

Charm- and bottom-quark masses from perturbative QCD

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(Received 4 May 2006; published 12 October 2006)

Using a new result for the first moment of the hadronic production cross section at order $\mathcal{O}(\alpha_s^3)$, and new data on the J/ψ and ψ' resonances for the charm quark, we determine the $\overline{\text{MS}}$ masses of the charm and bottom quarks to be $\bar{m}_c(\bar{m}_c) = 1.295 \pm 0.015$ GeV and $\bar{m}_b(\bar{m}_b) = 4.205 \pm 0.058$ GeV. We assume that the continuum contribution to the sum rules is adequately described by pQCD. While we observe a large reduction of the perturbative error, the shifts induced by the theoretical input are very small. The main change in the central value of m_c is related to the experimental data. On the other hand, the value of m_b is not changed by our calculation to the assumed precision.

DOI: [10.1103/PhysRevD.74.074006](https://doi.org/10.1103/PhysRevD.74.074006)

PACS numbers: 12.38.Bx, 14.65.Dw, 14.65.Fy

The strong coupling constant, α_s , and quark masses are the only input parameters of the QCD Lagrangian. Since at high energy masses become essentially negligible, α_s can be determined separately to a good precision. On the other hand, the determination of the masses is a much more cumbersome problem, where the details of the interaction can hardly be overcome. It is, therefore, not surprising that it has attracted a lot of attention. Particularly interesting are the charm and bottom quarks, where results can be derived from lattice, see e.g. [1], or from different types of sum rules, see e.g. [2–4]. In this paper, we shall follow the approach of [5], which requires, as input on the theoretical side, moments of the hadronic cross section. The first moment at $\mathcal{O}(\alpha_s^3)$ constitutes our main new result [6], which we use, together with new resonance data for J/ψ and ψ' [8], to derive the most up to date values of $\bar{m}_c(\bar{m}_c)$ and $\bar{m}_b(\bar{m}_b)$ in the $\overline{\text{MS}}$ scheme.

The use of QCD sum rules for the determination of the charm quark mass has originally been proposed in a series of papers gathered in [9]. The same approach has subsequently been applied to the bottom quark [10]. A specific feature of these early studies has been the use of higher moments of the hadronic cross section. The advantages of the lower moments have been studied in [11], and used at order $\mathcal{O}(\alpha_s^2)$ in [5] and subsequently in [12,13]. The latter analysis was based on the calculation of three-loop moments from [14] and the data of the cross section scan around the charm threshold given in [15]. An important conclusion of [5] is that the first moment is best suited for the analysis, since it has the weakest dependence on the nonperturbative effects and details of the threshold region, leading naturally to the smallest error.

Starting from $\mathcal{O}(\alpha_s^3)$, the perturbative cross section receives contributions from diagrams with a massless threshold, Fig. 1. Fortunately, gauge invariance restrictions on the photon-three-gluon vertex allow such diagrams only at $\mathcal{O}((q^2/m^2)^4 \log(q^2/m^2))$ [16,17]. Since we will use the first moment exclusively, we can safely neglect them in

our analysis and restrict the calculation to diagrams where a single heavy quark line connects both electromagnetic current vertices.

Let us now remind the reader of a few definitions. The hadronic ratio is given by

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{had.})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}, \quad (1)$$

where $\sigma(e^+e^- \rightarrow \mu^+\mu^-) = 4\pi\alpha^2/3s$. It can be related to the current-current correlator through

$$R(s) = 12\pi \text{Im}\Pi(q^2 = s + i\epsilon), \quad (2)$$

where

$$(-q^2 g_{\mu\nu} + q_\mu q_\nu)\Pi(q^2) = i \int dx e^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle. \quad (3)$$

For a given heavy quark of mass m , the contribution to the correlator can be expanded as

$$\Pi_h(q^2) = Q_h^2 \frac{3}{16\pi^2} \sum_{n>0} \bar{C}_n z^n, \quad z = \frac{q^2}{4m^2(\mu)}. \quad (4)$$

We perform all our calculations in the $\overline{\text{MS}}$ scheme, and expand the \bar{C}_n coefficients in the strong coupling as

$$\bar{C}_n = \sum_{k \geq 0} \left(\frac{\alpha_s(\mu)}{\pi} \right)^k \bar{C}_n^{(k)} \left(\log \left(\frac{m^2(\mu)}{\mu^2} \right) \right). \quad (5)$$

The coefficients of the expansion in the strong coupling constant are known exactly up to two-loops and all moments can be derived from a generating formula [18,19]. The first eight moments of the three-loop result have been given in [14]. Our new result is the four-loop correction to

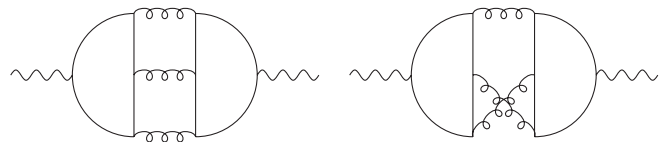


FIG. 1. Singlet diagrams with a three-gluon cut.

the first moment, i.e. the coefficient $\bar{C}_1^{(3)}$. We have obtained it by a direct Taylor expansion to second order in q^2 of the photon self-energy, followed by a reduction of the resulting four-loop single scale tadpoles using an approach inspired by the Laporta algorithm [20], see also [21]. We have kept

a single power of the linear gauge parameter in the 0th moment (value of $\Pi(0)$) to check for gauge invariance, but neglected it in the 1st moment to spare unnecessary computational complexity.

The purely bosonic contribution to $\bar{C}_1^{(3)}$ is given by

$$\begin{aligned}
 & + C_F^3 \left(\frac{45\,577\,758\,023\,909}{56\,435\,097\,600} - \frac{454}{2835} N_5 + \frac{173}{1080} N_4 + \frac{1\,078\,129}{3\,628\,800} N_3 + \frac{79}{1920} N_2 - \frac{8881}{604\,800} N_1 - \frac{3\,546\,972\,523}{4\,702\,924\,800} \pi^2 \right. \\
 & + \frac{1\,533\,371\,173}{653\,184\,000} \pi^4 + \frac{24\,802\,703}{342\,921\,600} \pi^6 + \frac{8192}{45} a_5 - \frac{252\,157}{4050} a_4 + \frac{79}{3360} \zeta_7 - \frac{5\,193\,373\,091}{27\,216\,000} \zeta_5 + \frac{79}{7200} \zeta_5 \pi^2 \\
 & - \frac{75\,721\,242\,853}{195\,955\,200} \zeta_3 - \frac{95\,128\,427}{48\,988\,800} \zeta_3 \pi^2 + \frac{79}{21\,600} \zeta_3 \pi^4 - \frac{10\,306\,609}{4\,082\,400} \zeta_3^2 + \frac{4352}{2025} l_2 \pi^4 + \frac{252\,157}{97\,200} l_2^2 \pi^2 + \frac{1024}{405} l_2^3 \pi^2 \\
 & - \frac{252\,157}{97\,200} l_2^4 - \frac{1024}{675} l_2^5 - \frac{6\,142\,460\,327}{22\,394\,880} l_m - \frac{173}{270} l_m N_3 - \frac{79}{480} l_m N_1 + \frac{12\,907\,273}{1\,866\,240} l_m \pi^2 + \frac{215\,987}{777\,600} l_m \pi^4 \\
 & + \frac{79}{45\,360} l_m \pi^6 - \frac{16\,384}{135} l_m a_4 + \frac{1\,962\,691}{32\,400} l_m \zeta_5 + \frac{6\,307\,081}{155\,520} l_m \zeta_3 + \frac{36\,163}{58\,320} l_m \zeta_3 \pi^2 + \frac{79}{540} l_m \zeta_3^2 + \frac{2048}{405} l_m l_2^2 \pi^2 \\
 & \left. - \frac{2048}{405} l_m l_2^4 - \frac{103}{90} l_m^2 + \frac{3}{5} l_m^3 \right) \\
 & + C_F^2 C_A \left(-\frac{4\,622\,898\,495\,103}{5\,374\,771\,200} + \frac{227}{945} N_5 - \frac{103}{288} N_4 - \frac{916\,241}{14\,515\,200} N_3 - \frac{87}{1280} N_2 - \frac{71\,941}{403\,200} N_1 + \frac{487\,467\,767}{3\,135\,283\,200} \pi^2 \right. \\
 & - \frac{3\,579\,168\,251}{1\,306\,368\,000} \pi^4 - \frac{36\,666\,499}{228\,614\,400} \pi^6 - \frac{224}{15} a_5 - \frac{313\,387}{16\,200} a_4 - \frac{87}{2240} \zeta_7 + \frac{10\,395\,883\,727}{54\,432\,000} \zeta_5 - \frac{29}{1600} \zeta_5 \pi^2 \\
 & + \frac{40\,758\,646\,507}{130\,636\,800} \zeta_3 + \frac{463\,655\,219}{97\,977\,600} \zeta_3 \pi^2 - \frac{29}{4800} \zeta_3 \pi^4 + \frac{15\,591\,581}{2\,721\,600} \zeta_3^2 - \frac{119}{675} l_2 \pi^4 + \frac{313\,387}{388\,800} l_2^2 \pi^2 - \frac{28}{135} l_2^3 \pi^2 \\
 & - \frac{313\,387}{388\,800} l_2^4 + \frac{28}{225} l_2^5 + \frac{4\,315\,007\,269}{4\,976\,640} l_m + \frac{103}{72} l_m N_3 + \frac{87}{320} l_m N_1 - \frac{5\,492\,587}{414\,720} l_m \pi^2 + \frac{1\,313\,957}{518\,400} l_m \pi^4 \\
 & - \frac{29}{10\,080} l_m \pi^6 + \frac{448}{45} l_m a_4 - \frac{2\,928\,779}{21\,600} l_m \zeta_5 - \frac{32\,483\,549}{103\,680} l_m \zeta_3 - \frac{61\,259}{38\,880} l_m \zeta_3 \pi^2 - \frac{29}{120} l_m \zeta_3^2 - \frac{56}{135} l_m l_2^2 \pi^2 \\
 & \left. + \frac{56}{135} l_m l_2^4 + \frac{259}{108} l_m^2 - \frac{11}{10} l_m^3 \right) \\
 & + C_F C_A^2 \left(\frac{25\,490\,069\,288\,387}{112\,870\,195\,200} - \frac{227}{2835} N_5 + \frac{1199}{8640} N_4 - \frac{413\,339}{9\,676\,800} N_3 + \frac{91}{3840} N_2 + \frac{3511}{37\,800} N_1 + \frac{1\,042\,284\,611}{9\,405\,849\,600} \pi^2 \right. \\
 & + \frac{361\,381\,739}{1\,306\,368\,000} \pi^4 + \frac{42\,598\,397}{685\,843\,200} \pi^6 - \frac{1712}{45} a_5 + \frac{272\,567}{10\,800} a_4 + \frac{13}{960} \zeta_7 - \frac{613\,974\,553}{9\,072\,000} \zeta_5 + \frac{91}{14\,400} \zeta_5 \pi^2 \\
 & + \frac{277\,861\,513}{195\,955\,200} \zeta_3 - \frac{15\,355\,283}{8\,164\,800} \zeta_3 \pi^2 + \frac{91}{43\,200} \zeta_3 \pi^4 - \frac{18\,234\,067}{8\,164\,800} \zeta_3^2 - \frac{1819}{4050} l_2 \pi^4 - \frac{272\,567}{259\,200} l_2^2 \pi^2 - \frac{214}{405} l_2^3 \pi^2 \\
 & + \frac{272\,567}{259\,200} l_2^4 + \frac{214}{675} l_2^5 - \frac{15\,657\,616\,211}{44\,789\,760} l_m - \frac{1199}{2160} l_m N_3 - \frac{91}{960} l_m N_1 + \frac{3\,652\,601}{746\,496} l_m \pi^2 - \frac{2\,078\,929}{1\,555\,200} l_m \pi^4 \\
 & + \frac{13}{12\,960} l_m \pi^6 + \frac{3424}{135} l_m a_4 + \frac{3\,411\,823}{64\,800} l_m \zeta_5 + \frac{42\,077\,383}{311\,040} l_m \zeta_3 + \frac{73\,807}{116\,640} l_m \zeta_3 \pi^2 + \frac{91}{1080} l_m \zeta_3^2 - \frac{428}{405} l_m l_2^2 \pi^2 \\
 & \left. + \frac{428}{405} l_m l_2^4 - \frac{13\,597}{14\,580} l_m^2 + \frac{121}{270} l_m^3 \right),
 \end{aligned}$$

where C_F and C_A are Casimir operators of the fundamental and adjoint representations, respectively. In the case of the $SU(N)$ group $C_F = (N^2 - 1)/2N$ and $C_A = N$. Furthermore, $l_m = \log(m^2(\mu)/\mu^2)$, $l_2 = \log 2$, ζ_i are Riemann zeta numbers, and $a_i = \text{Li}_i(1/2)$ with $\text{Li}_i(x)$ the polylogarithm function. The numerical constants N_i are defined by the ϵ expansion of the vacuum integrals given in

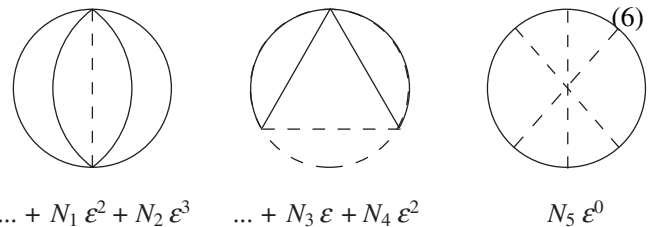


FIG. 2. Definition of the numerical constants N_i from the ϵ expansion of vacuum integrals.

Fig. 2, with the understanding that the integration measure per loop is $e^{\epsilon\gamma_E}/i\pi^{2-\epsilon} \int d^d k$

$$\begin{aligned}
 N_1 &= +2369.669\,517\,745\,751, \\
 N_2 &= +9090.208\,679\,977\,050, \\
 N_3 &= -764.094\,837\,335\,855\,8, \\
 N_4 &= -4647.352\,454\,831\,194, \\
 N_5 &= -1.808\,879\,546\,208\,335.
 \end{aligned} \tag{7}$$

The above numbers have been obtained by changing the normalization of the high precision results of [22]. In fact, we took the values of all the master integrals needed from there [23]. Relations between the numbers in Eq. (7) discovered in [25] have not been used in our analysis. We checked, however, the values of all integrals up to six lines with the help of Mellin-Barnes integral representations and the package [26].

The single fermion loop contribution to $\bar{C}_1^{(3)}$ with n_l massless quark species is given by

$$\begin{aligned}
 &+ C_F^2 T_F \left(-\frac{155\,640\,216\,647\,897}{526\,727\,577\,600} + \frac{30\,661}{322\,560} N_1 - \frac{391\,510\,309}{179\,159\,040} \pi^2 - \frac{288\,518\,747}{174\,182\,400} \pi^4 - \frac{30\,661}{30\,481\,920} \pi^6 + \frac{127\,252}{945} a_4 \right. \\
 &+ \frac{30\,661}{89\,600} \zeta_5 + \frac{318\,864\,713\,617}{1\,828\,915\,200} \zeta_3 + \frac{30\,661}{161\,280} \zeta_3 \pi^2 - \frac{30\,661}{362\,880} \zeta_3^2 - \frac{31\,813}{5670} l_2^2 \pi^2 + \frac{31\,813}{5670} l_2^4 - \frac{228\,583}{19\,440} l_m \\
 &\left. + \frac{139\,087}{12\,960} l_m \zeta_3 - \frac{22}{135} l_m^2 + \frac{2}{5} l_m^3 \right) \\
 &+ C_F^2 T_F n_l \left(\frac{1\,080\,947}{19\,440} + \frac{4793}{103\,680} N_3 - \frac{52\,723}{233\,280} \pi^2 + \frac{117\,811}{194\,400} \pi^4 - \frac{4793}{270} a_4 - \frac{857\,947}{194\,400} \zeta_5 - \frac{25\,009\,261}{466\,560} \zeta_3 \right. \\
 &\left. - \frac{4793}{69\,984} \zeta_3 \pi^2 + \frac{4793}{6480} l_2^2 \pi^2 - \frac{4793}{6480} l_2^4 - \frac{19\,801}{2160} l_m + \frac{120\,817}{12\,960} l_m \zeta_3 - \frac{22}{135} l_m^2 + \frac{2}{5} l_m^3 \right) \\
 &+ C_F C_A T_F \left(-\frac{11\,189\,587\,381\,799}{429\,185\,433\,600} + \frac{65}{2592} N_3 + \frac{11\,339}{645\,120} N_1 - \frac{188\,717\,291}{358\,318\,080} \pi^2 - \frac{6\,364\,753}{69\,672\,960} \pi^4 - \frac{11\,339}{60\,963\,840} \pi^6 \right. \\
 &+ \frac{8194}{945} a_4 + \frac{1\,724\,977}{43\,545\,600} \zeta_5 + \frac{360\,428\,176\,373}{40\,236\,134\,400} \zeta_3 - \frac{156\,623}{78\,382\,080} \zeta_3 \pi^2 - \frac{11\,339}{725\,760} \zeta_3^2 - \frac{4097}{11\,340} l_2^2 \pi^2 + \frac{4097}{11\,340} l_2^4 \\
 &\left. - \frac{775\,451}{349\,920} l_m - \frac{7013}{25\,920} l_m \zeta_3 + \frac{1051}{3645} l_m^2 - \frac{44}{135} l_m^3 \right) \\
 &+ C_F C_A T_F n_l \left(-\frac{31\,394\,101}{1\,049\,760} - \frac{4793}{207\,360} N_3 + \frac{52\,723}{466\,560} \pi^2 - \frac{26\,483}{388\,800} \pi^4 + \frac{4793}{540} a_4 + \frac{857\,947}{388\,800} \zeta_5 + \frac{7\,792\,237}{933\,120} \zeta_3 \right. \\
 &\left. + \frac{4793}{139\,968} \zeta_3 \pi^2 - \frac{4793}{12\,960} l_2^2 \pi^2 + \frac{4793}{12\,960} l_2^4 - \frac{627\,893}{116\,640} l_m + \frac{4183}{2880} l_m \zeta_3 + \frac{1051}{3645} l_m^2 - \frac{44}{135} l_m^3 \right), \tag{8}
 \end{aligned}$$

where T_F is the trace of the fundamental representation, which we take to have the standard value $T_F = 1/2$. Our result does not depend on the number of heavy quark species of mass m . It should be understood, however, that all terms without n_l come from the heavy quark loops.

The double fermion loop contribution to $\bar{C}_1^{(3)}$ has been published before in [27]. We found complete agreement with that result, and reproduce it here only for completeness

$$\begin{aligned}
 &C_F T_F^2 \left(\frac{163\,868}{98\,415} - \frac{3287}{2430} \zeta_3 - \frac{14\,483}{21\,870} l_m + \frac{203}{324} l_m \zeta_3 + \frac{236}{3645} l_m^2 + \frac{8}{135} l_m^3 \right) + C_F T_F^2 n_l \left(-\frac{782\,857}{65\,610} - \frac{29}{2592} N_3 + \frac{319}{5832} \pi^2 \right. \\
 &\left. - \frac{319}{4860} \pi^4 + \frac{116}{27} a_4 + \frac{5191}{4860} \zeta_5 + \frac{340\,313}{58\,320} \zeta_3 + \frac{145}{8748} \zeta_3 \pi^2 - \frac{29}{162} l_2^2 \pi^2 + \frac{29}{162} l_2^4 - \frac{3779}{21\,870} l_m + \frac{203}{324} l_m \zeta_3 + \frac{472}{3645} l_m^2 + \frac{16}{135} l_m^3 \right) \\
 &+ C_F T_F^2 n_l^2 \left(\frac{42\,173}{32\,805} - \frac{112}{135} \zeta_3 + \frac{1784}{3645} l_m + \frac{236}{3645} l_m^2 + \frac{8}{135} l_m^3 \right). \tag{9}
 \end{aligned}$$

Substituting the color factors and the numerical values of the constants, our complete result can be cast in the following form

$$\begin{aligned}
& 1.87882 - 2.79472n_l + 0.0961014n_l^2 + 18.4050l_m \\
& - 5.26881n_l l_m + 0.163146n_l^2 l_m - 0.660174l_m^2 \\
& + 0.474989n_l l_m^2 + 0.0215821n_l^2 l_m^2 + 0.656790l_m^3 \\
& - 0.256790n_l l_m^3 + 0.0197531n_l^2 l_m^3. \quad (10)
\end{aligned}$$

In order to determine the mass of a given quark from our perturbative result we shall use the moments of the hadronic cross section defined by

$$M_n = \int \frac{ds}{s^{n+1}} R(s). \quad (11)$$

With the help of a dispersion relation, this can be translated into

$$M_n = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \Pi_h(q^2)|_{q^2=0}. \quad (12)$$

If one compares the above with Eq. (4), then the mass is finally given by

$$m(\mu) = \frac{1}{2} \left(\frac{9}{4} Q_h^2 \frac{\bar{C}_n}{M_n^{\text{exp}}} \right)^{1/(2n)}. \quad (13)$$

As explained at the beginning, we are only interested in the first moment. The experimental value, Eq. (11), is made out of three constituents, the resonances, the threshold region and the continuum. We note that tiny changes in the central value of $\alpha_s(M_Z)$ [28] entering through the perturbative result used for the continuum have no impact on the final moments. We observe the same with respect to parts of the five-loop result [29]. Therefore, apart from the resonance data for J/ψ and ψ' [8], we adopt the moments from [5]. Note that an alternative set of experimental moments that would lead to a larger error estimate can be found in [12,13]. After correction for the resonances, we have

$$(M_1^{\text{exp}})_{\text{charm}} = 0.2087 \pm 0.0042, \quad (14)$$

$$(M_1^{\text{exp}})_{\text{bottom}} = 0.004456 \pm 0.000121. \quad (15)$$

Similarly to [5], we first determine the mass of the charm

quark at 3 GeV, and then run it down to $\bar{m}_c(\bar{m}_c)$. For consistency reasons the running is performed with three-loop accuracy, i.e. to $\mathcal{O}(\alpha_s^3)$. We use the package [30] and obtain

$$\bar{m}_c(\bar{m}_c) = 1.295 \pm 0.009_{\alpha_s} \pm 0.0003_{\mu} \pm 0.012_{\text{exp}} \text{ GeV}. \quad (16)$$

The subscripts denote the various sources of the error (variation of α_s within error bars, variation of μ in the range 3 ± 1 GeV and variation of the first moment). The combined error given in the abstract is obtained by adding the errors in squares. We note that the four-loop result gives a shift of -2 MeV whereas the new resonance data -10 MeV. Finally, we observe an almost threefold reduction of the perturbative error.

Performing a similar analysis for the b -quark, where the calculation is first done at 10 GeV and then the mass is run down to $\bar{m}_b(\bar{m}_b)$, we obtain

$$\bar{m}_b(\bar{m}_b) = 4.205 \pm 0.010_{\alpha_s} \pm 0.002_{\mu} \pm 0.057_{\text{exp}} \text{ GeV}. \quad (17)$$

The only change in the error determination with respect to the case of the c -quark is connected to the variation of μ in the range 10 ± 5 GeV. We note, that our result is exactly the same as that of the three-loop analysis in [5], apart from a reduction of the perturbative error by almost an order of magnitude. We would observe a shift of the central value, if the calculation were performed starting from low values of μ close to 5 GeV. This is contained in our error estimate from the variation of μ . As a final remark, let us mention that there is some discussion on the resonance contribution of $Y(4S)$ and $Y(5S)$ [12]. Adopting the values suggested by the latter paper, we observe a small shift of a few MeV in the central value, well within our error estimate.

The authors would like to thank J. Kühn for an interesting discussion on quark masses and sum rules. Parts of the presented calculations were performed on the DESY Zeuthen Grid Engine computer cluster. This work was supported by the Alexander von Humboldt Foundation.

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