

**Fivebranes from gauge theory**Hai Lin<sup>1</sup> and Juan Maldacena<sup>2</sup><sup>1</sup>*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*<sup>2</sup>*Institute for Advanced Study, Princeton, New Jersey 08540, USA*

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We study theories with 16 supercharges and a discrete energy spectrum. One class of theories has symmetry group  $SU(2|4)$ . They arise as truncations of  $\mathcal{N} = 4$  super Yang-Mills theory. They include the plane-wave matrix model,  $2 + 1$  super Yang-Mills theory on  $R \times S^2$  and  $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3/Z_k$ . We explain how to obtain their gravity duals in a unified way. We explore the regions of the geometry that are relevant for the study of some  $1/2$  BPS and near BPS states. This leads to a class of two dimensional  $(4,4)$  supersymmetric sigma models with nonzero  $H$  flux, including a massive deformed WZW model. We show how to match some features of the string spectrum with the Yang-Mills theory. The other class of theories are also connected to  $\mathcal{N} = 4$  super Yang-Mills theory and arise by making some of the transverse scalars compact. Their vacua are characterized by a 2D Yang-Mills theory or 3D Chern-Simons theory. These theories realize peculiar super-Poincaré symmetry algebras in  $2 + 1$  or  $1 + 1$  dimensions with “noncentral” charges. We finally discuss gravity duals of  $\mathcal{N} = 4$  super Yang-Mills theory on  $AdS_3 \times S^1$ .

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**I. INTRODUCTION**

In this paper we study an interconnected family of theories with 16 supercharges. All these theories share the common feature that they have a mass gap and a discrete spectrum of excitations. In most examples we have a dimensionless parameter which allows us to interpolate between weak and strong coupling. In the weakly coupled description we have a gauge theory. These theories have many vacua. We describe smooth gravity solutions corresponding to all these vacua. For some particular vacua we study the 't Hooft limit and we examine the properties of strings at large 't Hooft coupling.

In the first part of this paper we study theories with 16 supercharges whose symmetry algebra is an  $SU(2|4)$  supergroup. These theories are closely related to each other. Their BPS states can be conveniently studied by a Witten index [1,2]. The first example is the plane-wave matrix model [3]. The second example is  $2 + 1$  SYM on  $R \times S^2$  [4] and a third example is  $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3/Z_k$ . We construct their gravity duals. We give a general method for constructing the gravity solutions, and provide a few explicit solutions.

The plane-wave matrix model is a nice example of the gauge theory/gravity correspondence because it is an ordinary quantum mechanical system with a discrete energy spectrum. The theory has a large number of vacua, and a correspondingly large number of gravity solutions. Strictly speaking we can trust the gravity approximation only for a suitable subset of solutions. The generic solution, though formally smooth, has curvature of the order of the planck or string scale. In the 't Hooft large  $N$  limit we can focus on just one of these vacua at a time and ignore the tunneling to other vacua. The properties of single trace states (or single

string states) depend on the vacuum we are expanding around. All the theories in this family have an  $SO(6)$  symmetry. It is possible to consider half BPS states which carry  $SO(6)$  angular momentum  $J$ . We can count these BPS states precisely in each of these theories. In addition, we can consider near BPS states. Their description in the weakly coupled regime is similar to the one in four dimensional  $\mathcal{N} = 4$  super Yang-Mills theory and was studied in [5–10]. At large 't Hooft coupling, the spectrum of large charge near BPS states is obtained by considering pp-wave limits of the general solutions. In the simplest case we find a IIA plane wave [11,12]. In general, strings in light cone gauge are described by a massive field theory on the world sheet with  $(4,4)$  supersymmetry. The details of this theory depend on the vacuum we are expanding around. We study vacua associated to NS5 branes [4]. In these vacua we are led to strings propagating in the near horizon geometry of  $N_5$  fivebranes [13]. The field theory on the string is given by a massive deformation of the WZW model and linear dilaton theory that describes the near horizon region of NS5 branes. Depending on the value of  $N_5$  we get a different spectrum. We match some qualitative features of this spectrum with the weakly coupled gauge theory description. The energies of near BPS states have a nontrivial dependence on the 't Hooft coupling. So we expect a nontrivial interpolation between the weak and strong coupling results. In fact, for the plane-wave matrix model, at weak coupling, this interpolating function was computed to four loops in [10]. We show that this function has a physical interpretation in the strong coupling regime as the radius of a fivesphere in the geometry.

In the second part of our paper we consider field theories that have 16 supercharges and  $SO(4) \times SO(4)$  symmetry. These solutions are described by droplets of an incom-

pressible fluid as in [14]. When this fluid lives in an infinite two dimensional plane we find the gravity solutions corresponding to the 1/2 BPS states of  $\mathcal{N} = 4$  super Yang-Mills theory [14–16]. In this paper we discuss mainly the case where this fluid lives on a two torus. In addition we discuss the case of the cylinder. These are again theories that have 16 supercharges and many vacua. An interesting aspect is that these theories have Poincaré supersymmetry algebras in  $2 + 1$  or  $1 + 1$  dimensions which are such that the charges appearing on the right hand side are not central, a situation that cannot arise for Poincaré superalgebras in more than three dimensions [17,18]. This algebra was mentioned in the general classification in [19]. This is also the symmetry algebra that is linearly realized in the light cone gauge description of strings moving in the maximally supersymmetric IIB plane wave [20]. The theory associated to fermions on a torus gives rise to  $U(N)_K$  or  $U(K)_N$  Chern-Simons theory on  $R \times T^2$  in the IR. The full theory has explicit duality under  $K \leftrightarrow N$ .

We also discuss another family of smooth solutions that are obtained by doing an analytic continuation of the ansatz in [14]. The boundary conditions are different in this case. These solutions are associated to a certain Coulomb branch of the  $\mathcal{N} = 4$  super Yang-Mills theory on  $\text{AdS}_3 \times S^1$ .

This paper is organized as follows. In Sec. II we discuss theories with 16 supercharges and  $SU(2|4)$  symmetry group. We start by discussing various field theories and then proceed to write the gravity description for all of these examples. We also take the large  $J$  limit and analyze features of the BPS and near BPS spectrum of single trace (or single string) states. In Sec. III we discuss some features of theories with 16 supercharges that are obtained from free fermions on a  $T^2$  and also by analytic continuation of some of the formulas in [14]. Finally, various appendices give more details about some of the results.

## II. THEORIES WITH 16 SUPERCHARGES AND $SU(2|4)$ SYMMETRY GROUP

### A. The field theories

In this subsection we discuss various field theories with  $\widetilde{SU}(2|4)$  symmetry group.

It is convenient to start with  $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3$ . This theory is dual to  $\text{AdS}_5 \times S^5$  and its symmetry group is the superconformal group  $SU(2, 2|4)$ . The bosonic subgroup of the superconformal group is  $SO(2, 4) \times SO(6)$ . It is convenient to focus on an  $SU(2)_L \subset SO(4) \subset SO(2, 4)$ . This  $SU(2)_L$  is embedded in the  $SO(4)$  symmetry group that rotates the  $S^3$  on which the field theory is defined. If we take the full superconformal algebra and we truncate it to the subset that is invariant under  $SU(2)_L$  we clearly get a new algebra. This algebra forms the supergroup  $\widetilde{SU}(2|4)$ , where the tilde here denotes that we take its universal cover. In other words, the bosonic

subgroup is  $R \times SU(2) \times SU(4)$ .<sup>1</sup> This is the symmetry group of the theories we are going to consider below.<sup>2</sup>

We will get the theories of interest by quotienting  $\mathcal{N} = 4$  super Yang-Mills theory by various subgroups of  $SU(2)_L$ . For example, if we quotient by the whole  $SU(2)_L$  group we are left with the plane-wave matrix model [9]. We get a reduction to  $0 + 1$  dimensions because all Kaluza Klein modes on  $S^3$  carry  $SU(2)_L$  quantum numbers except for the lowest ones. The other theories are obtained by quotienting by  $Z_k$  and  $U(1)_L$  subgroups of  $SU(2)_L$ . We will discuss these theories in detail below.<sup>3</sup>

Since all theories have a common symmetry group they share some properties. One property that we will discuss in some detail are 1/2 BPS states carrying  $SO(6)$  angular momenta. These are states carrying energy  $E$  equal to the angular momentum under an  $SO(2) \subset SO(6)$  generated by  $J$ . The condition  $E = J$  is the BPS bound. The fact that these 1/2 BPS states are fully protected follows from the discussion in [7,8]. Moreover, the arguments in [7,8] allow us to count precisely these BPS states. Actually, to study BPS states it is convenient to define the index [2]

$$I(\beta_i) = \text{Tr} [ (-1)^F e^{-\mu(E-2S-J_1-J_2-J_3)} e^{-\beta_1(E-J_1)} e^{-\beta_2(E-J_2)} \times e^{-\beta_3(E-J_3)} ], \quad (2.1)$$

where  $S = S_3$  is one of the generators of  $SU(2)$ ,  $J_1 = M_{12}$ ,  $J_2 = M_{34}$  and  $J_3 = M_{56}$  are  $SO(6)$  Cartan generators. Let us explain why (2.1) is an index. Let us consider the supercharge  $Q^\dagger = Q_{-+++}^\dagger$ , where the indices indicate the charges under  $(S, J_1, J_2, J_3)$ . This supercharge has  $E = 1/2$ . This supercharge and its adjoint obey the anticommutation relation

$$\{Q, Q^\dagger\} = \mathcal{U} \equiv E - 2S - (J_1 + J_2 + J_3). \quad (2.2)$$

In addition the combinations  $E - J_i$  commute with the supercharges in (2.2). Using the standard arguments (see [1]) any state with nonzero values of  $\mathcal{U}$  does not contribute to (2.3). By evaluating (2.1) we will be able to find which BPS representations should remain as we change the coupling. The index (2.1) contains the same information as the indices defined in [7], see [2]. For further discussion see Appendix G. In order to count 1/2 BPS states we can use a simplified version of (2.1) obtained by taking the limit when  $\beta_3 \rightarrow \infty$ . In this limit the index depends only on  $q \equiv e^{-\beta_1 - \beta_2}$

<sup>1</sup>If we replace  $R$  by  $U(1)$  we get the compact form of  $SU(2|4)$ .

<sup>2</sup>This symmetry group also appears when we consider 1/2 BPS states in  $\text{AdS}_{4,7} \times S^{7,4}$   $M$ -theory solutions [14]. A closely related supergroup,  $SU(2, 2|2)$ , is the  $\mathcal{N} = 2$  superconformal group in 4 dimensions.

<sup>3</sup>Notice that this truncation procedure is a convenient way to construct the Lagrangian, but we cannot get the full quantum spectrum of the plane-wave matrix model by restricting to  $SU(2)_L$  invariant states of the full  $\mathcal{N} = 4$  super Yang-Mills theory.

$$I_N(q) \equiv \sum_{J=0}^{\infty} D(N, J) q^J = \lim_{\beta_3 \rightarrow +\infty} I(\beta_1, \beta_2, \beta_3), \tag{2.3}$$

$$q = e^{-\beta_1 - \beta_2}$$

where  $J = J_3$ . This partition function counts the number of 1/2 BPS states  $D(N, J)$  in the system. Below we will compute (2.3) for various theories. We will not compute (2.1) in this paper, but it could be computed using the techniques in [21].

In the limit that  $1 \ll J \ll N$  we will identify these states as massless geodesics in the geometric description. Notice that, even though we use some of the techniques in [14] to describe the *vacua* of these theories, we do not include backreaction when we consider 1/2 BPS states.<sup>4</sup> We are also going to study the near BPS limit, with  $J$  large and  $\hat{E} = E - J$  finite. For excitations along the  $S^5$  the one loop perturbative correction is the same as in the  $\mathcal{N} = 4$  parent Yang-Mills theory. On the gravity side, we will find that, at strong 't Hooft coupling, the result that differs from the naive extrapolation of the weak coupling results. This implies that there exist some interpolating functions in the sectors. We could similarly study other solutions with large quantum numbers under  $SO(6)$ , such as the configurations considered in e.g. [22,23] (see [24] for a review) which have several large quantum numbers. In this theory we could also have BPS and near BPS configurations with large  $SO(3)$  spin, which we will not study in this paper.

In these theories we have many vacua and, in principle, we can tunnel among the different vacua. In most of the discussion we will assume that we are in a regime in parameter space where we can neglect the effects of tunneling. This tunneling is suppressed in the 't Hooft regime where strings are weakly coupled. Note that despite tunneling the vacua remain degenerate since they all contribute positively to the index (2.3).

We will now discuss in more detail each theory individually.

**1.  $\mathcal{N} = 4$  SYM on  $R \times S^3/Z_k$**

Here we consider  $U(N)$   $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3/Z_k$ , with  $Z_k \subset SU(2)_L$ , and  $SU(2)_L$  as defined above (see also [25] for a more general discussion). We can also obtain this theory by starting with the free field content of  $\mathcal{N} = 4$ , projecting out all fields which are not invariant under  $Z_k$  and then considering the same interactions for the remaining fields as the ones we had for  $\mathcal{N} =$

4. Notice that we first project the elementary fields and we then quantize, which is not the same as retaining the invariant states of the original full quantum  $\mathcal{N} = 4$  theory. This is the standard procedure. The symmetry group of this theory is  $\widetilde{SU}(2|4)$ .

This theory is parametrized by  $N, k$ , and the original Yang-Mills coupling  $g_{YM}^2$ . Whereas  $\mathcal{N} = 4$  SYM on  $S^3$  has a unique vacuum, the theory on  $S^3/Z_k$  has many supersymmetric vacua. Let us analyze the vacua at weak coupling. Since all excitations are massive we can neglect all fields except for a Wilson line of the gauge field. More precisely, the vacua are given by the space of flat connections on  $S^3/Z_k$ . This space is parametrized by giving the holonomy of the gauge field  $U$  along the nontrivial generator of  $\pi_1(S^3/Z_k) = Z_k$ , up to gauge transformations. We can therefore diagonalize  $U$ , with  $U^k = 1$ . So the diagonal elements are  $k$ th roots of unity. Inequivalent elements are given by specifying how many roots of each kind we have. So the vacuum is specified by giving the  $k$  numbers  $n_1, n_2, \dots, n_k$ , with  $N = \sum_{l=1}^k n_l$ . Where  $n_l$  specifies how many times  $e^{i2\pi(l/k)}$  appears in the diagonal of  $U$ . We can also view these different vacua as arising from orbifolding the theory of  $D$ -branes on  $S^3 \times R$  and applying the rules in [26] with different choices for the embedding of the  $Z_k$  into the gauge group. The regular representation corresponds to  $n_l = N/k$  for all  $l$ , and we need to take  $N$  to be a multiple of  $k$ .

The total number of vacua is then

$$D(N, k) = \frac{(N + k - 1)!}{(k - 1)!N!}. \tag{2.4}$$

It is also interesting to count the total number of 1/2 BPS states with charge  $J$  under one of the  $SO(6)$  generators. These numbers are encoded conveniently in the partition function

$$I_{S^3/Z_k}(p, q) = \sum_{N=0}^{\infty} p^N I_N(q) = \sum_{N, J=0}^{\infty} D_{S^3/Z_k}(N, J) p^N q^J$$

$$= [I_{\mathcal{N}=4}(p, q)]^k = \frac{1}{\prod_{n=0}^{\infty} (1 - pq^n)^k}, \tag{2.5}$$

where  $I_{\mathcal{N}=4}(p, q)$  is the index for  $\mathcal{N} = 4$  super Yang-Mills theory. As an aside, note that the degeneracy of states in  $\mathcal{N} = 4$  super Yang-Mills theory can be written in various equivalent forms [16,27]<sup>5</sup>

$$\sum_{N, J=0}^{\infty} p^N q^{N^2/2} q^J D(N, J) = \prod_{n=1}^{\infty} (1 + pq^{n-(1/2)}), \tag{2.6}$$

$$\sum_{J=0}^{\infty} D(N, J) q^J = \frac{1}{\prod_{n=1}^N (1 - q^n)}, \tag{2.7}$$

<sup>4</sup>The 1/2 BPS states of the theories considered in this paper preserve less supersymmetry than the 1/2 BPS states that were considered in [14]. In other words, the 1/2 BPS states of [14] preserve the same amount of supersymmetry as the *vacua* (which have  $J = 0$ ) of the theories considered in this paper. Here we start with theories with 16 supercharges, while [14] started with theories with 32 supercharges.

<sup>5</sup>We have not seen the last equality (2.8) in recent papers, but it must be well known.

$$I_{\mathcal{N}=4}(p, q) \equiv \sum_{N, J=0}^{\infty} p^N q^J D(N, J) = \frac{1}{\prod_{n=0}^{\infty} (1 - pq^n)}. \quad (2.8)$$

In the first form we express it as a system of fermions in a harmonic oscillator potential. In the third form it looks like a system of bosons in a harmonic oscillator potential. In writing (2.5) we used the last representation in (2.8).

We see that even though we counted the vacua (2.4) at weak coupling, the result is still valid at strong coupling since they all contribute to the Witten index. In fact, setting  $q = 0$  in (2.5) we recover (2.4).

### 2. 2 + 1 SYM on $R \times S^2$

This field theory is constructed as follows. We start with  $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3$  and we truncate the free field theory spectrum to states that are invariant under  $U(1)_L \subset SU(2)_L$ , where  $SU(2)_L$  is one of the  $SU(2)$  factors in the  $SO(4)$  rotation group of the  $S^3$ . This results in a theory that lives in one less dimension. It is a theory living on  $R \times S^2$ . This theory was already considered in [4] by considering the fuzzy sphere vacuum of the plane-wave matrix model and then taking a large  $N$  limit that removed the fuzzyness and produced the theory on the ordinary sphere. Here we reproduce it as a  $U(1)_L$  truncation from  $\mathcal{N} = 4$  super Yang-Mills theory<sup>6</sup>

$$\begin{aligned} S = & \frac{1}{g_{\text{YM}2}^2} \int dt \frac{d^2\Omega}{\mu^2} \text{tr} \left( -\frac{1}{4} F^{mn} F_{mn} - \frac{1}{2} (D_m X^a)^2 \right. \\ & - \frac{1}{2} (D_m \Phi)^2 + \frac{i}{2} \bar{\Psi} \Gamma^m D_m \Psi + \frac{1}{2} \bar{\Psi} \Gamma^a [X^a, \Psi] \\ & + \frac{1}{2} \bar{\Psi} \Gamma^\Phi [\Phi, \Psi] + \frac{1}{4} [X_a, X_b]^2 + \frac{1}{2} [\Phi, X^a]^2 \\ & \left. - \frac{\mu^2}{8} X_a^2 - \frac{\mu^2}{2} \Phi^2 - \frac{3i\mu}{8} \bar{\Psi} \Gamma^{012\Phi} \Psi - \mu \Phi dt \wedge F \right) \end{aligned} \quad (2.9)$$

where  $m = 0, 1, 2$ ,  $a = 4, \dots, 9$  and  $(\Gamma^m, \Gamma^\Phi, \Gamma^a)$  are ten dimensional gamma matrices. We see that out of the seven transverse scalars of the maximally supersymmetric Yang-Mills theory we select one of them,  $\Phi$ , which we treat differently than the others. This breaks the  $SO(7)$  symmetry to  $SO(6)$  while still preserving 16 supercharges. The radius of  $S^2$  has size  $\mu^{-1}$  and we have used the two dimensional metric with this radius to raise and lower the indices in (2.9). For our purposes it is convenient to set  $\mu = 2$ , since this is the value we obtain by doing the  $U(1)_L$

<sup>6</sup>We write the metric of  $R \times S^3$  as  $ds^2 = -dt^2 + \frac{1}{4}[d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2]$ , where  $\theta \in [0, \pi]$ ,  $\phi \in [0, 2\pi]$ ,  $\psi \in [0, 4\pi]$ . We neglect the  $\psi$  dependence of all fields and we write the gauge field in  $N = 4$  SYM as  $A_{\mathcal{N}=4} = A + \Phi(d\psi + \cos\theta d\phi)$ , where  $A$  is the 2 + 1 dimensional gauge field.

truncation of  $\mathcal{N} = 4$  super Yang-Mills theory on an  $S^3$  of radius one.

The vacua are obtained by considering zero energy states. We write the field strength along the directions of the sphere as  $F = f d^2\Omega$ . We then see that  $\Phi$  and  $f$  combine into a perfect square in the Lagrangian

$$-\frac{1}{2}(f + \mu\Phi)^2. \quad (2.10)$$

For zero energy vacua this should be set to zero. Since the values of  $f$  are quantized, so are the values of the  $\Phi$  field at these vacua. We can first diagonalize  $\Phi$  and then we can see that its entries are integer valued. So a vacuum is characterized by giving the value of  $N$  integers  $n_1, \dots, n_N$ . The number of vacua is infinite, so we will not write an index. Nevertheless we will see that the gravity solutions reflect the existence of these vacua.

The dimensionless parameters characterizing this theory are  $N$  and the value of the 't Hooft coupling at the scale of the two sphere  $g_{\text{eff}}^2 N \equiv \frac{2\pi g_{\text{YM}2}^2 N}{\mu}$ , where  $\mu^{-1}$  is the size of the sphere. The size of the sphere is a dimensionful parameter which just sets the overall energy scale. We set  $\mu = 2$ , so that the energy of BPS states with angular momentum  $J$  in  $SO(6)$  is equal to  $E = J$ .

Notice that the large  $k$  limit of the theory analyzed in Sec. II A 1 gives us the theory analyzed here. The values of  $N$  are the same and

$$g_{\text{eff}}^2 N = \frac{2\pi g_{\text{YM}2}^2 N}{\mu} = g_{\text{YM}3}^2 N k, \quad (2.11)$$

where  $g_{\text{YM}3}$  is the Yang-Mills coupling in the original  $\mathcal{N} = 4$  theory in Sec. II A 1. So we see that the limit involves taking  $k \rightarrow \infty$ ,  $g_{\text{YM}3}^2 \rightarrow 0$  while keeping  $g_{\text{YM}2}^2$  fixed.

If one takes the strong coupling limit of this theory, by taking  $g_{\text{YM}2} \rightarrow \infty$ , we expect to get the theory living on M2 branes on  $R \times S^2$ . This theory has 32 supersymmetries and is the familiar theory associated with  $\text{AdS}_4 \times S^7$ . In this limit we find that the theory has full  $SO(8)$  symmetry. When we perform this limit we find that the energy  $E$  of the theory in this section goes over to  $\Delta - \bar{J}$ , where  $\Delta$  is the ordinary Hamiltonian for the M2 brane theory on  $R \times S^2$  and  $\bar{J}$  is the  $SO(2)$  generator in  $SO(8)$  which commutes with the  $SO(6)$  that is explicitly preserved by (2.9). For a single brane, the  $N = 1$  case, this can be seen explicitly by dualizing the gauge field strength into an eighth scalar. Then the vacua described around (2.10) are related to the 1/2 BPS states of the M2 brane theory. These should not be confused with the 1/2 BPS states of the 2 + 1 dimensional theory (2.9) which would be related to 1/4 BPS states from the M2 point of view.

### 3. Plane-wave matrix model

Finally we will discuss the plane-wave matrix model e.g. [3–10]. This arises by truncating the  $\mathcal{N} = 4$  theory to



$0 + 1$  dimensions by keeping all free field theory states that are invariant under  $SU(2)_L$  and keeping the same interactions for these states that we had in  $\mathcal{N} = 4$  super Yang-Mills theory [9]. We keep the zero modes for  $SO(6)$  scalars and truncate the gauge field to  $A_{\mathcal{N}=4} = X_1\omega_1 + X_2\omega_2 + X_3\omega_3$ , where  $\omega_i$  are three left invariant one-forms on  $S^3$ . Thus the  $X_i$  are the scalars that transform under  $SO(3)$ .

This theory has many vacua. These vacua are obtained by setting the scalars  $X_i$  equal to  $SU(2)$  Lie algebra generators. In fact the vacua are in one to one correspondence with  $SU(2)$  representations of dimension  $N$ . Suppose that we have  $N(n)$  copies of the irreducible representation of dimension  $n$  such that

$$N = \sum_n N(n)n. \quad (2.12)$$

Each choice of partition of  $N$  gives us a different vacuum. So the number of vacua is equal to the partitions of  $N$ ,  $P(N)$ .

$\mathcal{N} = 4$  super Yang-Mills theory has a unique vacuum. On the other hand, any solution of the plane-wave matrix model can be uplifted to a zero energy solution of  $\mathcal{N} = 4$  super Yang-Mills theory. What do the various plane-wave matrix model vacua correspond to in  $\mathcal{N} = 4$  super Yang-Mills theory? It turns out that these are simply large gauge transformations of the ordinary vacuum. The solutions uplift to  $A_{\mathcal{N}=4} = (\omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3) = -i(dg)g^{-1}$ , where  $g$  is an  $SU(2)$  group element in the same representation as the  $J_i$ . This  $SU(2)$  group is parameterizing the  $S^3$ . So they are pure gauge transformations from  $A_{\mathcal{N}=4} = 0$ . In summary, in  $\mathcal{N} = 4$  super Yang-Mills theory these different configurations are related by a gauge transformation. The gauge transformation is not  $SU(2)_L$  invariant, even though the actual configurations are  $SU(2)_L$  invariant. In the plane-wave matrix model they are gauge inequivalent.

As in [4], it is possible to get the  $2 + 1$  theory in Sec. II A 2 from a limit in which we take  $\tilde{N}$  copies of the representation of dimension  $n$  and we take  $n \rightarrow \infty$ . For finite  $n$  we get a  $U(\tilde{N})$  theory on a fuzzy sphere and in the  $n \rightarrow \infty$  limit the fuzziness goes away [4].

One can also count the total number of  $1/2$  BPS states with  $SO(6)$  charge  $J$ . These are given by the partition function

$$\begin{aligned} I_{\text{PWMM}}(p, q) &= \sum_{N, J=0}^{\infty} D_{\text{PWMM}}(N, J) p^N q^J \\ &= \prod_{m=1}^{\infty} I_{\mathcal{N}=4}(p^m, q) \\ &= \frac{1}{\prod_{m=1}^{\infty} \prod_{n=0}^{\infty} (1 - p^m q^n)}. \end{aligned} \quad (2.13)$$

Setting  $q = 0$  we get that the number of vacua are given by the partitions of  $N$ . It is interesting to estimate the large  $J$  and  $N$  behavior of this index. We obtain

$$D_{\text{PWMM}}(N, J) \sim e^{(3.189\dots)(NJ)^{1/3}}, \quad (2.14)$$

where we assumed  $J^2/N \gg 1$ ,  $N^2/J \gg 1$ . The fact that this is symmetric under  $N \leftrightarrow J$  follows from the fact that (2.13) is symmetric under  $p \leftrightarrow q$  up to the  $n = 0$  factor.

## B. Dual gravity solutions

All the theories that we have discussed above have the same supersymmetry group. All gravity solutions with this symmetry were classified in [14]. The bosonic symmetries,  $R \times SO(3) \times SO(6)$ , act geometrically. The first generator implies the existence of a Killing vector associated to shifts of a coordinate  $t$ . In addition we have an  $S^2$  and an  $S^5$  where the rest of the bosonic generators act. Thus the solution depends only on three variables  $x_1, x_2, y$ . The full geometry can be obtained from a solution of the 3 dimensional Toda equation

$$(\partial_{x_1}^2 + \partial_{x_2}^2)D + \partial_y^2 e^D = 0. \quad (2.15)$$

It turns out that  $y = R_{S^2} R_{S^5}^2 \geq 0$  where  $R_{S^i}$  are the radii of the two spheres. In order to have a nonsingular solution we need special boundary conditions for the function  $D$  at  $y = 0$ . In fact, the  $x_1, x_2$  plane could be divided into regions where the function  $D$  obeys two different boundary conditions

$$\begin{aligned} e^D \sim y \quad \text{for } y \rightarrow 0, & \quad S^5 \rightarrow 0 & \text{M5region} \\ \partial_y D = 0 \quad \text{at } y = 0, & \quad S^2 \rightarrow 0, & \text{M2region} \end{aligned} \quad (2.16)$$

see [14] for further details. The labels M2 and M5 indicate that in these two regions either a two sphere or a five sphere shrinks to zero in a smooth fashion. There are, however, no explicit branes in the geometry. We have a smooth solution with fluxes. However, we can think of these regions as arising from a set of M2 or M5 branes that wrap the contractible sphere. A bounded region of each type in the  $x_1, x_2$  plane implies that we have a cycle in the geometry with a flux related to the corresponding type of brane (see [14] for further details).

The different theories discussed above are related to different choices for the topology of the  $x_1, x_2$  plane. In addition, for each topology the asymptotic distribution of M2 and M5 regions can be different. See Fig. 1. Let us consider some examples. If we choose the  $x_1, x_2$  plane to be a two torus, then we get a solution that is dual to the vacua of the  $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3/Z_k$ , see Fig. 1(a). If the topology is a cylinder, with  $x_1$  compact and the M2 region is localized in the  $x_2$  direction, we have a solution dual to a vacuum of the  $2 + 1$  Yang-Mills theory on  $R \times S^2$ , see Fig. 1(b). If we choose a cylinder and we let the M2 region extend all the way to  $x_2 \rightarrow -\infty$ , and the M5 region extend to  $x_2 \rightarrow +\infty$ , and also there are localized M2, M5 strips in between, then we get a solution which is dual to a vacuum of the plane-wave

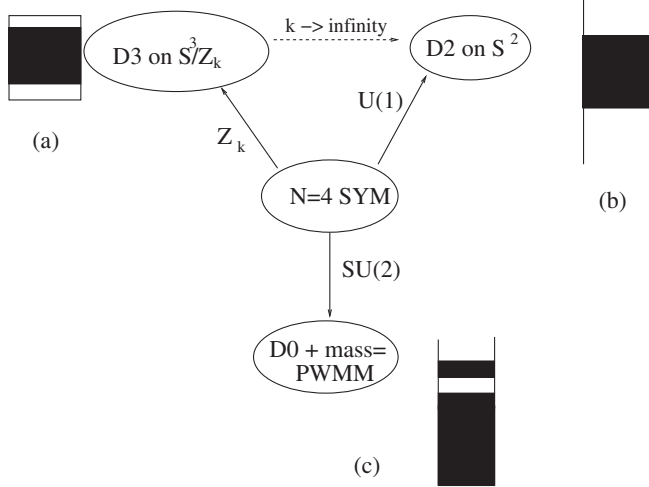


FIG. 1. Starting from four dimensional  $\mathcal{N} = 4$  super Yang-Mills theory and truncating by various subgroups of  $SU(2)_L$  we get various theories with  $\widetilde{SU}(2|4)$  symmetry. We have indicated the diagrams in the  $x_1, x_2$  space that determine their gravity solutions. The  $x_1, x_2$  space is a cylinder, with the vertical lines identified for (b) and (c) and it is a torus for (a).

matrix model, see Fig. 1(c). Finally, if we consider a cylinder and we have M5 regions that are localized [see Fig. 2(c)] then we get a solution that is dual to an NS5 brane theory on  $R \times S^5$ , we will come back to this case later.

In principle, we could consider configurations that are not translation invariant, as long as we consider configurations defined on a cylinder or torus as is appropriate. In this paper we will concentrate on configurations that are translation invariant along  $x_1$ . These will be most appropriate in the regime of parameter space where the 11th direction is small and we can go to a IIA description. So we focus on the region in parameter space where the string coupling is small and the effective 't Hooft coupling is large. If the configuration is translation invariant in the  $x_1$  direction we can transform the nonlinear Eq. (2.15) to a linear equation through the following change of variables [28]

$$y = \rho \partial_\rho V, \quad x_2 = \partial_\eta V, \quad e^D = \rho^2, \quad (2.17)$$

$$\frac{1}{\rho} \partial_\rho (\rho \partial_\rho V) + \partial_\eta^2 V = 0. \quad (2.18)$$

So we get the Laplace equation in three dimensions for an axially symmetric system.<sup>7</sup> The fact that one can obtain solutions in this fashion was observed in [14] and some singular solutions were explored in [29]. Below we will find the precise boundary conditions for  $V$  which ensure that we have a smooth solution.

<sup>7</sup>The angular direction of the three dimensional space is not part of the 10 or 11 dimensional spacetime coordinates.

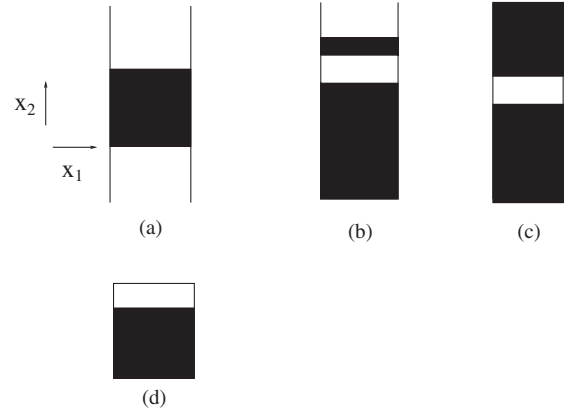


FIG. 2. Translational invariant configurations in the  $x_1, x_2$  plane which give rise to various gravity solutions. The shaded regions indicate M2 regions and the unshaded ones indicate M5 regions. The two vertical lines are identified. In (a) we see the configuration corresponding to the vacuum of the 2 + 1 Yang-Mills theory on  $R \times S^2$  with unbroken gauge symmetry. In (b) we consider a configuration corresponding to a vacuum of the plane-wave matrix model. In (c) we see a vacuum of the NS5 brane theory on  $R \times S^5$ . Finally, in (d) we have a droplet on a two torus in the  $x_1, x_2$  plane. This corresponds to a vacuum of the  $\mathcal{N} = 4$  super Yang-Mills theory on a  $R \times S^3/Z_k$ .

Let us now translate the boundary conditions (2.16) at  $y = 0$  into certain boundary conditions for the function  $V$ . In the region where  $e^D \sim y$  at  $y \sim 0$ , all that we require is that  $V$  is regular at  $\rho = 0$ , in the three dimensional sense. On the other hand if  $y = 0$  but  $\rho \neq 0$ , then we need to impose that  $\partial_y D = 0$ . This is proportional to

$$0 = \frac{1}{2} \partial_y D = \rho \frac{\partial \rho}{\partial y} = - \frac{\partial_\eta^2 V}{(\partial_\eta \partial_\rho V)^2 + (\partial_\eta^2 V)^2}. \quad (2.19)$$

We conclude that  $\partial_\eta^2 V = 0$ . Equation (2.18) then implies that  $\partial_\rho^2 V = 0$ . Therefore the curve  $y = 0, \rho \neq 0$ , or  $\partial_\rho V = 0$ , is at constant values of  $\eta$ , since the slope of the curve defined by  $\partial_\rho V = 0$  is  $\frac{\delta \eta}{\delta \rho} = \frac{-\partial_\eta^2 V}{\partial_\eta \partial_\rho V} = 0$ .

If we interpret  $V$  as the potential of an electrostatics problem, then  $-\partial_\rho V$  is the electric field along the  $\rho$  direction. The condition that it vanishes corresponds to the presence of a charged conducting surface. So the problem is reduced to an axially symmetric electrostatic configuration in three dimensions where we have conducting disks that are sitting at positions  $\eta_i$  and have radii  $\rho_i$ . See Fig. 3. These disks are in an external electric field which grows at infinity. If we considered such conducting disks in a general configuration we would find that the electric field would diverge as we approach the boundary of the disks. In our case this cannot happen, otherwise the coordinate  $x_2$  would be ill defined at the rim of the disks. So we need to impose the additional constraint that the electric field is finite at the rim of the disks. This implies

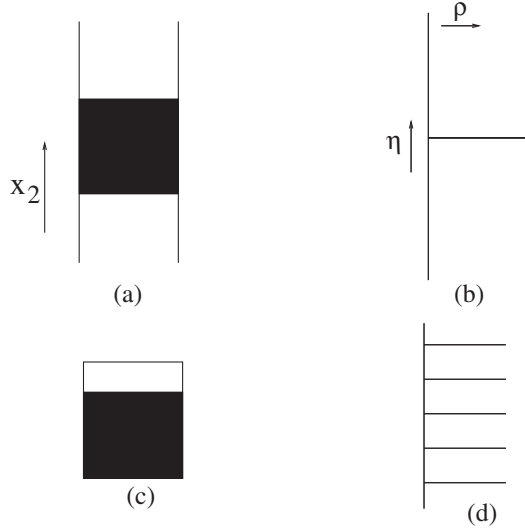


FIG. 3. Electrostatic problems corresponding to different droplet configurations. The shaded regions (M2 regions) correspond to disks and the unshaded regions map to  $\rho = 0$ . Note that the  $x_1$  direction in (a), (c) does not correspond to any variable in (b), (c). The rest of the  $\rho, \eta$  plane corresponds to  $y > 0$  in the  $x_2, y$  variables. In (a), (b) we see the configurations corresponding to a vacuum of 2 + 1 super Yang-Mills theory on  $R \times S^2$ . In (c), (d) we see a configuration corresponding to a vacuum of  $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3/Z_k$ . In (d) we have a periodic configuration of disks. The fact that it is periodic corresponds to the fact that we have also compactified the  $x_2$  direction.

that the charge density vanishes at the tip of the disks. This condition relates the charge on the disks  $Q_i$  to the radii of the disks  $\rho_i$ . So for each disk we can only specify two independent parameters, its position  $\eta_i$  and its total charge  $Q_i$ . The precise form of the background electric field depends on the theory we consider (but not on the particular vacuum) and it is fixed by demanding that the change of variable (2.17) is well defined. The relation between the translation invariant droplet configurations in the  $x_1, x_2$  plane and the disks can be seen in Fig. 3.

Since we are focusing on solutions which are translation invariant along  $x_1$  it is natural to compactify this direction and write the solution in IIA variables. This procedure will make sense as long as we are in a region of the solution where the IIA coupling is small (see [30] for a similar discussion).

The  $M$ -theory form of the solutions can be found in [14]. We obtain the string frame solution

$$ds_{10}^2 = \left( \frac{\ddot{V} - 2\dot{V}}{-V''} \right)^{1/2} \left\{ -4 \frac{\ddot{V}}{\dot{V} - 2\dot{V}} dt^2 + \frac{-2V''}{\dot{V}} (d\rho^2 + d\eta^2) + 4d\Omega_5^2 + 2 \frac{V''\dot{V}}{\Delta} d\Omega_2^2 \right\},$$

$$e^{4\Phi} = \frac{4(\ddot{V} - 2\dot{V})^3}{-V''\dot{V}^2\Delta^2}, \quad (2.20)$$

$$C_1 = - \frac{2\dot{V}'\dot{V}}{\dot{V} - 2\dot{V}} dt, \quad (2.21)$$

$$F_4 = dC_3, \quad C_3 = -4 \frac{\dot{V}^2 V''}{\Delta} dt \wedge d^2\Omega, \quad (2.22)$$

$$H_3 = dB_2, \quad B_2 = 2 \left( \frac{\dot{V}\dot{V}'}{\Delta} + \eta \right) d^2\Omega, \quad (2.23)$$

$$\Delta \equiv (\ddot{V} - 2\dot{V})V'' - (\dot{V}')^2, \quad (2.24)$$

where the dots indicate derivatives with respect to  $\log \rho$  and the primes indicate derivatives with respect to  $\eta$ .  $V(\rho, \eta)$  is a solution of the Laplace Eq. (2.18). For regular solutions, we need to supplement it by boundary condition specified by a general configuration of lines in  $(\rho, \eta)$  plane, like in Fig. 4.

Before we get into the details of particular solutions we would like to discuss some general properties. First note that if we take a random solution of (2.18) we will get singularities. In order to prevent them, we need to be a bit careful. As we explained above we need a solution of an electrostatic problem involving horizontal conducting disks. In addition we need to ensure the positive-definiteness of various metric components, i.e.  $\Delta \leq 0$  and  $V'' \leq 0$ ,  $\ddot{V} - 2\dot{V} \geq 0$ ,  $\dot{V} \geq 0$ . This is obeyed everywhere if we choose appropriate boundary conditions for the potential at large  $\rho, \eta$ . These boundary conditions imply that there is a background electric field that grows as we go to large  $\rho, \eta$ . For example, if we consider a configuration such as the one in Fig. 3(b), the disk is in the presence of a background potential of the form

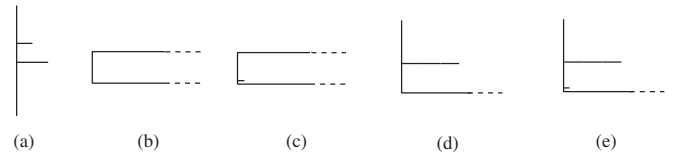


FIG. 4. In (a) we see a configuration which corresponds to a vacuum of 2 + 1 super Yang-Mills theory on  $R \times S^2$ . In (b) we see the simplest vacuum of the theory corresponding to the NS5 brane on  $R \times S^5$ . In this case we have two infinite conducting disks and only the space between them is physically meaningful. In (c) we have another vacuum of the same theory. If the added disk is very small and close to the top or bottom disks the solution looks like that of (b) with a few D0 branes added. In (d) we see a configuration corresponding to a vacuum of the plane-wave matrix model. In this case the disk at  $\eta = 0$  is infinite and the solution contains only the region with  $\eta \geq 0$ . In (e) we have another vacuum of the plane-wave matrix model with more disks.

$V_b \sim \rho^2 - 2\eta^2$ . This background electric field is the same for all vacua, e.g. it is the same in Figs. 3(b) and 4(a). For the plane-wave matrix model we have an infinite conducting surface at  $\eta = 0$  and only the region  $\eta \geq 0$  is physically significant. In this case the background potential is  $V_b \sim \rho^2 \eta - \frac{2}{3} \eta^3$ . In addition we have finite size disks as seen, for example, in Fig. 4(d) or 4(e). In Appendix A we show that for the configurations we talk about in this paper (2.20)–(2.24) gives a regular solution. We also show that the dilaton is nonsingular and that  $g_{tt}$  never becomes zero for the solutions we consider. This ensures that the solutions we have have a mass gap. This follows from the fact that the warp factor never becomes zero so that we cannot decrease the energy of a state by moving it into the region where the warp factor becomes zero. In principle, this argument does not rule out the presence of a small number of massless or tachyonic modes. The latter are, of course, forbidden by supersymmetry. A massless mode would not change the energy of the solution, so it would preserve supersymmetry. On the other hand, once we quantize the charges on the disks we do not have any continuous parameters in our solutions. So we cannot have any massless modes. Of course, this agrees with the field theory expectations since all theories we consider have a mass gap around any of the vacua.

Note that a rescaling of  $V$  leaves the ten dimensional metric and  $B$  field invariant but rescales the dilaton and the RR fields. This just corresponds to the usual symmetry of the IIA supergravity theory under rescaling of the dilaton and RR fields. There is second symmetry corresponding to rescaling  $\rho$ ,  $\eta$ , and  $V$  which corresponds to the usual scaling symmetry of gravity which scales up the metric and the forms according to their scaling dimensions. This allows us to put in two parameters in (2.20)–(2.24) such as an overall charge and the value of the dilaton at its maximum.

More interestingly, we can vary the number of disks, their charges and the distances between each other. See Fig. 4. These parameters are related to different choices of vacua for the different configurations.

All the solutions we are discussing, contain an  $S^2$  and an  $S^5$  and these can shrink to zero at various locations. Using these it is possible to construct three cycles and six cycles, respectively, by tensoring the  $S^2$  and  $S^5$  with lines in the  $\rho$ ,  $\eta$  plane. These translate into three cycles and six cycles in the IIA geometry. See Fig. 5. We can then measure the flux of  $H_3$  over the three cycle and call it  $N_5$  and we can measure the flux of  $*\tilde{F}_4$  on the six cycle and call it  $N_2$ . Using (2.20)–(2.24) or the formulas in [14] we can write them as

$$\begin{aligned}
 N_2 &= \frac{1}{\pi^3 l_p^6} \int e^D dx_2 \int dx_1 \\
 &= \frac{2}{\pi^2} \int_0^{\rho_i} \rho^2 \partial_\rho (\partial_\eta V|_{\eta_i^+} - \partial_\eta V|_{\eta_i^-}) d\rho = \frac{8Q_i}{\pi^2} \quad (2.25)
 \end{aligned}$$

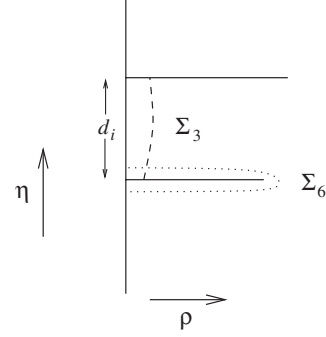


FIG. 5. We see a configuration associated to a pair of disks.  $d_i$  indicates the distance between the two nearby disks. The dashed line in the  $\rho$ ,  $\eta$  plane, together with the  $S^2$  form a three cycle  $\Sigma_3$  with the topology of an  $S^3$ . The dotted line, together with the  $S^5$  form a six cycle  $\Sigma_6$  with the topology of an  $S^6$ .

and

$$\begin{aligned}
 N_5 &= \frac{1}{2\pi^2 l_p^3} \int y^{-1} e^D dx_2 \int dx_1 \\
 &= \frac{1}{\pi} \int_{\eta_i+d_i}^{\eta_i} \frac{\rho}{\partial_\rho V} \partial_\eta^2 V|_{\rho=0} d\eta = \frac{2d_i}{\pi}. \quad (2.26)
 \end{aligned}$$

In deriving (2.26) we used that near  $\rho \rightarrow 0$  we can expand  $V = f_0(\eta) + \rho^2 f_1(\eta) + \dots$  and we used the equation for  $V$  (2.18) to relate  $f_1(\eta)$  to  $V''$ . We set  $\alpha' = 1$  and  $l_p = 1$  for convenience. The quantization conditions (2.25) and (2.26) show that  $N_5$  is proportional to the distance between neighboring disks  $d_i$  and that  $N_2$  is proportional to the total charge of each disk  $Q_i$ . When we solve the electrostatic problem we need to ensure that these parameters are quantized. Strictly speaking the flux given by  $N_2$  is quantized only after we quantize the four form field strength.

The topology of the solutions is related to the topology of the disk configurations. In other words, the number of six cycles and three cycles is related to the number of disks and the number of line segments in between, but is independent of the size of the disks or the distance between the disks.

As we discussed above we will be interested in BPS excitations with angular momentum on  $S^5$ . For large, but not too large, angular momentum these are well described by lightlike particles moving in the background (2.20)–(2.24) with angular momentum  $J$  along the  $S^5$ . In order to minimize their energy, these lightlike geodesics want to sit at a point in the  $\rho$ ,  $\eta$  space where

$$\frac{|g_{tt}|}{g_{55}} = \frac{\ddot{V}}{\dot{V} - 2\dot{V}} \geq 1 \quad (2.27)$$

is minimized, where  $\sqrt{g_{55}}$  is the radius of the five sphere. It turns out that this is minimized at the tip of the disks, where



the inequality in (2.27) is saturated.<sup>8</sup> This corresponds to saturating the BPS condition  $E \geq |J|$ . In fact, in order to minimize (2.27) we would like to set  $\dot{V} = 0$ . This occurs at  $\rho = 0$  and on the surface of the disks. However, in these cases, also  $\dot{V} = 0$ . Expanding the solutions near these regions we find that (2.27) actually diverges at  $\rho = 0$ , this is because  $S^5$  shrinks at  $\rho = 0$ . On the disks, (2.27) is bigger than one, except at the tip where it is one. See Appendix A for a more detailed discussion.

In order to find the behavior of the solution near these geodesics we expand the solution of the electrostatic problem near the tip of the disks. Near the tip of the disks we have a simple Laplace equation in two dimensions. Namely, we approximate the disk by an infinite half plane. We can then solve the problem by doing conformal transformations. Actually, we can do this whenever we are expanding around a solution at large  $\rho_0$  and we are interested in features arising at distances which are much smaller than  $\rho_0$ , but could be larger than the distances between disks, see Fig. 6. So let us first analyze this problem in general. We can define the complex coordinate

$$z \equiv \xi + i\eta \equiv \rho - \rho_0 + i\eta \quad (2.28)$$

so that we are expanding around the point  $(\rho, \eta) = (\rho_0, 0)$ . It is actually convenient for our problem to define a complex variable

$$w(z) = 2\partial_z V = \left( \frac{y}{\rho_0} - ix_2 \right), \quad (2.29)$$

where we also used an approximate form of (2.17). Equation (2.18) implies that  $w$  is a holomorphic function of  $z$ . We see that  $w$  is defined on the right half plane:  $\text{Re}(w) \geq 0$ . Equation (2.18) is simply the statement that the change of variables is holomorphic. Solutions are simply given by finding a conformal transformation that maps the  $w$  half-plane into a configuration in  $z$ -plane containing various cuts of lines specified by a general configuration, like those in Fig. 6.

For example we could take  $z = w^2$ . This maps the  $w$  half-plane into the  $z$  plane with a cut running on the negative real axis. More explicitly, this leads to  $V \sim \text{Re}(z^{3/2})$ . This is the solution near the tip of a disk, see Fig. 6(b).

Once we have found this map we can go back to the general ansatz (2.20)–(2.24) and write the resulting answer. When we do this we note that  $\dot{V} \sim \partial_\xi V / \rho_0$  and that  $\dot{V} \sim \partial_\xi^2 V / \rho_0^2$ . Since  $\rho_0$  is very large in our limit we keep only the leading order terms in  $\rho_0$ . After doing this we find the

<sup>8</sup>In the 11 dimensional description the point where (2.27) is minimized lies on the  $y = 0$  plane at a local maximum of  $e^D|_{y=0}$  in the  $x_1, x_2$  plane.

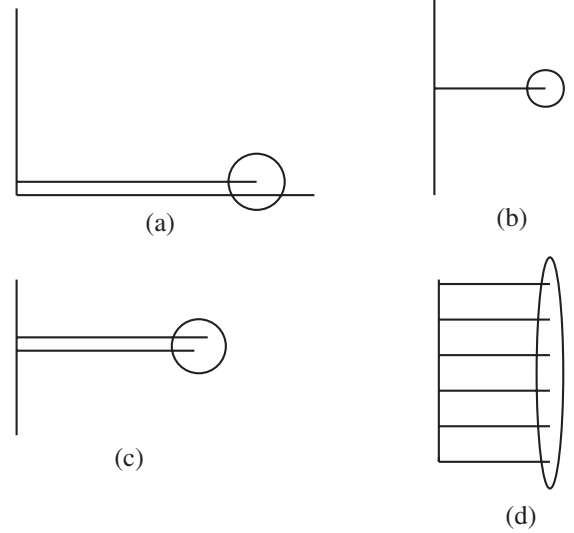


FIG. 6. In this figure we see the expansion around the region near the tip of the disks. In a generic situation the tip we focus on is isolated, see (b). In other cases, there are other disks nearby that sit close to the tip we are focusing on. In this case we can take a limit where we include the nearby tips. We see such situations in (a), (c) and (d). (d) corresponds to the periodic case. We can focus on a distance that is large compared to the period in  $\eta$  but small compared to the size of the disks.

approximate solution<sup>9</sup>

$$ds_{10}^2 \sim 4\rho_0 \left\{ - \left( 1 + \frac{1}{\rho_0} f^{-1} |\partial_w z|^2 \right) dt^2 + d\Omega_5^2 + \frac{f}{\rho_0} \left[ dw d\bar{w} + \left( \frac{w + \bar{w}}{2} \right)^2 d\Omega_2^2 \right] \right\}, \quad (2.30)$$

where  $f = \frac{\partial_w z + \partial_{\bar{w}} \bar{z}}{2(w + \bar{w})}$ .

Let us first consider the specific case where  $z = w^2$ . This describes the configuration near the tip of the disks. In this case we find that  $f = 1$  and the metric in the four dimensional space parametrized by  $w, \bar{w}$ ,  $\Omega_2$  is flat. In addition, we see that (2.27) is indeed saturated at  $w = 0$ .

Now let us go back to (2.30) and take a general pp-wave limit. We will take  $\rho_0 \rightarrow \infty$  and scale out the overall factor  $\rho_0$  away from the solution. In other words, we parameterize  $S^5$  as

$$d\Omega_5^2 = d\varphi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3^2 \sim d\varphi^2 \left( 1 - \frac{\vec{r}^2}{4\rho_0} \right) + \frac{1}{4\rho_0} d\vec{r}^2, \quad (2.31)$$

where we expanded around  $\frac{r}{\sqrt{4\rho_0}} = \theta \sim 0$  and kept  $\vec{r}$  finite in the limit. In addition, we set

<sup>9</sup>The rest of the fields, i.e. the dilaton and fluxes are the same as in (2.35)–(2.39), with  $t = x^+$ .

$$dt = dx^+, \quad d\varphi = dx^+ - \frac{1}{4\rho_0} dx^-, \quad (2.32)$$

$$-p_+ = E - J, \quad -p_- = \frac{J}{4\rho_0}, \quad (2.33)$$

$$4\rho_0 = R_{S^5}^2, \quad (2.34)$$

where the second line tells us how the generators transform and finally the last line is stating that the parameter  $\rho_0$  is physically the size of the  $S^5$  (we have set  $\alpha' = 1$ ).

After this pp-wave limit is taken for (2.20), (2.21), (2.23), and (2.24), the solution takes the form

$$ds_{10}^2 = 2dx^+ dx^- - (4f^{-1}|\partial_w z|^2 + \vec{r}^2)(dx^+)^2 + d\vec{r}^2 + 4f\left(dwd\bar{w} + \left(\frac{w + \bar{w}}{2}\right)^2 d\Omega_2^2\right),$$

$$e^{2\Phi} = 4f, \quad (2.35)$$

$$B_2 = i\left[\frac{(w + \bar{w})}{2}(\partial_w z - \partial_{\bar{w}} \bar{z}) - (z - \bar{z})\right] d^2\Omega\Delta, \quad (2.36)$$

$$C_1 = i(w + \bar{w})\frac{(\partial_w z - \partial_{\bar{w}} \bar{z})}{(\partial_w z + \partial_{\bar{w}} \bar{z})} dx^+, \quad (2.37)$$

$$C_3 = -(w + \bar{w})^3 f dx^+ \wedge d^2\Omega, \quad (2.38)$$

$$f \equiv \frac{\partial_w z + \partial_{\bar{w}} \bar{z}}{2(w + \bar{w})}, \quad (2.39)$$

where  $z$  is a holomorphic function of  $w$ . This is an exact solution of IIA supergravity. When a string is quantized in light cone gauge on this pp wave it leads to a (4,4) supersymmetric light cone Lagrangian, which will be discussed in Sec. IIC 3. One can also introduce two parameters by rescaling  $z$  and  $w$ . Similar classes of IIB pp-wave solutions and their sigma models were analyzed and classified in e.g. [31–33].

For the single tip solution

$$z = w^2 \quad (2.40)$$

we get

$$ds_{10}^2 = -2dx^+ dx^- - (\vec{r}^2 + 4\vec{u}^2)(dx^+)^2 + d\vec{r}^2 + d\vec{u}^2, \quad (2.41)$$

where  $\vec{r}$  and  $\vec{u}$  each parameterize  $R^4$ . This is a IIA plane wave with  $SO(4) \times SO(3)$  isometry and it was considered before in [11,12].

In conclusion, the expansion of the metric around the trajectories of BPS particles locally looks like a IIA plane wave (2.41) if the tip of the disk is far from other disks. When it is close to other disks we need to use the more general expression (2.35)–(2.39). We will analyze in detail specific cases in Sec. IIB 3. In the limit that we boost away the  $g_{++}$  component of the metric, the solution (2.35)–

(2.39) becomes  $R^{5,1}$  times a transverse four dimensional part of the solution which is a superposition of NS5 branes. Notice that  $f$  is a solution of the Laplace equation in the four dimensions parametrized by  $w, \bar{w}, \Omega_2$ . This is related to the fact that we should interpret the space between two closely spaced disks as being produced by NS5 branes. This will become more clear after we analyze specific solutions in e.g. Sec. IIB 4.

The rescaling of  $J$  in (2.33) has some physical significance since it will appear when we express the energy of near BPS states in terms of  $J$ . In other words, the light cone hamiltonian for a string on the IIA plane wave describes massive particles propagating on the world sheet. Four of the bosons have mass 1 and the other four have mass 2. The light cone energy for each particle of momentum  $n$  and mass  $m$  is

$$(E - J)_n = (-p_+)_n = \sqrt{m^2 + \frac{n^2}{p_-^2}} = \sqrt{m^2 + R_{S^5}^4 \frac{n^2}{J^2}},$$

$$\alpha' = 1, \quad (2.42)$$

where the masses of the world sheet fields are  $m = 1, 2$  depending on the type of scalar or fermion that we consider on the world sheet. The subindex  $n$  reminds us that this is the contribution from a particle with a given momentum along the string. Since the total momentum along the string should vanish, we need to have more than one particle carrying momentum, each giving rise to a contribution similar to (2.42). Note that the form of the spectrum is completely universal for all solutions, as long as the tip is far enough from other disks. On the other hand the value of  $\rho_0$  at the tip depends on the details of the solution. It depends not only on the theory we consider but also on the particular vacuum that we are expanding around. In the following sections we will compute the dependence of  $\rho_0$  on the particular parameters of each theory for some specific vacua.

When we can isolate a single disk we can always take pp-wave limit of the solution to the IIA plane wave (2.40) and (2.41) near the tip of this single disk. There are many other situations when nearby disks are very close, and we need to include also the region between disks, i.e. the region produced by NS5 branes. In these cases, the geometry parametrized by the second four coordinates  $w, \bar{w}, \Omega_2$  is more complicated. We will discuss it in following sections.

As is usual in the gravity/field theory correspondence one has to be careful about the regime of validity of the gravity solutions, and in our case, we should also worry about the following. In the field theory we have many vacua. So we can have tunnelling between the vacua. On the gravity side we have the same issue, we can tunnel between different solutions of the system. In order to understand this tunnelling problem it is instructive to consider vacua whose solutions are very close to the original

solution. Small deformations of a given solution that still preserve all the supersymmetries can be obtained, in the 11D language by considering small “ripples” in the regions connecting M2 and M5 regions. In the IIA description these become D0 branes. For very small excitations these D0 branes sit at  $\rho = 0$  at the position of the disks. At these positions it costs zero energy to add the D0 branes. In the electrostatic description we are adding a small disk close to the large disk, as in Fig. 4(c). In order to estimate the tunnelling amplitude we need to understand how we go from a configuration with no D0 branes to a configuration with D0 branes. In a region where we have a finite size three cycle  $\Sigma_3$  (see Fig. 5) with flux  $N_5$  we can create  $N_5$  D0 branes via a D2 instanton that wraps the  $\Sigma_3$  (see [34]). We see that such processes will be suppressed if the string coupling in this region is small and the  $\Sigma_3$  is sufficiently large.

In the following subsections we discuss specific solutions.

### 1. Solution for NS5 brane theory on $R \times S^5$

We start with this solution because it is the simplest from the gravity point of view. In this case we consider two infinite disks separated by some distance  $d \sim N$ , see Fig. 4(b). We find that the solution corresponds to  $N$  IIA NS5 branes wrapping a  $R \times S^5$ . The solution for  $V$  is

$$V = I_0(r) \sin\theta, \quad r = \frac{2\rho}{N\alpha'}, \quad \theta = \frac{2\eta}{N\alpha'}, \quad (2.43)$$

where  $I_0(r)$  is a modified Bessel function of the first kind. This leads to the ten dimensional solution<sup>10</sup>

$$ds_{10}^2 = N \left[ -2r \sqrt{\frac{I_0}{I_2}} dt^2 + 2r \sqrt{\frac{I_2}{I_0}} d\Omega_5^2 + \sqrt{\frac{I_2 I_0}{I_0 I_1}} (dr^2 + d\theta^2) + \sqrt{\frac{I_2}{I_0 I_0 I_2 s^2 + I_1^2 c^2}} d\Omega_2^2 \right],$$

$$B_2 = N \left[ \frac{-I_1^2 c s}{I_0 I_2 s^2 + I_1^2 c^2} + \theta \right] d^2\Omega, \quad (2.44)$$

$$e^\Phi = g_0 N^{3/2} 2^{-1} \left( \frac{I_2}{I_0} \right)^{3/4} \left( \frac{I_0}{I_1} \right)^{1/2} (I_0 I_2 s^2 + I_1^2 c^2)^{-(1/2)}, \quad (2.45)$$

$$C_1 = -g_0^{-1} \frac{1}{N} 4 \frac{I_1^2 c}{I_2} dt, \quad (2.46)$$

$$C_3 = -g_0^{-1} \frac{4I_0 I_1^2 s^3}{I_0 I_2 s^2 + I_1^2 c^2} dt \wedge d^2\Omega, \quad (2.47)$$

<sup>10</sup> $s = \sin\theta$ ,  $c = \cos\theta$ . We set  $\alpha' = 1$  in this paper. We used the convention in [35] that  $\frac{1}{2\pi\alpha'} \int_{\Sigma_3} H_3 = 2\pi N$ , to normalize  $H_3$ .

where  $I_n(r)$  are a series of modified Bessel functions of the first kind.

This solution is also a limit of the a solution analyzed in [14] using 7D gauged supergravity, except that here we solved completely the equations. The gauged-supergravity solution in [14] describes an elliptic M5 brane droplet on the  $x_1, x_2$  plane and we can take a limit that the long axis of the ellipse goes to infinity while keeping the short axis finite, this becomes a single M5 strip. This then corresponds to two infinite charged disks in the electrostatic configuration, see Figs. 4(b). We discuss more details of this relation in Appendix C.

The solution is dual to little string theory (see e.g. [36,37]) on  $R \times S^5$ . As we go to the large  $r$  region the solution (2.44)–(2.47) asymptotes to

$$ds_{10}^2 = N\alpha' [2r(-dt^2 + d\Omega_5^2) + dr^2 + (d\theta^2 + \sin^2\theta d\Omega_2^2)],$$

$$e^\Phi = g_s e^{-r}, \quad H_3 = 2N\alpha' \sin^2\theta d\theta \wedge d^2\Omega. \quad (2.48)$$

So we see that the solution asymptotes to IIA NS5 branes on  $R \times S^5$ . In addition we have RR fields which are growing exponentially when we go to large  $r$ . These fields break the  $SO(4)$  transverse rotation symmetry of the fivebranes to  $SO(3)$ . Since the coupling is also varying exponentially, it turns out that, in the end, the influence of the RR fields on the metric is suppressed only by powers of  $1/r$  relative to the terms that we have kept in (2.48) (relative to the  $H$  field terms, for example).

The solution is everywhere regular. When either  $S^5$  or  $S^2$  shrinks, it combines with  $r$  or  $\theta$  to form locally  $R^6$  or  $R^4$ . Note that at  $r = 0$  the solution has a characteristic curvature scale given by  $\mathcal{R} \sim \frac{1}{\alpha' N}$  and a string coupling of a characteristic size  $g_s \sim g_0 N^{3/2}$ . The string coupling decreases as we approach the boundary. Thus, if we take  $g_s$  small and  $N$  large we can trust the solution everywhere. On the other hand if we take  $g_s$  large, then we can trust the solution for large  $r$  but for small  $r$  we need to go to an 11 dimensional description, include  $x_1$  dependence and solve Eq. (2.15). It is clear from the form of the problem that for very large  $g_s$  we will recover  $\text{AdS}_7 \times S^4$  in the extreme IR if we choose a suitable droplet configuration. More precisely, as increase  $g_s$  we will need to go to the 11 dimensional description and include dependence on  $x_1$ . Then we can consider a periodic array of circular droplets. As  $g_s \rightarrow \infty$  each circle becomes the isolated circle that gives rise to  $\text{AdS}_7 \times S^4$  [14]. There is also a similar gravity picture for the relation between the 2 + 1 SYM on  $R \times S^2$  in Sec. II A 2 and the 3D superconformal M2 brane theory.

In addition we could consider other solutions in the disk picture that correspond to adding more small disks between the infinite disks, as in Fig. 4(c). These correspond to different vacua of this theory.

## 2. Solution for 2 + 1 SYM on $R \times S^2$

This solution corresponds to a single disk, as in Fig. 3(b). This disk is in the presence of a background field  $V_b \sim \rho^2 - 2\eta^2$ . The solution is a bit harder to obtain. We have obtained it by combining our ansatz with the results in [38], as explained in Appendix B. The resulting 10 dimensional solution is

$$ds_{10}^2 = \lambda^{1/3} \left[ -8(1+r^2)fdt^2 + 16f^{-1}\sin^2\theta d\Omega_5^2 + \frac{8rf}{r+(1+r^2)\arctan r} \left( \frac{dr^2}{1+r^2} + d\theta^2 \right) + \frac{2r[r+(1+r^2)\arctan r]f}{1+r\arctan r} d\Omega_2^2 \right], \quad (2.49)$$

$$B_2 = -\lambda^{1/3} \frac{2\sqrt{2}[r+(-1+r^2)\arctan r]\cos\theta}{1+r\arctan r} d^2\Omega, \quad (2.50)$$

$$e^\Phi = g_0\lambda^{1/2}8r^{1/2}(1+r\arctan r)^{-(1/2)} \times [r+(1+r^2)\arctan r]^{-(1/2)}f^{-(1/2)}, \quad (2.51)$$

$$C_1 = -g_0^{-1}\lambda^{-(1/3)} \frac{[r+(1+r^2)\arctan r]\cos\theta}{2r} dt, \quad (2.52)$$

$$C_3 = -g_0^{-1} \frac{r[r+(1+r^2)\arctan r]^2 f^2}{\sqrt{2}(1+r\arctan r)} dt \wedge d^2\Omega, \quad (2.53)$$

$$f \equiv \sqrt{\frac{2}{r}[r+(\cos^2\theta+r^2)\arctan r]}, \quad (2.54)$$

where  $\lambda$  and  $g_0$  are some constants. Here we have plugged in expression (B18) in Appendix B.

This solution is dual to the vacuum of the 2 + 1 SYM in Sec. II A 2, with  $\Phi = 0$  and unbroken  $U(N)$  gauge symmetry. The topology of this solution is  $R \times B^3 \times S^6$ , where the boundary of  $B_3$  is the  $S^2$  on which the field theory is defined. Solutions with other configurations of disks have different topology. The solution is also everywhere regular. Expanding for large  $r$  we find that (2.49)–(2.54) approaches the D2 brane solution<sup>11</sup> [30] on  $R \times S^2$

$$\frac{ds_{10}^2}{\alpha'} = (6\pi g_{\text{YM}2}^2 N)^{1/3} \left[ r^{5/2} \left( -dt^2 + \frac{1}{4} d\Omega_2^2 \right) + \frac{dr^2}{r^{5/2}} + r^{-1/2} (d\theta^2 + \sin^2\theta d\Omega_5^2) \right],$$

$$e^\Phi = g_{\text{YM}2}^2 (6\pi g_{\text{YM}2}^2 N)^{-1/6} r^{-5/4},$$

$$C_3 = -g_{\text{YM}2}^{-2} r^5 (6\pi g_{\text{YM}2}^2 N)^{2/3} \frac{1}{4} dt \wedge d^2\Omega. \quad (2.55)$$

<sup>11</sup>Here we have D2 brane on  $R \times S^2$ , where the radius of the  $S^2$  is  $\frac{1}{\mu}$ , and we set  $\mu = 2$ .

Comparing with (2.49)–(2.54) we can compute the value of  $\lambda$  and  $g_0$  in terms of Yang-Mills quantities. We can then compute the value of the radius of  $S^5$  at  $r = 0$ ,  $\theta = \pi/2$ . This is the point where the BPS geodesics moving along  $S^5$  sits. We find

$$\frac{R_{S^5}^2}{\alpha'} = \left( \frac{6\pi^3 g_{\text{YM}2}^2 N}{\mu} \right)^{1/3}, \quad \mu = 2. \quad (2.56)$$

The metric expanded around a geodesic with momentum along  $S^5$  is simply the plane wave (2.41). We can now insert (2.56) in the general expression (2.42) to derive the spectrum of near BPS excitations with large  $J$ .

Note that the leading correction to  $\hat{E} = E - J$  for fluctuations in the transverse directions in the  $S^5$ , which are parametrized by  $\vec{r}$  in (2.41), has the form

$$(E - J)_n = 1 + \frac{1}{2} \left( \frac{6\pi^3 g_{\text{YM}2}^2 N}{\mu} \right)^{2/3} \frac{n^2}{J^2} + \dots \quad (2.57)$$

This is the large coupling result from gravity approximation.

Under general principles we expect that the leading order correction in the large  $J$  limit in all regimes of the coupling constant should go like

$$(E - J)_n = 1 + f \left( \frac{g_{\text{YM}2}^2 N}{\mu} \right) \frac{n^2}{J^2} + \dots \quad (2.58)$$

At weak coupling we get basically the same answer we had for  $\mathcal{N} = 4$ , which at one loop order is  $f \left( \frac{g_{\text{YM}2}^2 N}{\mu} \right) = \frac{\pi g_{\text{YM}2}^2 N}{\mu}$ . So we see that in this case the function  $f$  has to be non-trivial. This is to be contrasted with the behavior in four dimensional  $\mathcal{N} = 4$  theory where the function  $f$  has the same form at weak and strong coupling [39], see also [40]. Of course it would be very nice to compute this interpolating function from the gauge theory side. We will see a similar phenomenon for the plane-wave matrix model in Sec. II B 5. This phenomenon is a generic feature of the strong/weak coupling problem, among many others observed in the literature, e.g. the 3/4 problem in the thermal Yang-Mills entropy [41], and the 3-loop disagreement of the near plane-wave string spectrum [42], which are results obtained in different regimes of couplings, and are probably explained by the presence of such interpolating functions.

We can have other more general solutions corresponding to multiple disks, as in Fig. 4(a). The different configurations in the disk picture match the different Higgs vacua for scalar  $\Phi$  as we discussed in Sec. II A 2. One can also consider strings propagating near the tip in a multidisk solution. In that case, the actual value of the interpolating function  $f$  in the strong coupling regime, which is related to the position of the tip of the disk, is not universal, in the



sense that it depends on the vacuum we expand around. What is universal, however, is the fact that the expansion around any of the tips gives us the IIA plane wave (2.41) as long as there are no other nearby disks. The situation when we consider many disks together will be discussed in the next section.

### 3. Solutions for two or more nearby tips

If there are nearby disks, then we can expand the solution near the tips of these disks and also include the fivebrane region between them. Consider, for example, a configuration with two nearby disks such as shown in Fig. 6(c). The holomorphic function  $z(w)$  in (2.35)–(2.39) is given by

$$\partial_w z = \frac{(w - ia)(w + ib)}{w} \quad (2.59)$$

with  $a, b$  real and positive. We see that for  $w \approx ia, -ib$  and for  $w \rightarrow \infty$  we recover the results we expect for single disks (2.40). This transformation maps the  $w$  right half plane (with  $\text{Re}(w) \geq 0$ ) to the  $z$  plane with two cuts. The points  $w = ia, -ib$  map to the two tips and  $w = 0$  maps to  $\text{Re}(z) \rightarrow -\infty$  between the two disks, which is expected to look like a fivebrane. In fact, we can check that the function  $f$  in (2.35)–(2.39) is given by

$$f = 1 + \frac{ab}{|w|^2}, \quad (2.60)$$

which means that we have a single center fivebrane solution. The 5-branes are located at  $w = 0$  as expected. One can also check that the fivebrane charge is proportional to the distance between two disks as in (2.26)

$$\text{Im}(\Delta z) = \text{Im} \int_{-ib}^{ia} \partial_w z dw = \pi ab. \quad (2.61)$$

In addition we find a contribution to  $g_{++}$  of the form

$$4f^{-1}|\partial_w z|^2 = 4 \frac{|w - ia|^2 |w + ib|^2}{|w|^2 + ab}. \quad (2.62)$$

When we consider a string moving on this geometry in light cone gauge we find that (2.62) appears as a potential for the world sheet fields. Notice that the minima of the potential are precisely at the two tips of the two disks corresponding to  $w = ia$  and  $w = -ib$  where we can take pp-wave limit.

When  $a = b$  we have a symmetric situation where the two disks have precisely the same length (same value of  $\rho_i$ ). In this case we see that the two minima are on the two sides of the fivebrane at equal distance between them. Notice that the throat region of the fivebrane corresponds to the region between the disks. This throat region is

singular in our approximation since the dilaton blows up as  $w \rightarrow 0$ . This is not physically significant since this lies outside the range of our approximation, since  $-\text{Re}(z)$  diverges. In fact, in the region between the disks we should actually match onto the fivebrane solution (2.44)–(2.47).

If  $a \neq b$ , say  $a > b$  for example, then we have an asymmetric configuration where one disk is larger than the other. The larger disk is the one whose tip is at  $w = a$ . If  $a \gg b$  then we find that the tip corresponding to the smaller disk is in the throat region of the fivebrane while the tip corresponding to the larger disk is in the region far from the fivebrane throat.

If we have  $n$  nearby disks, then the general solution is

$$\partial_w z = \frac{(w - ia_1)(w - ia_2) \cdots (w - ia_n)}{(w - ic_1)(w - ic_2) \cdots (w - ic_{n-1})} \quad (2.63)$$

with  $a_1 < c_1 < a_2 < c_2 < \cdots < c_{n-1} < a_n$ ,

where  $w = ia_i$  are the location of  $n$  tips and  $w = ic_i$  are the locations of  $n - 1$  sets of fivebranes. The resulting solution (2.35)–(2.39) describes a multicenter configuration of fivebranes on a plane wave. Boosting away the  $+$  components of all fields we find that we end up with a multi centered configuration of fivebranes where the  $SO(4)$  symmetry is broken to  $SO(3)$ , in fact all fivebranes are sitting along a line.

### 4. Solutions for $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3/Z_k$

In this section we consider some aspects of the gravity solutions describing  $\mathcal{N} = 4$  super Yang-Mills theory on  $R \times S^3/Z_k$ . This theory is particularly interesting since it is a very simple orbifold of  $\mathcal{N} = 4$  SYM, so that one could perhaps analyze in more detail the corresponding spin chains.

Let us start with the simplest solution, which is  $\text{AdS}_5/Z_k \times S^5$ . If the orbifold is an ordinary string orbifold, then there is a  $Z_k$  quantum symmetry. On the field theory side, this orbifold corresponds to considering a vacuum where the holonomy matrix  $U$  has  $n_l = N/k$  [see the notation around (2.4)] and we need to start with an  $N$  which is a multiple of  $k$ . This is the configuration which corresponds to the regular representation of the orbifold group action in the gauge group, see [25]. This is the simplest orbifold to consider from the string theory point of view. Other choices for the holonomy matrix  $U$ , such as  $U = 1$ , lead to an orbifold which is not the standard string theory orbifold. Such an orbifold can be obtained from the string theory one by turning on twisted string modes living at the singularity.

$\text{AdS}_5/Z_k \times S^5$  in type IIB can be dualized to an  $M$ -theory or IIA configuration which preserves the same supersymmetries as our ansatz. Let us first understand the  $M$ -theory description. Let us first single out the circle

where  $Z_k$  is acting. Then we lift IIB on this circle to  $M$ -theory on  $T^2$ . This  $T^2$  is parametrized by the coordinates  $x_1, x_2$  of the general  $M$ -theory ansatz in [14]. The solution obtained in this fashion is independent of  $x_1, x_2$ . The general solution of (2.15) with translation symmetry along  $x_1, x_2$  is<sup>12</sup>

$$e^D = c_1 y + c_2, \tag{2.64}$$

$$c_1 = \frac{g_s k}{2}, \quad c_2 = \frac{\pi g_s N}{4}. \tag{2.65}$$

Equivalently we can view the configuration as an electrostatic configuration where

$$V = -\frac{\pi N}{2k} \log \rho + V_b, \quad V_b = \frac{1}{g_s k} (\rho^2 - 2\eta^2), \tag{2.66}$$

which means that we have a line of charge at the  $\rho = 0$  axis in the presence of the external potential  $V_b$ .

These solutions are singular at  $y = 0$  since we are not obeying (2.16). At  $y = 0$  we find that  $4\rho_0 = R_{S^5}^2 = \sqrt{4\pi g_s N \alpha'^2}$ . In the IIB variables this singularity is simply the  $Z_k$  orbifold fixed point. We also find that the radius of the two torus is  $R_{x_1} = g_s$  and  $R_{x_2} = 1/g_s$ . This is as we expect when we go from IIB to  $M$  theory.

The map between the IIB and IIA solutions is simply a  $T$ -duality along the circle where  $Z_k$  acts by a shift  $\psi \sim \psi + \frac{4\pi}{k}$ . If  $k$  is sufficiently large it is reasonable to perform this  $T$  duality, at least for some region close to the singularity. Once we are in the IIA variables, we can allow the solution to depend on  $\eta$ . In fact, this dependence on  $\eta$  allows us to resolve the singularity and get smooth solutions. The electrostatic problem is now periodic in the  $\eta$  direction. We have a periodic configurations of disks, see Fig. 3(d), in the presence of an external potential of the form  $V_b$  in (2.66). Note that the external potential is not periodic in  $\eta$ . This is not a problem since the piece that determines the charge distribution on the disks is indeed periodic in  $\eta$ . Furthermore, the derivatives of  $V$  that appear in (2.20)–(2.24) are all periodic in  $\eta$ .<sup>13</sup> In the IIA picture the region between the disks can be viewed as originating from NS fivebranes. These NS fivebranes arise from the  $A_{k-1}$  singularity of the IIB solution after doing  $T$ -duality [43] (see also [44]). In fact, the period of  $\eta$  is proportional to  $k$ , so that we have  $k$  fivebranes  $N_5 = k$ . From this point of view

<sup>12</sup> $\alpha' = 1$ . This solution, if considered in the class of the analytically continued solutions in [14], describes  $\text{AdS}_5 \times S^5/Z_k$ .

<sup>13</sup>The  $\eta$  dependent piece in (2.26) ensures that as we go over the period of  $\eta$  we go over the period of  $x_2$  which is  $T$ -dual to the circle on which the  $Z_k$  acted.

the simplest situation is when all fivebranes are coincident. This corresponds to taking the matrix  $U$  proportional to the identity. On the other hand, the standard string theory orbifold corresponds to the case that we have  $k$  equally spaced disks separated by a unit distance. In other words, the fivebranes will all be equally spaced. In this case, since we have single fivebranes, we do not expect the geometric description to be accurate. Note that even though we are talking about these fivebranes, the full solution is non-singular. These fivebranes are a good approximation to the solution when we have large disks that are closely spaced, as we will see in detail below. But as we go to  $\rho \rightarrow 0$  the solution between the disks approaches the NS5 solution (2.44)–(2.47), which is nonsingular. The different vacua (2.4) correspond to the different ways of assigning charges  $n_l$  (see notation around (2.4)) to the disks that sit at positions labeled by  $\eta \sim l$ . There are  $k$  such special positions on the circle. Only in cases where we have coincident fivebranes can we trust the gravity description. This happens when some of the  $n_l$  are zero.

If we take the  $k \rightarrow \infty$  limit, keeping  $N$  finite, then the direction  $\eta$  becomes noncompact and we go back to the configurations considered in the previous section which are associated to the D2 brane theory (2 + 1 SYM) of Sec. IIA 2. This is also what we expected from the field theory description.

We were not able to solve the equations explicitly in this case. On the other hand, there are special limits that are explicitly solvable. These correspond to looking at the large  $N$  limit so that the disks are very large and then looking at the solution near the tip of the disks. Let us consider the case where we have a single disk per period of  $\eta$ . We can find the solution by using (2.63) and we get

$$\partial_w z = ik \prod_{n=-\infty}^{\infty} \frac{(w - ia n)}{(w - ia(n + \frac{1}{2}))} = k \tanh \frac{\pi w}{a}, \tag{2.67}$$

where  $k$  is the number of coincident fivebranes. When we insert this into (2.35)–(2.39) we find that the the solution corresponds to a periodic array of  $k$  NS fivebranes along spatial direction  $\chi$ .

$$f = \frac{k \sinh r}{2r(\cosh r + \cos \chi)} = \sum_{n=-\infty}^{\infty} \frac{k}{r^2 + (\chi + \pi + 2\pi n)^2}, \tag{2.68}$$

$$r + i\chi \equiv \frac{2\pi}{a} w, \quad \chi \sim \chi + 2\pi, \tag{2.69}$$

$$g_{++} = 8k \frac{r}{\sinh r} (\cosh r - \cos \chi). \tag{2.70}$$

The rim of the disks corresponds to  $w = ia$  or  $r = \chi = 0$  in (2.68). The  $g_{++}$  term in the metric (2.35)–(2.39) implies

that the light cone energy is minimized by sitting at these points. These points lie between the fivebranes, which sit at  $r = 0$ ,  $\chi = \pi$ . In flat space the  $T$ -dual of an  $A_{k-1}$  singularity corresponds to the near horizon region of a system of  $k$  fivebranes on a circle. Here we are getting a similar result in the presence of RR fields. As  $w \rightarrow \infty$  the solution (2.35)–(2.39) approaches the one that is the  $T$ -dual of the orbifold of a pp-wave with  $R^4 \times R^4/Z_k$  transverse dimensions

$$ds_{10}^2 = -2dx^+ dx^- - (\vec{r}^2 + \vec{u}^2)(dx^+)^2 + d\vec{r}^2 + du^2 + \frac{u^2}{4} d\Omega_2^2 + \frac{k^2}{u^2} d\chi^2. \quad (2.71)$$

At large  $u$  we can  $T$ -dual this back to the  $Z_k$  quotient of the IIB plane wave, a situation studied in [45,46].

Let us understand first the theory at the standard string theory orbifold point. This corresponds to the vacuum with  $n_l = N/k$ , for all  $l = 1, \dots, k$ . As we mentioned above, it is useful to view the Yang-Mills theory on  $R \times S^3/Z_k$  as the orbifold of the theory on the brane according to the rules in [26]. According to those rules we need to pick a representation of  $Z_k$  and embed it into  $U(N)$ . The regular representation then gives rise to the vacuum where all  $n_l$  are equal. For this particular choice we can use the inheritance theorem in [47] that, to leading order in the  $1/N$  expansion, the spectrum of  $Z_k$  invariant states in the orbifold theory is exactly the same as the spectrum of invariant states in  $\mathcal{N} = 4$ . This ensures that the matching between the string states on the orbifold and those of the Yang-Mills theory is the same as the corresponding matching in  $\mathcal{N} = 4$ . In the IIA description this regular orbifold goes over to a picture where we have  $k$  fivebranes uniformly spaced on the circle. In this case we cannot apply our gravity solutions near the fivebranes because we have single fivebranes. Furthermore, we expect that the orbifold picture should be the correct and valid description for string states even close to the orbifold point, as long as the string coupling is small. The spectrum of string states involving the second four dimensions (the orbifolded ones) can be thought of as arising from  $E - J = 1$  excitations which get a phase of  $e^{\pm i2\pi/k}$  under the generator of  $Z_k$ , but we choose a combination of these excitations that is  $Z_k$  invariant. This discussion is rather similar to the one in [48], where the  $\text{AdS}_5 \times S^5/Z_k$  orbifold (see e.g. [49]) was studied.

We can now consider other vacua. These are associated to different representations for the Wilson line. For example, we can choose  $n_k = N$  and  $n_i = 0$  for  $i \neq k$ . In this case the IIA gravity description can be trusted when we approach the origin as long as the 't Hooft coupling is large and  $k$  is large enough. Let us describe the physics in the pp-wave limit in more detail for this case. The pp-wave limit that we are considering consists in taking  $k$  fixed and somewhat large, so that the gravity description of the  $k$  coincident fivebranes is accurate, and then taking  $J$  and  $N$

to infinity with  $J^2/N$  fixed, exactly as in  $\mathcal{N} = 4$  super Yang-Mills theory [3]. In fact, we find that the world sheet theory in the first four directions is exactly the same as for  $\mathcal{N} = 4$  super Yang-Mills theory. In particular, the dispersion relation for light cone gauge world sheet excitations is precisely as in  $\mathcal{N} = 4$  super Yang-Mills theory [3], with the same numerical coefficient. The theory in the remaining four directions is more interesting. At large distance from the origin the world sheet field theory is just the orbifold of the standard IIB plane wave [20]. This is what we had for the regular representation vacuum that we discussed above. A string state whose world sheet is far from the origin, so that its IIB description is good, is a very excited string state. It is reasonable to expect that the spectrum of such states is not very sensitive to the vacuum we choose. This is what we are finding here, since the spectrum in this region is that of the vacuum of the regular representation we discussed above. On the other hand, as we consider string states where the string is closer to the minimum of its world sheet potential we should use the IIA description in terms of  $k$  coincident fivebranes, using the solution in (2.68). In this case the spectrum of excitations on the string world sheet is rather different than what we had at the standard orbifold point. In this case we have excitations of world sheet mass  $E - J = 2$  which are  $Z_k$  invariant. This spectrum matches with what we naively expect from considering impurities propagating on the string for the vacuum we are considering. This vacuum contains only single particle gauge theory excitations with  $E - J \geq 2$  for all fields that could be interpreted as excitations that are associated for the second four dimensions. Let us be a bit more explicit. We can identify some of these  $E - J = 2$  excitations as the Kaluza Klein modes of  $Z$  given by  $\epsilon^{\beta\beta'} \partial_{\alpha\beta} \partial_{\alpha'\beta'} Z$ . This gives a singlet under  $SU(2)_L$ , so that the  $Z_k \subset SU(2)_L$  acts trivially. So this Kaluza Klein mode survives the  $Z_k$  quotient. The  $\alpha, \alpha'$  indices give rise to a spin one mode under  $SU(2)_R \subset \widetilde{SU}(2|4)$ .<sup>14</sup> There is a spin zero excitation with  $E - J = 2$  which comes from the mode of the four dimensional gauge field along the  $\psi$  circle, the circle we are orbifolding. These elementary fields have  $E - J = 2$  and are associated to the  $E - J = 2$  excitations of the last four dimensions of the IIA plane wave. An analysis similar to the one we will discuss for the plane-wave matrix model and 2 + 1 SYM in Sec. II C shows that these excitations are exactly BPS and survive in the strong 't Hooft coupling limit.

Other gravity solutions which are asymptotic to  $\text{AdS}_5/Z_k$  were constructed in [50]. Those solutions have a form similar to that of the Eguchi-Hanson instanton [51] in the four spatial directions. In those solutions fermions are antiperiodic along the  $\psi$  direction. In our case, fermi-

<sup>14</sup>The spin zero mode under both  $SU(2)_{L,R}$  vanishes due to the equation of motion.

ons are periodic in the  $\psi$  circle. So, the solutions in [50] arise when we consider a slightly different field theory. Namely, when one considers Yang-Mills theory on  $R \times S^3/Z_k$  but where the fermions are antiperiodic along the circle on which  $Z_k$  acts. (One should also restrict to  $k$  even). This theory breaks supersymmetry. The solutions in [50] describe states (probably the lowest energy states) of these other theories. In such cases the orbifold is another state in the same theory, the theory with antiperiodic fermion boundary conditions along  $\psi$ . One then expects that localized tachyon condensation, of the form explored in [52], makes the orbifold decay into the solutions described in [50].

**5. Solutions for the plane-wave matrix model**

In this section we discuss some aspects of the gravity solutions corresponding to the plane-wave (or BMN) matrix model. In this case we should think of the electrostatic configuration as having an infinite disk at  $\eta = 0$  and the some finite number of disks of finite size at  $\eta_i > 0$ , see Fig. 4(b). The background electric potential is

$$V_b = \rho^2 \eta - \frac{2}{3} \eta^3. \tag{2.72}$$

The leading asymptotic form of the solution is

$$V = V_b + P \frac{\eta}{(\rho^2 + \eta^2)^{3/2}}, \tag{2.73}$$

where we have included the external potential plus the leading dipole moment produced by the disks. The leading contribution is a dipole moment because the conducting disk at  $\eta = 0$  gives an image with the opposite charge, so that there is no monopole component of the field at large  $\rho, \eta$ . The subleading terms in the asymptotic region are higher multipoles.

We can insert this into the general ansatz (2.20)–(2.24) and we find that in the UV region it goes over to the UV region of the solution for  $N_0$  D0 branes, where  $N_0$  is proportional to the dipole moment  $P$ . More details are in Appendix D. This dipole moment is given by

$$P = 2 \sum_i \eta_i Q_i \sim N_0 = \sum_i \left( \sum_{j < i} N_5^j \right) N_2^i, \tag{2.74}$$

where the index  $i$  runs over the various disks. Notice that the difference between neighboring disks  $d_i = \eta_{i+1} - \eta_i$  is proportional to the fivebrane charge. So the distances  $d_i$  are quantized. This formula, (2.74), should be compared to (2.12) by identifying  $n \sim d_i$  and  $N(n) = N_2^i$ .

In [53] this problem was analyzed using technique developed by Polchinski and Strassler in [54], which consists in starting with configurations of D0 branes smeared on two spheres. In our language, this is a limit when we replace the disks by point charges sitting at  $\rho = 0$ . This approximation is correct as long as the distance between

the disks is much bigger than the sizes of the disks and we look at the solution far away from the disks.<sup>15</sup>

From the field theory point of view it looks like the simplest vacuum is the one with all  $X = 0$ . This case corresponds to having  $N_0$  copies of the trivial (dimension one) representation of  $SU(2)$ . In the gravity description this corresponds to having a single disk at a distance of one unit from the conducting surface at  $\eta = 0$ , see Fig. 6(a). Unfortunately, since this vacuum corresponds to a single fivebrane, the gravity approximation will not be good near the fivebrane. We will focus on this situation in the next section. However, we can consider vacua corresponding to many copies of dimension  $N_5$  representations of  $SU(2)$ . These involve  $N_5$  fivebranes and we will be able to give interesting solutions, at least in the region relevant for the description of near BPS states. It should be possible to extrapolate these solutions to smaller values of  $N_5$  using conformal field theory. Let us now study the case that we have only a single disk at a small distance from the infinite disk at  $\eta = 0$ . So we consider a situation with  $N_5 \ll N_0$ . Based on the discussion in [4] we expect that the  $N_0$  D0 branes blow up into  $N_5$  NS5 branes. Of course, our solution will be smooth, but we will see that there is a sense in which we have  $N_5$  fivebranes. The appearance of fivebranes is probably connected with the picture in [55] for 1/2 BPS states in terms of eigenvalues that lie on a five sphere.

Unfortunately we were not able to find the full solution of Eq. (2.18). Nevertheless we can expand the solution near the tip of the disk. In fact, we can get the solution in a simple manner by starting from the solution corresponding to the region near the tip of two disks (2.59) and then letting one of the disks go to infinity. After a simple rescaling this produces

$$\partial_w z = i \frac{(w - ia)}{w}. \tag{2.75}$$

In this case, the function  $f$  in (2.35)–(2.39) becomes

<sup>15</sup>We can make this relation more precise as follows. Suppose that the potential in the asymptotic region behaves as  $V = \rho^2 \eta - \frac{2}{3} \eta^3 + \Delta$ , where will treat  $\Delta$  as a perturbation. Then from the IIA ansatz (2.20)–(2.24) we can write the solution as in [53] and find the warp factor  $Z$  in [53]. This gives  $Z = \frac{-1}{2\rho^2 \eta} \times (\frac{2}{\rho} \partial_\rho \Delta + \partial_\eta^2 \Delta)$  and  $B_2 = \frac{1}{\rho} \partial_\rho (\frac{4\eta^2}{\rho} \partial_\rho \Delta + 2\eta \partial_\eta \Delta - 2\Delta) d^2 \Omega$ . We get  $\Delta = \frac{P\eta}{(\rho^2 + \eta^2)^{3/2}}$  by comparing the leading order approximation  $Z = \frac{R^7}{r^7}$  and the fluxes  $H_3 = \alpha r^{-7} (T_3 - \frac{7}{3} V_3)$ ,  $G_6 = g_s^{-1} \alpha r^{-7} (\frac{1}{3} *_9 T_3 - \frac{7}{3} *_9 V_3)$ , which are dual to the mass terms (see [53]), where  $r = (\rho^2 + \eta^2)^{1/2}$ , and  $\rho$  and  $\eta$  are radial variables in  $SO(6)$  and  $SO(3)$  directions. If we replace  $Z$  by the expression corresponding to multicenter D0 branes uniformly smeared on several  $S^2$ s, then we get  $\Delta = \sum_i Q_i \left[ \frac{1}{[(\eta - \eta_0^{(i)})^2 + \rho^2]^{1/2}} - \frac{1}{[(\eta + \eta_0^{(i)})^2 + \rho^2]^{1/2}} \right]$ , this is precisely the limit when the disks above the  $\eta = 0$  plane are treated as point charges.



$$f = \frac{a}{2|w|^2} \quad (2.76)$$

and the contribution to  $g_{++}$  is

$$4f^{-1}|\partial_w z|^2 = \frac{8}{a}|w - ia|^2. \quad (2.77)$$

So we see that we get the near horizon region of fivebranes. The contribution (2.77) to the  $g_{++}$  metric component gives rise to a potential on the light cone gauge string world sheet. This potential localizes the string at some particular position along the throat. Writing  $w = ia e^{\phi + i\theta}$ , the 10 dimensional solution is<sup>16</sup>

$$\begin{aligned} ds_{10}^2 = & -2dx^+ dx^- + d\vec{r}^2 - \vec{r}^2 dx^{+2} \\ & - 4N_5(e^{2\phi} + 1 - 2e^\phi \cos\theta)dx^{+2} \\ & + N_5(d\phi^2 + d\theta^2 + \sin^2\theta d\Omega_2^2), \end{aligned} \quad (2.78)$$

$$e^\Phi = g_s e^{-\phi}, \quad (2.79)$$

$$C_1 = -\frac{1}{g_s} 2\sqrt{N_5}(e^{2\phi} - e^\phi \cos\theta)dx^+, \quad (2.80)$$

$$C_3 = \frac{1}{g_s} N_5^{3/2} 2e^\phi \sin^3\theta dx^+ \wedge d^2\Omega, \quad (2.81)$$

$$H_3 = 2N_5 \sin^2\theta d\theta \wedge d^2\Omega, \quad (2.82)$$

where  $g_s$  is the value of the dilaton at the tip. By performing a boost  $x^\pm \rightarrow \lambda^{\pm 1} x^\pm$  with  $\lambda \rightarrow 0$  we set to zero all nontrivial terms involving  $dx^+$  and we recover the usual

fivebrane near horizon geometry [13]. By taking a limit of small  $\phi$  and  $\theta$  we find the IIA plane-wave in (2.41).

An important parameter is the size of the  $S^5$  in string theory units at the tip of the disks. This can be approximated as<sup>17</sup> (see Appendix D)

$$\frac{R_{S^5}^2}{\alpha'} = 4\pi \left( \frac{g_{\text{YM}0}^2 N_2}{2m^3} \right)^{1/4}, \quad N_2 = \frac{N_0}{N_5}, \quad m = 1, \quad (2.83)$$

where  $m$  is the mass of the  $SO(6)$  scalars and is set to 1.  $N_0$  is the number of D0 branes or the rank of the gauge group in the plane-wave matrix model. Our gravity approximation is good when we are in the regime of interest,  $N_5 \ll N_0$ , and the size of  $S^5$  in string unit is large. From this result we can compute the spectrum of near BPS excitations with large angular momentum  $J$ . For fluctuations in the directions parametrized by  $\vec{r}$  in (2.78) the spectrum is

$$\begin{aligned} (E - J)_n &= \sqrt{1 + (4\pi)^2 \left( \frac{g_{\text{YM}0}^2 N_0}{2m^3 N_5} \right)^{1/2} \frac{n^2}{J^2}} \\ &= 1 + 4\pi^2 \left( \frac{2g_{\text{YM}0}^2 N_0}{m^3 N_5} \right)^{1/2} \frac{n^2}{J^2} + \dots \end{aligned} \quad (2.84)$$

Under general principles, in the  $t'$  't Hooft limit, with  $N_5$  fixed, we expect the spectrum to be of the form

$$(E - J)_n = 1 + f\left(\frac{g_{\text{YM}0}^2 N_0}{m^3 N_5}, N_5\right) \frac{n^2}{J^2} + \dots \quad (2.85)$$

in the large  $J$  limit.

The  $N_5 = 1$  case has been analyzed perturbatively up to four loops in [10]. In our conventions<sup>18</sup> their result reads

$$f_{\text{pert}}\left(\frac{g_{\text{YM}0}^2 N_0}{m^3}, N_5 = 1\right) = \frac{2\pi^2 g_{\text{YM}0}^2 N_0}{m^3} \left[ 1 - \frac{7}{8} \frac{g_{\text{YM}0}^2 N_0}{m^3} + \frac{71}{32} \left( \frac{g_{\text{YM}0}^2 N_0}{m^3} \right)^2 - \frac{7767}{1024} \left( \frac{g_{\text{YM}0}^2 N_0}{m^3} \right)^3 + \dots \right]. \quad (2.86)$$

Of course we expect that the function  $f$  interpolates smoothly between the weak coupling result (2.86) and the strong coupling result (2.84).

Our gravity solutions are not valid for  $N_5 = 1$ , especially in the region relevant for this computation. On the other hand, we see that the quantity that determines  $f$  is the radius of the fivesphere. We can think of this solution as follows. Let us first use an approximation similar to that used by Polchinski and Strassler [53,54]. In this case we approximate the solution by smearing D0 branes on a fivesphere, which we interpret as a fivebrane which carries D0 brane charge. We then determine the size of the fivebrane by coupling it to the external fields that are responsible for inducing the mass on the D0 worldvolume. This gives the radius of the fivebrane. In fact, this was computed

in [4] where the formula similar to (2.83) was found (the precise numerical factors were not computed in [4]). So it is natural to believe that (2.84) will still be the correct answer for  $N_5 = 1$ . In other words, the coupling constant “renormalization” that was found in [10] is interpreted here as a physical quantity giving us the size of the fivebrane in the gravity description at strong coupling. This is the situation for the first four coordinates. The fact that a single fivebrane has no near horizon region also suggests that something drastic happens to the second four direc-

<sup>17</sup>Our normalization of the action is  $S = \frac{1}{g_{\text{YM}0}^2} \int \text{Tr} \{ \frac{1}{2} (D_0 Y_i)^2 - \frac{1}{2} m^2 Y_i^2 + \frac{1}{4} [Y_i, Y_j]^2 + \dots \}$  where  $Y_i$  are the  $SO(6)$  scalars. The dimensionless parameter is  $g_{\text{YM}0}^2/m^3$ .

<sup>18</sup>The relations between their variables and ours are  $4\Lambda = \left(\frac{2}{M}\right)^3 N = \frac{g_{\text{YM}0}^2 N_0}{m^3}$ ,  $8\pi^2 \Lambda_r = f$ .

<sup>16</sup>We set  $\alpha' = 1$  in this paper.

tions that are transverse to the single fivebrane. We observed this feature for  $N_5 = 1$  also from gauge theory side and will explain it in the next section.

Finally, let us discuss the issue of tunneling between different vacua. In general we can tunnel between the different vacua of the matrix model. But the tunneling can be suppressed in some regimes. For example, let us consider the case we discussed above where we consider the vacuum corresponding to a single large disk at a distance  $N_5$  from the  $\eta = 0$  plane, see Fig. 6(a). From the gravity point of view we can take one unit of charge from the large disk and put some other disks. Charge is not conserved in the process, but  $N_0$  should be conserved. Reducing the charge of the large disk by one unit we are left with  $N_5$  D0 branes to distribute in the geometry. So, for example, we can put another disk at a distance of one unit from the  $\eta = 0$  plane with  $N_5$  units of charge. In the geometry this transition is mediated by a D-brane instanton. The geometry between the original disk and the  $\eta = 0$  plane can be approximated by the solution in Sec. II B 1. That solution contains a noncontractible  $\Sigma_3$ , see Fig. 5. If we wrap a Euclidean D2 brane on this  $\Sigma_3$  we find that, since there is flux  $N_5$  through it, we need  $N_5$  D0 branes ending on it [34]. Thus, this instanton describes the creation of  $N_5$  D0 branes. Its action is proportional to the action of the Euclidean D2 brane. This process describes the tunneling between the vacua in Figs. 4(d) and 4(e). If the volume of the  $\Sigma_3$  is sufficiently large and the string coupling is sufficiently small this process will be suppressed. In order for this to be the case we need to arrange the field theory parameters appropriately. Notice that there is no instanton that produces a smaller number of D0 branes. This also agrees with the field theory. If we start with the vacuum with many copies of the  $N_5$  dimensional representation of  $SU(2)$ , then we can take one of these representations and partition those  $N_5$  D0 branes into lower dimensional representations. This is basically the process described by the above instanton. In other words, the fact that the D-brane instanton produces  $N_5$  D0 branes matches with what we get in the field theory.

**C. Further analysis of near BPS states**

In previous sections we have mainly analyzed the near BPS states associated to string oscillations in the the first four dimensions, which are described by free massive fields on the world sheet. In this section we mainly focus on the second four dimensions which are associated to fivebrane geometries. Since the spectrum depends on the vacuum we expand around, we will focus on the large  $J$  near BPS excitations around some particular vacua of the plane-wave matrix model. We will consider first the  $N_5 = 1$  vacuum and then the  $N_5 > 1$  vacua, both from the gauge theory and gravity points of view. We also make some remarks about the simplest vacuum of the 2 + 1 super Yang-Mills theory on  $R \times S^2$ .

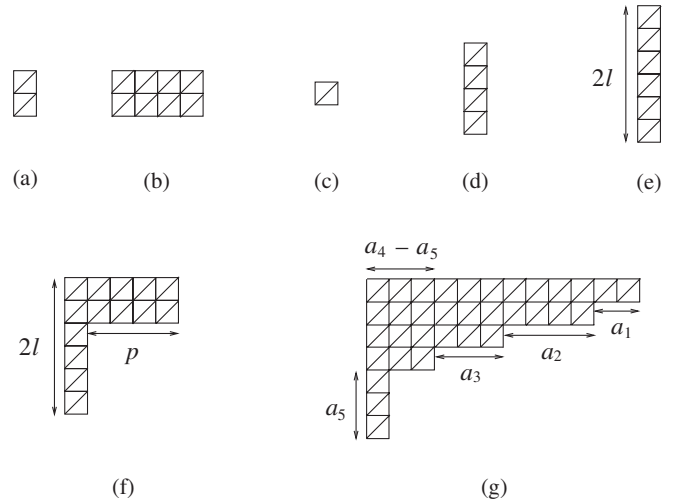


FIG. 7. Young supertableaux corresponding to various representations of  $SU(2|4)$  or  $SU(2|2)$  discussed in the text. In (e) and (f),  $2(l - 2) = a_5$ ,  $p = a_2$  for  $SU(4|2)$  Dynkin labels. Figure (g) shows the correspondence between supertableau and Dynkin labels for a general physically allowed representation  $(a_1, a_2, a_3|a_4|a_5)$  of  $SU(4|2)$ , see also [7,56].

**1.  $N_5 = 1$  vacua of the plane-wave matrix model**

Let us start by discussing the trivial vacuum of the matrix model, where we expand around the classical solution where all  $X = 0$ . This is the vacuum we denote by  $N_5 = 1$  and which should correspond to a single fivebrane. When we expand around this vacuum we have 9 bosonic and 8 fermionic excitations which form a single representation of  $\widetilde{SU}(2|4)$ , corresponding to the Young supertableau in Fig. 7(a). Our notation for  $\widetilde{SU}(2|4)$  representations follows the one in [7,56]. We are interested in forming single trace excitations which should correspond to single string states in the geometry. For example, we can consider the state created by the field  $Z$  of the form  $\text{Tr}[Z^J]$ ,<sup>19</sup> where  $Z = Y^5 + iY^6$ .<sup>20</sup> This state is BPS and it belongs to the doubly atypical (or very short) representation whose Young supertableau is shown in Fig. 7(b). As in [3] we can consider near BPS states by writing states of roughly the form  $\sum_l \text{Tr}[Y^i Z^l Y^j Z^{J-l}] e^{i2\pi(ln/J)}$  where  $i, j = 1, \dots, 4$ . We can view each insertion of the field  $Y^i$  as an ‘‘impurity’’ that propagates along the chain formed by the  $Z$  oscillators. These impurities are characterized by the momentum  $p = n/J$  and a dispersion relation  $\epsilon(p)$ , where  $\epsilon$  is the contribution of this impurity to  $\hat{E} \equiv E - J$ . Here we are thinking about a situation where we have an infinitely long chain where boundary effects can be neglected. These fields have  $\epsilon(p = 0) = 1$ , we can think of this as the ‘‘mass’’ of the

<sup>19</sup>We denote the field  $Z$  and its creation operator by the same letter.

<sup>20</sup>In this section  $Y^i, i = 1, \dots, 6$  are the scalars that transform under  $SO(6)$  and  $X^i$  are the ones transforming under  $SO(3)$ .

particles. This is an exact result and can be understood as a consequence of the Goldstone theorem. Namely, when we pick the field  $Z$  and we construct the ground state of the string with powers of  $Z$  we are breaking  $SO(6)$  to  $SO(4)$ . The excitations  $Y^i$ ,  $i = 1, \dots, 4$  correspond to the action of the broken generators. This is a fact that does not even require supersymmetry. In other words, we are simply rotating the state  $\text{tr}[Z^J]$ . It is also useful to consider the supersymmetry that is preserved by this chain. Out of the supergroup  $\widetilde{SU}(2|4)$  our choice of  $Z$  leaves an  $SU(2)_G \times \widetilde{SU}(2|2)$  subgroup<sup>21</sup> that acts on the excitations that propagate along the string. The group  $SU(2)_G$  together with one of the  $SU(2)$  subgroups in  $\widetilde{SU}(2|2)$  forms the  $SO(4)$  in  $SO(6)$  that rotates the first four dimensions. The second  $SU(2)$  subgroup of  $\widetilde{SU}(2|2)$  is the  $SU(2)$  factor in  $\widetilde{SU}(2|4)$  and rotates the three scalars  $X^i$ . We can use  $\widetilde{SU}(2|2)$  to classify these excitations. The noncompact  $U(1)$  in  $\widetilde{SU}(2|2)$  corresponds to the generator  $\hat{E} = E - J$  and gives us the mass of the particle. The fields  $Y^i$  belong to the fundamental representation of  $\widetilde{SU}(2|2)$  whose Young supertableau is in Fig. 7(c). In addition they transform in the spin one half representation of  $SU(2)_G$ . We can think of these excitations as “quasiparticles” that propagate along the string. The properties of these quasiparticles were studied in great detail in [10] where the dispersion relation and particular components of the  $S$  matrix were computed to four loops. These quasiparticles contain four bosons and four fermions.

So far we have been discussing mainly the fields  $Y^i$  and the fermions which have  $E - J = 1$ . What about the other fields in the theory? There are four other elementary fields which have  $\Delta - J = 2$ . These are the three scalars  $X^i$  of  $SO(3)$  and the field  $\bar{Z}$  plus four fermions. Naively, we might think that these would lead to mass two impurities that propagate along the string. This is not the case. Actually, what happens is that they mix with the fields that we have already described and do not lead to new quasiparticles [57]. For example an insertion of the field  $\bar{Z}$ , such as  $\text{tr}[\bar{Z}Z^{J+1}]$  mixes with the states  $\text{tr}[Y^i Z^l Y^i Z^{J-l}]$ . The result of this mixing is such that the resulting spectrum can be fully understood in terms of two quasiparticles of mass one that propagate along the string. Something similar happens with the insertion of the  $SO(3)$  scalar  $X^i$ , which mixes with the insertion of two fermions of individual mass one. In fact, the one loop Hamiltonian in this sector is a truncation of the one loop Hamiltonian of  $\mathcal{N} = 4$  SYM in [58]. So the results we are mentioning here follow in a direct way from the explicit diagonalization undertaken in [57]. The final conclusion of this discussion, is that in perturbation theory we have a chain which contains impurities with mass one, that transform in the fundamental of

$\widetilde{SU}(2|2)$  and fundamental of  $SU(2)_G$ . We have four bosons and four fermions, which can be viewed as the Goldstone modes of the symmetries broken by the BPS operator  $\text{tr}[Z^J]$ . This spectrum is compatible with the index (2.1) evaluated on single trace states.

Let us now discuss what happens at large 't Hooft coupling. The radius of the fivebrane is given by (2.83) (with  $N_5 = 1$ ). In addition, we have seen that the near BPS states are described by the pp-wave geometry (2.78)–(2.82) which corresponds to the near horizon region of  $N_5$  fivebranes. The first four transverse dimensions correspond to the motion of the string in the direction of the fivebranes and the spectrum contains particles that transform in the fundamental of  $\widetilde{SU}(2|2)$  and the fundamental of  $SU(2)_G$  as we had in the weak coupling analysis. The dispersion relation is given by the usual relativistic formula (2.84).

On the other hand, when we consider the fate of the last four transverse dimensions we run into trouble with the geometric description. We see that the solution (2.78)–(2.82) does not make sense for  $N_5 = 1$  since a single fivebrane is not supposed to have a near horizon region [13]. The reason is that the near horizon region involves a bosonic WZW model with level  $k = N_5 - 2$  and this theory is unitary only if  $N_5 - 2 \geq 0$ . In our context, we also have RR fields that try to push the string into the near horizon region. Since for  $N_5 = 1$  we do not have such a region, the simplest assumption is that the second four dimensions are somehow not present in our pp-wave limit. This would agree with what we saw in the weak coupling analysis above, where we did not have any quasiparticles propagating along the string corresponding to the second four dimensions. Of course, a string quantized in light cone gauge is not Lorentz invariant in six dimensions. But perhaps this is not a problem in this case, since the presence of RR fields breaks Lorentz invariance. Nevertheless, one would like to understand the background in a more precise way in the covariant formalism, so that one can ensure that we have a good string theory solution.

## 2. $N_5 > 1$ vacua of the plane-wave matrix model

In order to find a better defined string theory we need to consider  $N_5 > 1$ . So, let us consider what happens when we expand around the vacuum of the plane-wave matrix model corresponding to  $N_5 > 1$ . This is the vacuum where the matrices  $X_i$  are the generators of the dimension  $N_5$  representation of  $SU(2)$ . We would like to understand the similarities and differences between these vacua and the  $N_5 = 1$  vacuum. When we expand around these vacua we find that we have  $N_5$   $\widetilde{SU}(2|4)$  supermultiplets, the ones whose Young supertableaux are given in Fig. 7(e) with  $l = 1, \dots, N_5$  [7]. We can view them as the Kaluza Klein modes on a fuzzy  $S^2$ . The subsector of this theory where we consider only excitations of the first Kaluza Klein mode is the same as the one we had in the  $N_5 = 1$  sector. In fact,

<sup>21</sup>The  $_G$  subindex indicates that it is global symmetry that commutes with supersymmetry.

the one loop Hamiltonian for these excitations is exactly the same as the one we had for the  $N_5 = 1$  case. This can be seen as follows. Since these modes are proportional to the identity matrix in the  $N_5 \times N_5$  space that gives rise to the fuzzy sphere we see that their interactions are the same as the ones we had around the  $N_5 = 1$  vacuum. The only difference could arise when we consider diagrams that come from one loop propagator corrections. But the value of these propagator corrections is determined by the condition that the energy of the state  $\text{tr}[Z^J]$  is not shifted, since it is a BPS state. One difference, relative to the expansion around the  $N_5 = 1$  vacuum is that the one loop Hamiltonian is proportional to  $g_{\text{YM}0}^2 N_2 / N_5$  as opposed to  $g_{\text{YM}0}^2 N_0$  (where  $N_0 = N_2 N_5$ ). More precisely, we find that the function  $f$  in (2.85) has the form

$$f\left(\frac{g_{\text{YM}0}^2 N_0}{m^3 N_5}, N_5\right) = 2\pi^2 \frac{g_{\text{YM}0}^2 N_0}{m^3 N_5^2} + \dots \quad (2.87)$$

for small 't Hooft coupling. We obtain this result as follows. First we notice that  $\hat{E} = 1$  excitations are given by diagonal matrices in the  $N_5 \times N_5$  blocks that produce the fuzzy sphere. These matrices are  $N_2 \times N_2$  matrices. In other words, the relevant fields can be expressed as  $Y^i = \mathbf{1}_{N_5 \times N_5} \otimes \tilde{Y}^i$  where  $\tilde{Y}^i$  are  $N_2 \times N_2$  matrices, with  $N_2 \equiv N_0 / N_5$ . Then the action truncated to the  $\tilde{Y}^i$  fields looks like the  $N_5 = 1$  action except that we get an extra factor of  $N_5$  from the trace over the diagonal matrix  $\mathbf{1}_{N_5 \times N_5}$ . This effectively changes the coupling constant  $g_{\text{YM}0}^2 \rightarrow \tilde{g}_{\text{YM}0}^2 = g_{\text{YM}0}^2 / N_5$ . Since the  $\tilde{Y}^i$  fields are  $N_2 \times N_2$  matrices we see that corrections in this subsector will be proportional to  $\tilde{g}_{\text{YM}0}^2 N_2$ . Notice that (2.87) it involves a different combination of  $N_0$  and  $N_5$  than the one that appears at strong coupling (2.84). So the interpolating function in (2.85) should have a nontrivial  $N_5$  dependence. In summary, at one loop, the excitations built out of impurities in the first Kaluza Klein harmonic on the fuzzy  $S^2$  give rise to four bosonic and fermionic quasiparticles of mass  $\hat{E} = 1$  as we had in the  $N_5 = 1$  case.

Let us now focus on the second Kaluza Klein mode, given by the supermultiplet of  $\widetilde{SU}(2|4)$  in Fig. 7(d). This multiplet contains four bosonic and four fermionic states of mass  $\hat{E} = E - J = 2$ . These eight states transform in the  $\widetilde{SU}(2|2)$  representation of Fig. 7(a). Let us describe how the bosonic states arise. We expand  $Z = \tilde{Z} + \mathcal{J}^i Z_i + \dots$  in fuzzy sphere Kaluza Klein harmonics using the  $N_5 \times N_5$  matrices  $\mathcal{J}^i$  which give a representation of  $SU(2)$  (see [59]). Three of the states correspond to the impurities  $Z_i$  and they are in the  $(1, 0)$  representations of  $SU(2) \times SU(2) \subset \widetilde{SU}(2|2)$  and they are singlets of  $SU(2)_G$ . The fourth state, denoted by  $\Phi$ , arises when we expand  $X^i = \mathcal{J}^i (\mathbf{1} + \Phi) + \dots$ . This has  $E = 2$  and spin zero under all  $SU(2)$ s. It gives rise to an excitation with  $\hat{E} = E - J = 2$

and spin zero. In addition to these four bosonic states we have their fermionic partners. When we consider BPS states with  $E - 2S - \sum_i J_i = 0$ , the only bosonic state that contributes is  $Z_+$ , which has  $S = 1$ . Thus the state  $\text{tr}[Z_+ \tilde{Z}^J]$  is BPS. In order to ensure that its energy is not corrected we need to check that it cannot combine with other BPS states. The analysis in [7] tells us which representations this could combine with. By looking explicitly at the ones arising when we construct single trace states we can see that these other representations are not present. This is a result that is exact in the planar limit. In Appendix G we use the index defined in (2.1) to prove the above statement.

What we learned is that for  $N_5 > 1$ , as opposed to the case with  $N_5 = 1$ , we have a new quasiparticle of mass two propagating along the string. In fact, the same argument would go through for the case of  $2 + 1$  SYM on  $R \times S^2$  in Sec. II A 2, expanded around the trivial vacuum, and the new supermultiplet correspond to the three derivatives  $D_i$ ,  $i = 0, 1, 2$  and the seventh scalar  $\Phi$ , and their fermionic partners. They have mass 2 and correspond to the second four coordinates of the IIA plane wave. In all these cases we have extra quasiparticles propagating along the string. This agrees with the fact that in string theory we have eight transverse directions for the string. The first four dimensions behave as we discussed above (in Sec. II C 1) and its presence is ensured by the  $SO(6)$  symmetry. The details of the second four dimensions depend on the vacuum we expand around. So let us concentrate more on these second four dimensions.

### 3. Comparison between world sheet theory and gauge theory

We will now discuss the two dimensional field theory that describes the second set of four transverse dimensions for a string in light cone gauge moving in the pp-wave geometry (2.78)–(2.82). The target space for this two dimensional theory is  $R \times S^3$  with an  $H_3$  flux on the  $S^3$  equal to  $N_5$  and a linear dilaton in the  $R$  direction. These are the dimensions parametrized by  $\phi, \theta, \Omega_2$  in (2.78). In addition we have a potential which localizes the string at some point along the throat and at a point in  $S^3$ . This potential arises from the  $g_{++}$  component of the metric in (2.78). Ignoring the potential for a moment we see that we have a the conformal field theory describing the throat of  $N_5$  fivebranes [13]. The potential breaks the  $SO(4)$  rotation symmetry of the throat region to  $SO(3)$ . The resulting sigma model has (4,4) supersymmetry on the world sheet. When the potential is nonzero the supersymmetry in the  $1 + 1$  dimensional world sheet theory is of a peculiar kind [19]. In ordinary global (4,4) supersymmetry the supercharges transform under an  $SU(2) \times SU(2)$   $R$ -symmetry but those symmetries do not appear in the right hand side of



the supersymmetry algebra.<sup>22</sup> Let us denote the supercharges by  $Q_{\pm}^i$ , where  $i = 1, \dots, 4$  are  $SO(4) = SU(2) \times SU(2)$  indices, and  $\pm$  indicates two dimensional chirality. The anticommutators of these supercharges have the form

$$\begin{aligned} \{Q_+^i, Q_+^j\} &= \delta^{ij}(E + P), & \{Q_-^i, Q_-^j\} &= \delta^{ij}(E - P), \\ \{Q_+^i, Q_-^j\} &= m\epsilon^{ijkl}J^{kl}, \end{aligned} \quad (2.88)$$

where  $J^{kl}$  are the  $SO(4)$  generators and  $m$  is a dimensionful parameter which we can set to one. This parameter is related to the scale entering in the potential and determines the mass of BPS particles which carry  $SO(4)$  quantum numbers. When the potential is set to zero we set  $m = 0$  and we get the ordinary commutation relations we expect for the usual  $(4,4)$  supersymmetry algebra. Let us denote the algebra (2.88) by  $(4, 4)_m$ . Notice that this is a Poincaré superalgebra which contains non-Abelian charges in the right hand side. This is possible in total spacetime dimension  $d \leq 3$  [19] but not in  $d > 3$  [18]. This algebra is a dimensional reduction of a Poincaré superalgebra in  $2 + 1$  dimensions that we discuss in more detail in Appendix E.

Note that the potential implies that the light cone energy is minimized (and it is zero) when the string sits at  $\phi = \theta = 0$ . There is just a finite energy gap of the order of  $N_5|p_-|$  preventing it from going into the region  $\phi \rightarrow -\infty$  where the pp-wave approximations leading to (2.78)–(2.82) break down.<sup>23</sup> Potentials in models preserving  $(4,4)$  supersymmetry were studied in [60,61] for models based on hyperkahler manifolds. Here we are interested in models with nonzero  $H$  flux. In fact, for the general solution (2.35)–(2.39) we can write down the string theory in light cone gauge

$$S = S_1 + S_2, \quad (2.89)$$

$$S_1 = \int dt \int_0^{2\pi\alpha'|p_-|} d\sigma d^2\theta \frac{1}{2} [D_+ R^i D_- R^i + R^i R^i], \quad (2.90)$$

$$\begin{aligned} S_2 = \int dt \int_0^{2\pi\alpha'|p_-|} d\sigma d^2\theta & \left\{ \frac{1}{2} f(W, \bar{W})(D_+ W D_- \bar{W} \right. \\ & + D_+ \bar{W} D_- W) + z(W) + \bar{z}(\bar{W}) \\ & + [f(W, \bar{W})(W + \bar{W})^2 g_{ij}(\Theta) \\ & \left. + B_{ij}(\Theta, W, \bar{W})] D_+ \Theta^i D_- \Theta^j \right\}, \end{aligned} \quad (2.91)$$

where  $S_1$  describes the first four coordinates and consists of four free massive superfields.  $S_2$  is the action describing

<sup>22</sup>Notice that here we are talking about the global  $(4,4)$  supersymmetry. These are the modes  $G_0^i$  of the superconformal algebra generated by  $G_n^i$ . Some of the  $SU(2)$  currents do appear in the anticommutators of some of the  $G_n^i$ ,  $n \neq 0$ .

<sup>23</sup>In the region  $\phi \rightarrow -\infty$  we need to use the fivebrane solution in Sec. II B 1.

the second four coordinates. We have written the action in  $\mathcal{N} = 1$  superspace, by picking one special supercharge. Note that this particular supercharge, say  $Q_{\pm}^1$ , obeys the usual super-Poincaré algebra, therefore we can use the usual superspace formalism. The  $B$  field and the function  $f$  are simply the ones in (2.35)–(2.39). The theory has  $(4, 4)_m$  supersymmetry. We have not shown this explicitly from the Lagrangian (2.91) but we know this from the supergravity analysis. Compared to the usual WZW action for a system of fivebranes, the only new term is the potential term. Note that RR fields in (2.35)–(2.39) are such that four of the fermions are free, which are the ones included in  $S_1$ , and the remaining four are interacting and appear in  $S_2$  in (2.91).

Let us first study the theory (2.91) for large  $N_5$ . In that case, we can expand the fields around the minimum of the potential. If we keep only quadratic fluctuations we have four free bosons and fermions. In order to characterize these particles we go to their rest frame. Setting  $P = 0$  we find that (2.88) reduces to the  $\widetilde{SU}(2|2)$  algebra. These particles transform in the representation with two boxes as in Fig. 7(a) (but now viewed as a representation of  $\widetilde{SU}(2|2)$ ). In terms of  $SU(2) \times SU(2)$  quantum numbers we have  $(1, 0) + (1/2, 1/2) + (0, 0)$  where particles with half integer spin are fermions. This is a short representation, with energy  $\hat{E} = 2$ . In fact, if we consider a closed string and a superposition of two such particles with zero momentum we can form states that transform in the representations given in Fig. 7(e), which are also protected. As we make  $N_5$  smaller these protected representations have to continue having the same energy. Of course, this argument only works perturbatively in  $1/N_5$  since  $N_5$  is not a continuous parameter and we can have jumps in the number of protected states as we change  $N_5$ . In order to figure out more precisely which representations are protected it is convenient to introduce an index defined by

$$I(\gamma) = \text{Tr}[(-1)^F 2S_3 e^{-\hat{\mu}(\hat{E} - S_3 - \tilde{S}_3)} e^{-\gamma \hat{E}}], \quad (2.92)$$

where  $S_3$  and  $\tilde{S}_3$  are generators in each of the two  $SU(2)$  groups. We use the letter  $I$  to distinguish (2.92) from (2.1). One can argue that only short representations contribute and that the final answer is independent of  $\hat{\mu}$ , see Appendix G. We can compute this for large  $N_5$  using the free world sheet theory and we obtain

$$I(\gamma)_{N_5=\infty} = \sum_{n=1}^{\infty} e^{-2n\gamma}. \quad (2.93)$$

Since  $N_5$  is not a continuous parameter we see that as we make  $N_5$  smaller (2.93) could change but only by terms that are nonperturbative in the  $1/N_5$  expansion. Thus for  $N_5$  fixed and large we expect that the corrections would affect only terms of the form  $e^{-(\text{const})N_5\gamma}$ .

Now, let us compare this with the expectations from the gauge theory side. In order to find protected representa-

tions on the gauge theory side it is convenient to use the index (2.1). Since we are focusing on single trace states we can compute (2.1) just for single trace states. For the case that we expand around the vacuum corresponding to  $N_2$   $SU(2)$  representations of dimension  $N_5$  we get

$$I_{\text{s.t.}N_5} = I_{\text{s.t.}N_5=1} + \frac{e^{-2N_5(\beta_1+\beta_2+\beta_3)}}{(1 - e^{-2N_5(\beta_1+\beta_2+\beta_3)})} - \frac{e^{-2(\beta_1+\beta_2+\beta_3)}}{(1 - e^{-2(\beta_1+\beta_2+\beta_3)})}, \quad (2.94)$$

$$I_{\text{s.t.}N_5=1} = \frac{e^{-\beta_2-\beta_1}}{1 - e^{-\beta_2-\beta_1}} + \frac{e^{-\beta_3-\beta_1}}{1 - e^{-\beta_3-\beta_1}} + \frac{e^{-\beta_3-\beta_2}}{1 - e^{-\beta_3-\beta_2}}. \quad (2.95)$$

We describe the details of this computation in Appendix G. Let us summarize here some of the results. In Appendix G we show that for the  $N_5 = 1$  case we simply get the contributions expected from summing over the representations in Fig. 7(b). These contributions have the form expected from the BPS states on the string theory side coming from the first four transverse dimensions, the dimensions along the fivebrane. So we expect that the extra contribution in (2.94) should correspond to the contribution of the second set of four dimensions. In other words, it should be related to the BPS states in the two dimensional field theory [(2.91) with (2.75)] describing the second four transverse dimensions. In order to extract that contribution it is necessary to match the extra contribution we observe in (2.94) to the contributions we expect from protected representations. In other words, we can compute the index  $I$  for various protected representations and we can then match them (2.94). In Appendix G we compute this index for atypical (short) representations and we show that (2.94) can be reproduced by summing over representations of the form shown in Fig. 7(f). In terms of the notation introduced in [7] [see Fig. 7(g)], which uses the Dynkin labels, we expect representations with  $(a_1, a_2, a_3|a_4|a_5) = (0, p, 0|a_5 + 1|a_5)$  with  $p \geq 0$  and  $a_5 = 2(n - 1)$ ,  $n = 1, \dots$  but  $n \neq 0 \pmod{N_5}$ . All values of  $p$  and  $n$  that are allowed appear once. Representations with various values of  $p$  contribute with states that can be viewed as arising from the product of representations of the form in Fig. 7(b) and 7(e). The ones in Fig. 7(b) were identified with the first four transverse dimensions. So we interpret the sum over  $p$  as producing strings of various lengths given by the total powers of  $Z$ , plus the BPS states which are associated to the first four (free) dimensions on the string. So we conclude that the BPS states that should be identified with the second four dimensions should be associated to the sum over  $n$ . Thus we expect from gauge theory side that the field theory on the string associated to the second four dimensions should have an index given by

$$I_{\text{expected}} = \sum_{n=1}^{\infty} e^{-2n\gamma} - \sum_{n=1}^{\infty} e^{-2nN_5\gamma}. \quad (2.96)$$

We include the details of derivation in Appendix G. So we see that this differs from (2.93) by a nonperturbative terms in  $1/N_5$  of the form  $e^{-2N_5\gamma}$ . We view (2.96) as the gauge theory prediction for BPS states on the string theory side. Here we have checked that this matches the string theory in a  $1/N_5$  expansion, but it would be nice to obtain the second term in (2.96) [which could be viewed as a nonperturbative correction to (2.93)] from an analysis of the two dimensional field theory based on the WZW model plus linear dilaton theory with a potential. These theories have a large group of symmetries and the theories with no potential are solvable. It would be nice to see whether (2.91) is integrable.

### III. THEORIES WITH 16 SUPERCHARGES AND $U(1) \times SO(4) \times SO(4)$ SYMMETRY GROUP

In this section we will discuss another class of theories with 16 supercharges. In this case the supersymmetry group has a  $U(1) \times SO(4) \times SO(4)$  bosonic symmetry, where the two  $SO(4)$  act on the supercharges. The general form of type IIB supergravity solutions with these symmetry was found in [14], and its form is rewritten in Appendix H. Solutions depend nontrivially on three coordinates  $x_1, x_2, y$ , where  $y \geq 0$ . The solution is parametrized by a function  $z(x_1, x_2, y)$  which obeys a linear equation

$$\partial_i \partial_i z + y \partial_y \left( \frac{\partial_y z}{y} \right) = 0. \quad (3.1)$$

Regular solutions are in one to one correspondence with droplets of an incompressible fluid in the  $x_1, x_2$  plane. These droplets correspond to two possible boundary conditions  $z = \pm \frac{1}{2}$  at  $y = 0$  which geometrically are associated to one of two  $S^3$ s shrinking to zero smoothly at  $y = 0$ . These solutions are much easier to obtain than the solutions discussed in the previous sections because the problem is precisely linear and the boundary conditions are very simple. In the special case that the  $x_1, x_2$  plane is infinite and we have finite size droplets, the solutions correspond to  $1/2$  BPS states in  $\text{AdS}_5 \times S^5$  [14].

We can now also consider cases where we compactify the  $x_1, x_2$  plane. Since the asymptotic structure of the  $x_1, x_2$  plane has changed, these solutions are dual to other field theories. The case that  $x_1$  is compact and  $x_2$  is noncompact was discussed in [14]. Let us summarize those results. If we have a droplet that is bounded in the  $x_2$  direction, like in Fig. 8(b)–8(d), then the dual boundary theory can be thought of as the theory of  $N$  M5 branes on  $R \times S^1 \times S^1 \times S^3$ . When one of the  $S^1$ s is very small we can think of this as a theory of D4 branes on  $R \times S^1 \times S^3$ . A simple way to understand this theory is as follows. We consider one of the complex transverse scalars of  $\mathcal{N} = 4$  super Yang-Mills theory. When the Yang-Mills theory is on  $R \times S^3$  the

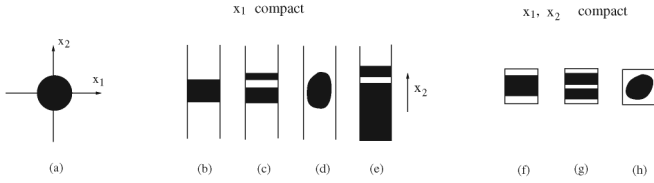


FIG. 8. In (a) we see a circular droplet in the uncompactified  $x_1, x_2$  plane which corresponds to the vacuum of  $\mathcal{N} = 4$  super Yang-Mills theory. In (b,c,d) we show different vacua in the case that we compactify the  $x_1$  coordinate. This “uplifts”  $\mathcal{N} = 4$  super Yang-Mills theory to a  $4 + 1$  dimensional gauge theory, or more precisely to the  $(0,2)$  six dimensional field theory that lives on  $M5$  branes. Figure (d) shows the limit to the  $M5$  brane theory when the  $x_1$  dependence recovers. If we compactify also  $x_2$ , as in (f,g,h) we get a little string theory whose low energy limit is a Chern-Simons theory. If the sizes of  $x_1$  and  $x_2$  are finite, we get the theory on  $R \times T^2$  and figures (f,g,h) show different vacua. As we take both sizes to zero, we obtain the theory on  $R^{1+2}$ . The configuration in (e) corresponds to a vacuum of the theory of  $M2$  branes with a mass deformation.

Lagrangian contains a term of the form  $-\frac{1}{2}(|DZ|^2 + |Z|^2)$ . We can now write  $Z = e^{iY}(Y + iX)$ . Then the Lagrangian becomes  $-\frac{1}{2}(DX)^2 - \frac{1}{2}(DY)^2 - 2YD_0X$ . We now see that the problem is translational invariant in  $X$ . Actually, the problem looks like a particle in a magnetic field.<sup>24</sup> Note that the Hamiltonian associated to this Lagrangian is equal to  $H' = H - J$  where  $H$  is the original Hamiltonian which is conjugate to translations in the time direction and  $J$  is the generator of  $SO(6)$  that rotates the field  $Z$ . If one compactifies the direction  $X$ , using the procedure in [62], we get the five dimensional gauge theory living on  $D4$  branes, see Appendix F. This description of the theory is appropriate at weak coupling or long distances on the  $D4$  branes. The proper UV definition of this theory is in terms of the six dimensional  $(0,2)$  theory that lives on  $M5$  branes. So we have the theory on  $M5$  branes on  $R^{1,1} \times S^1 \times S^3$ . We could, of course, decompactify this theory and consider the theory of  $M5$  branes on  $R^{2,1} \times S^3$ . These theories preserve 16 supercharges. The process of compactifying the coordinate  $X$  broke the 32 supersymmetries to 16. When this theory is on  $R^{1,1} \times S^1 \times S^3$  or  $R^{1+2} \times S^3$ , the size of  $x_1$  should be taken to zero and the solutions correspond to those in Fig. 8(b) and 8(c).

Let us consider the  $D4$  theory on  $R \times S^1 \times S^3$  (or the  $M5$  theory on  $R \times T^2 \times S^3$  in the UV limit). This theory has a large number of supersymmetric vacua. The structure of these vacua is captured by the  $1 + 1$  dimensional Lagrangian

$$\int \text{Tr} \left[ -\frac{1}{4}F^2 - \frac{1}{2}(DY)^2 - YF \right]. \quad (3.2)$$

<sup>24</sup>Note that the  $(x_1, x_2)$  coordinates appearing in the gravity solution correspond to the coordinates  $(X, Y)$  in the field theory.

The space of vacua is the same as the Hilbert space of 2D Yang-Mills theory on a cylinder [14]. All these vacua have zero energy. At first sight we might expect the theory on  $R^{1,1} \times S^3$  to have a continuum family of vacua related to possible expectation values for  $Y$ . Note, however, that the electric field is given by  $E_1 \sim F_{01} + 2Y$ . For zero energy configurations  $F_{01} = 0$ . So the quantization condition for the electric field will quantize the values of  $Y$ . This is good, since, as we explain in the Appendix E the supersymmetry algebra does not allow massless particles. In fact, the spectrum of states around each of these vacua has a mass gap. The explicit gravity solutions were derived in [14] and are written in the Appendix. In Appendix we show that the dilaton  $\Phi$ , as well as the warp factor are bounded in the IR region for any droplet configuration of this type. They never go to zero and the solution is everywhere regular. This is related to the fact that the dual field theory has a mass gap.

Another configuration we can consider in the case that we have a cylinder in the  $x_1, x_2$  plane is shown in Fig. 8(e). In this case we fill the lower half of the cylinder. In this case we get the  $M2$  brane theory in  $2 + 1$  dimensions with a mass deformation. We get this theory in  $R^{1+2}$  after setting the radius of  $x_1$  to zero and taking the strong coupling limit (and doing the obvious U-duality transformations<sup>25</sup>). If the size of  $x_1$  and the string coupling are finite, then we get the theory on  $R \times T^2$ . This theory was discussed in [14,65,66], and has an interesting vacuum structure, corresponding to  $M2$  branes polarized into  $M5$  branes wrapping two possible  $S^3$ s.

Let us discuss the situation when the  $x_1, x_2$  plane is compactified into a two torus, as in Fig. 8(f)–8(h). We have a 2 dimensional array of periodic droplets.<sup>26</sup> Let us start first with a description of the gravity solution. An important first step is to find the asymptotic behavior of the solution. The function  $z$ , which obeys (3.1), goes to a constant at large  $y$ . We can find the value of the constant by integrating  $z$  over the two torus at fixed  $y$ . The result of this integral is independent of  $y$ , and we can compute it easily at  $y = 0$  where it is given by the difference in areas between the two possible boundary conditions,  $z = \pm \frac{1}{2}$ . So we find  $z = \frac{1}{2} \frac{N-K}{N+K}$  asymptotically, where we used that the areas are quantized due to the flux quantization condition [14], so that  $N, K$  are the areas of the fermions and the holes, respectively. After doing  $T$ -dualities on both circles of the  $T^2$  and an  $S$  duality we find that the solution is asymptotic to

<sup>25</sup>This theory via IIB/M duality corresponds to the DLCQ of IIB plane-wave string theory [63], see also [64].

<sup>26</sup>In this case, in the large  $y$  region, the solution looks similar to the solution we would obtain if we take the full  $x_1, x_2$  plane and we consider a “gray” configuration filled with a fractional density. It shares some similarity but is different from the situation considered in e.g. some of the references in [67] where “gray” regions are finite.

$$ds_{10}^2 = -dt^2 + du_1^2 + du_2^2 + N\alpha'd\Omega_3^2 + K\alpha'd\tilde{\Omega}_3^2 + \frac{NK}{N+K}\alpha'd\rho^2, \quad (3.3)$$

$$e^\Phi = g_s\sqrt{NK}\sqrt{N+K}\alpha'^{3/2}e^{-\rho}, \quad (3.4)$$

$$H_3 = 2N\alpha'd^3\Omega + 2K\alpha'd^3\tilde{\Omega}. \quad (3.5)$$

We can view this as a little string theory in 1 + 2 dimensions. These (asymptotic) solutions are not regular as  $\rho \rightarrow -\infty$  since the dilaton increases. In that region we should do an  $S$  duality and then  $T$ -dualities back to the original type IIB description. Then, once we choose a droplet configuration, the solution is regular. This procedure works only if the coordinates  $u_1, u_2$  in (3.3) are compact. Of course, we could also consider the situation when these coordinates are noncompact. In that case we have Poincaré symmetry in 2 + 1 dimensions. In fact, such a solution appears as the near horizon limit of two intersecting five-branes<sup>27</sup> [68,69] and was recently studied in [70]. Note that the asymptotic geometry (3.3)–(3.5) is symmetric under

$$K \leftrightarrow N, \quad (3.6)$$

which is associated with the symmetry  $z \leftrightarrow -z$ . So we expect that this is a precise symmetry of the field theory.

It is interesting to start from the D4 brane theory that we discussed above and then compactify one of its transverse directions, the direction  $Y$  in (3.2). The Lagrangian in (3.2) is not invariant under infinitesimal translations of  $Y$ , but it is invariant under discrete translations if the period of  $Y$  is chosen appropriately. Following the standard procedure, [62], (see Appendix F for details) we obtain a theory in six dimensions which can be viewed as the theory arising on  $N$  D5 branes that are wrapping an  $R^{1,1} \times S^1 \times S^3$  with  $K$  units of RR 3 form flux on  $S^3$ . This RR flux induces a level  $K$  three dimensional Chern-Simons term. In fact by compactifying  $Y$  from (3.2) we get a 2 + 1 action  $\frac{K}{4\pi} \times \int \text{Tr}[-\frac{1}{4}F^2 + \omega_{\text{cs}}]$  on  $R \times T^2$  with Chern-Simons term. It turns out that the gauge coupling constant is also set by  $K$ . Perhaps a simple way to understand this is that the mass of the gauge bosons, which is due to the Chern-Simons term is related by supersymmetry to the mass scale set by the radius of the threesphere, which we can set to one. This implies that  $g^2K \sim 1$ . This derivation makes sense only when  $K/N$  is large and we could be missing finite  $K/N$  effects. Notice that in this limit the  $\tilde{S}^3$  that is interpreted as the worldvolume of  $N$  D5 branes is larger than the other  $S^3$  in (3.3)–(3.5). The gauge theory description is valid in the IR but the proper UV definition of this theory is in terms of

<sup>27</sup>One can make a change of variables  $e^{2\rho} = \sqrt{N+K}\alpha'^{1/2}r_1r_2$ ,  $u_2 = \frac{\alpha'^{1/2}}{\sqrt{N+K}}(N \log r_1 - K \log r_2)$  and then  $r_1$  and  $r_2$  become the transverse radial directions of the two sets of fivebranes (intersecting on  $R^{1,1}$ ), respectively, in the near horizon geometry, where the number of supersymmetries is doubled.

the little string theory in (3.3). The theory has a mass gap for propagating excitations but is governed by a  $U(N)_K$  Chern-Simons theory at low energies. The  $U(1)$  factor is free and it should be associated to a “singleton” in the geometric description. On the other hand, it seems necessary to find formulas that are precisely symmetric under  $K \leftrightarrow N$ . More precisely, in the limit  $N/K$  large we get a  $U(K)_N$  Chern-Simons theory by viewing the theory as coming from  $K$  D5 branes wrapping the other  $S^3$ . Interestingly, these two Chern-Simons theories are dual to each other [71],<sup>28</sup> which suggests that this is the precise low energy theory for finite  $N$  and  $K$ . Similar conclusions were reached in [70]. Of course, in our problem we do not have just this low energy theory, we have a full massive theory, with a mass scale set by the string scale. We do not have an independent way to describe it other than giving the asymptotic geometry (3.3)–(3.5), as is the case with little string theories. On the other hand one can show that the symmetry algebra implies that the theory has a mass gap, see Appendix E.

We can compute the number of vacua from the gravity side. There we have Landau levels on a torus where we have total flux  $N + K$  and we have  $N$  fermions and  $K$  holes. This gives a total number of vacua

$$D_{\text{grav}}(N, K) = \frac{(N+K)!}{K!N!} \quad (3.7)$$

and the filling fraction  $\frac{N}{N+K}$ . Actually, to be more precise, we derive this Landau level picture as follows. We start from the gravity solutions which are specified by giving the shape of droplets on the torus. We should then quantize this family of gravity solutions. This was done in [72] (see also [73]), who found that the quantization is the same as the quantization for the incompressible fluid we have in the lowest Landau level for  $N$  fermions in a magnetic field. We now simply compactify the plane considered in [72]. This procedure is guaranteed to give us the correct answer for large  $N$  and  $K$ . The number of vacua computed from  $U(N)_K$  agrees with (3.7) up to factors going like  $N, K$  or  $N + K$  which we have not computed. These factors are related to the precise contribution of the  $U(1)$ .<sup>29</sup> In order to compare the field theory answer to the gravity answer one would have to understand properly the role of “singletons,” which could give contributions of order  $N, K$ , etc. We leave a precise comparison to the future but it should be noted that we have a precise agreement for large  $N$  and  $K$  where the gravity answers are valid.

We have nonsingular gravity solutions if we choose simple configurations for these fermions where they form

<sup>28</sup>Level rank duality, as analyzed in [71], holds up to pieces which comes from free field correlators. This means that we have not checked whether the  $U(1)$  factor, as we introduced it here, leads to a completely equivalent theory.

<sup>29</sup>The number of vacua for  $SU(N)_K$  Chern-Simons theory is given by  $\frac{(N+K-1)!}{K!(N-1)!}$ .



well defined droplets. The particle hole duality of the Landau problem is the level rank duality in Chern-Simons theory, and is  $K \leftrightarrow N$  duality of the full configuration.

### A. Supersymmetry algebra

An unusual property of all the theories we discussed above is that their supersymmetry algebra in  $2 + 1$  (or  $1 + 1$ ) dimensions is rather peculiar. In ordinary Poincaré supersymmetry the generators appearing in the right hand side of the supersymmetry algebra commute with all other generators. This is actually a theorem for  $d \geq 4$  [18]. For this reason they are called *central* charges. In our case the superalgebra has anticommutators of the form

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\tilde{\gamma}_{\alpha\beta}^{\mu} p_{\mu} \delta_{ij} + 2m\epsilon_{\alpha\beta}\epsilon_{ijkl}M_{kl}, \quad (3.8)$$

where  $m$  is a constant of dimension of mass,  $i, j$  are  $SO(4)$  indices and  $M_{ij}$  are  $SO(4)$  generators. This superalgebra appeared in the general classification in [19]. In this paper we have set  $m = 1$  for convenience. This choice is related to the choice of mass scales (e.g. radius of  $S^3$ ) appearing in the various theories. These generators do not commute with the supercharges. So this is a Poincaré superalgebra with *noncentral* charges.<sup>30</sup> The  $SO(4)$  that appears in (3.8) is the product of an  $SU(2)$  that acts on the first  $S^3$  times another  $SU(2)$  which acts on the second  $S^3$ , where the  $S^3$ s we mention here are the three spheres in the geometric description. There are other supercharges which transforms under another  $SO(4)$ . In Appendix E we write down this algebra more explicitly and we write down various Lagrangians with this symmetry. The truncation of this algebra to  $1 + 1$  dimensions is written in (2.88).

All these theories have interesting BPS particles. Again for large  $J$  we have simple plane-wave limits. In this case the plane-wave geometry is basically the one corresponding to the standard IIB plane-wave. As before, it is interesting to find out where the BPS geodesics lie in the geometry. Let us suppose that we consider a particle carrying spin under a generator  $J = J_{12}$  in the  $SO(4)$  which rotates the first sphere. Using the metric in [14] we can see that

$$\frac{E^2}{J^2} = \frac{1}{\frac{1}{2} + z}. \quad (3.9)$$

In the solutions we consider here  $|z| \leq \frac{1}{2}$ . So we find that the energy is minimized when  $z = \frac{1}{2}$ . This corresponds to the regions in the  $y = 0$  plane where the other  $S^3$  shrinks to zero size. In addition we have to sit at a point where  $V_i = 0$ . Where this point is depends on the distribution of the other droplets, but one can see that within each droplet

<sup>30</sup>This situation, is of course, common in anti-de-Sitter superalgebras. It has also been observed before in some deformations of Euclidean Poincaré superalgebras [74].

there is a point where  $V_i = 0$ . This implies that in a configuration with many droplets one will have as many BPS geodesics as droplets of the type we are considering. One could probably derive exact indices, or partition functions, that count these BPS particles. These are BPS versions of the field theory objects considered in [75].

### B. Solutions with $SO(2, 2) \times SO(4) \times U(1)$ symmetry

By performing a simple analytic continuation of the type considered in [14] it is possible to write an ansatz of the form

$$\begin{aligned} ds_{10}^2 &= y\sqrt{\frac{2z+1}{2z-1}}ds_{\text{AdS}_3}^2 + y\sqrt{\frac{2z-1}{2z+1}}d\tilde{\Omega}_3^2 \\ &\quad + \frac{2y}{\sqrt{4z^2-1}}(d\chi + V)^2 \\ &\quad + \frac{\sqrt{4z^2-1}}{2y}(dy^2 + dx^i dx^i) \\ F_5 &= -\frac{1}{4}\left\{d\left[y^2\frac{2z+1}{2z-1}(d\chi + V)\right] + y^3 *_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right)\right\} \\ &\quad \wedge d\text{Vol}_{\text{AdS}_3} - \frac{1}{4}\left\{d\left[y^2\frac{2z-1}{2z+1}(d\chi + V)\right] \right. \\ &\quad \left. + y^3 *_3 d\left(\frac{z-\frac{1}{2}}{y^2}\right)\right\} \wedge d^3\tilde{\Omega}, \end{aligned} \quad (3.10)$$

$$dV = \frac{1}{y} *_3 dz, \quad (3.11)$$

where  $z$  obeys

$$\partial_i \partial_i z + y \partial_y \left(\frac{\partial_y z}{y}\right) = 0. \quad (3.12)$$

We can now look for solutions where the  $\text{AdS}_3$  factor does not shrink but where the  $S^3$  factor could shrink. Regularity requires  $z = 1/2$  at  $y = 0$ . We look for solutions where  $z \geq 1/2$  everywhere. In order to obtain non-trivial solutions we put charged sources on the right hand side of (3.12). Let us consider a source that is localized at  $y = y_0, \vec{x} = \vec{x}_0$ . We take  $\vec{x}_0 = 0$  for the time being. It turns out that if we introduce the right amount of charge, the circle parametrized by  $\chi$  shrinks in a smooth way, combining with  $y, \vec{x}$  to give a space that locally looks like the origin of  $R^4$ . More precisely, this occurs if the function  $z$  behaves near  $y = y_0$  as

$$z \approx \frac{y_0}{2\sqrt{(y-y_0)^2 + |\vec{x}|^2}}, \quad (3.13)$$

so we see the charge  $Q_0$  at  $y_0$  is equal to  $y_0/2$ . To summarize, we have the following equation and boundary condition for regular solutions

$$\partial_i \partial_i z + y \partial_y \left( \frac{\partial_y z}{y} \right) = - \sum_{l=1}^n \frac{y_l}{2} (4\pi) \delta(y - y_l) \delta^2(\vec{x} - \vec{x}_l), \quad (3.14)$$

$$z|_{y=0} = \frac{1}{2}, \quad (3.15)$$

where we can have several charges located at  $(y_l, \vec{x}_l)$ . If we do not have coincident points in three dimensions the solution is smooth.

Notice that for large  $y$  and large  $y_l$  these solutions reduce to the usual Gibbons Hawking metrics [76] times  $R^6$ , where the  $R^6$  comes from the large radius limit of  $\text{AdS}_3$  and  $S^3$  in (3.10).

The simplest and most symmetric solution corresponds to a single point charge of strength  $y_0/2$  at  $(y_0, 0)$ , see Fig. 9(a), which corresponds to

$$z = \frac{r^2 + y_0^2 + y^2}{2\sqrt{(r^2 + y_0^2 + y^2)^2 - 4y^2 y_0^2}}, \quad (3.16)$$

$$V_\phi = \frac{r^2 - y_0^2 + y^2}{2\sqrt{(r^2 + y_0^2 + y^2)^2 - 4y^2 y_0^2}}. \quad (3.17)$$

It turns out that this solution is  $\text{AdS}_5 \times S^5$ . We can see this by the coordinate change

$$x_1 + ix_2 = r e^{i\phi}, \quad (3.18)$$

$$y = y_0 \cosh u \cos \beta, \quad (3.19)$$

$$r = y_0 \sinh u \sin \beta, \quad (3.20)$$

$$\psi = \chi - \phi/2, \quad (3.21)$$

$$\alpha = \chi + \phi/2. \quad (3.22)$$

So that we get

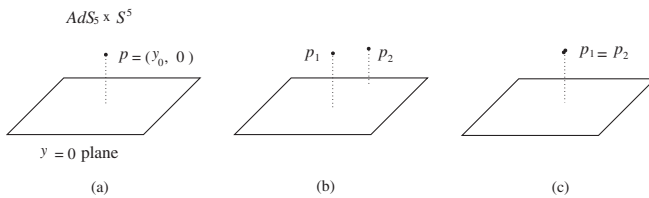


FIG. 9. (a) In the analytically continued IIB ansatz, the  $\text{AdS}_5 \times S^5$  solution corresponds to boundary conditions  $z = 1/2$  at the  $y = 0$  plane and a point charge  $Q_0 = y_0/2$  located at  $p = (y_0, 0)$ , away from the  $y = 0$  plane. In (b) there is a more general smooth configuration where there are two point charges located at different points  $p_1 = (y_1, \vec{x}_1)$  and  $p_2 = (y_2, \vec{x}_2)$ . Their charges are  $y_1/2, y_2/2$  respectively. In (c) two such point charges merge at the same point, and they develop a  $R^4/Z_2$  singularity. If there were  $k$  coincident such charges, then they give rise to  $R^4/Z_k$  singularity.

$$ds_{10}^2 = y_0 [(\cosh^2 u ds_{\text{AdS}_3}^2 + du^2 + \sinh^2 u d\psi^2) + (\cos^2 \beta d\tilde{\Omega}_3^2 + d\beta^2 + \sin^2 \beta d\alpha^2)], \quad (3.23)$$

$$F_5 = 4y_0^2 [\cosh^3 u \sinh u du \wedge d\psi \wedge d\text{Vol}_{\text{AdS}_3} + \cos^3 \beta \sin \beta d\beta \wedge d\alpha \wedge d^3 \tilde{\Omega}], \quad (3.24)$$

where

$$y_0 = R_{\text{AdS}_5}^2 = R_{S^5}^2 = \sqrt{4\pi g_s N \alpha'^2}. \quad (3.25)$$

It is  $\text{AdS}_5 \times S^5$  written with an  $\text{AdS}_3 \times S^1$  slicing. This particular solution has more symmetry and more supersymmetry than other generic solutions in the family. It is perhaps useful to note that the  $\text{AdS}_3 \times S^1$  boundary is conformally related to  $R \times S^3$

$$ds_{\text{AdS}_3 \times S^1}^2 = [-\cosh^2 v dt^2 + dv^2 + \sinh^2 v d\varphi^2 + d\psi^2], \quad (3.26)$$

$$= \frac{1}{\sin^2 \theta} [-dt^2 + \cos^2 \theta d\varphi^2 + d\theta^2 + \sin^2 \theta d\psi^2], \quad (3.27)$$

$$= \frac{1}{\sin^2 \theta} ds_{R \times S^3}^2; \quad \sin \theta = \frac{1}{\cosh v}. \quad (3.28)$$

Notice that the conformal factor blows up on a circle of  $S^3$ . The blown up position corresponds to the boundary of  $\text{AdS}_3$ .

Now we consider situations for many point charges. Consider a charge  $Q_l$  at a position  $y_l$  and integrate  $F_5$  over the  $S^3$  and an  $S^2$  surrounding the charge in the  $(y, \vec{x})$  space. From (3.10) the result is proportional to  $y_l Q_l \sim N_l$ , where we used that the flux is quantized. For smooth solutions  $Q_l = y_l/2$  and we obtain a relation  $y_l = \sqrt{4\pi g_s \alpha'^2 N_l}$ , the same as (3.25). Notice that (3.14) describes a family of solutions where we can change continuously the values of  $\vec{x}_l$  but we cannot change continuously the values of  $y_l$  due to the flux quantization condition.

Let us now start with a smooth solution that has two equal charges, at  $y_1 = y_2$ , and we take the limit when these two charges lie on top of each other (i.e. we set  $\vec{x}_1 = \vec{x}_2$ ). We get a singular solution since the total charge is twice the value that would make the solution regular. Such a solution has a  $Z_2$  singularity and near the position of the charge it looks like  $R^4/Z_2$ . Similarly if we take  $k$  equal charges coincident we get a space that locally is  $R^4/Z_k$  (or an  $A_{k-1}$  singularity). See Fig. 9(b) and 9(c).

Let us now consider the corresponding field theory. These solutions are related to  $\mathcal{N} = 4$  super Yang-Mills theory on  $\text{AdS}_3 \times S^1$ . Let us focus on a complex combination,  $Z$ , of two of the six scalar fields. Let us expand this

field in Kaluza Klein modes on  $S^1$ . The constant mode leads to a field with negative mass on  $\text{AdS}_3$ . This negative mass arises from the conformal coupling of the scalar fields.<sup>31</sup> The Lagrangian of this theory contains the term  $\frac{1}{2}(|D_\psi Z|^2 - \frac{\mathcal{R}}{6}|Z|^2)$ , and the scalar curvature  $\mathcal{R} = -6$  for  $\text{AdS}_3$  with unit radius. We can obtain a massless field on  $\text{AdS}_3$  if we take the first Kaluza Klein mode of  $Z$  on the  $\psi$  circle. Namely, we can consider  $Z = \hat{z} e^{i\psi}$ . We can take  $\hat{z}$  to be a diagonal matrix with eigenvalues  $\hat{z}_i$  where the multiplicity of each eigenvalue is  $N_i$ , where  $\sum_i N_i = N$ . It is natural to conjecture that this state is related to the gravity configuration with  $x_i^1 + ix_i^2 = \hat{z}_i$  and  $y_i^2 \sim N_i$ . This seems to give rise to a picture where the symmetries match on the two sides. On the other hand, it seems puzzling that as we take  $\hat{z}_i \rightarrow \hat{z}_j$  we do not get the same solution as the one corresponding to the situation where we have a single eigenvalue with multiplicity  $N_i + N_j$ . In fact we get a smooth configuration if  $N_i \neq N_j$  and a singular one (with a  $Z_2$  singularity) if  $N_i = N_j$ . More generally we get a  $Z_k$  singularity if  $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_k$  are coincident and their original multiplicities are the same. Since the field theory is on  $\text{AdS}_3 \times S^1$  we will need boundary conditions for the fields at the boundary of  $\text{AdS}_3$ , so perhaps the gauge symmetry is broken by the boundary conditions when we have multicenter solutions. In other words, perhaps the multicenter solutions only exist when the gauge symmetry is already broken at the boundary. Thus we cannot restore it by taking  $\hat{z}_i \rightarrow \hat{z}_j$ . Clearly a better understanding of this point is needed. Other gravity solutions with an  $\text{AdS}_3 \times S^1$  boundary were recently considered in [78]. It is possible that those solutions are related to a subset of the ones considered in this paper. Similar, but different, solutions were analyzed in [79].

#### IV. CONCLUSIONS

In this paper we have studied various theories with 16 supercharges. These theories are interesting because they have a dimensionless parameter that allows us to interpolate continuously between strong and weak coupling. These theories have simple observables, such as the spectrum of gauge invariant states. In this respect they are rather similar to  $\mathcal{N} = 4$  Yang-Mills theory on  $R \times S^3$ . In fact, they arise as truncations of  $\mathcal{N} = 4$ . It is therefore interesting to study the similarities and differences between these theories and  $\mathcal{N} = 4$ . From some points of view these theories are simpler. For example, the plane-wave matrix model is just an ordinary quantum mechanical theory with a finite number of degrees of freedom. On the other hand they are more complicated because they have less symmetry than  $\mathcal{N} = 4$  Yang-Mills theory. For example, in the theories we studied here symmetry alone

does not determine the gravity solutions. These theories have many vacua, as opposed to  $\mathcal{N} = 4$  which has only one. In addition, the physics around different vacua can have different qualitative features.

The first set of theories that we studied has  $\widetilde{SU}(2|4)$  symmetry group. We discussed some features of these field theories. We gave explicit formulas for the counting of 1/2 BPS states (2.5) and (2.13) and we explained how to construct an index (2.1) carrying information about more general BPS states. The single trace contribution to the index was computed in (2.94). This can, in turn, be translated into an index for the two dimensional world sheet theory (2.92) and (2.96) describing near BPS string states. The general form of the gravity solutions is written in (2.20)–(2.24) (from [14]) with the boundary conditions corresponding to an electrostatic problem involving conducting disks in three dimensions. For given asymptotic boundary conditions there are many possible disk configurations. The number of disk configurations matches with the number of expected vacua in the field theory. Full explicit solutions were given in a couple of cases (2.44)–(2.54). The solution in (2.49)–(2.54) is dual to 2 + 1 Yang-Mills theory on  $R \times S^2$  for the vacuum with unbroken  $U(N)$  gauge symmetry. All solutions are smooth in the IR region and they have no horizons. We have then focused on states with large  $J$ , where  $J$  is an  $SO(6)$  generator. We treat these large  $J$  states in the 't Hooft limit, where  $J$  is large but kept finite in the large  $N$  limit, so that we can neglect back reaction. In this limit we can think of the BPS states as massless geodesics moving along a circle inside an  $S^5$  and sitting at some point in the rest of the coordinates. These geodesics sit at the points corresponding to the tip of the disks. For a given vacuum there are as many distinct geodesics as there are disks in the electrostatic picture. Looking at the spacetime near these geodesics we found the general pp-wave solution (2.35)–(2.39). We then used these metrics to study the spectrum of near BPS states. In the string theory side, at large 't Hooft coupling, we can quantize the string in light cone gauge. Four of the transverse dimensions are described by free massive fields. These are associated to oscillations of the string in the  $S^5$  directions. The near BPS spectrum associated to stringy oscillations along these directions is characterized by a single parameter which corresponds to the radius of the  $S^5$  at the position of the BPS geodesic. This parameter is nonuniversal, in the sense that it depends on the theory we consider, the vacuum that we pick and also the particular BPS geodesic that we are expanding around, e.g. see (2.56) and (2.83). On the other hand, the metric *very* close to each massless geodesic has the form of the IIA plane wave (2.41). So the form of this metric is a universal feature of the near BPS limit in these geometries. There is however an important subtlety. Even though *very* near the massless geodesic the metric behaves as in (2.41) it can happen that the geometry has other features that are at distances

<sup>31</sup>Of course, this mass obeys the Breitenlohner-Freedman bound [77].

comparable to the string scale. This happens when we consider the vacua of the plane-wave matrix model that correspond to  $N_5$  coincident fivebranes (with relatively small values of  $N_5$ ). In this case the correct string theory description involve a massive deformation of the WZW model plus linear dilaton theory. This 1 + 1 dimensional field theory has  $(4, 4)_m$  supersymmetry (2.88) which has the peculiar feature of having noncentral charges. It would be interesting to see if this theory is integrable. We considered the weak coupling spectrum of single trace states around various vacua of this field theory. We found that the number of transverse oscillators depends on the vacuum. At weak coupling, for the single NS5 vacuum (the trivial vacuum) we have only four transverse oscillation modes while for  $N_5 > 1$  we have eight transverse oscillation modes. To be precise, on the gauge theory side, we only proved that the BPS spectrum of oscillations is consistent with eight modes, there could be more modes that do not contribute to the index. It would be nice to perform the complete one loop analysis of this model in order to find out precisely how many we have. We computed the number of 1/4 BPS single trace BPS states (2.96) which lead to 1/2 BPS states on the string world sheet. In other words, there are 1/2 BPS states of the field theory on the string. We did not count exactly these BPS states independently for the 1 + 1 dimensional field theory but we did show that we get the right answer for large  $N_5$ .

We then considered theories that arise when we take the solutions in [14] associated to free fermions and put them on a two torus. If we shrink the two torus to zero we get a little string theory (3.3)–(3.5) with Poincaré invariance in 2 + 1 dimensions. If we keep the torus finite, then we get this 2 + 1 dimensional little string theory on a two torus. This little string theory is characterized by two integers  $N$  and  $K$ . In the large  $K$  limit we can argue that the low energy description is given by a  $U(N)_K$  Chern-Simons theory, see (F2). We expect that the low energy theory should be exactly that of  $U(N)_K$  Chern-Simons theory. This low energy theory is level rank dual to  $U(K)_N$  Chern-Simons theory<sup>32</sup> in the large  $N, K$  limit. The  $K \leftrightarrow N$  symmetry is a full symmetry of the little string theory. The solutions that we described, which are associated to fermion droplets on a two torus, give a semiclassical description for the various vacua of  $U(N)_K$  Chern-Simons theory on a two torus. These solutions are relevant only when the 2 + 1 dimensional little string theory is on a two torus. This theory, as well as other theories that arise in similar ways, such as the theory living on the mass deformed M2 branes [14,65,66] have the supersymmetry algebra discussed in Appendix E which contains noncentral terms such as (E3).

In addition we discussed a curious family of solutions that is obtained by doing an analytic continuation of the

<sup>32</sup>We did not check that the  $U(1)$  factor indeed works as we are describing here.

ansatz in [14]. These solutions are related to  $\mathcal{N} = 4$  super Yang-Mills theory on  $\text{AdS}_3 \times S^1$ . They seem to correspond to a peculiar Coulomb branch where the field  $Z$  has an expectation value. But some further study is needed to elucidate the precise relation to the field theory.

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## APPENDIX A: DETAILED ANALYSIS OF THE REGULARITY OF THE SOLUTIONS

In this appendix we prove general properties of the solutions we consider in this paper. We can show, using equations for  $V$ , that  $\dot{V}V'' - (\dot{V}')^2 = -[\rho^2(V'')^2 + (\dot{V}')^2]$  is always negative. The dot denotes  $\rho \partial_\rho$  and the prime denotes  $\partial_\eta$ . We also need that everywhere  $\dot{V} = y \geq 0$  and  $V'' \leq 0$ ,  $\dot{V} - 2V' \geq 0$ . In this section it is a bit more convenient conceptually to formulate the problem in terms of a new variable  $Z = \dot{V}$ . We see that we can express all the functions in (2.20)–(2.24) in terms of  $Z, Z'$  and  $\dot{Z}$  since we can express  $V''$  in terms of  $\dot{V}$  using the Laplace equation. The variable  $Z$  obeys the equation

$$Z'' + \rho \partial_\rho \left( \frac{\partial_\rho Z}{\rho} \right) = 0. \quad (\text{A1})$$

This equation has the same form as the equation in the IIB solution (H2) when we have an additional isometry. Note, however, that we have different boundary conditions. Notice that we want to show that  $Z$  is non-negative. The boundary conditions at  $\rho = 0$  and on the disks imply that there  $Z$  is zero. The positivity conditions constrain the allowed asymptotic boundary conditions. They allow only  $V \sim \rho^2 - 2\eta^2$ , or  $Z \sim \rho^2$  we have the whole  $\rho, \eta$  plane. If we only have  $\eta > 0$ , we could also allow  $V = \eta\rho^2 - \frac{2}{3}\eta^3$ , or  $Z = \eta\rho^2$ . Note that  $Z$  needs to grow at infinity in both cases, since we want to impose, in addition, that  $\dot{Z} \geq 0$ . Notice that the structure of the Eq. (A1) is such that if  $Z$  is zero at the disks and  $Z$  is positive at infinity, then  $Z$  is positive everywhere. Now we want to ensure that  $\dot{Z}$  is positive everywhere. For this purpose it is convenient to define  $Y = \dot{Z}/\rho^2 = (\partial_\rho Z)/\rho = -V''$ , which obeys the same equation as the variable  $V$  itself. The boundary conditions are such that  $Y$  is positive far away. In addition,  $Y$  is zero on the disks. At the origin  $Y$  is required to be regular. So it is intuitively clear that it will be positive everywhere, except on the disks.

In order to find a nonsingular solution we need an additional condition. We need to ensure that



$$0 \leq \frac{2\dot{V}}{\ddot{V}} \leq 1. \quad (\text{A2})$$

The first inequality is obeyed automatically and is a strict inequality away from the disks and the origin. The second inequality can be analyzed as follows. Choose a function  $U = (\partial_\rho V)/\rho$ . Then we need to show that  $\partial_\rho U \geq 0$ . The equation that  $U$  obeys has the form

$$U'' + \frac{1}{\rho^3} \partial_\rho(\rho^3 \partial_\rho U) = 0. \quad (\text{A3})$$

It the Laplace equation in five dimensions for a system that is  $SO(4)$  rotationally symmetric. The boundary conditions at infinity are such that  $U$  is positive. At  $\rho = 0$  and on the disks we have that  $U$  is zero. As long as we have some finite disks, then we see that  $\dot{U}$  must be strictly positive everywhere, except on the disks and possibly at  $\rho = 0$ . Note that in the case that we have only an infinite disk at  $\eta = 0$ , so that we have the solution that corresponds to the 11 dimensional plane wave, with no excitations, then we get that  $\dot{U} = 0$  and the solution (2.20)–(2.24) is singular. This is expected since we are doing a reduction on a circle that is null everywhere.

Our discussion so far has ensured that the solution is nonsingular and the dilaton is finite everywhere except, possibly at  $\rho = 0$  and on the disks where the various inequalities that we have discussed are saturated. In order to show that the solution is nonsingular also in these regions, we need a more detailed analysis. For example, near  $\rho = 0$ , we have that  $\dot{V} \sim \rho^2$  and that  $\ddot{V} - 2\dot{V} \sim a\rho^4$ , with  $a > 0$  and  $-V'' > 0$  due to our previous arguments. These conditions ensure that the dilaton stays finite and that the solution (2.20)–(2.24) is nonsingular.

Doing a similar analysis near a disk we also find that the solution is regular at the disk positions.

### Regularity of the solutions coming from D4 on $R^{1,1} \times S^3$

Here we analyze the regularity property of the D4 brane solutions. In order to characterize the solution we need to give the numbers  $a_j, b_j$  which obey  $a_j < b_j < a_{j+1} \dots$ . These numbers are the values of  $x_2$  at the boundaries of the black strips, see Fig. 8(b) and 8(c). We have a black strip between  $a_j$  and  $b_j$ . Then the solution is given by [14]

$$2z = -1 + \sum_j \frac{x - a_j}{\sqrt{(x - a_j)^2 + y^2}} - \frac{x - b_j}{\sqrt{(x - b_j)^2 + y^2}}, \quad (\text{A4})$$

$$2yV_1 = \sum_j \frac{y}{\sqrt{(x - a_j)^2 + y^2}} - \frac{y}{\sqrt{(x - b_j)^2 + y^2}}, \quad (\text{A5})$$

$$2z + i2yV_1 = -1 + \sum_j (w_j - z_j), \quad (\text{A6})$$

$$w_j = \frac{x - a_j + iy}{\sqrt{(x - a_j)^2 + y^2}}, \quad z_j = \frac{x - b_j + iy}{\sqrt{(x - b_j)^2 + y^2}}. \quad (\text{A7})$$

We see that the complex numbers  $w_j$  and  $z_j$  lie on the unit circle in the upper half plane.

The ten dimensional solution is

$$ds_{IIA}^2 = e^{2\Phi}(-dt^2 + d\tilde{x}_1^2) + \frac{\sqrt{1 - 4z^2}}{2y}(dy^2 + dx_2^2) + y\sqrt{\frac{1 + 2z}{1 - 2z}}d\tilde{\Omega}_3^2 + y\sqrt{\frac{1 - 2z}{1 + 2z}}d\tilde{\Omega}_3^2, \quad (\text{A8})$$

$$e^{-2\Phi} = \frac{1 - 4z^2 - 4y^2V_1^2}{2y\sqrt{1 - 4z^2}}, \quad (\text{A8})$$

$$F_4 = -\frac{e^{-2\Phi}}{4} \left[ \frac{(1 - 2z)^{3/2}}{(1 + 2z)^{3/2}} *_2 d\left(y^2 \frac{1 + 2z}{1 - 2z}\right) \wedge d\tilde{\Omega}_3 + \frac{(1 + 2z)^{3/2}}{(1 - 2z)^{3/2}} *_2 d\left(y^2 \frac{1 - 2z}{1 + 2z}\right) \wedge d\Omega_3 \right], \quad (\text{A9})$$

$$B_2 = -\frac{4y^2V_1}{1 - 4z^2 - 4y^2V_1^2} dt \wedge d\tilde{x}_1. \quad (\text{A9})$$

Note that  $g_{00}$  is determined in terms of the dilaton. This is related to the fact that the 11 dimensional lift of this solution is lorentz invariant in 2 + 1 dimensions. We will now show that  $e^{-2\Phi}$  remains finite and nonzero in the IR region. Of course, in the UV region  $\Phi \rightarrow \infty$  and we need to go to the 11 dimensional description. Note that away from  $y = 0$  the denominator in (A8) is nonzero. The fact that the numerator is nonzero follows from the representation (A6) and the fact that  $w_j, z_j$  in (A7) are ordered points on the unit circle on the upper half plane, so the norm  $|2z + i2yV_1| < 1$ . As we take the  $y \rightarrow 0$  limit we see that both the numerator and denominator in (A8) vanish. We can then expand in powers of  $y$  and check that indeed we get a finite, nonzero result, both for  $a_j < x_2 < b_j$  and  $x_2 = a_j, b_j$ .

### APPENDIX B: DERIVATION OF THE D2 SOLUTION

In this appendix we explain how we obtained the solution (2.49)–(2.54). We start with the configuration of four dimensional gauged supergravity in [38], which has four commuting angular momenta in  $SO(8)$ . We consider the special case when only one of the angular momenta is nonzero. This is a half BPS state of  $M$ -theory on  $AdS_4 \times S^7$  and, as such, it can be described in terms of the general  $M$ -theory ansatz in [14]. By comparing the expressions in

[14,38] we find that the solution corresponds to an elliptical droplet of M2 boundary conditions (2.16). The solution can be written in the following parametric form<sup>33</sup>

$$e^D = 4\sin^2\theta(1 + z^2H(z))\sinh\varphi, \quad (\text{B1})$$

$$x_2 + ix_1 = (e^{-(1/2)\varphi} \cos\phi + ie^{(1/2)\varphi} \sin\phi) \frac{\cos\theta}{\sqrt{\sinh\varphi}}, \quad (\text{B2})$$

$$y = z\sin^2\theta, \quad (\text{B3})$$

$$\partial_z\varphi = \frac{-z \sinh\varphi}{1 + z^2H(z)}, \quad \partial_z(zH(z)) = \cosh\varphi. \quad (\text{B4})$$

The last two first order Eqs. (B4) are equivalent to a second order equation for  $zH(z)$

$$(z^{-1} + zH(z))\partial_z^2(zH(z)) = 1 - (\partial_z(zH(z)))^2. \quad (\text{B5})$$

Note that we still need to solve this equation to find a full solution.

The elliptic droplet in the  $x_1, x_2$  plane is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad a = \frac{1}{\sqrt{\sinh\varphi}} e^{\varphi/2}|_{z=0}, \quad (\text{B6})$$

$$b = \frac{1}{\sqrt{\sinh\varphi}} e^{-(\varphi/2)}|_{z=0}.$$

If we take a limit that  $a \rightarrow \infty, b = 1$ , we can let

$$\sinh\varphi \approx \frac{1}{2}e^\varphi \approx \cosh\varphi \quad (\text{B7})$$

and  $\varphi_0 = \varphi(z=0)$  goes to infinity. This limit corresponds to dropping the 1 in the right hand side of (B5).

If we expand around  $\cos\phi \approx 1$ , we can neglect the dependence of  $x_1$  and we find a solution of the 2D Toda equation. We can now write  $\varphi = \varphi_0 + \tilde{\varphi}$  where  $\tilde{\varphi}$  is stays constant in the limit. We can remove the  $\varphi_0$  dependence performing a simple symmetry transformation of the Toda equation which does not affect the 11 dimensional solution:  $e^D \rightarrow e^{-2\varphi_0}e^D, x_2 + ix_1 \rightarrow (x_2 + ix_1)e^{\varphi_0}, y \rightarrow y$ . Now we have the solution corresponding to a single M2 brane strip in  $x_1, x_2$  plane.

$$e^D = 4\sin^2\theta(1 + z^2H(z))e^{\tilde{\varphi}}, \quad x_2 = e^{-\tilde{\varphi}} \cos\theta, \quad (\text{B8})$$

$$y = z\sin^2\theta,$$

<sup>33</sup>We can also write the solution corresponding to the 1/2 BPS extremal one-charge limit of the AdS<sub>4</sub> black hole, e.g. [80], in the Toda form. This solution and the solution for M2 strip both belong to the more general solution:  $e^D = 4\sin^2\theta(1 + z^2H(z))/\tilde{F}(z)^2, x_2 + ix_1 = (e^{-(1/2)\varphi} \cos\phi + ie^{(1/2)\varphi} \sin\phi)\tilde{F}\cos\theta, y = z\sin^2\theta$ , where  $\partial_z\tilde{F} = \frac{z\tilde{F}}{2(1+z^2H(z))} \cosh\varphi$  and  $H$  obeys (B5). The solution that corresponds to the extremal AdS<sub>4</sub> black hole is  $H = 1 + \frac{2Q}{z}$ . We can plug this in the previous equations and integrate to get  $\log\tilde{F} = \int \frac{z}{2(1+z^2H(z))} dz$ .  $z$  is related to the radius  $r$  of AdS<sub>4</sub> black hole as  $z = 2r$ .

where  $\tilde{\varphi}$  is defined through  $e^{\tilde{\varphi}} = 2\partial_z(zH(z))$ , with  $H(z)$  obeying Eq. (B5) without the one in the right hand side. We also have a boundary condition  $e^{\tilde{\varphi}(0)} = 2C$ .

One can see that the single strip of M2 branes in the electrostatic problem corresponds to a single charged conducting disk in the external potential  $\frac{\rho^2 - 2\eta^2}{8\beta}$ . The solution for the whole potential is

$$V = -z + \sin^2\theta \left( zH(z)e^{-\tilde{\varphi}} + \frac{z}{2} \right), \quad (\text{B9})$$

$$\rho = 2\sin\theta\sqrt{(1 + z^2H(z))e^{\tilde{\varphi}}}, \quad \eta = -2zH(z)\cos\theta. \quad (\text{B10})$$

The  $S^5$  shrinks when  $\sin\theta = 0$ , which corresponds to the  $\rho = 0$  axis, and the  $S^2$  shrinks when  $z = 0$ , which is a disk in  $\eta = 0$  plane centered around the origin and extends to finite  $\rho_0$ . The size of the disk is

$$\rho_0 = 2\sqrt{2C}. \quad (\text{B11})$$

The charge density  $\sigma(\rho)$  on the disk is proportional to the jump of the  $\eta$  component of the electric field  $-\partial_\eta V = -x_2$ , so we have

$$\sigma(\rho) = \frac{1}{4\pi C\rho_0} \sqrt{\rho_0^2 - \rho^2}, \quad (\text{B12})$$

which has maximum at the center and vanishes at the edge. The full potential can be expressed in integral form

$$V = \frac{\rho^2 - 2\eta^2}{8\beta} + \int_0^{2\pi} \int_0^{\rho_0} \frac{1}{4\pi C\rho_0} \frac{\sqrt{\rho_0^2 - r^2} r dr d\phi}{\sqrt{\rho^2 + \eta^2 - 2\rho r \cos\phi + r^2}} \quad (\text{B13})$$

up to a constant shift.

Now we will compare the two expressions (B9) and (B13) and solve the Eq. (B5) in the limit that we drop the 1 in the right hand side of (B5). Let us look at  $V$  along the  $\rho = 0$  line above the  $\eta = 0$  plane. This corresponds to  $\cos\theta = -1$ . We now integrate (B13) at  $\rho = 0$  and we impose the condition that  $z = 0$  when  $\eta = 0$ . Comparing with (B9) we find a relation between  $z$  and  $\eta$

$$z = \frac{\eta^2}{4\beta} - \frac{1}{4C\rho_0} \left[ (\rho_0^2 + \eta^2) \arctan \frac{\rho_0}{\eta} - \rho_0\eta - \frac{\pi}{2} \rho_0^2 \right]. \quad (\text{B14})$$

This is for  $\rho = 0, \eta > 0$ . In addition, we know the expression for  $zH(z)$  in terms of  $\eta$  from (B9)

$$zH(z) = \frac{\eta}{2}. \quad (\text{B15})$$

Thus (B14) and (B15) give a solution of  $H(z)$  in a parametric form. We also find the relation between  $\beta$  and  $C$

$$\beta = \frac{4\sqrt{2}}{\pi} C^{3/2}. \quad (\text{B16})$$

One may now write the solution in a simpler form by introducing  $r = \eta/\rho_0$

$$zH(z) = \sqrt{2C}r, \quad z = \frac{1}{\sqrt{2C}}[r + (1 + r^2) \arctan r], \quad (\text{B17})$$

where  $r$  ranges from 0 to  $\infty$ . This is a solution for Eq. (B5) when we drop the 1 in the right hand side.

We end up with the solution corresponding to single strip of M2 branes

$$e^D = 8C(1 + r^2)\sin^2\theta, \quad x_2 = \frac{1}{2C}[1 + r \arctan r] \cos\theta, \\ y = \frac{1}{\sqrt{2C}}[r + (1 + r^2) \arctan r] \sin^2\theta, \quad (\text{B18})$$

and  $C$  is a simple rescaling parameter that is associated to the charge of the solution.

### APPENDIX C: SOLUTION FOR NS5 BRANES ON $R \times S^5$

In this appendix, we write the solution for for NS5 branes on  $R \times S^5$  in the Toda form. This is not necessary for anything we did in this paper but connects it to the gauged-supergravity solution of [14]. Let us start with the 7D gauged-supergravity solution which corresponds to an elliptic M5 droplet in  $x_1, x_2$  plane [14]. This solution for the 3D Toda equation is [14]<sup>34</sup>

$$e^D = m^2 r^2 f \sinh 2\rho, \quad y = m^2 r^2 \sin\theta, \quad (\text{C1})$$

$$x_2 + ix_1 = (e^{-\rho} \cos\phi + ie^{\rho} \sin\phi) \frac{\cos\theta}{\sqrt{\sinh 2\rho}}, \quad (\text{C2})$$

$$\cosh 2\rho = F', \quad f = 1 + \frac{F}{2\sqrt{x}}, \quad x \equiv 4m^4 r^4, \quad (\text{C3})$$

$$(2\sqrt{x} + F)F'' = 1 - (F')^2, \quad (\text{C4})$$

where prime is the derivative with respect to  $x$ . The elliptic droplet in the  $x_1, x_2$  plane has axis  $a = \frac{e^{\rho}}{\sqrt{\sinh 2\rho}}|_{r=0}$ ,  $b = \frac{e^{-\rho}}{\sqrt{\sinh 2\rho}}|_{r=0}$  and we take the limit similar to Appendix B, that  $a \rightarrow \infty$ ,  $b = 1$ . Then we can approximate  $\sinh 2\rho \approx \frac{1}{2}e^{2\rho} \approx \cosh 2\rho$  and  $\rho_0 = \rho|_{r=0}$  will goes to infinity. This is equivalent to dropping the 1 in (C4). After a simple rescaling we find the solution to the 2D Toda equation

<sup>34</sup>The extremal limit of AdS<sub>7</sub> black hole, e.g. [81,82], can be written in the Toda form: We start from a more general solution  $e^D = m^2 r^2 f / \tilde{F}^2$ ,  $y = m^2 r^2 \sin\theta$ ,  $x_2 + ix_1 = (e^{-\rho} \cos\phi + ie^{\rho} \sin\phi) \tilde{F} \cos\theta$ , where  $\partial_r \tilde{F}(r) = \frac{2m^2 r \tilde{F}(r)}{f} \cosh 2\rho$ , see [14]. The extremal AdS<sub>7</sub> black hole corresponds to solution  $F = x + Q$ , we can integrate to get  $\log \tilde{F} = \int \frac{2m^2 r}{f} dr$ , and plug in these into the more general solution.

$$e^D = m^2 r^2 f e^{2\rho} \approx \left(\sqrt{x} + \frac{1}{2}F\right)F', \quad (\text{C5})$$

$$x_2 = e^{-2\rho} \cos\theta \approx (2F')^{-1} \cos\theta, \quad (\text{C6})$$

$$y = m^2 r^2 \sin\theta = \frac{\sqrt{x}}{2} \sin\theta. \quad (\text{C7})$$

where  $F$  obeys Eq. (C4) without the 1. Comparing this to (2.17) and (2.43) we can write

$$x = \frac{1}{4}C^{-2}\rho^2 I_1^2(\rho), \quad F = \frac{1}{2}C^{-1}\rho^2 I_2(\rho), \\ F' = CI_0^{-1}(\rho), \quad (\text{C8})$$

where  $C$  is a trivial overall scale. This gives a solution to (C4) (without the 1) in a parametric form.

### APPENDIX D: CHARGE AND ASYMPTOTICS OF THE D0 BRANE SOLUTIONS

In this appendix, we discuss the charge  $N_2$  and  $N_5$  and asymptotic matching of the solutions dual to vacua of the plane-wave matrix model. We then discuss the interpolating function  $f$  in the leading gravity approximation in Sec. II B 5.

We now consider the boundary conditions that correspond to the solutions dual to the plane-wave matrix model and we consider a vacuum corresponding to a single large disk at distance  $d \sim N_5$  from the  $\eta = 0$  plane. These are the vacua corresponding to  $N_5$  fivebranes. We write the leading solution of the potential in asymptotic region

$$V = \alpha(\rho^2 \eta - \frac{2}{3}\eta^3) + \tilde{\Delta}, \quad \tilde{\Delta} = \frac{P\eta}{(\eta^2 + \rho^2)^{3/2}}. \quad (\text{D1})$$

Using the coordinate  $r = 4\sqrt{\rho^2 + \eta^2}$  and  $t = x_0$  we find that the leading order solution at large  $r$  in (2.20)–(2.24) is the standard D0 brane solution [30] at large  $r$ , with warp factor

$$Z = \frac{2^8 15P}{r^7 \alpha}, \quad \alpha = \frac{8}{g_s}. \quad (\text{D2})$$

We now need to compute  $P$ . We compute the charge and the distance. Since we have images we have  $P = 2dQ$ . The distance is given in terms of  $N_5$  by (2.26). In order to compute the charge we note that if we have a large disk with a size  $\rho_0 \gg N_5$  then the configuration at large distances looks like a single conducting disk at  $\eta = 0$  with some extra sources localized near  $(\rho, \eta) = (\rho_0, 0)$ . We can thus approximate the induced charge on the disk to be the induced charge we would have on the conducting plane at  $\eta = 0$  if we had not introduced the disk. This induced charge is given simply by the external potential which is the first term in (D1). We can thus approximate

$$Q = \frac{1}{4\pi} \int \partial_\eta V_{\text{ext}} = \frac{\alpha \rho_0^4}{8}, \quad d = \frac{\pi}{2} N_5. \quad (\text{D3})$$

Now we can go back to the expression for  $Z$  and write it as

$$Z = \frac{2^5 15 \pi \rho_0^4 N_5}{r^7}, \quad (\text{D4})$$

where we are in the regime where the disk is very close to the  $\eta = 0$  plane.

We can now compare with the result in [30]

$$Z = \frac{2^7 \pi^{9/2} \Gamma(7/2) g_{\text{YM}0}^2 N_0}{r^7} = \frac{2^4 15 \pi^5 g_{\text{YM}0}^2 N_0}{r^7}. \quad (\text{D5})$$

Comparing the two we find

$$\rho_0^4 = \frac{1}{2} \pi^4 g_{\text{YM}0}^2 N_2. \quad (\text{D6})$$

We find also that the function  $f$  in Sec. II B 5 is

$$E - J \sim 1 + f \frac{n^2}{J^2}, \quad f = \frac{1}{2} R_{S^5}^4, \quad R_{S^5}^2 = 4\rho_0. \quad (\text{D7})$$

Finally we obtain

$$\frac{R_{S^5}^2}{\alpha'} = 4 \left( \frac{\pi^4 g_{\text{YM}0}^2 N_2}{2m^3} \right)^{1/4}, \quad (\text{D8})$$

$$f = 4\pi^2 \left( \frac{2g_{\text{YM}0}^2 N_2}{m^3} \right)^{1/2} \quad (\text{D9})$$

in the strong coupling regime.

## APPENDIX E: POINCARÉ SUPER ALGEBRAS WITH NONCENTRAL CHARGES

In this appendix we discuss two Poincaré superalgebras with mass deformations [19] which appeared in our discussion. First we present an algebra with 8 supercharges and then an algebra with 16 supercharges.

Let us define  $(\gamma^\mu)_\alpha^\beta$  as

$$\gamma^0 = i\sigma^2, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^3, \quad (\text{E1})$$

where  $\sigma^i$  are Pauli matrices. We also define

$$\begin{aligned} \tilde{\gamma}_{\alpha\beta}^\mu &= (\gamma^\mu)_\alpha^\gamma \epsilon_{\gamma\beta}, & \tilde{\gamma}^0 &= -\delta^{\alpha\beta}, \\ \tilde{\gamma}^1 &= -\sigma^3, & \tilde{\gamma}^2 &= \sigma^1, \end{aligned} \quad (\text{E2})$$

and we see that  $(\tilde{\gamma}^\mu)_{\alpha\beta}$  is symmetric in the indices  $\alpha, \beta$ .

## 1. Superalgebras with 8 supercharges

We define supercharges  $Q_{\alpha i}$  with  $i$  an  $SO(4)$  index and  $\alpha$  is the 2 + 1 Lorentz index (spinor of  $SO(2, 1)$ ). We can impose the reality condition  $Q_{\alpha i}^\dagger = Q_{\alpha i}$ .

We start by considering a superalgebra with 8 supercharges given by

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\tilde{\gamma}_{\alpha\beta}^\mu p_\mu \delta_{ij} + 2m\epsilon_{\alpha\beta} \epsilon_{ijkl} M_{kl}, \quad (\text{E3})$$

$$[p_\mu, Q_{\alpha i}] = 0, \quad [p_\mu, p_\nu] = 0, \quad (\text{E4})$$

$$[\Sigma_{\mu\nu}, Q_{\alpha i}] = \frac{1}{2} (\tilde{\gamma}_{\mu\nu})_\alpha^\beta Q_{\beta i}, \quad (\text{E5})$$

$$[M_{jl}, Q_{\alpha i}] = i(\delta_{ij} Q_{\alpha l} - \delta_{il} Q_{\alpha j}), \quad (\text{E6})$$

$$[M_{ij}, M_{kl}] = i(\delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{jk} M_{il} - \delta_{il} M_{jk}), \quad (\text{E7})$$

$$[\Sigma_{\mu\nu}, p_\lambda] = i(\eta_{\nu\lambda} p_\mu - \eta_{\mu\lambda} p_\nu), \quad (\text{E8})$$

$$[\Sigma_{\mu\nu}, \Sigma_{\lambda\rho}] = i(\eta_{\nu\lambda} \Sigma_{\mu\rho} + \eta_{\mu\rho} \Sigma_{\nu\lambda} - \eta_{\mu\lambda} \Sigma_{\nu\rho} - \eta_{\nu\rho} \Sigma_{\mu\lambda}), \quad (\text{E9})$$

$$[M_{jl}, p_\mu] = 0, \quad [M_{jl}, \Sigma_{\mu\nu}] = 0, \quad (\text{E10})$$

where  $M_{ij}$  are  $SO(4)$  generators.  $M_{ij}$  are noncentral charges in the super-Poincaré algebra in 2 + 1 dimensions.  $\Sigma_{\mu\nu}$  is the Lorentz generator in  $SO(2, 1)$ . Notice that the first line is the only nonobvious commutator and is the one stating that we have *noncentral* charges.

In order to check the closure of the superalgebra we need to check the Jacobi identity. The identities involving one bosonic generator will be automatically obeyed since they are just simply stating that objects transform covariantly under the appropriate symmetries.

So the only nontrivial identity that we need to check is the one involving three odd generators. The Jacobi identity is

$$\begin{aligned} [Q_{\alpha i}, \{Q_{\beta j}, Q_{\gamma l}\}] + [Q_{\beta j}, \{Q_{\gamma l}, Q_{\alpha i}\}] + [Q_{\gamma l}, \{Q_{\alpha i}, Q_{\beta j}\}] &= i\epsilon_{\beta\gamma} \epsilon_{liab} (\delta_{bi} Q_{\alpha a} - \delta_{ai} Q_{\alpha b}) + i\epsilon_{\gamma\alpha} \epsilon_{liab} (\delta_{bj} Q_{\beta a} - \delta_{aj} Q_{\beta b}) \\ &\quad + i\epsilon_{\alpha\beta} \epsilon_{ijab} (\delta_{bl} Q_{\gamma a} - \delta_{al} Q_{\gamma b}) \\ &= -i\epsilon_{ijla} [\epsilon_{\beta\gamma} Q_{\alpha a} + \epsilon_{\gamma\alpha} Q_{\beta a} + \epsilon_{\alpha\beta} Q_{\gamma a}] \\ &\quad - i\epsilon_{ijlb} [\epsilon_{\beta\gamma} Q_{\alpha b} + \epsilon_{\gamma\alpha} Q_{\beta b} + \epsilon_{\alpha\beta} Q_{\gamma b}] \equiv 0. \end{aligned} \quad (\text{E11})$$

It is interesting to study the particle spectrum for theories based on this superalgebra (E3). This theory cannot have massless propagating particles. This can be seen as follows. We assume that the massless particle has  $p_- = 0$ ,  $p_+ \neq 0$  and  $p_2 = 0$ . In this case the supersymmetry algebra

implies that  $Q_-^i$  and  $M_{ij}$  annihilate all states in the supermultiplet. On the other hand the  $Q_+^i$  generators arrange themselves into creation and annihilation operators and change the  $SO(4)$  quantum numbers in the multiplet. Thus we reached a contradiction. This argument allows



Chern-Simons interactions since that is a topological theory. So all propagating particles are massive. Let us go to the rest frame of the massive particle, with  $p_1 = p_2 = 0$ . Then the “little group” (i.e. the truncation of (E3) to the generators that leave this choice of momenta invariant) is the  $\widetilde{SU}(2|2)$  supergroup. The tilde represents the fact that we take the corresponding  $U(1)$  to be noncompact. The representation theory of this algebra was studied in [56,83,84]. As usual, there are short representations when the BPS bound is obeyed when the mass of the particle is  $M = 2m(j_1 + j_2)$ , where  $m$  is the mass parameter in (E3).

The superalgebra (E3)–(E10) can be reduced to 1 + 1 dimensions in a trivial fashion, we just set  $p_2 = 0$  and remove two of the Lorentz generators. This is the symmetry algebra (2.88) of the sigma model considered in (2.91). The reason this superalgebra arises is the following. Suppose we start with a theory with supergroup  $\widetilde{SU}(2|4)$  and we pick a 1/2 BPS state with charge  $J$  under generator  $J$  in  $SO(2) \subset SO(6)$ . The supercharges that annihilate this state form the supergroup  $\widetilde{SU}(2|2)$ . The light cone string Lagrangian (2.91) describes small fluctuations around these BPS states so that the supergroup  $\widetilde{SU}(2|2)$  should

act on them linearly. Since the world sheet action is boost invariant along the world sheet, we find that this supergroup should be extended to (E3).

Let us give some further examples of theories with this superalgebra. We can construct a 1 + 1 dimensional SYM with this superalgebra from the plane-wave matrix model via matrix theory compactification techniques [62] (also [85]). In fact this 1 + 1 SYM was constructed in this way by e.g. [11,12,86]. Here we will reproduce this result and we will use  $SO(9, 1)$  gamma matrices and the fermions are  $SO(9, 1)$  spinors.<sup>35</sup> We will then compactify a scalar of the 1 + 1 SYM and get a 2 + 1 super Yang-Mills Chern-Simons theory satisfying the above superalgebra.

One starts from the plane-wave matrix model whose mass terms for the  $SO(6)$  scalars takes the form  $-\frac{1}{2}(X_a)^2$ , where  $a = 1, 2, \dots, 6$ . We have set the mass for the  $SO(6)$  scalar to 1. We should write the action so that it is translation invariant in one of the transverse scalars. We can make a field-redefinition for two  $SO(6)$  scalars  $X_1 + iX_2 = e^{it}(Y + i\phi)$  and for fermions  $\Psi = e^{(1/2)\Gamma_{12}t}\theta$ . Then the action of plane-wave matrix model is

$$S = \frac{1}{g_{\text{YM}0}^2} \int dx_0 \text{Tr} \left( -\frac{1}{2}(D_0 X_I)^2 - \frac{1}{2}(D_0 Y)^2 - \frac{1}{2}(D_0 \phi)^2 - \frac{i}{2} \bar{\theta} \Gamma^0 D_0 \theta - \frac{1}{2} \bar{\theta} \Gamma_I [X_I, \theta] - \frac{1}{2} \bar{\theta} \Gamma_1 [\phi, \theta] - \frac{1}{2} \bar{\theta} \Gamma_2 [Y, \theta] \right. \\ \left. + \frac{1}{2} [\phi, X_I]^2 + \frac{1}{2} [\phi, Y]^2 + \frac{1}{2} [Y, X_I]^2 + \frac{1}{4} [X_I, X_J]^2 - \frac{1}{2} (X_a)^2 - \frac{1}{2} 2^2 (X_i)^2 + \frac{3}{2} i \bar{\theta} \Gamma_{789} \theta + 2i \epsilon^{ijk} X_i X_j X_k \right. \\ \left. - \frac{1}{2} i \bar{\theta} \Gamma_0 \Gamma_{12} \theta - 2Y D_0 \phi \right). \tag{E12}$$

We have 3 + 4 + 2 scalars, where the first seven scalars with indices  $I = 3, 4, \dots, 9$  are split into  $a = 3, 4, 5, 6$  and  $i = 7, 8, 9$  and the rest two scalars are  $Y$  and  $\phi$ .

Then the action becomes translation invariant in the  $\phi$  direction. We now compactify  $\phi$  by replacing  $\phi$  with gauge covariant derivative  $\phi \rightarrow i \frac{\partial}{\partial x_1} + A_1$ ,  $-i[\phi, O] \rightarrow \partial_1 O - i[A_1, O]$  [62] (also [85]). Plugging this into the original action (E12) one gets the 1 + 1 dimensional super Yang-Mills theory on  $R^{1,1}$  with a mass deformation

$$S = \frac{1}{g_{\text{YM}1}^2} \int dx_0 dx_1 \text{Tr} \left( -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (D_\mu X_I)^2 - \frac{1}{2} (D_\mu Y)^2 - \frac{i}{2} \bar{\theta} \Gamma^\mu D_\mu \theta - \frac{1}{2} \bar{\theta} \Gamma_I [X_I, \theta] - \frac{1}{2} \bar{\theta} \Gamma_2 [Y, \theta] + \frac{1}{2} [Y, X_I]^2 \right. \\ \left. + \frac{1}{4} [X_I, X_J]^2 - \frac{1}{2} (X_a)^2 - \frac{1}{2} 2^2 (X_i)^2 + \frac{3}{2} i \bar{\theta} \Gamma_{789} \theta + 2i \epsilon^{ijk} X_i X_j X_k - \frac{1}{2} i \bar{\theta} \Gamma_{012} \theta - Y \epsilon^{\mu\nu} F_{\mu\nu} \right). \tag{E13}$$

We have 3 + 4 + 1 scalars, with the seven scalars whose indices are  $I = 3, 4, \dots, 9$ , where  $a = 3, 4, 5, 6$  and  $i = 7, 8, 9$ , and another scalar  $Y$ , and  $\mu = 0, 1$ . The theory has super-Poincare algebra on  $R^{1,1}$  with  $SU(2) \times SU(2)$   $R$  symmetry. The first  $SU(2)$  rotates the first three scalars  $i = 7, 8, 9$  and the second  $SU(2)$  is one of the  $SU(2)$  factors in the  $SO(4)$  rotating the four scalars  $a = 3, 4, 5, 6$ . In addition, the theory has an  $SU(2)$  global symmetry, which is the second  $SU(2)$  factor in the  $SO(4)$  we have just mentioned. Compactifying along  $x_1$  and taking the compactification size to zero we get back to the plane-wave matrix model which has a larger symmetry group. The

parameters in the two theories are related by  $g_{\text{YM}1}^2 = 2\pi R_{x_1} g_{\text{YM}0}^2$ , where  $R_{x_1}$  is the radius of the  $x_1$  circle. The 1 + 1 SYM constructed from the plane-wave matrix model coincides with the DLCQ of the IIA plane wave [11,12], which was first obtained by [11,12] from matrix regularization (double dimensional reduction) of the supermembrane action under kappa-symmetry fixing condition on 11D maximal plane wave. The action we reproduce here

<sup>35</sup>Our convention is different from that of [85] or [3], which use  $SO(9)$  gamma matrices.

(E13) is written manifestly Lorentz invariant in  $1 + 1$  dimensions.

We pointed out that this theory can be uplifted again making  $Y$  periodic. We make the replacement  $Y \rightarrow i \frac{\partial}{\partial x_2} - A_2$ ,  $-i[Y, O] \rightarrow \partial_2 O + i[A_2, O]$  [62]. The coupling  $YF_{01}$ , becomes a Chern-Simons term in  $2 + 1$  dimensions. The quantization of the level of the Chern-Simons action im-

$$\begin{aligned}
 S = \frac{k}{4\pi} \left\{ \int \text{Tr} \left[ -\frac{1}{2} F \wedge *F + A \wedge dA + \frac{2}{3} A \wedge A \wedge A - \frac{i}{12} \bar{\psi} \Gamma_{\mu\nu\lambda} \psi dx^\mu \wedge dx^\nu \wedge dx^\lambda \right] \right. \\
 + \int d^3x \text{Tr} \left[ -\frac{1}{2} (D_\mu X_I)^2 - \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi - \frac{1}{2} \bar{\psi} \Gamma^I [X_I, \psi] + \frac{1}{4} [X_I, X_J]^2 - \frac{1}{2} (X_a)^2 - \frac{1}{2} 2^2 (X_i)^2 + 2i \epsilon^{ijk} X_i X_j X_k \right. \\
 \left. \left. + \frac{i}{4} \epsilon^{ijk} \bar{\psi} \Gamma_{ijk} \psi \right] \right\}, \quad (\text{E14})
 \end{aligned}$$

where we have  $3 + 4$  scalars with indices  $I = 3, 4, \dots, 9$  split into  $a = 3, 4, 5, 6$  and  $i, j, k = 7, 8, 9$ , and the worldvolume indices are  $\mu, \nu, \lambda = 0, 1, 2$ . The coupling constants is related to the  $1 + 1$  SYM by  $g_{\text{YM}2}^2 = 2\pi R_{x_2} g_{\text{YM}1}^2$ ,  $\frac{k}{4\pi} = \frac{1}{g_{\text{YM}2}^2}$  and  $k \in \mathbb{Z}$ . So we see that  $k$  is the only coupling constant in the theory. When  $k$  is large the theory is weakly coupled.

## 2. Superalgebras with 16 supercharges

Finally, let us turn our attention to the superalgebra for theories with 16 supercharges. Now we have two  $SO(4)$  groups and a second set of supercharges  $\tilde{Q}_{\alpha m}$ . We add the anticommutators

$$\{\tilde{Q}_{\alpha m}, \tilde{Q}_{\beta n}\} = 2\tilde{\gamma}_{\alpha\beta}^\mu p_\mu \delta_{mn} + 2m' \epsilon_{\alpha\beta} \epsilon_{mnr s} \tilde{M}_{rs}, \quad (\text{E15})$$

where  $\tilde{M}_{rs}$  is generator of the second  $SO(4)$ . The anticommutator of  $Q_{\alpha i}$  with  $\tilde{Q}_{\alpha m}$  is zero. The rest of the algebra is rather obvious and is just given by the covariance properties of the indices as in (E3). In principle we can have  $m' \neq m$  in (E3). In the theories studied here we have  $m' = m$ . If we want to have BPS states under both  $Q$  and  $\tilde{Q}$  then we need that  $m/m'$  to be a rational number. Note that the little group for a massive particle is  $S\tilde{U}(2|2) \times S\tilde{U}(2|2)$ .

Let us be a little more precise about these  $SO(4)$  groups. The ansatz in [14] above has two three spheres on which two  $SO(4)$  group act. Let us call them  $SO(4)_i$  with  $i = 1, 2$ . Each of these two groups are  $SO(4)_i = SU(2)_{Li} \times SU(2)_{Ri}$ . The supercharges  $Q_{\alpha i}$  in (E3) transform under  $SU(2)_{L1} \times SU(2)_{L2}$ . The supercharges  $\tilde{Q}_{\alpha i}$  transform under  $SU(2)_{R1} \times SU(2)_{R2}$ . If we quotient any of these theories by a  $Z_k$  in  $SU(2)_{Ri}$ , we get a theory that only has 8 generators as in (E3).

$$\begin{aligned}
 S = \frac{2}{g_{\text{YM}4}^2} \int d^2x d^3\Omega \text{Tr} \left( -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} D_M X_a D^M X_a - \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi - \frac{1}{2} \bar{\Psi} \Gamma^a [X_a, \Psi] - \frac{1}{2} \bar{\Psi} \Gamma_5 [Y, \Psi] + \frac{1}{4} [X_a, X_b]^2 \right. \\
 \left. + \frac{1}{2} [Y, X_a]^2 + \frac{\mu}{2} i \bar{\Psi} \Gamma_{045} \Psi - \frac{1}{2} \mu^2 X_a^2 - 2\mu Y F_{04} \right) \quad (\text{F1})
 \end{aligned}$$

where  $M, N = 0, 1, \dots, 4$ ;  $a = 6, 7, 8, 9$ ;  $\Gamma_M, \Gamma_5, \Gamma_a$  are ten dimensional gamma matrices. The theory does not have

plies that the compactification radius of  $Y$  is quantized. This quantization condition also follows from the fact that the coupling  $YF_{01}$  is not invariant under arbitrary shifts of  $Y$ , and  $e^{iS}$  is periodic only if we shift  $Y$  by the right amount. Finally we get the  $2 + 1$  dimensional super Yang-Mills Chern-Simons theory

This algebra with 16 generators is the one that appeared on the worldvolume of theories related to the IIB constructions of Sec. III. In the case of the M2 brane theory the two  $SO(4)$ s are global  $R$ -symmetries of the theory. In the case that we consider an M5 on  $R^{2,1} \times S^3$  one of the  $SO(4)$  groups is a symmetry acting on the worldvolume. When the size of  $S^3$  becomes infinity, the  $SO(4)$  that acts on the worldvolume is contracted to  $ISO(3)$  and only the translation generators remain in the right hand side of the supersymmetry algebra. Thus, we do not get into trouble with the Haag-Lopuszanski-Sohnius theorem [17] in total spacetime dimension  $d \geq 4$ .

The dimensional reduction of this algebra to  $1 + 1$  dimensions gives the linearly realized symmetries on the light cone world sheet of a string moving in the maximally supersymmetric IIB plane wave [20].

## APPENDIX F: 4 + 1 D SYM AND 5 + 1 D SYM WITH CHERN-SIMONS TERM FROM $\mathcal{N} = 4$ SYM

In this section we discuss in more detail the Lagrangian on the D4 brane and the D5 brane that we obtained by starting from  $\mathcal{N} = 4$  super Yang-Mills theory and compactifying the transverse scalars. The procedure is identical to the one used in Appendix E.

We start from the  $\mathcal{N} = 4$  SYM on  $R \times S^3$  with mass terms for the 6 scalars  $-\frac{1}{2} \mu^2 X_a^2$ , where  $a = 4, 5, 6, 7, 8, 9$ . We redefine two of the scalars  $X_4 + iX_5 = e^{i\mu t} (Y + \phi)$  and fermions  $\Psi_{\text{old}} = e^{(1/2)\mu \Gamma_{45} t} \Psi$ . We then make the replacement  $\phi \rightarrow i \frac{\partial}{\partial x_4} + A_4$ ,  $-i[\phi, O] \rightarrow \partial_4 O - i[A_4, O]$  [62]. We obtain a  $4 + 1$  super Yang-Mills theory on  $R^{1,1} \times S^3$  with a mass deformation

Poincaré invariance in 4 + 1 dimensions, but it has Poincaré invariance in the  $R^{1,1}$  subspace  $x_0, x_4$ . It has  $SO(4)$   $R$  symmetry. When it is truncated by keeping states that are invariant only under the  $SU(2)_L$  which acts on  $S^3$  it gives the 1 + 1 SYM in (E13). When we reduce it on  $S^1$ , it gives back the  $\mathcal{N} = 4$  super Yang-Mills theory. The parameters are related by  $g_{\text{YM}4}^2 = 2\pi R_{x_1} g_{\text{YM}3}^2$ .

This 4 + 1 super Yang-Mills theory can be uplifted again by compactifying  $Y$ , and making replacement  $Y \rightarrow i \frac{\partial}{\partial x_5} - A_5, -i[Y, O] \rightarrow \partial_5 O + i[A_5, O]$ , similar to Appendix E. The uplifted action is a 5 + 1 super Yang-Mills theory on  $R^{2,1} \times S^3$  which contains a Chern-Simons term for the 2 + 1 dimensional gauge fields

$$S = \frac{2}{g_{\text{YM}5}^2} \int \text{Tr} \left[ -\frac{1}{4} F_{MN} F^{MN} \right] + \frac{K}{4\pi} \int \text{Tr} \left[ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right] \wedge \frac{d^3 \Omega}{\text{Vol}_{S^3}} + \frac{2}{g_{\text{YM}5}^2} \int d^3 x d^3 \Omega \text{Tr} \left[ -\frac{1}{2} D_M X_a D^M X_a + \frac{1}{4} [X_a, X_b]^2 - \frac{1}{2} \mu^2 X_a^2 - \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi - \frac{1}{2} \bar{\Psi} \Gamma^a [X_a, \Psi] + \frac{\mu}{2} i \bar{\Psi} \Gamma_{045} \Psi \right], \quad (\text{F2})$$

where  $M, N = 0, 1, \dots, 4, 5$ ;  $a = 6, 7, 8, 9$ . The coupling constant is  $g_{\text{YM}5}^2 = 2\pi R_{x_2} g_{\text{YM}4}^2, \frac{K}{4\pi} = \frac{2\mu \text{Vol}_{S^3}}{g_{\text{YM}5}^2}, K \in \mathbb{Z}$ , where  $\text{Vol}_{S^3}$  is the volume of the  $S^3$ . This is the  $S^3$  on which the original  $\mathcal{N} = 4$  is defined. Notice that the coupling constant is given in terms of  $K$ . This implies the weak coupling limit corresponds to large  $K$ . The theory only has Poincaré invariance on  $R^{2,1}$  subspace  $x_0, x_4, x_5$ . Truncating this theory by keeping only states invariant under the  $SU(2)_L$  that acts on  $S^3$  we recover the 2 + 1 dimensional Yang-Mills Chern-Simons in (E14). This 5 + 1D theory can also be reduced to a 4 + 1 dimensional super Yang-Mills theory on  $R^{2,1} \times S^2$  if we replace  $S^3$  with  $S^3/Z_k$  and reduce on the fiber direction of the latter in the similar way in Sec. II A 2.

Similarly, the 4 + 1 dimensional Yang-Mills theory (F1) can be reduced to a 3 + 1 dimensional Yang-Mills theory on  $R^{1,1} \times S^2$  by truncating by  $U(1)_L \subset SU(2)_L$  which acts on  $S^3$ . Alternatively, this theory can be obtained through the uplifting procedure applied to the D2 brane theory on  $R \times S^2$  that we discussed in Sec. II A 2.

## APPENDIX G: COMPUTATION OF THE INDEX COUNTING BPS STATES

In this appendix we compute the index (2.1) for various situations. We are interested in computing this index for single trace states in the 't Hooft  $N = \infty$  limit. Since the index is basically a counting problem we can use Polya theory, as explained in [21,87]. What we want to do is the following. We have a set of ‘‘letters’’ which are the various oscillator modes. This set depends on the vacuum we expand around. We define the single particle partition function

$$z = \sum_{\text{bosons}} e^{-\beta_i Q_i} - \sum_{\text{fermions}} e^{-\beta_i Q_i}, \quad (\text{G1})$$

where  $Q_i$  are various charges. The single trace states are counted by

$$Z_{\text{s.t}} = - \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log[1 - z(n\beta_i)]. \quad (\text{G2})$$

Where  $\varphi(n)$  is the Euler Phi function which counts number of integers less than  $n$  that are relatively prime with  $n$ .  $\varphi(1) \equiv 1, \varphi(2) = 1, \varphi(3) = 2$ , etc.

When we compute (2.1) only states with  $\mathcal{U} \equiv H_4 \equiv E - 2S - \sum_i J_i = 0$  contribute. Let us first consider the states in the first Kaluza Klein mode, which is in the representation in Fig. 7(a). For convenience we will use  $Y_j$  to denote  $SO(6)$  scalars and  $X_i$  for  $SO(3)$  scalars in this appendix. The bosons that contribute are given by  $Y^j + iY^{j+1}, j = 1, 3, 5$  and  $X^+ = X^1 + iX^2$ . The fermions that contribute have the indices  $\psi_{+,+++}, \psi_{+,++-}, \psi_{+,-++}$  where the indices indicate the charges under  $(S, J_1, J_2, J_3)$  and  $S$  is one of the generators of  $SU(2) \subset SU(2|4)$ . Then we find

$$z_1 = e^{-\beta_2 - \beta_3} + e^{-\beta_1 - \beta_3} + e^{-\beta_1 - \beta_2} + e^{-2(\beta_1 + \beta_2 + \beta_3)} - e^{-2\beta_1 - \beta_2 - \beta_3} - e^{-\beta_1 - 2\beta_2 - \beta_3} - e^{-\beta_1 - \beta_2 - 2\beta_3} (1 - z_1) = (1 - e^{-\beta_1 - \beta_2})(1 - e^{-\beta_2 - \beta_3})(1 - e^{-\beta_1 - \beta_3}). \quad (\text{G3})$$

We can now use the formula

$$- \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log(1 - q^n) = \frac{q}{(1 - q)} \quad (\text{G4})$$

in order to write (G2) in terms of (G3) to obtain

$$I_{\text{s.t.} N_5=1} = \frac{e^{-\beta_2 - \beta_1}}{1 - e^{-\beta_2 - \beta_1}} + \frac{e^{-\beta_3 - \beta_1}}{1 - e^{-\beta_3 - \beta_1}} + \frac{e^{-\beta_3 - \beta_2}}{1 - e^{-\beta_3 - \beta_2}}. \quad (\text{G5})$$

In order to read off which representations are contributing it is useful to compute the index for the doubly atypical representations of the form  $(a_1, a_2, a_3 | a_4 | a_5) = (0, p, 0 | 0 | 0)$ . This notation refers to the Dynkin labels, see Fig. 7(g) and [7] for further details. We obtain

$$I_{(0,p,0|0|0)} = e^{-p(\beta_1 + \beta_2)} \frac{(1 - e^{-\beta_2 - \beta_3})(1 - e^{-\beta_1 - \beta_3})}{(1 - e^{\beta_1 - \beta_3})(1 - e^{\beta_2 - \beta_3})} + \text{cyclic}. \quad (\text{G6})$$

We can see that if we sum this over  $p$  we obtain

$$I_{\text{s.t. } N_5=1} = \sum_{p=1}^{\infty} I_p. \quad (\text{G7})$$

This discussion implies that all the BPS representations that contribute to the index for the  $N_5 = 1$  case are the ones we expect from the doubly atypical representations to which the string ground state  $\text{tr}[Z^J]$  belongs to. Of course, in order to show that these representations are protected we do not need any of this technology, since doubly atypical representations cannot be removed [7]. All we are showing here is that we find no evidence of further BPS representations for the  $N_5 = 1$  vacuum. This result is not totally trivial since we can certainly construct individual single trace states in other atypical representations. These are singly atypical representations. But we find that they always come in pairs that could combine into long representations. Of course, the explicit analysis we described in Sec. II C shows that they all do combine. Before we leave this simple case, let us understand how we connect these results to the spectrum of the string theory in light cone gauge. It is convenient to focus on the  $\widetilde{SU}(2|2)$  subgroup in  $\widetilde{SU}(2|4)$ , the energy  $\hat{E}$  in  $\widetilde{SU}(2|2)$  is the same as  $\hat{E} = E - J_3$  in  $SU(2|4)$ . So we are interested in taking a limit where  $\beta_3$  is large and  $\beta_1 + \beta_2$  is small. We place no constraint on  $\beta_1 - \beta_2$ . Actually, to be more precise, note that the choice of generator  $J_3$ , or field  $Z = Y^5 + iY^6$  leaves a subgroup  $\widetilde{SU}(2|2) \times SU(2)_G \subset \widetilde{SU}(2|4)$  unbroken. The chemical potential  $\beta_1 - \beta_2$  couples to the generator in the global  $SU(2)_G$  which is not part of the  $\widetilde{SU}(2|2)$  supergroup. Our goal is to relate (G5) to an index we can compute on the string world sheet of the form

$$I(\gamma, \tilde{\gamma}) = \text{Tr}[(-1)^F 2S_3 e^{-\tilde{\mu}(\hat{E} - S_3 - \tilde{S}_3)} e^{-\gamma \hat{E}} e^{-\tilde{\gamma} J_3^G}] \quad (\text{G8})$$

which is the same as (2.92) except that we have added a chemical potential for the generator  $J_3^G$  in  $SU(2)_G$ . It is clear that we should identify  $\gamma = \beta_3$  and  $\tilde{\gamma} = \beta_1 - \beta_2$ . Let us state the final result and then we will justify it. We have

$$\lim_{\beta_1 + \beta_2 \rightarrow 0} \left[ I_{\text{s.t.}}(\beta_i) - \frac{q}{(1-q)} \right] = -I(\gamma = \beta_3, \tilde{\gamma} = \beta_1 - \beta_2). \quad (\text{G9})$$

Let us explain how we obtained this. The states giving rise to string world sheets in the plane-wave limit have very large values of  $E$ . So we need to isolate from (G5) the contribution from states with large values of  $E$ . The first idea is to isolate from (G5) terms with large powers of  $q = e^{-\beta_1 - \beta_2}$  but low powers of  $e^{-\beta_3}$ . The only such states are the ones in the first term in (G5). Unfortunately, such states have no  $\beta_3$  dependence at all and correspond to the ground states. This is related to the fact that (G8) would vanish if we had not inserted  $J_3$ . In (G5) the absence of high powers

of  $q$  in the  $\beta_3$  dependent terms is due to the fact that each representation with large  $p$  contributes with a factor of  $(1 - q)$  to terms with finite powers of  $e^{-\beta_3}$ . This factor arises as follows. Among the supercharges with  $\mathcal{U} = 0$  we have one which has zero  $\hat{E}$ . It has quantum numbers  $Q_{+, --+}^\dagger$ . This supercharge does not annihilate  $\beta_3$  dependent terms and gives rise to the  $(1 - q)$  factor. This can be seen more explicitly by rewriting the first term in (G6) as

$$q^{-p} \left\{ 1 + (1 - q) \times \frac{q^{-1/2} e^{-\beta_3} (e^{\tilde{\gamma}/2} + e^{-\tilde{\gamma}/2}) - e^{-2\beta_3} (1 + 1/q)}{(1 - q^{-1/2} e^{\tilde{\gamma}/2 - \beta_3}) (1 - q^{-1/2} e^{-\tilde{\gamma}/2 - \beta_3})} \right\}, \quad (\text{G10})$$

$$\tilde{\gamma} = \beta_1 - \beta_2, \quad q = e^{-\beta_1 - \beta_2}. \quad (\text{G11})$$

The term independent of  $\beta_3$  is the contribution to the ground state of the string and is explicitly subtracted in (G9). The other terms in (G10) as well as the second and third terms in (G6) contain a factor of  $(1 - q)$ . So these contributions would vanish if we took the  $q \rightarrow 1$  limit. In order to avoid this problem we introduce a factor  $w^{2S}$  when we compute the contribution of each BPS representation. We then take a derivative with respect to  $w$ , set  $w = 1$  and take  $q \rightarrow 1$ . This gives us a finite answer for each  $p$  in (G6). In fact we get to strip off a factor of  $(1 - wq)$  and replace it by  $(-1)$ , since we are interested in taking the large  $p$  limit. We can equivalently obtain this limit by simply starting from the full expression and taking the limit in (G7) since terms which involve finite powers of  $q$  will cancel out due to the  $(1 - q)$  factor, while the sum over many terms involving a power of  $q^p$  will give a  $1/(1 - q)$  cancelling the explicit factor of  $(1 - q)$ . In other words, if we were to truncate the sum (G7) to a finite number of terms and then take  $q \rightarrow 1$  then we would get zero for all  $\beta_3$  dependent terms.

The limit (G9) gives us

$$I = -\frac{e^{-\beta_3 - \tilde{\gamma}/2}}{1 - e^{-\beta_3 - \tilde{\gamma}/2}} - \frac{e^{-\beta_3 + \tilde{\gamma}/2}}{1 - e^{-\beta_3 + \tilde{\gamma}/2}}. \quad (\text{G12})$$

This is indeed the result we get for  $I$  if we compute the contribution in the string theory side for a  $(4, 4)_m$  world sheet theory with fields in the fundamental representation of  $\widetilde{SU}(2|2)$  and the fundamental of  $SU(2)_G$ . These are precisely the fields coming from the first four directions of the string world sheet.

Now let us now consider vacua with  $N_5 > 1$ . In order to compute the index in these cases it is useful to write a general formula for the index for an arbitrary  $SU(2|4)$  representation. If the representation is typical then the index vanishes. We can understand this as follows. On a typical  $SU(2|4)$  representation we find that, for the purposes of counting the states, the supercharges act like fermionic creation and annihilation operators. In fact,



when we look at the expression for the characters in [84] we find that there is a factor of the form

$$\prod_j (1 - e^{-\theta_i H_i^j}), \quad (\text{G13})$$

where  $j$  runs over half of the supercharges and  $H_i^j$  are the Cartan charges of this supercharge. The index we have defined is simply a character evaluated for special values of  $\theta_i$  which are such that a particular supercharge,  $Q_{-,+++}^\dagger$ , gives a contribution of the form  $(1 - 1) = 0$ . This ensures that the index vanishes for long (or typical) representations. In atypical representations one finds that the character does not contain the full factor (G13). In fact, the index will receive contributions only from states with  $\mathcal{U} = 0$  [see (2.2)]. So we can truncate the  $SU(2|4)$  superalgebra to the elements that have  $\mathcal{U} = 0$ . This gives a  $\widetilde{SU}(1|3)$  superalgebra. So the states contributing to the index form  $\widetilde{SU}(1|3)$  representations. The index is the same as the character of the  $\widetilde{SU}(1|3)$  representation. It turns out that if we consider an atypical representation of  $\widetilde{SU}(2|4)$  of the form  $(a_1, a_2, a_3|a_5 + 1|a_5)$  then states with  $\mathcal{U} = 0$  form a typical representation of  $\widetilde{SU}(1|3)$ . Doubly atypical representations of  $\widetilde{SU}(2|4)$  give rise to atypical representations of  $\widetilde{SU}(1|3)$ . Let us be more explicit. Let us start with the atypical  $\widetilde{SU}(2|4)$  representation  $r$  labeled by  $(a_1, a_2, a_3|a_5 + 1|a_5)$ . The index evaluated on this representation gives us

$$I_{(a_1, a_2, a_3|a_5 + 1|a_5)} = -(-1)^{a_5} e^{-Q(\sum_i \beta_i)} (1 - e^{-\beta_1 - \beta_2})(1 - e^{-\beta_1 - \beta_3})(1 - e^{-\beta_2 - \beta_3}) \chi_{(a_1, a_2)}(\mathbf{g}), \quad (\text{G14})$$

where

$$Q = 2 + a_5 + a_3 + \frac{2}{3}a_2 + \frac{1}{3}a_1, \quad (\text{G15})$$

and  $\chi_{(a_1, a_2)}$  is a character of an  $SU(3)$  representation with  $(a_1, a_2)$  Dynkin labels and evaluated on an  $SL(3)$  matrix of the form  $\mathbf{g} = \text{diag}e^{(1/3)\sum_i \beta_i} (e^{-\beta_1}, e^{-\beta_2}, e^{-\beta_3})$ . When we derived (G14) we used the fact that we obtain a typical representation of  $\widetilde{SU}(1|3)$  and we used the typical character formulas in [84] to write the character in terms of  $SU(3)$  representations. Notice that in (G14) we see the factor of the form (G13) which comes from the supercharges in  $\widetilde{SU}(1|3)$  [84].<sup>36</sup>

<sup>36</sup>The formulas in [84] say that the character of typical representation of the  $SU(n|m)$  supergroup are given by the product of the characters of the  $U(1) \times SU(n) \times SU(m)$  representations associated to the highest weight state times a factor of the form (G13) which arises from half of the supercharges (the supercharges are split into raising and lowering and we get (G13) from the lowering ones).

Returning to our problem, we want to evaluate the single particle contribution to the index from the additional Kaluza Klein multiplets that we have for a fuzzy sphere. These multiplets transform in the representations  $(0, 0, 0|2(l-2) + 1|2(l-2))$ ,  $l \geq 2$ , where  $l = 2$  correspond to the multiplet with a Young supertableau with four vertical boxes as in Fig. 7(d). The vacuum associated to  $N_5$  fivebranes has multiplets with  $l = 2, \dots, N_5$ , in addition to the multiplet present for the trivial vacuum which is given by Fig. 7(a). Using (G14) we can evaluate the single particle contribution from each of these multiplets as

$$z_l = -e^{-2(l-1)(\sum_i \beta_i)} (1 - e^{-\beta_1 - \beta_2})(1 - e^{-\beta_1 - \beta_3}) \times (1 - e^{-\beta_2 - \beta_3}), \quad (\text{G16})$$

where  $l > 1$ . Then we see that we can represent the full single particle contribution as

$$1 - z_1 - \sum_{l=2}^{N_5} z_l = (1 - e^{-\beta_1 - \beta_2})(1 - e^{-\beta_1 - \beta_3}) \times (1 - e^{-\beta_2 - \beta_3}) \frac{(1 - e^{-2N_5(\beta_1 + \beta_2 + \beta_3)})}{1 - e^{-2(\beta_1 + \beta_2 + \beta_3)}}. \quad (\text{G17})$$

Inserting (G17) into (G2) and using (G4) we obtain (2.94). The final result (2.94) contains all the information about possible surviving BPS representations for the single string case which can be obtained by group theory alone. Notice that this was obtained purely from representation theory and no assumptions were made on the dynamics, other than the planar approximation, which implies that single trace states do not mix with multiple trace states. It could well be possible that by using more detailed properties of the dynamics one might be able to obtain more detailed information about BPS states. In other words, the index gives us a lower bound on the number of BPS states, but the actual number could be bigger.

We see that the structure of the index is such that we get one contribution that is common to all vacua, which is what we had for the trivial vacuum at  $N_5 = 1$ , plus some extra terms that arise only for  $N_5 > 1$ , which are written in (2.94). It is clear that these extra terms are the ones that contain the information about the extra four dimensions of the string. Focusing on these terms and taking the limit (G9) we obtain the value of the world sheet index over the last four coordinates (2.96). As expected, we do not get any terms involving  $\tilde{\gamma}$  since we expect that the  $SU(2)_G$  symmetry only acts on the first four dimensions.

Let us be more explicit about the atypical representations that contribute to the index. We find

$$I_{\text{s.t. } N_5} - I_{\text{s.t. } N_5=1} = \sum_{n=1, n \neq 0 \pmod{N_5}}^{\infty} e^{-2n(\beta_1 + \beta_2 + \beta_3)}. \quad (\text{G18})$$

And each term can be written as

$$- e^{-2n(\beta_1 + \beta_2 + \beta_3)} = \sum_{p=0}^{\infty} I_{(0,p,0|2(n-1)+1|2(n-1))}, \quad (\text{G19})$$

where

$$I_{(0,p,0|2(n-1)+1|2(n-1))} = -e^{-2n(\sum \beta_i)} (1 - e^{-\beta_1 - \beta_2}) (1 - e^{-\beta_1 - \beta_3}) (1 - e^{-\beta_2 - \beta_3}) \times \left[ \frac{e^{-p(\beta_1 + \beta_2)}}{(1 - e^{\beta_1 - \beta_3})(1 - e^{\beta_2 - \beta_3})} + \text{cyclic} \right] \quad (\text{G20})$$

is the contribution of an atypical representation with the Young supertableau in Fig. 7(f). We interpret the sum over  $p$  as indicating that we can add any number of Zs. So the resulting BPS states could be matched by thinking that we have the usual BPS states in the first four dimensions, the same we had for  $N_5 = 1$  plus BPS states along the other four dimensions with  $\hat{E} = 2n$ . In conclusion, we interpret each term of the form (G19) as giving rise to a BPS state in the second four dimensions with  $\hat{E} = 2n$  and  $S_3 = \tilde{S}_3 = n$ . This is the contribution we would get from an  $\widetilde{SU}(2|2)$  supermultiplet with a single column of  $2n$  boxes.

Finally, let us explain why (2.92) is an index that counts BPS states for the theory on the string. We start by defining

$$\chi = \text{Tr}[(-1)^F w^{2S_3} e^{-\hat{\mu}(\hat{E} - S_3 - \tilde{S}_3)} e^{-\gamma \hat{E}}], \quad (\text{G21})$$

which is a character of  $\widetilde{SU}(2|2)$ . Let us denote by  $\mathcal{V} \equiv \hat{E} - S_3 - \tilde{S}_3$  the generator that is conjugate to  $\hat{\mu}$ . Then we see that there are two supercharges  $Q_{+,-}^\dagger, Q_{-,+}^\dagger$  (and their complex conjugates) which have  $\mathcal{V} = 0$  eigenvalue. In addition, in  $\widetilde{SU}(2|2)$  all supercharges have  $\hat{E} = 0$  eigenvalues.<sup>37</sup> On a long representation these two supercharges give rise to a factor of the form  $(1 - w)(1 - 1/w)$ . We see that for  $w = 1$  long representations do not contribute. However, we also see that short representations do not contribute either because they typically have one factor of  $(1 - w^{\pm 1})|_{w=1}$ . The solution to this problem is to take the derivative of (G21) with respect to  $w$  and then set  $w = 1$ . Then long representations will not contribute but short representations will contribute. This proves that (2.92) is an

index. Short representations of  $SU(2|2)$  are, for example, those that have a single column or a single row.

If we take the free theory that is associated to the second four coordinates of the IIA pp wave we find that the single particle excitations transform in a short representation of  $\widetilde{SU}(2|2)$  given by a single column of two boxes [as in Fig. 7(a)]. This representation contains two BPS states, with  $\mathcal{V} = 0$  contributing to (2.92). These are a boson of spins  $(S_3, \tilde{S}_3) = (1, 0)$  and a fermion with spins  $(\frac{1}{2}, \frac{1}{2})$ . Only when these particles have zero momentum in the spatial dimension can they contribute to the index. We can thus evaluate the index in the Fock space by simply writing

$$\begin{aligned} \mathcal{I}_{\text{Fock}} &= \partial_w \left[ \frac{1 - w e^{-2\gamma}}{1 - w^2 e^{-2\gamma}} \right]_{w=1} = \sum_{n=1}^{\infty} e^{-2n\gamma} \\ &= \frac{e^{-2\gamma}}{(1 - e^{-2\gamma})}, \end{aligned} \quad (\text{G22})$$

where we used that only a single bosonic and fermionic oscillator contribute. We see that this expression contains the contributions expected from  $\widetilde{SU}(2|2)$  states with Young supertableaux with a column of  $2n$  boxes. Such multiplets contain two BPS states with  $(S_3, \tilde{S}_3) = (n, 0), (n - 1/2, 1/2)$  and energy  $\hat{E} = 2n$ .

## APPENDIX H: FORMULAS FROM [14]

For completeness we review the formulas for the general form of the solutions in [14].

The IIB ansatz is

$$\begin{aligned} ds_{10}^2 &= -\frac{2y}{\sqrt{1-4z^2}}(dt+V)^2 + y\sqrt{\frac{1+2z}{1-2z}}d\Omega_3^2 + y\sqrt{\frac{1-2z}{1+2z}}d\tilde{\Omega}_3^2 + \frac{\sqrt{1-4z^2}}{2y}(dy^2 + dx^i dx^i) \\ F_5 &= -\frac{1}{4}\left\{d\left[y^2\frac{1+2z}{1-2z}(dt+V)\right] + y^3 *_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right)\right\} \wedge d^3\Omega - \frac{1}{4}\left\{d\left[y^2\frac{1-2z}{1+2z}(dt+V)\right] + y^3 *_3 d\left(\frac{z-\frac{1}{2}}{y^2}\right)\right\} \wedge d^3\tilde{\Omega}, \end{aligned} \quad (\text{H1})$$

where  $dV = \frac{1}{y} *_3 dz$ ,  $i = 1, 2$  and  $*_3$  is the flat space epsilon symbol in the three dimensions parametrized by  $y, x_1, x_2$ . The function  $z$  obeys the equation

<sup>37</sup>In fact all generators have  $\hat{E} = 0$  eigenvalues, so one can truncate this algebra to  $PSU(2|2)$ . We are not interested in doing this here.

$$\partial_i \partial_i z + y \partial_y \left( \frac{\partial_y z}{y} \right) = 0. \quad (\text{H2})$$

The  $M$  theory ansatz is

$$ds_{11}^2 = -4e^{2\lambda}(1 + y^2 e^{-6\lambda})(dt + V)^2 + 4e^{2\lambda} d\Omega_3^2 + y^2 e^{-4\lambda} d\tilde{\Omega}_2^2 + \frac{e^{-4\lambda}}{1 + y^2 e^{-6\lambda}} (dy^2 + e^D dx^i dx^i) \quad (\text{H3})$$

$$G_4 = \left\{ -4d[y^3 e^{-6\lambda}(dt + V)] + 2\tilde{*}_3 \left[ y^2 \partial_y \left( \frac{1}{y} \partial_y e^D \right) dy + y \partial_i \partial_y D dx^i \right] \right\} \wedge d^2 \tilde{\Omega},$$

where  $V_i = \frac{1}{2} \epsilon_{ij} \partial_j D$ ,  $e^{-6\lambda} = \frac{\partial_y D}{y(1-y\partial_y D)}$  and  $\tilde{*}_3$  is the 3D flat space  $\epsilon$  symbol. The function  $D$  obeys

$$\partial_i \partial_i D + \partial_y^2 e^D = 0. \quad (\text{H4})$$

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