be rewritten as the following constraint equation:

$$\frac{\sinh\mu}{\cosh\mu - \sqrt{1-\frac{2M}{\psi}}} = \frac{\dot{\psi}}{1-\frac{2M}{\psi}}. \tag{3.14}$$

IV. VARIATIONAL PRINCIPLE

In order to obtain the dynamics of the above "matter + gravity" system consider first the Hilbert action $\mathcal{A}$ composed of the two parts: the gravitational part $\mathcal{A}_{\rm grav}$ being the integral over the four-dimensional domain $D = \{t_1 \le t \le t_2\}$ of the scalar curvature and the matter part $\mathcal{A}_{\rm mat}$ concentrated on the hypersurface and carrying the information about the matter content of the shell:

$$\mathcal{A} = \mathcal{A}_{\rm grav} + \mathcal{A}_{\rm mat} = \int_D \mathcal{L}_{\rm grav} + \int_{D\cap\Sigma}\mathcal{L}_{\rm mat}. \tag{4.1}$$

The gravitational Lagrangian splits into two parts: a regular part outside of the shell and a singular (Dirac deltalike) part on the shell

$$\mathcal{L}_{\rm grav} = \frac{1}{16\pi}\sqrt{\det g}\mathbf{R} = \mathcal{L}^{\rm sing}_{\rm grav} + \mathcal{L}^{\rm reg}_{\rm grav} \tag{4.2}$$

with $\mathbf{R} = \mathbf{R}_{\rm reg} + \mathbf{R}_{\rm sing}$.

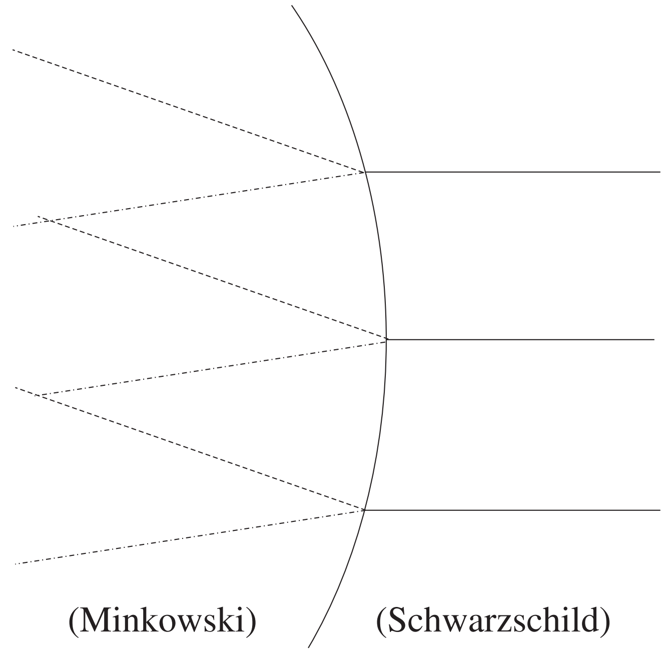

FIG. 1. Given the shell position ψ and velocity $\dot{\psi}$, there are two possible values of the angle μ, satisfying constraint (3.14). Consequently, the Schwarzschild spacetime (see foliation $\{t = \text{const}\}$ on the right-hand side) may be tailored with the Minkowski spacetime in two different ways (see possible two foliations $\{\tau = \text{const}\}$ on the left-hand side).

Taking into account only configurations obtained by tailoring the external Schwarzschild solution with the internal Minkowski, we automatically annihilate the regular part of the curvature: $\mathbf{R}_{\rm reg} = 0$. What remains is, therefore, only the singular part, which we calculate according to the identity $\mathcal{G}^\mu{}_\mu = \sqrt{\det g}(\mathbf{R}^\mu{}_\mu - 2\mathbf{R}) = -\sqrt{\det g}\mathbf{R}$. Hence, using (2.1) and (2.2), we obtain

$$\sqrt{\det g}\mathbf{R}_{\rm sing} = -\mathbf{G}^\mu{}_\mu\boldsymbol{\delta}_\Sigma = -[Q^a{}_a]\delta_\Sigma. \tag{4.3}$$

Calculating the jump of the extrinsic mean curvature $Q^a{}_a$ of Σ, it can be shown (see Appendix B for the proof) that the following formula for the total value of the gravitational part of the Hilbert action holds:

$$\mathcal{A}_{\rm grav} = \int_{t_1}^{t_2}\left(2M\frac{\cosh\mu}{\cosh\mu - \sqrt{1-\frac{2M}{\psi}}} + -2\psi + \psi\dot{\psi}\mu + \frac{3M}{2}\right)dt - \left[\frac{\psi^2}{2}\mu\right]_{t_1}^{t_2}, \tag{4.4}$$

where $\mu = \mu(\psi, \dot{\psi})$ is any of the two solutions of the constraint equation (3.14).

To calculate the matter part of the action we assume the simplest—hydrodynamical—model for matter. Consequently (cf. [4,11]), the matter Lagrangian is equal to the rest-frame energy density of the matter. The dust case corresponds, e.g., to the constant function $m(\nu) = m_0$, where the *specific* rest-frame energy m_0 of the matter (i.e. calculated *per unit amount of matter*) does not depend upon its specific volume ν, because it is equal to the sum of the rest-frame masses of the noninteracting dust particles. For a generic fluid the total energy contains also the interaction energy of the fluid particles, depending upon the local density of the fluid or, equivalently, upon ν. We assume, therefore, that the rest-frame energy of the fluid is a given function $m = m(\nu)$, which plays the role of the constitutive equation and implies all the mechanical properties of the fluid composing the shell.

The fluid is homogeneous and, therefore, its specific volume ν equals the total (two-dimensional) volume $4\pi\psi^2$ of the shell, divided by the total amount of the fluid. To simplify further our notation, we choose units in such a way that the total amount of the fluid contained in the shell equals 8π. This leads to the following formula:

$$\nu := \tfrac{1}{2}\psi^2.$$

To calculate the matter part of the action, we multiply the specific rest-frame energy $m(\nu)$ by the matter density J^0 and integrate it over the shell (cf. [11]). Because $m(\nu)$ does not depend upon angular variables (θ, φ) and the integration of the matter density over the shell gives the total amount of the fluid which is constant in time, we conclude that, *modulo* an appropriate choice of units, we have to integrate $m(\nu)$ with respect to the time only:

$$\begin{aligned}\mathcal{A}_{\text{mat}} &= \int_{D\cap\Sigma} J^0 m(\nu) d\theta d\varphi dt = \int_{t_1}^{t_2} m(\nu) ds \\ &= \int_{t_1}^{t_2} m \left\| \frac{\partial}{\partial t} + \dot{\psi}\frac{\partial}{\partial r}\right\| dt \\ &= \int_{t_1}^{t_2} m \sqrt{\left(1-\frac{2M}{\psi}\right) - \frac{\dot{\psi}^2}{(1-\frac{2M}{\psi})}} dt. \end{aligned} \tag{4.5}$$

Finally, adding (4.4) and (4.5) we obtain

$$\mathcal{A} = \int_{t_1}^{t_2} \mathrm{L}\, dt + F(t_2) - F(t_1), \tag{4.6}$$

with

$$\begin{aligned}\mathrm{L} &= m(\nu)\sqrt{\left(1-\frac{2M}{\psi}\right) - \frac{\dot{\psi}^2}{1-\frac{2M}{\psi}}} + \frac{3M}{2} \\ &\quad + \frac{2M\cosh\mu}{\cosh\mu - \sqrt{1-\frac{2M}{\psi}}} - 2\psi + \psi\dot{\psi}\mu, \end{aligned} \tag{4.7}$$

where $F(t) = -\frac{1}{2}\psi^2\mu$. Of course, the boundary term $F(t_2) - F(t_1)$ can be neglected.

In the above variational principle the status of the (arbitrarily chosen) value M of the Schwarzschild mass is highly unclear. Physically, being equal to the ADM mass, it describes the total mass (i.e. the total energy) of the interacting "matter + gravity" system and, therefore, we would expect it to be equal to the Hamiltonian of the system, which is not the case here. Indeed, to obtain the corresponding Hamiltonian formulation, we first evaluate the momentum canonically conjugate to the variable ψ from $p_\psi = \frac{\partial \mathrm{L}}{\partial \dot{\psi}}$. From (3.14) we get

$$\frac{\partial\dot{\psi}}{\partial\mu} = \left(1-\frac{2M}{\psi}\right)\frac{1-\sqrt{1-\frac{2M}{\psi}}\cosh\mu}{(\cosh\mu - \sqrt{1-\frac{2M}{\psi}})^2}, \tag{4.8}$$

and, consequently, formula (4.7) implies

$$\begin{aligned} p_\psi &= \psi\mu - \frac{\psi\sinh\mu}{\sqrt{1-\frac{2M}{\psi}}} \\ &\quad \times \left(\frac{m(\nu)}{\psi}\frac{1}{\sqrt{2-\frac{2M}{\psi} - 2\cosh\mu\sqrt{1-\frac{2M}{\psi}}}} + 1\right). \end{aligned} \tag{4.9}$$

The Hamiltonian function is then obtained via the usual Legendre transformation: $\mathcal{H}(\psi, p_\psi) = p_\psi\dot{\psi} - \mathrm{L}$. Hence, instead of the total energy M we obtain the following value of the Hamiltonian function:

$$\begin{aligned}\mathcal{H} &= -\frac{m(\nu)\sqrt{1-\frac{2M}{\psi}}}{\sqrt{1-\frac{\dot{\psi}^2}{\{1-[(2M)/\psi]\}^2}}} + \psi - \psi\sqrt{1-\frac{2M}{\psi}}\cosh\mu \\ &= -m(\nu)\sqrt{1-\frac{2M}{\psi}}\frac{\cosh\mu - \sqrt{1-\frac{2M}{\psi}}}{\sqrt{2-\frac{2M}{\psi} - 2\cosh\mu\sqrt{1-\frac{2M}{\psi}}}} \\ &\quad + \psi - \psi\sqrt{1-\frac{2M}{\psi}}\cosh\mu. \end{aligned} \tag{4.10}$$

We conclude that fixing *a priori* the value of M does not lead to a true Hamilton principle but rather to an analog of the Maupertuis-Lagrange variational principle in classical mechanics, where the total energy of the system is given in advance. However, this is not a genuine Maupertuis-Lagrange approach since the relation between $(\psi, \dot{\psi})$ and the energy M is still missing and cannot be retrieved from the above formula. Many authors have noticed these difficulties, but a correct variational formula for matter shells has never been derived from first principles. In the following, we are going to prove that only those configurations for which the parameter M equals a well-defined function $M = \mathcal{H}(\psi, \dot{\psi})$ [see formula (4.4)], with H playing the role of Hamiltonian, are physical.

V. IMPROVED VARIATIONAL PRINCIPLE

We propose the following, simple remedy for all these problems, which leads to the correct variational and Hamiltonian formulation of the model. Our method is based on an analysis of the boundary phenomena (usually neglected) arising in the Hilbert variational principle, which was proposed in [9] (see also references herein). It was noticed that the variation of the gravitational Lagrangian (4.2) contains, besides the standard volume part responsible for the field equations, also the boundary part, containing the variation of the extrinsic curvature of the boundary ∂D of the domain D in question. More precisely, taking into account also the matter Lagrangian, we have

$$\delta \mathcal{A} = \int_D \frac{\delta \mathcal{L}}{\delta g}\delta g_{\mu\nu} + \int_{D\cap\Sigma} \frac{\delta \mathcal{L}}{\delta \varphi}\delta\varphi + -\frac{1}{16\pi} \times \int_{\partial D} g_{ab}\delta Q^{ab}, \tag{5.1}$$

where φ denotes a generic matter field on the shell, $Q^{ab} = \sqrt{|\det \hat{g}|}(L\hat{g}^{ab} - L^{ab})$, L_{ab} is the extrinsic curvature of ∂D, and $\hat{g}^{ab}$ is its three-dimensional inverse metric.

For purposes of the present paper, we apply the above formula to a spatially bounded, cylindric domain $D_R = \{t_1 \le t \le t_2; r \le R\}$ and denote $t = x^0$ and $r = x^1$, whereas x^A are angular coordinates, $A, B = \{2, 3\}$. The value of the parameter R (later irrelevant, because the limit $R \to \infty$ will be considered) is chosen in such a way that D_R contains our shell, i.e. $\psi(t) < R$ for $t_1 \le t \le t_2$ (see Fig. 2).

The boundary ∂D_R splits into: (i) the cylindric surface $\mathcal{C}_R := \{r = R\}$, and (ii) the two covers $K_i := \{t = t_i\}$, $i = 1, 2$. Moreover, due to the deltalike singularity of the external curvature Q^{ab} at the 2-dimensional "corners" $\mathcal{C}_R \cap K_i$, the last (surface) integral in (5.1) contains also two corner integrals. To simplify our notation, we marry the corner terms together with the cover terms and write

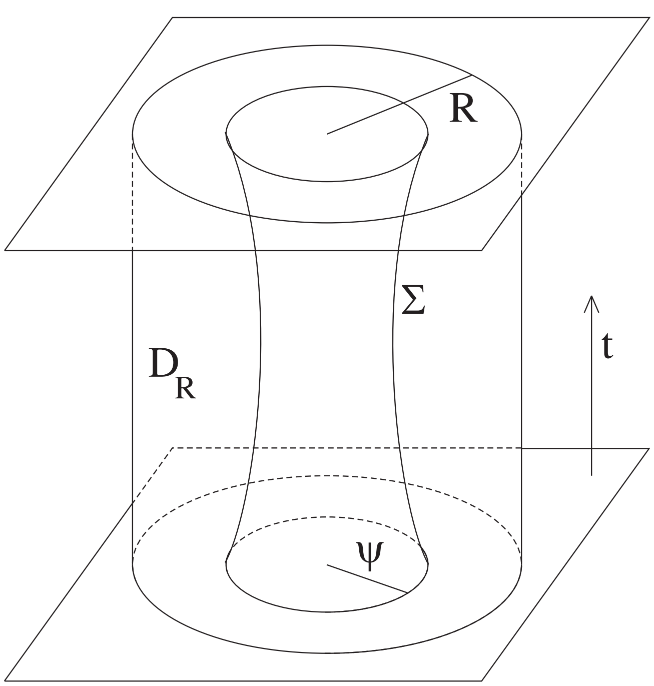

FIG. 2. Schematic view of the variation region D_R.

$$\int_{\partial D_R} = \int_{\mathcal{C}_R} + \int_{K_2} - \int_{K_1}. \tag{5.2}$$

We do not change the last two ("cover + corner") terms because their variations vanish if we keep fixed the initial and the final field configurations (see also [9] for the discussion of the canonical content of these integrals). Instead, we perform a Legendre transformation on the cylinder $\mathcal{C}_R$. For this purpose observe that, because of the spherical symmetry of the field configuration, we have $g_{0A} = 0$, $Q^{0A} = 0$, and, consequently,

$$g_{ab}\delta Q^{ab} = g_{00}\delta Q^{00} + g_{AB}\delta Q^{AB} = g_{00}\delta Q^{00} - Q^{AB}\delta g_{AB} + \delta(g_{AB}Q^{AB}).$$

Now, we define a new improved action by adding to the Einstein-Hilbert action (5.1), a boundary term, given by the last term above:

$$\mathcal{A}_{\text{tot}} = \int_{D_R} \mathcal{L}_{\text{grav}} + \int_{D_R\cap\Sigma} \mathcal{L}_{\text{mat}} + \int_{\mathcal{C}_R} \mathcal{L}_{\text{boundary}}, \tag{5.3}$$

where

$$\mathcal{L}_{\text{boundary}} = \frac{1}{16\pi} g_{AB}Q^{AB}. \tag{5.4}$$

Of course, the new action implies the same field equations, because the volume part of (5.1) does not change. However, the quantity $g_{ab}\delta Q^{ab}$ in the last (boundary) term of (5.1) is now replaced on the cylinder $\mathcal{C}_R$ by $g_{00}\delta Q^{00} - Q^{AB}\delta g_{AB}$. Hence, "variation with fixed boundary" of $\mathcal{A}_{\text{tot}}$ means something different than before, for $\mathcal{A}$. Indeed, to kill the boundary term when varying $\mathcal{A}_{\text{tot}}$, the following quantities must be kept fixed on the cylinder $\mathcal{C}_R$: Q^{00} [as before, in (5.1)] and the two-geometry g_{AB} [instead of Q^{AB} as (5.1)]. As will be seen in the sequel, this approach gives us freedom to consider a much less restricted family of field configurations. Consequently, we shall have more Euler-Lagrange equations, one of them providing the missing relation $M = \mathcal{H}(\psi, \dot{\psi})$ between configuration variables $(\psi, \dot{\psi})$ and the total energy (the ADM mass) of the "shell + gravity" system.

For this purpose we consider now in $\mathcal{M}_+$, instead of the external Schwarzschild geometry (3.2), a "Schwarzschild-like" geometry parametrized by an arbitrary function of time $M = M(t)$:

$$ds_+^2 = -\left(1 - \frac{2M(t)}{r}\right)dt^2 + \left(1 - \frac{2M(t)}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{5.5}$$

for $r \ge \psi(t)$. This family of configurations is useless for the previous variational principle because the extrinsic curvature Q^{AB} of $\mathcal{C}D_R$ depends upon the function $M(t)$ and, therefore, does not correspond to the "variation with fixed boundary values" Q^{AB}. Instead, the quantity $r^2 d\Omega^2$, i.e. the angular part g_{AB} of the above metric, is fixed. Miraculously, also the value Q^{00} on $\mathcal{C}_R$, calculated for the above metric, is fixed and equal to the corresponding value for the flat metric $M = 0$. We conclude that the boundary term of $\delta\mathcal{A}_{\rm tot}$ vanishes and we really obtain this way a genuine variation principle for two arbitrary functions $\psi(t)$ and $M(t)$.

It is easy to see that the matching condition (3.7) and the constraint equation (3.14) remain unchanged for the time-depending parameter $M = M(t)$. Moreover, we prove (Appendix C) the following formula:

$$\mathcal{A}_{\rm tot} = \int_{t_1}^{t_2} \mathrm{L}_{\rm tot} dt + F(t_2) - F(t_1) + G(t_2) - G(t_1) + \frac{R}{2}(t_2 - t_1), \tag{5.6}$$

where

$$\mathrm{L}_{\rm tot} = m(\nu)\sqrt{\left(1 - \frac{2M}{\psi}\right) - \frac{\dot\psi^2}{1 - \frac{2M}{\psi}}} + M + \frac{2M\cosh\mu}{\cosh\mu - \sqrt{1 - \frac{2M}{\psi}}} - 2\psi + \psi\dot\psi\mu, \tag{5.7}$$

and

$$G(t) = \dot M\left\{\frac{R^2}{2(1 - \frac{2M}{R})^2} - \frac{\psi^2}{2(1 - \frac{2M}{\psi})^2} + 2M(R - \psi) + 12M^2\ln\left(\frac{2M - R}{2M - \psi}\right) - 2M^2\left(\frac{4}{(1 - \frac{2M}{R})} + \frac{1}{(1 - \frac{2M}{R})^2} - \frac{4}{(1 - \frac{2M}{\psi})} - \frac{1}{(1 - \frac{2M}{R})^2}\right)\right\}. \tag{5.8}$$

We see that the dependence upon the (arbitrarily chosen) radius R enters only in the boundary (both spatial and temporary) terms which may be simply neglected. Since the bulk term (5.7) does not depend upon R, we may pass to the limit $R \to \infty$. As a result we obtain $\mathrm{L}_{\rm tot}$ as the total Lagrangian function of the "shell + gravity" system.

A simple but remarkable feature of this Lagrangian is that it does not depend upon the time derivative $\dot M$, since it enters only into the boundary term (5.8), which is neglected. Hence, variation with respect to the function $M(t)$ leads to an algebraic (instead of differential) equation:

$$\frac{\delta \mathrm{L}_{\rm tot}}{\delta M} = \frac{\partial \mathrm{L}_{\rm tot}}{\partial M} = 0.$$

We show in Appendix D that this equation *can be solved explicitly* with respect to M. As a result we obtain

$$M(\psi, \mu) = \frac{\psi}{2}\left\{1 - \left(\cosh\mu - \sqrt{\frac{m(\frac{\psi^2}{2})^2}{\psi^2} + \sinh^2\mu}\right)^2\right\}. \tag{5.9}$$

Unlike in the previous approach, the total mass M of the system, seen by an observer at infinity, is not fixed *a priori* but is now defined as a function of the configuration variables $\psi, \dot\psi$ [remember that μ is also treated as a function of ψ and $\dot\psi$ defined implicitly by (3.14)]. Substituting (5.9) into (5.7) we express the total Lagrangian only in terms of ψ and $\dot\psi$. The following final result is proved in Appendix D:

$$\mathrm{L}_{\rm tot}(\psi, \dot\psi) = \psi\dot\psi\mu(\psi, \dot\psi) - M(\psi, \mu(\psi, \dot\psi)). \tag{5.10}$$

VI. HAMILTONIAN FORMULATION OF THE DYNAMICS

To obtain the Hamiltonian version of the above model, we first calculate the momentum p_ψ canonically conjugate to the variable ψ. The following result is proved in Appendix D:

$$p_\psi := \frac{d\mathrm{L}_{\rm tot}}{d\dot\psi} = \psi\mu. \tag{6.1}$$

Performing the standard Legendre transformation and using (5.10), we conclude that the numerical value of the Hamiltonian function of the system is equal to its ADM mass at infinity:

$$\mathcal{H}(\psi, p_\psi) := p_\psi\dot\psi - \mathrm{L}_{\rm tot} = M. \tag{6.2}$$

Its conservation $M(t) = \text{const}$ is now implied by the energy conservation in Hamiltonian mechanics. We stress that the total mass conservation is not postulated *a priori* in our approach, but *derived* as a consequence of the dynamics of the total "shell + gravity" system.

The phase space of the theory is thus globally parametrized by the configuration variable ψ and the corresponding canonical momentum p_ψ. We remember that the Lagrangian variables $(\psi, \dot\psi)$ were not global since we have two different values of $\mu = \mu(\psi, \dot\psi)$ satisfying Eq. (3.14). The canonical structure of the phase space may also be described in terms of the symplectic form:

$$\omega = dp_\psi \wedge d\psi = d(\psi\mu) \wedge d\psi = d\mu \wedge d(\tfrac{1}{2}\psi^2) = d\mu \wedge d\nu. \tag{6.3}$$

We see that the hyperbolic angle μ can be interpreted as the momentum canonically conjugate to the proper volume $\nu = \frac{1}{2}\psi^2$ of the shell. Because of (5.9) and (6.2), we may write the Hamiltonian in terms of these canonical variables:

$$\mathcal{H}(\mu, \nu) = \sqrt{\frac{\nu}{2}}\left\{1 - \left(\cosh\mu - \sqrt{\frac{m(\nu)^2}{2\nu} + \sinh^2\mu}\right)^2\right\}. \tag{6.4}$$

For the dust matter $[m(\nu) = m_0]$ the canonical structure (6.3) and the Hamiltonian (6.4) have been already derived in [5] by a different method. Namely, the Hamiltonian structure for a generic "shell + gravity" system (not necessarily spherically symmetric) was first obtained (cf. also [7]); next, this structure was reduced to the spherical symmetry. (The dependence of the Hamiltonian structure upon a specific choice of the time parametrization was later discussed in [8].) A direct derivation of the correct Lagrangian (5.10) for the spherical shell, starting from the standard Hilbert variational principle reduced to the spherical case, however, was never performed before. Moreover, the present result is much more general because it is true for any constitutive equation $m = m(\nu)$ of the fluid and not only for dust.

Formula (6.4) implies that the evolution equations of the system may be written as

$$\dot{\nu} = \frac{\partial \mathcal{H}}{\partial \mu}, \tag{6.5}$$

$$\dot{\mu} = -\frac{\partial \mathcal{H}}{\partial \nu}, \tag{6.6}$$

the first of which being nothing but the constraint equation (3.14), while the second one gives the remaining information about the dynamics of the system.

APPENDIX A: CONSTRUCTION OF REGULAR COORDINATES

Consider internal space $\mathcal{M}_-$ equipped with the flat Minkowski metric (3.1) for $\rho \leq \phi(\tau)$ and the external space $\mathcal{M}_+$ equipped with the "Schwarzschild-like" geometry (5.5) for $r \geq \psi(t)$. It is immediately seen that the compatibility condition for the internal three-metric g_{ab} on $\Sigma = \partial\mathcal{M}_- = -\partial\mathcal{M}_+$ implies that formulas (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), and (3.14) remain valid also for the case of M which is no longer constant in time.

To construct a regular coordinate system on the entire spacetime $\mathcal{M}$, we first introduce in (3.2) a new variable $x = r - \psi(t)$ for which we have $\Sigma = \{x = 0\}$. Under this coordinate transformation, the Schwarzschild-like metric (5.5) assumes the following form:

$$ds_+^2 = \frac{1}{(1 - \frac{2M(t)}{(x+\psi)})}\left\{\left(\dot{\psi}^2 - \left(1 - \frac{2M(t)}{(x+\psi)}\right)^2\right)dt^2 + dx^2 + 2\dot{\psi}dtdx\right\} + (x+\psi)^2 d\Omega^2, \tag{A1}$$

Now, we want to extend coordinates (x^μ), where $x^0 = t$, $x^1 = x$, $x^2 = \vartheta$, and $x^3 = \varphi$, to $\mathcal{M}_-$ in such a way that the metric $g_{\mu\nu}$ is continuous. Because of the spherical symmetry, coordinates (x^A), $A = 2, 3$ may be left unchanged. Hence, our coordinate transformation is described by two functions:

$$\tau = A(t, x), \tag{A2}$$

$$\rho = B(t, x), \tag{A3}$$

where we may choose arbitrarily the value of x corresponding to the center of $\mathcal{M}_-$. The Minkowski metric (3.1) assumes now the following form:

$$ds_-^2 = -(\dot{A}^2 - \dot{B}^2)dt^2 + (B_{,x}^2 - A_{,x}^2)dx^2 + 2(\dot{B}B_{,x} - \dot{A}A_{,x})dxdt + B^2 d\Omega^2. \tag{A4}$$

Continuity of g_{AB} at $x = 0$ implies

$$\phi(\tau) = B(t, 0) = \psi(t). \tag{A5}$$

Denoting

$$A(t, 0) = f(t) \tag{A6}$$

we recover equation $\dot{f}\,\dot{\phi} = \dot{\psi}$. Now, denoting

$$\frac{\partial A}{\partial x}(t, 0) = a(t), \tag{A7}$$

$$\frac{\partial B}{\partial x}(t, 0) = b(t), \tag{A8}$$

the matching conditions for g_{00}, g_{11}, and g_{01} (the coefficients multiplying dt^2, dx^2, and $dxdt$, respectively) between (A1) and (A4) read

$$\dot{\psi}^2 - \dot{f}^2 = \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}} - \left(1 - \frac{2M}{\psi}\right), \tag{A9}$$

$$b^2 - a^2 = \frac{1}{1 - \frac{2M}{\psi}}, \tag{A10}$$

$$\dot{\psi}b - \dot{f}a = \frac{\dot{\psi}}{1 - \frac{2M}{\psi}}, \tag{A11}$$

of which the first one is equivalent to the constraint equation (3.7).

In fact if we obtain a and $\dot{f}$ from (A10) and (A11) and substitute them into (A9) we obtain

$$\left(1 - \frac{2M}{\psi}\right)^2 = \frac{(b-1)^2}{b^2 - \frac{1}{1-[(2M)/\psi]}}\dot{\psi}^2. \tag{A12}$$

Now the hyperbolic angle μ between the constant time slices on the Minkowski side and the constant time slices on the Schwarzschild side can be easily calculated in Minkowskian coordinates as the angle between the vector tangent to both foliations

$$\mu = (\mathrm{sign}\,\dot{\psi})\mathrm{arcosh}\frac{(\frac{\partial}{\partial r}|\frac{\partial}{\partial \rho})}{\|\frac{\partial}{\partial r}\|\|\frac{\partial}{\partial \rho}\|}. \tag{A13}$$

But $\frac{\partial}{\partial r} = \frac{\partial \tau}{\partial r}\frac{\partial}{\partial \tau} + \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = a\frac{\partial}{\partial \tau} + b\frac{\partial}{\partial \rho}$, evaluated on shell in Minkowskian coordinates. Now $(\frac{\partial}{\partial r}|\frac{\partial}{\partial \rho}) = b$, $\|\frac{\partial}{\partial \rho}\| = 1$, and $\|\frac{\partial}{\partial r}\| = \sqrt{b^2 - a^2} = \frac{1}{\sqrt{1-[(2M)/\psi]}}$ so that

$$\mu = (\mathrm{sign}\,\dot{\psi})\mathrm{arcosh}\left(\sqrt{1 - \frac{2M}{\psi}}b\right). \tag{A14}$$

Once μ is given as a solution (3.14), then the values of $\dot{f}$, a and b are uniquely determined from Eqs. (A10), (A11), and (A14). Now every pair of functions A and B satisfying (A7) and (A8) may be chosen as regular coordinates (t, x) on the Minkowski side of the shell.

APPENDIX B: GRAVITATIONAL LAGRANGIAN FOR $M = \text{const}$

To calculate the second fundamental form of $\Sigma \subset \mathcal{M}_+$, we use the new variable $x = r - \psi(t)$ for which the shell is placed at $x = 0$. Consequently, the Schwarzschild metric reduces to (A1) with $M = \text{const}$. It has, therefore, the following form:

$$ds^2 = g_{tt}(t, x)dt^2 + g_{xt}(t, x)dtdx + g_{xx}(t, x)dx^2 + g_{AB}(t, x)dx^A dx^B, \tag{B1}$$

with $A, B = \vartheta, \varphi$ and the extrinsic curvature of Σ can be written as

$$K_{ab} = -\frac{1}{\sqrt{g^{xx}}}\Gamma^x_{ab}, \tag{B2}$$

with $a, b = t, \vartheta, \varphi$ so that the ADM momentum of K_{ab} is

$$Q^{ab} = \sqrt{|\det g_{cd}|}(\hat{g}^{ab}K - K^{ab}), \tag{B3}$$

where $\hat{g}^{ab} = g^{ab} - \frac{g^{xa}g^{xb}}{g^{xx}}$ is the three-dimensional inverse of the induced metric g_{ab} on the shell and $K^{ab} = \hat{g}^{ac}\hat{g}^{bd}K_{cd}$. So we obtain

$$Q = Q^{ab}g_{ab} = -2\sqrt{|\det g_{cd}|}\hat{g}^{ab}K_{ab} = 2\sqrt{|\det g_{\mu\nu}|}\hat{g}^{ab}\Gamma^x_{ab}. \tag{B4}$$

Plugging (B1) into (B4), we obtain

$$Q = \frac{1}{\sqrt{|J|}}\{(g_{\vartheta\vartheta}g_{tt,x} + 2g_{tt}g_{\vartheta\vartheta,x}) + -(g_{\vartheta\vartheta}g_{tx,t} + 2g_{tx}g_{\vartheta\vartheta,t})\}\sin\theta, \tag{B5}$$

with $J = g_{tt}g_{xx} - g_{tx}^2$. It is easy to check that $g_{\vartheta\vartheta}g_{tx,t}$ and $g_{tx}g_{\vartheta\vartheta,t}$ are the same both on the Minkowski and the Schwarzschild side so that we have $[g_{\vartheta\vartheta}g_{tx,t}] = [g_{tx}g_{\vartheta\vartheta,t}] = 0$. Consequently, the jump of Q reduces to

$$[Q] = \left[\frac{1}{\sqrt{|J|}}(g_{\vartheta\vartheta}g_{tt,x} + 2g_{tt}g_{\vartheta\vartheta,x})\right]\sin\theta. \tag{B6}$$

But, on the Schwarzschild side we have

$$g_{\theta\theta}\partial_x g_{tt} = -2M\left(1 + \frac{\dot{\psi}^2}{(1 - \frac{2M}{\psi})^2}\right), \tag{B7}$$

$$g_{tt}\partial_x g_{\theta\theta} = \frac{2\psi}{(1 - \frac{2M}{\psi})}\left(\dot{\psi}^2 - \left(1 - \frac{2M}{\psi}\right)^2\right), \tag{B8}$$

and $J = -1$. On the Minkowski side we take the metric given by (A4) and obtain

$$g_{\theta\theta}\partial_x g_{tt} = 2\psi^2\frac{\dot{b}}{a} - \frac{2M(b - \frac{1}{(1-\frac{2M}{\psi})})}{(b - 1)^2}, \tag{B9}$$

$$g_{tt}\partial_x g_{\theta\theta} = \frac{2b\psi}{(1 - \frac{2M}{\psi})}\left(\dot{\psi}^2 - \left(1 - \frac{2M}{\psi}\right)^2\right), \tag{B10}$$

and $J = -(\dot{B}A_{,x} - \dot{A}B_{,x})^2 = -(a\dot{\psi} - b\dot{f})^2 = -1$. The jump of Q across Σ is then easily evaluated as

$$[Q] = 2\sin\theta\left(\psi^2\frac{\dot{b}}{a} - 3M\frac{2b - 1}{b - 1} + 4\psi\right). \tag{B11}$$

Consider

$$\cosh\mu = \sqrt{1 - \frac{2M}{\psi}}b, \tag{B12}$$

for which

$$-\dot{\mu}\sinh\mu = \sqrt{1 - \frac{2M}{\psi}}\dot{b} + \frac{M}{\psi^2}\frac{\dot{\psi}}{(1 - \frac{2M}{\psi})}\cosh\mu. \tag{B13}$$

From (A10) we have

$$a = \sqrt{b^2 - \frac{1}{(1 - \frac{2M}{\psi})}} = \frac{\sinh\mu}{\sqrt{1 - \frac{2M}{\psi}}}. \tag{B14}$$

Hence, from (B11), we get

$$[Q] = 2\sin\theta\left(-\psi^2\dot{\mu} - \frac{4M\cosh\mu}{\cosh\mu - \sqrt{1 - \frac{2M}{\psi}}} - 3M + 4\psi\right). \tag{B15}$$

The gravitational part of the Lagrangian is given only by its singular part:

$$\mathcal{A}^{\text{sing}}_{\text{grav}}|_M = -\frac{1}{16\pi}\int_{D\cap\Sigma}[Q]dtd\vartheta d\varphi. \tag{B16}$$

So that integrating by parts we finally obtain (4.4).

APPENDIX C: GRAVITATIONAL LAGRANGIAN FOR $M(t)$

As seen in Sec. V, the total Hilbert action is given by

$$\mathcal{A}_{\text{tot}} = \int_{D_R}\mathcal{L}_{\text{grav}} + \int_{D_R\cap\Sigma}\mathcal{L}_{\text{mat}} + \int_{\mathcal{C}_R}\mathcal{L}_{\text{boundary}}. \tag{C1}$$

The gravitational Lagrangian is now divided into its singular part, given once again by $[Q]$, and the regular part which in this case is not vanishing:

$$\int_{D_R} \mathcal{L}_{\mathrm{grav}} = \int_{D_R\cap\Sigma} \mathcal{L}^{\mathrm{sing}}_{\mathrm{grav}} + \int_{D_R} \mathcal{L}^{\mathrm{reg}}_{\mathrm{grav}} = \int_{D_R\cap\Sigma} [Q] + \int_{D_R} \sqrt{\det g}\,\mathbf{R}^{\mathrm{reg}}. \tag{C2}$$

Evaluation of $[Q]$ is analogous to the case $M = \text{const}$ and the only term which shows dependence on $\dot{M}$ is (B9)

$$g_{\theta\theta}\partial_x g_{tt} = 2\psi^2\frac{\dot{b}}{a} - \frac{2M}{(b-1)^2}\Big(b - \frac{1}{(1-\frac{2M}{\psi})}\Big) + 2\frac{\dot{M}\psi}{(1-\frac{2M}{\psi})}\frac{(b - \frac{1}{[1-(2M)/\psi]})}{a(b-1)}, \tag{C3}$$

so that

$$[Q] = 2\sin\theta\Big(\psi^2\frac{\dot{b}}{a} - 3M\frac{2b-1}{b-1} + 4\psi + \frac{\dot{M}\psi}{(1-\frac{2M}{\psi})^2}\frac{b(1-\frac{2M}{\psi})-1}{a(b-1)}\Big). \tag{C4}$$

Now substituting again $\mu = \operatorname{arcosh}\sqrt{1-\frac{2M}{\psi}}b$ we get

$$\mathcal{A}^{\mathrm{sing}}_{\mathrm{grav}} = \int_{D_R\cap\Sigma} \mathcal{L}^{\mathrm{sing}}_{\mathrm{grav}} = \mathcal{A}^{\mathrm{sing}}_{\mathrm{grav}}|_M - \frac{1}{2}\int_{t_1}^{t_2} \frac{\dot{M}\,\dot{\psi}\,\psi}{(1-\frac{2M}{\psi})^2}dt. \tag{C5}$$

For the regular part of the gravitational Lagrangian we get

$$\mathbf{R}^{\mathrm{reg}} = -\frac{2}{(1-\frac{2M(t)}{r})^3}\Big(4\frac{\dot{M}(t)^2}{r^2} + \Big(1-\frac{2M(t)}{r}\Big)\frac{\ddot{M}(t)}{r}\Big), \tag{C6}$$

so that

$$\int_{D_R}\sqrt{\det g}\,\mathbf{R}^{\mathrm{reg}} = \int_{D_R} r^2\sin\theta\,\mathbf{R}^{\mathrm{reg}}\,dt\,dr\,d\theta\,d\varphi = 4\pi\int_{t_1}^{t_2}dt\int_{\psi(t)}^{R} r^2\mathbf{R}^{\mathrm{reg}}dr, \tag{C7}$$

and so

$$\mathcal{A}^{\mathrm{reg}}_{\mathrm{grav}} = -\frac{1}{16\pi}\int_{D_R}\sqrt{\det g}\,\mathbf{R}^{\mathrm{reg}} = \frac{1}{2}\int_{t_1}^{t_2}\frac{\dot{M}\,\dot{\psi}\,\psi}{(1-\frac{2M}{\psi})^2}dt - [G(t)]_{t_1}^{t_2}, \tag{C8}$$

where $G(t)$ is given by (5.8). Finally, evaluating

$$g^{AB}Q_{AB} = -2R^2\sin\theta(\Gamma^r_{tt}g^{tt} + \Gamma^r_{\theta\theta}g^{\theta\theta}) = 2\sin\theta(R - M(t)), \tag{C9}$$

we see that the boundary contribution to the action equals

$$\int_{\mathcal{C}_R}\mathcal{L}_{\mathrm{boundary}} = \frac{1}{2}\int_{t_1}^{t_2}(R - M(t))dt = \frac{R}{2}(t_2-t_1) - \frac{1}{2}\int_{t_1}^{t_2}M(t)dt. \tag{C10}$$

Adding together different contributions to the Hilbert action we get Eq. (5.6).

APPENDIX D: MOMENTUM CANONICALLY CONJUGATE TO ψ

Considering μ as an implicit function of $(\psi, \dot{\psi}, M, \dot{M})$ given by the constraint Eq. (3.14), we have

$$\frac{\partial\mu}{\partial M} = \frac{1}{\psi}\frac{\sinh\mu(2\cosh\mu - \sqrt{1-\frac{2M}{\psi}})}{(1-\frac{2M}{\psi})(1-\sqrt{1-\frac{2M}{\psi}}\cosh\mu)}, \tag{D1}$$

so that the equation of motion $\frac{\partial\mathcal{L}_{\mathrm{tot}}}{\partial M} = 0$ gives

$$\frac{\frac{m(\nu)}{\psi}(1+\frac{\dot{\psi}^2}{\{1-[(2M)/\psi]\}^2})}{\sqrt{1-\frac{2M}{\psi}}\sqrt{1-\frac{\dot{\psi}^2}{\{1-[(2M)/\psi]\}^2}}} = -\frac{\cosh\mu - \sqrt{1-\frac{2M}{\psi}}}{\sqrt{1-\frac{2M}{\psi}}} \times\Big(1+\frac{\dot{\psi}^2}{(1-\frac{2M}{\psi})^2}\Big). \tag{D2}$$

But we have

$$\sqrt{1-\frac{\dot{\psi}^2}{(1-\frac{2M}{\psi})^2}} = \frac{\sqrt{1-2\cosh\mu\sqrt{1-\frac{2M}{\psi}}+1-\frac{2M}{\psi}}}{\cosh\mu - \sqrt{1-\frac{2M}{\psi}}}, \tag{D3}$$

whence

$$-\frac{m(\nu)}{\psi} = \sqrt{1-2\cosh\mu\sqrt{1-\frac{2M}{\psi}}+1-\frac{2M}{\psi}}, \tag{D4}$$

from which (5.9) easily follows. Equation (5.10) is then obtained substituting (D3) and (D4) into Eq. (5.7).

Finally Eq. (6.1) is simply obtained from (5.7) noting that

$$p_\psi = \frac{\partial \mathrm{L}_{\mathrm{tot}}}{\partial\dot{\psi}} + \Big(\frac{\partial \mathrm{L}_{\mathrm{tot}}}{\partial\mu} + \frac{\partial \mathrm{L}_{\mathrm{tot}}}{\partial M}\frac{\partial M}{\partial\mu}\Big)\frac{\partial\mu}{\partial\dot{\psi}}, \tag{D5}$$

with $\frac{\partial \mathrm{L}_{\mathrm{tot}}}{\partial\mu} = \psi\dot{\psi}$, $\frac{\partial \mathrm{L}_{\mathrm{tot}}}{\partial\dot{\psi}} = \psi\mu$, $\frac{\partial \mathrm{L}_{\mathrm{tot}}}{\partial M} = -1$, and $\frac{\partial M}{\partial\mu} = \psi\dot{\psi}$.

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