PHYSICAL REVIEW D **74,** 066001 (2006)

Bubbling supertubes and foaming black holes

Iosif Bena[1] and Nicholas P. Warner[2]

[1]*Department of Physics and Astronomy, University of California, Los Angeles, California 90095, USA*
[2]*Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089-0484, USA*

(Received 13 April 2006; published 8 September 2006)

We construct smooth BPS three-charge geometries that resolve the zero-entropy singularity of the $U(1) \times U(1)$ invariant black ring. This singularity is resolved by a geometric transition that results in geometries without any branes sources or singularities but with nontrivial topology. These geometries are both ground states of the black ring, and nontrivial microstates of the D1-D5-P system. We also find the form of the geometries that result from the geometric transition of N zero-entropy black rings, and argue that, in general, such geometries give a very large number of smooth bound-state three-charge solutions, parametrized by $6N$ functions. The generic microstate solution is specified by a four-dimensional hyper-Kähler geometry of a certain signature, and contains a ‘‘foam’’ of nontrivial two-spheres. We conjecture that these geometries will account for a significant part of the entropy of the D1-D5-P black hole, and that Mathur's conjecture might reduce to counting certain hyper-Kähler manifolds.

DOI: 10.1103/PhysRevD.74.066001 PACS numbers: 11.25.−w, 04.65.+e

I. INTRODUCTION

Mathur and collaborators have proposed a bold solution to the black hole information paradox [1–3]. By fully analyzing the implications of the AdS/CFT correspondence to the physics of the D1-D5 system, they have argued that each vacuum of the CFT is dual to a smooth bulk solution that has neither a horizon nor a loss of information. These geometries thus account for the rather large entropy of the D1-D5 system. The success of this endeavor for the D1-D5 system has led to the speculation that one might similarly find solutions that account for the entropy of the D1-D5-P system. If this were possible, then the AdS/CFT correspondence would compel one to accept that the D1-D5-P black hole should be thought of as an ‘‘ensemble’’ of geometries; this would open a new and fascinating window into the understanding of black holes in string theory.

Most of the progress in understanding whether the D1-D5-P microstates are dual to bulk solutions has occurred on two apparently distinct fronts, which this paper will unify. The first has involved finding individual smooth solutions carrying D1-D5-P charges, and analyzing them in the CFT [4–8]. This has shown that indeed some CFT microstates are dual to three-charge bulk geometries, and has highlighted interesting features of the bound-state geometries.

The second has involved understanding the D-brane physics behind the existence of these solutions, and analyzing these configurations from a string theory perspective. In particular, in [9] it was argued that there exists a very large class of brane configurations, with three charges and three dipole charges, that can have arbitrary shape, and generalize the two-charge supertubes of [10]. Since the entropy of the D1-D5 system comes from the arbitrary shapes of the two-charge supertubes, it is natural to expect that the arbitrary shapes of three-charge supertubes account for a sizable part of the entropy of the D1-D5-P black hole.

Finding the supergravity solutions of these three-charge supertubes of arbitrary shape is quite involved but in [11] it was shown that one can solve the equations underlying these solutions [11–13] in a linear fashion, and reduce the whole problem of finding three-charge BPS solutions to electromagnetism in four dimensions. A side effect of the study of three-charge supertubes was the prediction [9,14] and subsequent discovery [11,15–17] of BPS black rings, which by themselves have opened up new windows into black-hole physics [18–29].

For the purpose of finding three-charge geometries dual to microstates of the D1-D5-P CFT, one is not so much interested in black rings with a regular event horizon, but rather in the zero-entropy limit of these rings, which for simplicity we refer to as three-charge supertubes. The general three-charge supertube solution is given by six arbitrary functions: four determine the shape of the object, three describe the charge density profiles, but there is one functional constraint coming from setting the event horizon area to zero [11,20]. The near-tube geometry is of the form $\mathrm{AdS}_3 \times S^2$ and, since the size of the S^2 is finite, the curvature is low everywhere. However, since the AdS_3 is periodic around the ring, these solutions have a null orbifold singularity. In order to obtain smooth, physical geometries corresponding to supertubes given by six arbitrary functions, one must learn how to resolve this singularity.

To do this, we use the fact that singularities coming from wrapped branes are resolved in string theory via geometric transitions [30–33], which result in a topology change. The cycle wrapped by the branes shrinks to zero size while the dual cycle becomes large. The branes thus disappear from the space and the naive solution ‘‘transitions’’ to one of a different topology in which the branes have been replaced by a flux through a nontrivial dual cycle. After the transition the number of branes is encoded in the integral of the field strength over this new topologically nontrivial cycle. In the limit when the number of branes

 © 2006 The American Physical Society

become small, the region where the topology change occurs becomes small and the solution approaches the naive supertube geometry with its null orbifold singularity.

Unfortunately there is, as yet, no systematic way to find the geometries that result from a geometric transition of a supertube of arbitrary shape. What we can do however is to use the fact that we know the topology of the solution after the transition to determine completely the solutions with $U(1) \times U(1)$ invariance. Having done this, we can easily extend our analysis to solutions that only have (triholomorphic) $U(1)$ symmetry, and this leads to obvious conjectures as to the appropriate backgrounds when there is no symmetry.

Consider the $\mathbb{R}^4$ base that contains the supertube/black ring. This base can be written as a trivial Gibbons-Hawking space with one center of unit Gibbons-Hawking (GH) charge. The singularity of the supertube is resolved by the nucleation of a pair of GH centers, with equal and opposite charges, $-Q$ and Q, near the location of the supertube. Despite the fact that the signature of the new base can change from $(+,+,+,+)$ to $(-,-,-,-)$, the overall geometry is regular. One should also note that if $|Q| \neq 1$ then there will be ordinary, $\mathbb{Z}_{|Q|}$, spatial orbifold singularities at the corresponding GH centers. Such spatial orbifolds are well understood in string theory and in the underlying conformal field theory, and are therefore harmless. So when we say the solution is regular, we will mean up to such spatial orbifolds at the GH centers. The new solution has a nontrivial topology, with two two-cycles, but no branes. A schematic of this transition is depicted in Fig. 1. The three dipole charges of the naive solution are now given by the integrals on the newly nucleated S^2 of the three twoform field strengths on the base.[1] The size of this two-sphere is determined by the balance between the fluxes wrapping it, and the attraction of the $-Q$ and Q GH centers. When the fluxes are very small, or Q very large, these GH centers become very close, and the solution approaches the naive supertube solution.

The physics of the singularity resolution we propose here is very similar to, and was inspired by, the one observed in the recent paper of Lin, Lunin and Maldacena [33], where the bubbling solutions reduce to naive giant gravitons in the small dipole charge limit, and have a topological transition in which branes are replaced by fluxes. As in [33], when the dipole charges become large the bubbling solution has no obvious brane interpretation.

There are two very nontrivial confirmations that our solutions are the correct resolving geometries. First, one can put the bubbling supertube in Taub-NUT, and move it into the four-dimensional region. The resulting four-dimensional solution is in the same class as the solutions

[1]These twoform field strengths come from reducing the M-theory fourform field strength on the three T^2's

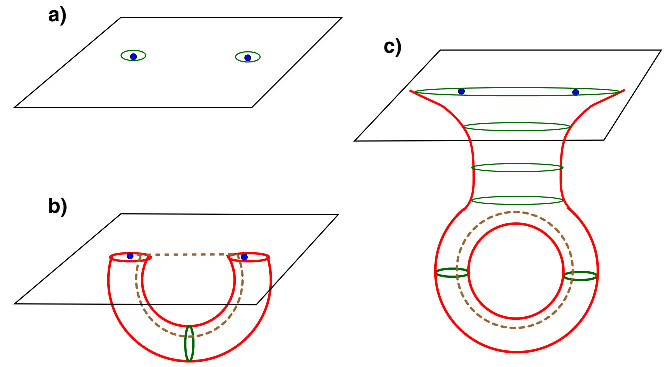

FIG. 1 (color online). The geometric transition: This figure shows a section through the transition geometry in which an S^2 is depicted as an S^1, and the S^1 of the supertube as a pair of two blue points. The naive geometry is shown in (a), and the resolved geometry is shown in (b). After the transition the green S^2 Gaussian surface around the supertube becomes noncontractible and a new two-cycle (depicted by the brown dotted S^1) appears. In (c) we display the transition for large dipole charges, when the link with the naive D-brane solution (a) becomes much less obvious.

[22] that resolve the singularity of the zero-entropy four-dimensional black hole. One can also take the limit of our solutions in which the $+Q$ GH center is moved onto the center with a GH charge of $+1$ at the origin. In this limit, our solutions reduce to the bound-state solutions constructed by Mathur and collaborators [5–7] by taking novel extremal limits of the nonextremal rotating three-charge black hole [34].

Our solutions suggest quite a few nontrivial features of the three-charge geometries that are dual to microstates of the three-charge black hole. We can argue that several concentric rings are resolved by the nucleation of several pairs of GH centers, one pair for each ring, and that such a solution is a bound state. This indicates that the most general bound-state solution with a GH base should have a collection of GH centers of positive and negative charges at arbitrary positions inside the $\mathbb{R}^3$ base of the GH space, with the sum of the charges equal to one. The solutions have nonvanishing fluxes through the nontrivial two-cycles of the base, and have no localized brane charge.

One also expects the geometric transition we present here to resolve the singularities of the three-charge supertubes of arbitrary shape. The resolved solutions should have the same topology as the $U(1) \times U(1)$ solution. However, their bases will no longer be multiple-center GH spaces but more general four-dimensional hyper-Kähler manifolds. Even if such base manifolds have changing signature, we expect the overall solutions to be regular, just as for the $U(1) \times U(1)$ invariant solutions. A configuration containing N supertubes of arbitrary shapes is determined by $6N$ functions, and after the transition should give $6N$ functions worth of smooth geometries. This is a huge number of smooth solutions, which might as well be large enough to account for a significant part of the entropy of the three-charge black hole. It is also possible that a significant part of this entropy comes from degrees of freedom along the three T^2's of the solution, which probably cannot be described by supergravity.

Our results indicate that proving or disproving the strong form of Mathur's conjecture—that black hole microstates are dual to *smooth supergravity solutions*[2]—reduces to a well-defined mathematical problem: classifying and counting asymptotically flat four-dimensional hyper-Kähler manifolds that have regions of signature $(+,+,+,+)$ and regions of signature $(-,-,-,-)$. If the conjecture is correct, it indicates that the black hole is an ensemble of hyper-Kähler geometries involving foams of a very large number of topologically nontrivial 2-spheres, threaded by fluxes. More generally, our results have potentially interesting consequences for the structure of the supersymmetric vacuum states of string theory. We will discuss this further in the last section of this paper.

The BPS black rings have two microscopic interpretations: one in terms of the D1-D5-P CFT [18] and another in terms of a four-dimensional black hole CFT [18,19,27]. Hence, our solutions have two microscopic interpretations. On one hand, they are dual to microstates of the black ring CFT, and should be thought of as the ground states of the BPS black ring, in the same way that the solutions of [35,36] give the ground state of the five-dimensional three-charge black hole, and the solutions of [22] give the ground state of the four-dimensional four-charge black hole. On the other hand, they are dual to vacua of the D1-D5-P CFT. Our analysis does not establish to which CFT vacua our solutions are dual, and we leave this very interesting question to future work. However, based on the microscopic description of supertubes [18,35,36] we expect the solutions that correspond to multiple supertubes to be dual to CFT states with longer effective strings than the solutions that come from only one supertube. Hence, the solutions with the largest number of bubbles should correspond to the CFT states with the longest effective strings, which are the ones that give the D1-D5-P black hole entropy.

In Sec. II we explain the features of the geometric transition that resolves the singularity of the zero-entropy black rings. These features are very similar to those observed in [33] to give bubbling AdS geometries. In Sec. III we investigate the general form of solutions with a Gibbons-Hawking base with an arbitrary distribution of centers. In Sec. IV we consider solutions in which there are no point-charge sources and only topologically nontrivial fluxes. We also discuss some simple examples. The reader who is already familiar with the construction of metrics with Gibbons-Hawking base and is primarily interested in the solution that resolves the singularity of the zero-entropy black ring should skip to Sec. V. Indeed, in Sec. V we explicitly construct the bubbling solutions that come from the geometric transition of zero-entropy black rings and then compare their features to those of the naive supertube solutions. We also place the bubbling solutions in Taub-NUT and relate them to the ground states of four-dimensional black holes constructed in [22]. Section VI contains some final remarks.

While working on this paper we became aware of another group that is working on similar issues [37]. Our paper and theirs will appear simultaneously on the archive.

II. SINGULARITY RESOLUTION AND GEOMETRIC TRANSITIONS

As is well known, D-branes warp the geometry by shrinking it along the longitudinal directions and expanding it along the transverse directions. Hence, solutions sourced by branes that wrap a closed curve generally have a singularity because the tension in the brane causes this curve to shrink to zero size. Perhaps the best known example of such a singularity is Poincaré AdS space with a periodic direction. For example, if one takes the standard $AdS_5 \times S^5$ solution corresponding to D3-branes, then it is regular but if one periodically identifies one of the spatial directions of the D3-branes then those directions collapse to zero size as one goes down the Ads throat.

An even more interesting example of such a singularity comes from studying M2 branes polarized into M5 branes by a transverse field [38]. The M5 branes are wrapped on a topologically trivial S^3 and are stabilized against collapse by the transverse field. The half-BPS supergravity solution describing this system [39,40] looks like $AdS_4 \times S^7$ far away from the polarization shell, and like $AdS_7 \times S^4$ near the shell. However, because the M5 branes that source the $AdS_7 \times S^4$ solution are wrapped on a three-sphere, the naive near-shell geometry has a singularity.

In [33] it was realized that this singularity is resolved by a geometric transition. The S^3 wrapped by the branes shrinks to zero size at the position of the branes. The S^4 that links this S^3 is a "Gaussian surface" for the M5-brane charge and necessarily becomes large. Moreover, since the S^3 also shrinks to zero size at the origin of the space, this results in the creation of another topologically nontrivial S^4. The integral of the fourform flux on the second S^4 is equal to $\frac{N_2}{N_5}$. Hence, before the transition the solution had a shell of M2 branes polarized into M5 branes, and after the transition the solution has a nontrivial topology, two S^4's threaded by fluxes N_5 and $\frac{N_2}{N_5}$, and no branes. The M2 brane charge measured at infinity comes from the nontrivial fluxes through the two S^4's; these fluxes combine via the supergravity Chern-Simons term to generate the electric charge.

The three-charge supertubes that one obtains from the zero-entropy limit of black rings also have a similar singu-

[2]There also exists a weak form of Mathur conjecture—that black-hole microstates are dual to string theory configurations with unitary scattering that are not necessarily smooth in supergravity.

TABLE I. Layout of the branes that give the supertubes and black rings in an M-theory duality frame. Vertical bars |, indicates the directions along which the branes are extended, and horizontal lines, —, indicate the smearing directions. The functions, $x^\mu(\phi)$, indicate that the brane wraps a simple closed curve that gives the supertube profile. A $\star$ indicates that a brane is smeared along the supertube profile, and pointlike on the other three directions.

	1	2	3	4	5	6	7	8	9	10	11
M2	|	|	|	—	—	—	—	$\star$	$\star$	$\star$	$\star$
M2	|	—	—	|	|	—	—	$\star$	$\star$	$\star$	$\star$
M2	|	—	—	—	—	|	|	$\star$	$\star$	$\star$	$\star$
M5	|	—	—	|	|	|	|	$x^\mu(\phi)$			
M5	|	|	|	—	—	|	|	$x^\mu(\phi)$			
M5	|	|	|	|	|	—	—	$x^\mu(\phi)$			

larity. The brane content of these supertubes is shown in Table I. The tubes are "wrapped" on a topologically trivial S^1 that sits in the spatial $\mathbb{R}^4$ base. The integral of the field strength $F_{23ij}dx^i \wedge dx^j$ on the S^2 that surrounds this S^1 gives the number of M5 branes that wrap the 4567 directions and extend along the S^1 of the tube. Similarly, the integrals of $F_{45ij}dx^i \wedge dx^j$ and $F_{67ij}dx^i \wedge dx^j$ measure the other two dipole charges of the solution. After the geometric transition, the $S^1 \subset \mathbb{R}^4$ of the tube shrinks to zero size, and the S^2 around the supertube becomes fat. Moreover, since this S^1 also shrinks to zero size at the origin of $\mathbb{R}^4$, this will give another topologically nontrivial S^2. The resulting four-geometry, $\mathcal{M}^4$, will therefore have two noncontractible two-spheres, S^2_A and S^2_B, and no brane sources. The product of the integrals of the fluxes over these nontrivial two-spheres, S^2_A and S^2_B, will give the M2-brane charges measured at infinity and induced through the supergravity Chern-Simons term. For example, the M2 brane charge along the 23 directions should be given by

$$Q^{M2}_{23} = \frac{1}{2}\int_{\mathcal{M}^4\times T^2_{45}\times T^2_{67}} F\wedge F = \mathcal{I}^{-1}_{AB}\int_{S^2_A} F_{45ij} \times \int_{S^2_B} F_{67ij}, \tag{2.1}$$

where $\mathcal{I}_{AB}$ is the intersection matrix of the cycles A and B. After the transition there are no more brane sources and so the solution should be completely determined by the base space and by the fluxes. Moreover, in order for the solution to preserve the same supersymmetries as three sets of M2 branes, the base space must be hyper-Kähler [11–13]. For the transition of supertubes of arbitrary shapes, this information is not enough to fully determine the solution, however for the $U(1) \times U(1)$ invariant supertubes one can completely characterize the resulting geometry.

First, the solution after the transition will still have the $U(1) \times U(1)$ symmetry. One can now use a theorem[3] that states [41] that if a four-dimensional hyper-Kähler manifold has a $U(1) \times U(1)$ symmetry then a linear combination of the two $U(1)$'s must be triholomorphic[4] and hence the metric must have Gibbons-Hawking form. After the geometric transition, the solution has two independent two-cycles and so the Gibbons-Hawking space must have three centers. The base space before and after the transition must be asymptotic to $\mathbb{R}^4$ and this means that the sum of the GH charges at the three centers must be equal to one. In order to avoid singularities at the GH centers, the GH charges must be integers, and so one center must have a negative charge. Moreover, in the limit when the dipole charges are small the solution must approach the supertube in a flat base; hence the center at the origin of the coordinate system must have charge 1. The other two centers have therefore charges Q and $-Q$. In this limit we expect these centers to be located very close to each other, near the position of the ring in the naive supertube geometry. Furthermore, the three centers must also be collinear in order to preserve the $U(1) \times U(1)$ symmetry.

Thus, by using just a few facts about geometric transitions (which came by a trivial extension of the physics seen in [33]) and the fact that four-dimensional hyper-Kähler manifolds with $U(1) \times U(1)$ symmetry are Gibbons-Hawking [41], we have reached the conclusion that the singularity of the zero-entropy black ring is resolved by the nucleation of a pair of oppositely charged Gibbons-Hawking centers at the location of the ring.

One can now extend this argument to several concentric rings, and observe that, if one ring is resolved by the nucleation of one pair of centers, several rings should be resolved by the nucleation of several pairs of such points. If the rings are concentric then the GH centers should also be collinear. One also expects that a solution with several GH centers that are not collinear should be a simple deformation of a bubbling supertube, and so it should also be dual to a CFT microstate. In this way one could expect the solutions to contain pairs of equal but opposite GH charges; however, it is also possible to deform such solutions so as to separate or combine the GH centers. Thus, the class of solutions that have a GH base and are physically interesting should have any collection of GH centers of positive and negative (integer) charges at arbitrary positions inside the $\mathbb{R}^3$ base of the Gibbons-Hawking space, with the constraint that the GH charges sum to one. More generally, if there is no symmetry one should expect a general hyper-Kähler metric in which the metric can flip from positive definite to negative definite.

We should also note that we expect these general multicenter solutions to be bound states. One way to see this is to consider an n-tube solution, which after the transition has $(2n + 1)$ GH centers, $(n + 1)$ of which have positive

[3] We thank Harvey Reall for mentioning this paper to us.

[4] Triholomorphic means that the $U(1)$ preserves all three complex structures of the hyper-Kähler base.

charge. Having resolved the geometry, and perhaps separated GH charges still further, there is no canonical way to pair up GH points and decide which pair forms a particular tube. Moreover, in the limit when all the positively charged centers coincide and all the negatively charged centers coincide, this reproduces the bound-state geometries of [5–7]. Another way to see that the multicenter geometries are bound states comes from the fact that one cannot generically separate the centers into separated clusters because of the fluxes wrapping the nontrivial S^2's of the base.

In the next two sections we analyze this general solution. In Sec. V we construct the solution outlined above for the single bubbling supertube and then we put this solution in a Taub-NUT background and show that the singularity resolution mechanism derived in this section reproduces the one found in the case of the zero-entropy four-dimensional black hole [22].

III. THREE-CHARGE SOLUTIONS WITH A GIBBONS-HAWKING BASE

A. The solutions in terms of harmonic functions

In the M-theory frame, a background that preserves the same supersymmetries as three sets of M2-branes can be written as [11,12]

$$ds_{11}^2 = -\left(\frac{1}{Z_1 Z_2 Z_3}\right)^{2/3}(dt+k)^2 + (Z_1 Z_2 Z_3)^{1/3} h_{mn} dx^m dx^n + \left(\frac{Z_2 Z_3}{Z_1^2}\right)^{1/3}(dx_1^2 + dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2}\right)^{1/3}(dx_3^2 + dx_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2}\right)^{1/3}(dx_5^2 + dx_6^2), \tag{3.1}$$

$$\mathcal{A} = A^{(1)} \wedge dx_1 \wedge dx_2 + A^{(2)} \wedge dx_3 \wedge dx_4 + A^{(3)} \wedge dx_5 \wedge dx_6, \tag{3.2}$$

where $A^{(I)}$ and k are oneforms in the five-dimensional space transverse to the T^6. The metric, h_{mn}, is four-dimensional and hyper-Kähler.

When written in terms of the "dipole field strengths" Θ^I,

$$\Theta^{(I)} \equiv dA^{(I)} + d\left(\frac{dt+k}{Z_I}\right), \tag{3.3}$$

the BPS equations simplify to [11,12]

$$\Theta^{(I)} = \star_4 \Theta^{(I)} \qquad \nabla^2 Z_I = \tfrac{1}{2} C_{IJK} \star_4 (\Theta^{(J)} \wedge \Theta^{(K)}) \qquad dk + \star_4 dk = Z_I \Theta^{(I)}, \tag{3.4}$$

where $\star_4$ is the Hodge dual taken with respect to the four-dimensional metric h_{mn}, and $C_{IJK} \equiv |\epsilon_{IJK}|$. If the T^6 is replaced by a more general Calabi-Yau manifold, the C_{IJK} change accordingly.

We will take the base to have a Gibbons-Hawking metric:

$$h_{mn} dx^m dx^n = V(dx^2 + dy^2 + dz^2) + \frac{1}{V}(d\psi + \vec{A} \cdot d\vec{y})^2, \tag{3.5}$$

where we write $\vec{y} = (x, y, z)$ and where

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} V. \tag{3.6}$$

The solutions of (3.4) with a Gibbons-Hawking base have been derived before in [13,17]. Here we derive them again using the linear algorithm outlined in [11] because we need some of the intermediate results. We consider a completely general base with an arbitrary harmonic function, V. We will denote the oneform, $\vec{A} \cdot d\vec{y} \equiv A$. One should also recall that the coordinate ψ has the range $0 \le \psi \le 4\pi$.

This metric has a natural set of frames:

$$\hat{e}^1 = V^{-(1/2)}(d\psi + A), \qquad \hat{e}^{a+1} = V^{1/2} dy^a, \qquad a = 1, 2, 3. \tag{3.7}$$

There are also two natural sets of twoforms:

$$\Omega_{\pm}^{(a)} \equiv \hat{e}^1 \wedge \hat{e}^{a+1} \pm \tfrac{1}{2}\epsilon_{abc} \hat{e}^{b+1} \wedge \hat{e}^{c+1}, \qquad a = 1, 2, 3. \tag{3.8}$$

The $\Omega_{-}^{(a)}$ are anti-self-dual and harmonic, defining the hyper-Kähler structure on the base. The forms, $\Omega_{+}^{(a)}$, are self-dual, and we can take the self-dual field strengths, $\Theta^{(I)}$, to be proportional to them:

$$\Theta^{(I)} = -\sum_{a=1}^{3} (\partial_a (V^{-1} K^I)) \Omega_{+}^{(a)}. \tag{3.9}$$

For $\Theta^{(I)}$ to be closed, the functions K^I have to be harmonic in $\mathbb{R}^3$. Potentials satisfying $\Theta^{(I)} = dB^I$ are then

$$B^I \equiv V^{-1} K^I (d\psi + A) + \vec{\xi}^I \cdot d\vec{y}, \tag{3.10}$$

where

$$\vec{\nabla} \times \vec{\xi}^I = -\vec{\nabla} K^I. \tag{3.11}$$

Hence, $\vec{\xi}^I$ are vector potentials for magnetic monopoles located at the poles of K^I.

The three self-dual Maxwell fields $\Theta^{(I)}$ are thus determined by the three harmonic functions K^I. Inserting this result in the right-hand side of (3.4) we find

$$Z_I = \tfrac{1}{2} C_{IJK} V^{-1} K^J K^K + L_I, \tag{3.12}$$

where L_I are three more independent harmonic functions.

We now write the oneform, k, as

$$k = \mu(d\psi + A) + \omega, \tag{3.13}$$

and then the last equation in (3.4) becomes

$$\vec{\nabla} \times \vec{\omega} = (V\vec{\nabla}\mu - \mu\vec{\nabla}V) - V\sum_{i=1}^{3} Z_I \vec{\nabla}\left(\frac{K^I}{V}\right). \quad (3.14)$$

Taking the divergence yields the following equation for μ:

$$\nabla^2\mu = 2V^{-1}\vec{\nabla}\cdot\left(V\sum_{i=1}^{3} Z_I\vec{\nabla}\frac{K^I}{V}\right), \quad (3.15)$$

which is solved by

$$\mu = \frac{1}{6}C_{IJK}\frac{K^IK^JK^K}{V^2} + \frac{1}{2V}K^IL_I + M, \quad (3.16)$$

where M is yet another harmonic function. Indeed, M determines the anti-self-dual part of dk that cancels out of the last equation in (3.4). Substituting this result for μ into (3.14) we find that ω satisfies

$$\vec{\nabla} \times \vec{\omega} = V\vec{\nabla}M - M\vec{\nabla}V + \tfrac{1}{2}(K^I\vec{\nabla}L_I - L_I\vec{\nabla}K^I). \quad (3.17)$$

The solution is therefore characterized by the eight harmonic functions K^I, L_I, V and M. Moreover, as observed in [27], the solutions are invariant under the shifts:

$$K^I \to K^I + c^IV,$$
$$L_I \to L_I - C_{IJK}c^JK^K - \frac{1}{2}C_{IJK}c^Jc^KV,$$
$$M \to M - \frac{1}{2}c^IL_I + \frac{1}{12}C_{IJK}(Vc^Ic^Jc^K + 3c^Ic^JK^K), \quad (3.18)$$

where the c^I are three arbitrary constants.

The eight functions that give the solution may be identified with the eight independent parameters that make up the $E_{7(7)}$ invariant as follows:

$$x_{12} = L_1, \quad x_{34} = L_2, \quad x_{56} = L_3, \quad x_{78} = -V,$$
$$y_{12} = K^1, \quad y_{34} = K^2, \quad y_{56} = K^3, \quad y_{78} = 2M. \quad (3.19)$$

With these identifications, one can identify the right-hand side of (3.17) in terms of the symplectic invariant of the **56** of $E_{7(7)}$:

$$\vec{\nabla} \times \vec{\omega} = \frac{1}{4}\sum_{A,B}(y_{AB}\vec{\nabla}x_{AB} - x_{AB}\vec{\nabla}y_{AB}). \quad (3.20)$$

For future reference, we note that the quartic invariant of the **56** of $E_{7(7)}$ is determined by

$$\begin{aligned} J_4 = &-\tfrac{1}{4}(x_{12}y^{12} + x_{34}y^{34} + x_{56}y^{56} + x_{78}y^{78})^2 \\ &- (x_{12}x_{34}x_{56}x_{78} + y^{12}y^{34}y^{56}y^{78}) + x_{12}x_{34}y^{12}y^{34} \\ &+ x_{12}x_{56}y^{12}y^{56} + x_{34}x_{56}y^{34}y^{56} + x_{12}x_{78}y^{12}y^{78} \\ &+ x_{34}x_{78}y^{34}y^{78} + x_{56}x_{78}y^{56}y^{78}. \end{aligned} \quad (3.21)$$

B. Dirac-Misner strings and closed timelike curves

To look for the presence of closed timelike curves in the metric, one considers the space-space components of the metric given by (3.1) and (3.5) in the direction of the base. If we denote $W \equiv (Z_1Z_2Z_3)^{1/6}$, and use the expression for k in (3.13) then we find

$$\begin{aligned} ds_4^2 &= -W^{-4}(\mu(d\psi + A) + \omega)^2 + W^2V^{-1}(d\psi + A)^2 + W^2V(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \\ &= W^{-4}(W^6V^{-1} - \mu^2)\left(d\psi + A - \frac{\mu\omega}{W^6V^{-1} - \mu^2}\right)^2 - \frac{W^2V^{-1}}{W^6V^{-1} - \mu^2}\omega^2 + W^2V(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \\ &= \frac{\mathcal{Q}}{W^4V^2}\left(d\psi + A - \frac{\mu V^2}{\mathcal{Q}}\omega\right)^2 + W^2V\left(r^2\sin^2\theta d\phi^2 - \frac{\omega^2}{\mathcal{Q}}\right) + W^2V(dr^2 + r^2d\theta^2), \end{aligned} \quad (3.22)$$

where we have introduced the quantity

$$\mathcal{Q} \equiv W^6V - \mu^2V^2 = Z_1Z_2Z_3V - \mu^2V^2. \quad (3.23)$$

We have also chosen to write the metric on $\mathbb{R}^3$ in terms of a generic set of spherical polar coordinates, (r, θ, ϕ).

Upon evaluating $\mathcal{Q}$ as a function of the eight harmonic functions that determine the solution, one obtains a beautiful result:

$$\begin{aligned} \mathcal{Q} = &-M^2V^2 - \tfrac{1}{3}MC_{IJK}K^IK^JK^k - MVK^IL_I \\ &- \tfrac{1}{4}(K^IL_I)^2 + \tfrac{1}{6}VC^{IJK}L_IL_JL_K \\ &+ \tfrac{1}{4}C^{IJK}C_{IMN}L_JL_KK^MK^N \end{aligned} \quad (3.24)$$

with $C^{IJK} = C_{IJK}$. We see that $\mathcal{Q}$ is nothing other than the $E_{7(7)}$ quartic invariant (3.21) where the x's and y's are identified as in (3.19).

From (3.1) and (3.22) we see that to avoid closed timelike curves (CTC's), the following inequalities must be true everywhere:

$$\mathcal{Q} \ge 0, \qquad W^2V \ge 0,$$
$$(Z_JZ_KZ_I^{-2})^{1/3} = W^2Z_I^{-1} \ge 0, \qquad I = 1, 2, 3. \quad (3.25)$$

The last two conditions can be subsumed into

$$VZ_I = \tfrac{1}{2}C_{IJK}K^JK^K + L_IV \ge 0, \qquad I = 1, 2, 3. \quad (3.26)$$

The obvious danger arises when V is negative. We will show in the next subsection that all these quantities remain finite and positive in a neighborhood of $V = 0$, despite the fact that W blows up. Nevertheless, these quantities could possibly be negative away from the $V = 0$ surface. While we will, by no means, make a complete analysis of the positivity of these quantities, we will discuss it further in Sec. IV, and show that (3.26) does not present a significant problem in a simple example. One should also note that $\mathcal{Q} \geq 0$ requires $\prod_I (VZ_I) \geq \mu^2 V^4$, and so generically the constraint $\mathcal{Q} \geq 0$ is stronger than the constraints (3.26).[5]

Having imposed these conditions there is evidently another potentially dangerous term: $r^2 \sin^2\theta d\phi^2 - \frac{\omega^2}{\mathcal{Q}}$. There will be CTC's if the first term does not always dominate over the second. In particular, one will have CTC's if ω remains finite as one moves onto the polar axis where $\theta = 0, \pi$. This happens precisely when there is a Dirac-Misner string[6] in the metric. Thus to avoid CTC's we must make sure that the solution has no Dirac-Misner strings anywhere.

C. Ergospheres

As we have seen, the general solutions we will consider have functions, V, that change sign on the $\mathbb{R}^3$ base of the GH metric. Our purpose here is to show that such solutions are completely regular, with positive definite metrics, in the regions where V changes sign. As we will see the surfaces $V = 0$ amount to a set of completely harmless ergospheres.

The most obvious issue is that if V changes sign, then the overall sign of the metric (3.5) changes and there might be a large number of closed timelike curves when $V < 0$. However, we remarked above that the warp factors, in the form of W, prevent this from happening. Specifically, the expanded form of the complete, eleven-dimensional metric when projected onto the GH base yields (3.22). In particular, one has

$$W^2 V = (Z_1 Z_2 Z_3 V^3)^{1/3} \sim ((K_1 K_2 K_3)^2)^{1/3} \qquad (3.27)$$

on the surface $V = 0$. Therefore, $W^2 V$ is regular and positive on this surface.

There is still the danger of singularities at $V = 0$ for the other background fields. We first note that there is no danger of such singularities being hidden implicitly in the $\vec{\omega}$ terms. Even though (3.14) suggests that the source of $\vec{\omega}$ is singular at $V = 0$, we see from (3.17) that the source is regular at $V = 0$ and thus there is nothing hidden in $\vec{\omega}$. We therefore need to focus on the explicit inverse powers of V in the solution.

First, the factors of V cancel in the torus warp factors, which are of the form $(Z_I Z_j Z_K^{-2})^{1/3}$. The coefficient of $(dt + k)^2$ is W^{-4}, which vanishes as V^2. The singular part of the cross term, dtk, is the $\mu dt(d\psi + A)$, which, from (3.16), diverges as V^{-2}, and so the cross term remains finite at $V = 0$. So the metric, and the spatial parts of the inverse metric, are regular at $V = 0$. This surface is therefore not an event horizon. It is, however, a Killing horizon or, more specifically, an ergosphere: The timelike Killing vector defined by translations in t becomes null when $V = 0$.

At first sight, it does appear that the Maxwell fields are singular on the surface $V = 0$. Certainly the "magnetic components," Θ^I, in (3.9) are singular when $V = 0$. However, one knows that the metric is nonsingular and so one should expect the singularity in the Θ^I to be unphysical. This intuition is correct: One must remember that the complete Maxwell fields are the $A^{(I)}$, and these are indeed nonsingular at $V = 0$. One finds that the singularities in the "magnetic terms" of $A^{(I)}$ are canceled by singularities in the "electric terms" of $A^{(I)}$, and this is possible at $V = 0$ precisely because it is an ergosphere and the magnetic and electric terms can communicate. Specifically, one has, from (3.3) and (3.10),

$$dA^{(I)} = d\left(B^{(I)} - \frac{(dt + k)}{Z_I}\right). \qquad (3.28)$$

Near $V = 0$ the singular parts of this behave as

$$\begin{aligned} dA^{(I)} &\sim d\left(\frac{K^I}{V} - \frac{\mu}{Z_I}\right)(d\psi + A) \\ &\sim d\left(\frac{K^I}{V} - \frac{K^1 K^2 K^3}{\frac{1}{2} V C_{IJK} K^J K^K}\right)(d\psi + A) \sim 0. \end{aligned} \qquad (3.29)$$

The cancellations of the V^{-1} terms here occur for much the same reason that they do in the metric (3.22).

Therefore, even if V vanishes and changes sign and the base metric becomes negative definite, the complete 11-dimensional solution is regular and well-behaved around the $V = 0$ surfaces. It is this fact that gets us around the uniqueness theorems for asymptotically Euclidean self-dual (hyper-Kähler) metrics in four dimensions, and as we will see, there are now a vast number of candidates for the base metric.

IV. CONSTRUCTING EXPLICIT SOLUTIONS

A. The harmonic functions

We now specify the type of harmonic functions that will underlie our solutions. In particular, we will consider functions with a finite set of isolated sources. Let $\vec{y}^{(j)}$ be the positions of the source points in the $\mathbb{R}^3$ of the base, and let $r_j \equiv |\vec{y} - \vec{y}^{(j)}|$. We take

[5]There might of course exist some solutions where two of the VZ_I change sign on exactly the same codimension one surface, but these are nongeneric.

[6]In terms of vector fields, Dirac strings and Dirac-Misner strings are the same thing, but we use the former term for vector potentials of Maxwell fields, and we use the latter when the vector is part of the metric. The latter is a potentially dangerous physical singularity, unless it can be unwound by a nontrivial $U(1)$ fibration.

$$V = \varepsilon_0 + \sum_{j=1}^{N} \frac{q_j}{r_j}, \tag{4.1}$$

where ε_0 can be chosen to be 1 if the base is asymptotically Taub-NUT and $\varepsilon_0 = 0$ for an asymptotically Euclidean (AE) space. If $q_j \in \mathbb{Z}$ then the metric at $r_j = 0$ has a (spatial) $\mathbb{Z}_{|q_j|}$ orbifold singularity, but this is benign in string theory, and so we will view such backgrounds as regular. It is convenient to define

$$q_0 \equiv \sum_{j=1}^{N} q_j, \tag{4.2}$$

and note that the metric is asymptotic to $\mathbb{R}^4$ if and only if $|q_0| = 1$. By convention we will take $q_0 > 0$. This means that V is positive for large r. Moreover, the fact that $q_j \in \mathbb{Z}$ means that the only nontrivial backgrounds will have some negative q_j's, and thus the function V will be negative in the vicinity of these GH centers. As we saw in Sec. III, the surfaces $V = 0$ when V changes sign.

We can choose the harmonic functions K^I, L_I and M to be localized anywhere on the base. These solutions have localized brane sources, and include, for example, the supertube and the black ring in Taub-NUT [22,25–27]. However, as explained in Sec. II, we are interested in the solutions without localized branes, so we consider harmonic functions K^I, L^I and M whose singularities are localized at the GH centers:

$$K^I = k_0^I + \sum_{j=1}^{N} \frac{k_j^I}{r_j}, \qquad L^I = \ell_0^I + \sum_{j=1}^{N} \frac{\ell_j^I}{r_j}, \qquad M = m_0 + \sum_{j=1}^{N} \frac{m_j}{r_j}. \tag{4.3}$$

B. Cycles and fluxes

The multiple-center GH base (4.1) has many noncontractible two-cycles, Δ_{ij}, that run between the GH centers. These two-cycles can be defined by taking any curve, γ_{ij}, between $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ and considering the $U(1)$ fiber of (3.5) along the curve. This fiber collapses to zero at the GH centers, and so the curve and the fiber sweep out a 2-sphere (up to $\mathbb{Z}_{|q_j|}$ orbifolds). There are $(N-1)$ linearly independent homology two-spheres, and the set $\Delta_{i(i+1)}$ represents a basis. These spheres intersect one another at the points $\vec{y}^{(k)}$.

The fluxes that thread these two-cycles depend on the behavior of the functions, K^I at the GH centers. To determine the fluxes we need the explicit forms for the vector potentials, B^I, in (3.10), and to find these we first need the vector fields, $\vec{v}_i$, that satisfy

$$\vec{\nabla} \times \vec{v}_i = \vec{\nabla}\left(\frac{1}{r_i}\right). \tag{4.4}$$

One then has

$$\vec{A} = \sum_{j=1}^{N} q_j \vec{v}_j, \qquad \vec{\xi}^I = \sum_{j=1}^{N} k_j^I \vec{v}_j. \tag{4.5}$$

If we choose coordinates so that $\vec{y}^{(i)} = (0, 0, a)$, $(y_1, y_2, y_3) = (x, y, z)$ and let ϕ denote the polar angle in the (x, y)-plane, then

$$\vec{v}_i \cdot d\vec{y} = \left(\frac{(z-a)}{r_i} + c_i\right) d\phi, \tag{4.6}$$

where c_i is a constant. The vector field, $\vec{v}_i$, is regular away from the z-axis, but has a Dirac string along the z-axis. By choosing c_i we can cancel the string along the positive or negative z-axis, and by moving the axis we can arrange these strings to run in any direction we choose, but they must start or finish at some $\vec{y}^{(i)}$, or run out to infinity.

Now consider what happens to B^I in the neighborhood of $\vec{y}^{(i)}$. Since the circles swept out by ψ and ϕ are shrinking to zero size, the string singularities near $\vec{y}^{(i)}$ are of the form

$$B^I \sim \frac{k_i^I}{q_i}\left(d\psi + q_i\left(\frac{(z-a)}{r_i} + c_i\right) d\phi\right) - k_i^I\left(\frac{(z-a)}{r_i} + c_i\right) d\phi \sim \frac{k_i^I}{q_i} d\psi. \tag{4.7}$$

This shows that the vector, $\vec{\xi}^I$, in (3.10) cancels the string singularities in the $\mathbb{R}^3$ for each of the complete vector fields, B^I. The singular components of B^I thus point along the $U(1)$ fiber of the GH metric.

If we choose any curve, γ_{ij}, between $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ then the vector fields, B^I, are regular over the whole Δ_{ij} except at the end points, $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. Let $\hat{\Delta}_{ij}$ be the cycle Δ_{ij} with the poles excised. Since $\Theta^{(I)}$ is regular at the poles, we have

$$\Pi_{ij}^{(I)} \equiv \frac{1}{4\pi}\int_{\Delta_{ij}} \Theta^{(I)} = \frac{1}{4\pi}\int_{\hat{\Delta}_{ij}} \Theta^{(I)} = \frac{1}{4\pi}\int_{\partial\hat{\Delta}_{ij}} B^{(I)} = \frac{1}{4\pi}\int_0^{4\pi} d\psi (B^{(I)}|_{y(j)} - B^{(I)}|_{y(i)}) = \left(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i}\right). \tag{4.8}$$

We have normalized these periods for later convenience, and they give the Ith flux threading the cycle Δ_{ij}. As we will see, these fluxes are directly responsible for holding up the cycle.

C. Solving for ω

Since everything is determined by the eight harmonic functions (4.3), all that remains is to solve for ω in Eq. (3.17). The right-hand side of (3.17) has two kinds of terms:

$$\frac{1}{r_i}\vec{\nabla}\frac{1}{r_j} - \frac{1}{r_j}\vec{\nabla}\frac{1}{r_i} \quad \text{and} \quad \vec{\nabla}\frac{1}{r_i}. \tag{4.9}$$

Hence ω will be built from the vectors $\vec{v}_i$ of (4.4) and some new vectors, $\vec{w}_{ij}$, defined by

$$\vec{\nabla} \times \vec{w}_{ij} = \frac{1}{r_i}\vec{\nabla}\frac{1}{r_j} - \frac{1}{r_j}\vec{\nabla}\frac{1}{r_i}. \tag{4.10}$$

To find a simple expression for $\vec{w}_{ij}$ it is convenient to use the coordinates outlined above with the z-axis running through $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. Indeed, choose coordinates so that $\vec{y}^{(i)} = (0, 0, a)$ and $\vec{y}^{(j)} = (0, 0, b)$ and one may take $a > b$. Then the explicit solutions may be written very simply:

$$w_{ij} = -\frac{(x^2 + y^2 + (z-a)(z-b))}{(a-b)r_i r_j} d\phi. \tag{4.11}$$

This is then easy to convert to a more general system of coordinates. One can then add up all the contributions to ω from all the pairs of points.

There is, however, a more convenient basis of vector fields that may be used instead of the w_{ij}. Define

$$\begin{aligned}\omega_{ij} &\equiv w_{ij} + \frac{1}{(a-b)}(v_i - v_j + d\phi) \\ &= -\frac{(x^2 + y^2 + (z - a + r_i)(z - b - r_j))}{(a-b)r_i r_j} d\phi.\end{aligned} \tag{4.12}$$

These vector fields then satisfy

$$\vec{\nabla} \times \vec{\omega}_{ij} = \frac{1}{r_i}\vec{\nabla}\frac{1}{r_j} - \frac{1}{r_j}\vec{\nabla}\frac{1}{r_i} + \frac{1}{r_{ij}}\left(\vec{\nabla}\frac{1}{r_i} - \vec{\nabla}\frac{1}{r_i}\right), \tag{4.13}$$

where

$$r_{ij} \equiv |\vec{y}^{(i)} - \vec{y}^{(j)}| \tag{4.14}$$

is the distance between the ith and jth center in the Gibbons-Hawking metric.

We then see that the general solution for $\vec{\omega}$ may be written as

$$\vec{\omega} = \sum_{i,j}^{N} a_{ij}\vec{\omega}_{ij} + \sum_{i}^{N} b_i \vec{v}_i, \tag{4.15}$$

for some constants a_{ij}, b_i.

The important point about the ω_{ij} is that they have *no string singularities whatsoever*, and thus they can be used to solve (3.17) with the first set of source terms in (4.9), without introducing Dirac-Misner strings, but at the cost of adding new source terms of the form of the second term in (4.9). If there are N source points, $\vec{y}^{(j)}$, then using the w_{ij} suggests that there are $\frac{1}{2}N(N-1)$ possible string singularities associated with the axes between every pair of points $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. However, using the ω_{ij} makes it far more transparent that all the string singularities can be reduced to those associated with the second set of terms in (4.9) and so there are at most N possible string singularities and these can be arranged to run in any direction from each of the points $\vec{y}^{(j)}$.

However, for nonsingular solutions and, as we have seen, to avoid CTC's, we must find solutions without Dirac-Misner strings. The vector potentials, $\vec{v}_i$, necessarily have such singularities, and therefore string singularities will arise through the second term in (4.15). These strings originate from each $\vec{y}^{(j)}$, and while they can be arranged to coincide and cancel in some places, there will always be regions that have nontrivial strings. We therefore have to require that the solution for ω be constructed entirely out of the ω_{ij} in (4.13). That is, we must require that $b_i = 0$ in (4.15). This yields a set of N constraints that relate the charges and distances, r_{ij}. We will refer to these as the "bubble equations."

D. The nonsingular solutions

We have seen that the constants q_j and k_i^I determine the geometry and the fluxes in the solution. We now fix the remaining constants, ℓ_i^I and m_i, by requiring that the solutions have no sources for the brane charge. With but a few exceptions, nonzero sources for the brane charge will lead to singularities or black hole horizons, and are better avoided if one wants to construct microstate solutions.

As one approaches $r_j = 0$ one finds

$$Z_I \sim \left(\frac{1}{2}C_{IJK}\frac{k_j^J k_j^K}{q_j} + \ell_j^I\right)\frac{1}{r_j}. \tag{4.16}$$

We thus remove the brane sources by choosing

$$\ell_j^I = -\frac{1}{2}C_{IJK}\frac{k_j^J k_j^K}{q_j}, \qquad j = 1, \ldots, N. \tag{4.17}$$

Since there are no brane sources, μ cannot be allowed to diverge at $r_j = 0$, which determines

$$m_j = \frac{1}{12}C_{IJK}\frac{k_j^I k_j^J k_j^K}{q_j^2} = \frac{1}{2}\frac{k_j^1 k_j^2 k_j^3}{q_j^2}. \tag{4.18}$$

The constant terms in (4.3) determine the behavior of the solution at infinity. If the asymptotic geometry is Taub-NUT, all these term can be nonzero, and they correspond to combinations of the moduli. (A more thorough investigation of these parameters can be found in the last section of [27].) However, in order to obtain solutions that are asymptotic to five-dimensional Minkowski space, $\mathbb{R}^{4,1}$, one must take $\varepsilon_0 = 0$ in (4.1), and $k_0^I = 0$. Moreover, μ must vanish at infinity, and this fixes m_0. For simplicity we also fix the asymptotic values of the moduli that give the size of the three T^2's, and take $Z_I \to 1$ as $r \to \infty$. Hence, the solutions that are asymptotic to five-dimensional Minkowski space have

$$\varepsilon_0 = 0, \qquad k_0^I = 0, \qquad l_0^I = 1, \qquad m_0 = -\frac{1}{2} q_0^{-1} \sum_{j=1}^N \sum_{I=3}^N k_j^I. \tag{4.19}$$

It is straightforward to generalize our results to solutions with different asymptotics, and, in particular, to Taub-NUT.

As we observed above, we must find a solution with no Dirac-Misner strings, and thus ω must be made out of the ω_{ij} in (4.13). To match the $\frac{1}{r_i}\vec{\nabla}\frac{1}{r_j}$ terms on the right-hand side of (3.17) we must take

$$\begin{aligned}\vec{\omega} &\equiv \frac{1}{2}\sum_{i,j=1}^N \Big(\Big(\frac{1}{2}\sum_{I=1}^3 (k_i^I l_j^I - l_i^I k_j^I)\Big) + (q_i m_j - m_i q_j)\Big)\vec{\omega}_{ij} \\ &= \frac{1}{4}\sum_{i,j=1}^N q_i q_j \Big(\prod_{I=1}^3 \Big(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i}\Big)\Big)\vec{\omega}_{ij} \\ &= \frac{1}{4}\sum_{i,j=1}^N q_i q_j \Pi_{ij}^{(1)}\Pi_{ij}^{(2)}\Pi_{ij}^{(3)}\vec{\omega}_{ij}.\end{aligned} \tag{4.20}$$

This then satisfies

$$\begin{aligned}&\vec{\nabla}\times\vec{\omega} - \Big(V\vec{\nabla}M - M\vec{\nabla}V + \frac{1}{2}(K^I\vec{\nabla}L_I - L_I\vec{\nabla}K^I)\Big) \\ &= \sum_{i=1}^N \Big(m_0 q_i + \frac{1}{2}\sum_{I=1}^3 k_i^I\Big)\vec{\nabla}\Big(\frac{1}{r_i}\Big) \\ &\quad + \frac{1}{4}\sum_{i,j=1}^N q_i q_j \Pi_{ij}^{(1)}\Pi_{ij}^{(2)}\Pi_{ij}^{(3)}\frac{1}{r_{ij}}\Big(\vec{\nabla}\frac{1}{r_i} - \vec{\nabla}\frac{1}{r_j}\Big),\end{aligned} \tag{4.21}$$

and the absence of CTC's means that the right-hand side of this must vanish. Collecting terms in $\vec{\nabla}(r_j^{-1})$ and requiring that each of them vanish leads to a system of "bubble equations," relating r_{ij} to the fluxes:

$$\sum_{\substack{j=1\\ j\neq 1}}^N \Pi_{ij}^{(1)}\Pi_{ij}^{(2)}\Pi_{ij}^{(3)}\frac{q_i q_j}{r_{ij}} = -2\Big(m_0 q_i + \frac{1}{2}\sum_{I=1}^3 k_i^I\Big). \tag{4.22}$$

The solution for ω in (4.20), and the equations (4.22) can be trivially extended to more complicated $U(1)^n$ supergravity theories by replacing $\Pi_{ij}^{(1)}$ $\Pi_{ij}^{(2)}$ $\Pi_{ij}^{(3)}$ with $\frac{1}{6}C_{IJK}\Pi_{ij}^{(I)}\Pi_{ij}^{(J)}\Pi_{ij}^{(K)}$, and replacing products of quantities like $\prod_{I=1}^3 X^I$ by $\frac{1}{6}C_{IJK}X^I X^J X^K$.

Note that if one sums (4.22) over all values of i then the skew symmetry of the left-hand side causes it to vanish. The result is

$$\sum_{i=1}^N \Big(m_0 q_i + \frac{1}{2}\sum_{I=1}^3 k_i^I\Big) = 0, \tag{4.23}$$

which is the last equation in (4.19). Thus there are generically only $(N-1)$ independent "bubble equations" in (4.22).

We see from (4.22) that the r_{ij} are related directly to the fluxes, but for $N > 2$, the r_{ij} are not fixed by a choice of fluxes: There are moduli, and we will discuss this below. Also note that if *any one* of the fluxes Π_{ij}^I, $I = 1, 2, 3$ vanishes, then the r_{ij} drops out of the equations completely.

The other important constraint is (3.25). We are not going to make a complete analysis of this, but we note that the only obvious danger points are when $r_j = 0$ for some j. For the nonsingular solutions considered here we have Z_i going to a constant at $r_j = 0$, and from (3.23) we see that $\mathcal{Q}$ will become negative unless

$$\mu(\vec{y} = \vec{y}^{(j)}) = 0, \qquad j = 1, \ldots, N. \tag{4.24}$$

It turns out that this set of constraints is exactly the same as the set of Eqs. (4.22). We have checked this explicitly, but it is also rather easy to see from (3.14). The string singularities in $\vec{\omega}$ potentially arise from the $\vec{\nabla}(r_j^{-1})$ terms on the right-hand side of (3.14). We have already arranged that the Z_i and μ go to finite limits at $r_j = 0$, and the same is automatically true of $K^I V^{-1}$. This means that the only term on the right-hand side of (3.14) that could, and indeed will, source a string is the $\mu\vec{\nabla}V$ term. Thus removing the string singularities is equivalent to (4.24). Moreover, the fact that the sum of the resulting equations reduces to (4.23) is simply because the condition $\mu \to 0$ as $r \to \infty$ means that there is no Dirac-Misner string running out to infinity.

For the nonsingular solutions it is easy to check that

$$VZ_I = \sum_{i=1}^N \frac{q_i}{r_i} - \frac{1}{4}C_{IJK}\sum_{i,j=1}^N \Pi_{ij}^J\Pi_{ij}^K\frac{q_i q_j}{r_i r_j}, \tag{4.25}$$

and, as we noted in (3.26), this must be positive. While we have not been able to show this is true in general, we suspect that the positivity of these functions will follow from the bubble equations, (4.22), triangle inequalities between r_i, r_j and r_{ij}, and some simple constraints on the charges. We will consider a very simple example below.

At this point it is very instructive to count parameters. There are $3N$ charges, k_j^I, and the set of points, $\vec{y}^{(j)}$, have $3N$ parameters. The Euclidean $\mathbb{R}^3$ of the base has three translational symmetries and three rotational symmetries, which means that, for $N \geq 3$, the generic solution has $6(N-1)$ parameters. The equations impose $(N-1)$ constraints, leaving $5(N-1)$ free parameters. For $N = 2$ there is a residual axisymmetry which means that there are $5N - 4 = 6$ parameters, which correspond to the choice of the k_j^I. We should also remember that three combinations of the k_j^I do not affect the final solution, because of the gauge invariance (3.18).

Physically, it is natural to fix the charges and solve (4.22) to determine some of the r_{ij} in terms of other r_{lm}. Note that doing this turns (4.22) into a linear system for the r_{ij}^{-1}, which is elementary to invert. In finding the solutions, one must remember to impose the triangle inequalities:

$$r_{ij} + r_{jk} \geq r_{ik}, \quad \forall\ i, j, k. \tag{4.26}$$

It is easy to see that there is always a physical solution for some ranges of the r_{ij}. Indeed, if put all the points $\vec{y}^{(j)}$ on a single axis, which means the complete solution preserves a $U(1) \times U(1)$ symmetry overall, then the only geometric parameters are the $(N-1)$ separations of the points on the axis. The triangle inequalities are all trivially satisfied. Thus only $(N-1)$ of the r_{ij} are independent, and they are uniquely fixed[7] by (4.22) in terms of the choice of flux parameters, k_j^I. One can then vary the r_{ij} about this solution and one finds the complete family with all allowed ranges of r_{ij} consistent with the triangle inequality.

It is rather easy to understand the physical picture of the solution set. The fluxes, determined by k_j^I, are holding up the blown up cycles, whose sizes are determined by the r_{ij}. If one puts all the cycles in a straight line, then their sizes are fixed uniquely by the magnitudes of the fluxes through the cycles. One can think of these cycles as being characterized by "rods" of length $r_{i(i+1)}$ along the axis. However, the cycles can move around, and so the rods can pivot about their end points while remaining connected to one another. The rod lengths can vary, but they generically vary only a small amount: Their length is set by the fluxes through the cycles, and these are modified only when neighboring cycles get close enough that they interfere with each others fluxes. Rods can, however, combine and break when a point $\vec{y}^{(\ell)}$ crosses the axis $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. Put more mathematically, by imposing axisymmetry one gets a preferred homology basis with fixed scales. This basis will generically undergo Weyl reflections when the difference of two cycles collapses.

E. Examples

Consider V with three charges:

$$q_1 = 1, \qquad q_2 = 1, \qquad q_3 = -1. \tag{4.27}$$

Since we have $q_0 \equiv \sum q_j = 1$, the base is asymptotic to $\mathbb{R}^4$. The Eqs. (4.22) reduce to a system of the form

$$-\frac{A_{12}}{r_{12}} + \frac{A_{13}}{r_{13}} = (B_2 + B_3), \qquad \frac{A_{12}}{r_{12}} + \frac{A_{23}}{r_{23}} = (B_1 + B_3),$$
$$\frac{A_{13}}{r_{13}} + \frac{A_{23}}{r_{23}} = (B_1 + B_2 + 2B_3), \tag{4.28}$$

where

$$A_{12} \equiv \prod_{I=1}^{3}(k_1^I - k_2^I), \qquad A_{13} \equiv \prod_{I=1}^{3}(k_1^I + k_3^I),$$
$$A_{23} \equiv \prod_{I=1}^{3}(k_2^I + k_3^I), \qquad B_j \equiv \sum_{I=1}^{3} k_j^I, \qquad j = 1, 2, 3. \tag{4.29}$$

If we impose the condition that the Gibbons-Hawking centers are collinear, for example $r_{13} = r_{12} + r_{23}$, then the forgoing equations reduce to a quadratic equation. The ordinary, zero-entropy black ring emerges as $r_{23} \to 0$, and so we know there is certainly a family of solutions in this limit.

Now suppose we have a solution, and we want to find the family to which it belongs. We only need use two of the equations in (4.28), and so consider the first two equations. Choose r_{12} to have the value for the known solution. The first two equations in (4.28) tell us that the third charge $(j = 3)$ must be located at a determined distance from each of the other two charges, i.e. r_{13} and r_{23} are fixed. This determines the location of the third charge up to rotations about the axis through the first two charges. Suppose we now decrease r_{12} by a small amount. The first two equations in (4.28) tell us that if $\pm\frac{A_{13}}{A_{12}} > 0$ then $\pm r_{13}$ must decrease and if $\pm\frac{A_{23}}{A_{12}} > 0$ then $\pm r_{23}$ must increase. Thus the third charge must move around to compensate, and if r_{13} and r_{23} change in opposite senses then the third charge will move in an orbit around the first or second charge. If r_{13} and r_{23} change in the same sense then the third charge will move toward or away from the axis between the first and second charges. For generic k_j^I, there will only be a range of values for r_{12} for which a solution is possible. The distances r_{13} and r_{23} can only compensate for limited changes in r_{12} without violating triangle inequalities.

Perhaps the most instructive case to consider is when every K-charge is equal: $k_i^I = k$, for all i, I, $|q_j| = 1$ for all j, and $q_0 = 1$. Decompose the Gibbons-Hawking points, q_j into two sets:

$$\mathcal{S}_\pm \equiv \{j:\ q_j = \pm 1\}. \tag{4.30}$$

Define the electrostatic potentials:

$$\mathcal{V}_\pm(\vec{y}) \equiv \sum_{j\in\mathcal{S}_\pm} \frac{8k^2}{3}\frac{1}{|\vec{y} - \vec{y}^{(j)}|}. \tag{4.31}$$

Then the equations (4.22) reduce to

$$\mathcal{V}_+(\vec{y}^{(i)}) = (N+1), \quad \forall\ i \in \mathcal{S}_-;$$
$$\mathcal{V}_-(\vec{y}^{(i)}) = (N-1), \quad \forall i \in \mathcal{S}_+. \tag{4.32}$$

These also have the redundancy arising from (4.23):

$$\sum_{i\in\mathcal{S}_-} \mathcal{V}_+(\vec{y}^{(i)}) = \sum_{i\in\mathcal{S}_+} \mathcal{V}_-(\vec{y}^{(i)}). \tag{4.33}$$

Thus we see that the positive Gibbons-Hawking points

[7] If, in solving (4.22), one finds one of the independent r_{ij} to be negative, one can render it positive by reordering the points.

must be on a specific equipotential of $\mathcal{V}_-$, and do not care where the other positive charges are. Similarly, the negative Gibbons-Hawking points must be on a specific equipotential of $\mathcal{V}_+$, and do not care where the other negative charges are.

Suppose we have $N = 3$ with labeling (4.27), then (4.32) tells us that

$$r_{13} = r_{23} = \frac{4k^2}{3}, \tag{4.34}$$

and that r_{12} is a free parameter. However, the triangle inequality limits r_{12} to $0 \leq r_{12} \leq \frac{8k^2}{3}$.

For $N = 5$ things are a little more complicated. Let a be the separation of the two negative charges. Having fixed a there should be a five-parameter family of solutions to (4.32): One must locate the positive charges on the equipotentials of $\mathcal{V}_-$, which leads to six parameters. There are then apparently two constraints coming from the first equation (4.32) but one of them is redundant via (4.33).

There are some obvious solutions that can be obtained using symmetry. The potential, $\mathcal{V}_-$, has a symmetry axis, $\mathcal{A}$, through the two Gibbons-Hawking points and the equipotentials come either as a single surface, $\mathcal{B}$, of revolution about $\mathcal{A}$, or they consist of two disconnected deformed spheres about each point. If the two negative charges are close enough together then the equipotentials $\mathcal{V}_- = (N-1) = 4$ is a single surface, $\mathcal{B}$, which has a well-defined equator, $\mathcal{E}$, midway between the two negative charges. (See Fig. 2.) One obvious three-parameter solution to (4.32) is to put all three positive charges anywhere on $\mathcal{E}$. The remaining two parameters come from moving the charges off $\mathcal{E}$: One can move two of them in any way one wishes, and the third one's position is fixed by the first equation in (4.32). As for $N = 3$, there will be limits on the range of motion of the three points.

Finally, we note that, for this example, (4.25) collapses to yield the condition

$$VZ_I = \frac{3}{8k^2}\left(\mathcal{V}_+ - \mathcal{V}_- + \frac{3}{2}\mathcal{V}_+\mathcal{V}_-\right) \geq 0. \tag{4.35}$$

For $N = 3$ this inequality is simply

$$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} + \frac{4k^2}{r_3}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \frac{1}{r_1} + \frac{(r_3 - r_2 + r_{23})}{r_2 r_3} + \frac{1}{r_3}\left(\frac{4k^2}{r_1} + \frac{(4k^2 - r_{23})}{r_2}\right) \geq 0. \tag{4.36}$$

This is trivially satisfied because of the triangle inequality $r_3 + r_{23} \geq r_2$ and because $r_{23} = \frac{4k^2}{3}$.

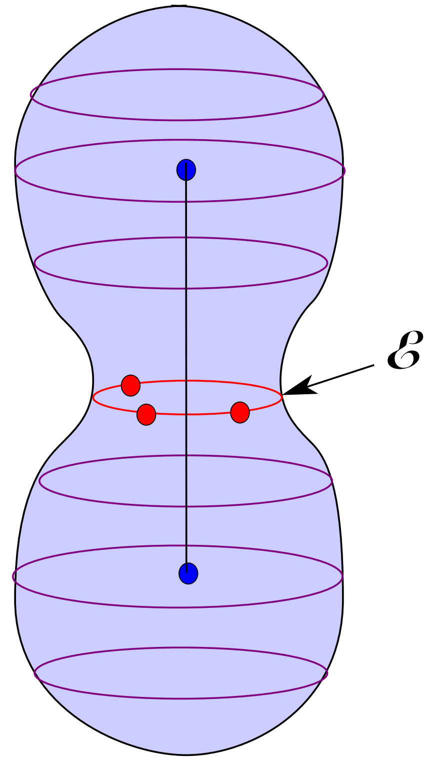

FIG. 2 (color online). This shows the equipotential, $\mathcal{V}_- = 4$. The three positive charges can be located anywhere on the equator, $\mathcal{E}$.

V. BUBBLING SUPERTUBES

A. The resolved solution

In this section we investigate the form of the solutions that resolve the singularity of the three-charge black ring. We are also going to require the resolved solution to have the same $U(1) \times U(1)$ symmetry as the black ring. The geometric resolution we describe here is depicted in Fig. 1.

As discussed in Sec. III, the metric on the base is given by

$$V = \frac{1}{r} - \frac{Q}{r_a} + \frac{Q}{r_b}, \tag{5.1}$$

where r_a and r_b denote the distance to the points a and b. If a, b and the origin are collinear then the solution has a $U(1)$ invariance from the $\mathbb{R}^3$ of the GH metric and another $U(1)$ invariance from the GH fiber.

We want the nonsingular solution outlined in Sec. IV. We can choose the K-charges freely, but we only allow them to be sourced at the GH centers:

$$K^I = \frac{k_0^I}{r} + \frac{k_a^I}{r_a} + \frac{k_b^I}{r_b}. \tag{5.2}$$

Note that k_0^I now denotes the charge at $r = 0$, and not the additive constant in (4.3). As before, we want this solution to be pure geometry with fluxes, with charges coming from fluxes, and we require the harmonic functions Z_i to have no divergences at the points 0, a, b. Together with asymptotic flatness, this completely determines the functions L_I:

$$L_I = 1 + \frac{1}{2}C_{IJK}\left(-\frac{k_0^J k_0^K}{r} + \frac{k_a^J k_a^K}{Q r_a} - \frac{k_b^J k_b^K}{Q r_b}\right), \tag{5.3}$$

where, as before, $C_{IJK} = C^{IJK} \equiv |\epsilon_{IJK}|$. Similarly, requiring μ to be regular leads to

$$M = \frac{1}{12}C_{IJK}\left(\frac{k_0^I k_0^J k_0^K}{r} + \frac{k_a^I k_a^J k_a^K}{Q^2 r_a} + \frac{k_b^I k_b^J k_b^K}{Q^2 r_b}\right) - \frac{1}{2}\sum_{I,j=0,a,b} k_j^I. \tag{5.4}$$

To remove Dirac-Misner strings and CTC's leads to three more relations of the form (4.22) and only two of these relations are independent. Since we are taking the centers to be collinear, these equations determine a and b as a function of the k_j^I. Thus the complete solution is fixed by the choice of the k_j^I, which also fix the fluxes through the two S^2's of the base. We will give these equations below but, as we noted in Sec. IV, these equations are equivalent to requiring that $\vec{\omega}$ can be written solely in terms of the $\vec{\omega}_{ij}$ of (4.12). Here we have[8]

$$\vec{\omega} = \frac{1}{2Q^2}\prod_{I=1}^{3}(k_a^I + Qk_0^I)\vec{\omega}_{0a} - \frac{1}{2Q}\prod_{I=1}^{3}(k_a^I + k_b^I)\vec{\omega}_{ab} + \frac{1}{2Q^2}\prod_{I=1}^{3}(k_b^I - Qk_0^I)\vec{\omega}_{0b}. \tag{5.5}$$

B. Orthogonal cycles and diagonalization

As we have discussed in Sec. III, our solutions have a gauge invariance (3.18), and therefore only two combinations of the three parameters in K_I appear in the solution. This gauge invariance could be used to eliminate the k_0^I for example; however, in order to simplify the solution it is better to introduce other variables that make the relation between the bubbling solutions and the supertubes more direct.

To avoid unnecessary normalization conventions between the charges in the supergravity solution and the number of branes, we work in a convention where they are equal; this happens when the three T^2's have equal size and $G_5 = \frac{\pi}{4}$ [16,27]. The three M2-brane charges are then

$$N_K = 2C_{KIJ}\left(\left(1 + \frac{1}{Q}\right)k_a^I k_a^J + 2k_a^I k_b^J + \left(1 - \frac{1}{Q}\right)k_b^I k_b^J + 2k_a^I k_0^J + 2k_b^I k_0^J\right). \tag{5.6}$$

[8]The solution in a more complicated $U(1)^n$ supergravity theory is obtained by simply replacing the products of the form $\prod_{I=1}^{3}(k_a^I + k_b^I)$ with $\frac{1}{6}C_{IJK}(k_a^I + k_b^I)(k_a^J + k_b^J)(k_a^K + k_b^K)$.

If we now introduce new, physical variables n_I and f_I,

$$n_I \equiv 2(k_a^I + k_b^I), \qquad f_I \equiv \frac{1}{Q}[k_a^I(1 + Q) + k_b^I(Q - 1) + 2Qk_0^I], \tag{5.7}$$

the charges become

$$N_K = C^{KIJ} n_I f_J. \tag{5.8}$$

As expected from (3.18), all the components of the solution only depend on the f's and n's. The n_I and f_I are simple combinations of fluxes through the nontrivial cycles. Indeed, n_I is proportional to the flux through the cycle between a and b that resolves the supertube. The interpretation of f_I is a bit more obscure; however, when $Q = 1$, then $n = 2(k_a + k_b)$ and $f = 2(k_a + k_0)$, and so the cycles that give n and f run between the point with a minus GH charge and the two points with positive GH charge.

C. Solving the bubble equations

Since we are restricting the GH centers to lie on an axis there are *a priori* several distinct cases determined by the signs of a, b and $b - a$. However, for the bubbling supertube we will only need consider the regime with $b > a > 0$. The bubble equations are

$$\frac{1}{aQ^2}\prod_{I=1}^{3}\left(\frac{1}{2}(Q-1)n_I - Qf_I\right) - \frac{1}{bQ^2}\prod_{I=1}^{3}\left(\frac{1}{2}(Q+1)n_I - Qf_I\right) + 4\sum_{I=1}^{3} n_I = 0, \tag{5.9}$$

$$\frac{1}{(b-a)Q}\prod_{I=1}^{3} n_I - \frac{1}{bQ^2}\prod_{I=1}^{3}\left(\frac{1}{2}(Q+1)n_I - Qf_I\right) - 4\sum_{I=1}^{3}\left(\frac{1}{2}(Q-1)n_I + Qf_I\right) = 0. \tag{5.10}$$

These equations are also trivially generalized to the case of a more complicated $U(1)^n$ supergravity theory.

The general solutions of these equations are quite involved, however they have some rather interesting features. The simplest case to analyze is to take $|Q| = 1$ and $n_I = f_I = 4k$ for all I as we did in the previous section. For $Q = 1$ we learn from (4.34), or (5.9), that

$$a = \frac{4k^2}{3}, \qquad b = \frac{8k^2}{3}, \tag{5.11}$$

thus the negative GH charge is exactly in the middle of the two positive charges. For $Q = -1$ we have

$$a = 0, \qquad b = \frac{4k^2}{3}, \tag{5.12}$$

and so the two positive GH charges coincide at the origin.

Of more interest physically is the limit in which the distance between a and b becomes very small compared with the distance from the origin to the points a and b. As we will see in the next subsection, the latter distance asymptotes to the radius of the naive supertube, and the solution resembles the naive supertube solution. The distance between a and b is given by the balance between the attraction of the Q and $-Q$ charges, and the tendency of the fluxes wrapped on the two-cycle between a and b to expand this cycle and make this distance larger. Therefore, one expects to obtain a solution that matches the naive supertube solution *both* in the small flux limit (small n) and in the large Q limit.

When a and b are very close to each other it is relatively easy to find an approximate solution to (5.9) and (5.10). Equation (5.10) determines the separation between a and b, and Eq. (5.9) gives the separation between these two points and the origin. The leading part of (5.9) (obtained by setting $a = b$) then gives

$$a \approx b \approx \frac{2C^{IJK}(f_I f_J n_K - n_I n_J f_K) + (3 + \frac{1}{Q^2})\frac{1}{6}C^{IJK} n_I n_J n_K}{16\sum_{I=1}^{3} n_I}. \tag{5.13}$$

As we will see in the next subsection, this matches the radius of the naive supertube solution both in the small n and in the large Q limits. One can also estimate the size of the bubble, $a - b$; we will give the complete formula for this size when we discuss bubbling supertubes in Taub-NUT in Sec. V E.

We have also numerically checked that $\mathcal{Q} > 0$ and $r^2\sin^2\theta d\phi^2 - \frac{\omega^2}{\mathcal{Q}} \geq 0$ in an example in which a and b are large compared to $a - b$. Thus, the bubbling of the supertube does not generate CTC's.

The angular momenta of the bubbling supertube are easy to find. The complicated form of the solutions of the bubble equations might have worried one that the angular momenta, which depend explicitly on a and b, would be rather horrible. However, a pleasant surprise awaits.

To read off the angular momenta we first do a change of coordinates to write the asymptotic form of the GH base as $\mathbb{R}^4$. If one chooses asymptotically $\vec{A}\cdot d\vec{y} = (1 + \cos\theta)d\phi$ in (3.5), then the coordinate change,

$$\phi = \tilde{\phi}_2 - \tilde{\phi}_1, \qquad \psi = 2\tilde{\phi}_1, \qquad r = \frac{1}{4}\rho^2, \tag{5.14}$$

makes the $\mathbb{R}^4$ form of the asymptotic solution explicit. The two angular momenta are then determined from the asymptotic behavior of μ and ω via

$$\mu(1 - \cos\theta) - \omega \approx \frac{J_1\sin^2\frac{\theta}{2}}{4r} \tag{5.15}$$

$$\mu(1 + \cos\theta) + \omega \approx \frac{J_2\cos^2\frac{\theta}{2}}{4r}. \tag{5.16}$$

One combination of the angular momenta is independent of a and b, and is given by

$$(J_1 + J_2) = -\frac{(Q^2 - 1)}{4Q^2}\frac{1}{6}C^{IJK} n_I n_J n_K + \frac{1}{2}C^{IJK}(n_I n_J f_K + f_I f_J n_K). \tag{5.17}$$

Despite the very complicated form of the solution of (5.9) and (5.10), the other combination of the angular momenta is also very simple. The final results are

$$J_1 = -\frac{(Q - 1)}{2Q}\frac{1}{6}C^{IJK} n_I n_J n_K + \frac{1}{2}C^{IJK} n_I n_J f_K, \tag{5.18}$$

$$J_2 = \frac{(Q - 1)^2}{4Q^2}\frac{1}{6}C^{IJK} n_I n_J n_K + \frac{1}{2}C^{IJK} f_I f_J n_K. \tag{5.19}$$

As we will see in the next subsection, these will again match the naive supertube angular momenta, both in the large Q limit and in the small n_I limit.

D. Matching the naive supertube solution

We first write the naive solution describing the zero-entropy black ring by rewriting the $\mathbb{R}^4$ base as a GH metric with a single GH center of charge one. This single center can be taken to be at $r = 0$, and then the supertube generically has sources for K, L and M at $r = 0$ and at $r_a = 0$. The radius, R, of the supertube is then the distance, r_{0a}, between the two source points. We can use the gauge invariance, (3.18), to set the K-charge at $r = 0$ to zero, and then the supertube solution is given by [27]

$$V = \frac{1}{r}, \qquad K^I = \frac{n_I}{2r_a}, \qquad L_I = 1 + \frac{\bar{N}_I}{4r_a}, \qquad M = -\frac{J_T}{16}\left(\frac{1}{R} - \frac{1}{r_a}\right). \tag{5.20}$$

In this expression, n_I and $\bar{N}_I$ are the (integer) numbers of M5-branes and M2-branes that make up the supertube, and J_T is the angular momentum of the tube alone. The "tube" angular momentum, J_T, is related to the radius by

$$J_T = 4R_T(n_1 + n_2 + n_3). \tag{5.21}$$

In the case of the zero-entropy black ring, this angular momentum is completely determined in terms of the n_I and $\bar{N}_I$:

$$J_T = \frac{2n_1 n_2 \bar{N}_1 \bar{N}_2 + 2n_1 n_3 \bar{N}_1 \bar{N}_3 + 2n_2 n_3 \bar{N}_2 \bar{N}_3 - n_1^2\bar{N}_1^2 - n_2^2\bar{N}_2^2 - n_3^2\bar{N}_3^2}{4n_1 n_2 n_3}, \tag{5.22}$$

where $\bar{N}_1 = N_1 - n_2 n_3$, and similarly for $\bar{N}_2$ and $\bar{N}_3$. The angular momenta of the naive supertube solution are

$$J_1 = \frac{n_1\bar{N}_1 + n_2\bar{N}_2 + n_3\bar{N}_3}{2} + n_1 n_2 n_3$$
$$J_2 = J_T + \frac{n_1\bar{N}_1 + n_2\bar{N}_2 + n_3\bar{N}_3}{2} + n_1 n_2 n_3. \tag{5.23}$$

As one can see both from the form of the K^I when a and b are very close, and from the integral of the fluxes on the cycle that runs between a and b, the n_I of the bubbling solutions are identical to the M5 dipole charges n_i. If one then interprets (5.8) as a change of variables between the N_I and the f_I, one can express the supertube angular momenta and radius in terms of the n_I and f_I:

$$J_T = \tfrac{1}{2}C^{IJK}(f_I f_J n_K - n_I n_J f_K) + \tfrac{1}{8}C^{IJK} n_I n_J n_K, \tag{5.24}$$

$$J_1 = \frac{1}{2}C^{IJK}(n_I n_J f_K) - \frac{1}{12}C^{IJK} n_I n_J n_K,$$
$$J_2 = \frac{1}{2}C^{IJK}(f_I f_J n_K) + \frac{1}{24}C^{IJK} n_I n_J n_K, \tag{5.25}$$

which gives

$$R_T = \frac{2C^{IJK}(f_I f_J n_K - n_I n_J f_K) + \frac{1}{2}C^{IJK} n_I n_J n_K}{16\sum_{I=1}^{3} n_I}. \tag{5.26}$$

As we have advertised in the previous subsection, these match exactly the bubbling supertube radius and angular momenta both in the limit when the dipole charges n_I are small, and in the limit when Q is large.

We should also note that the nnf and nff combinations that appear in the angular momentum formulas (5.18) and (5.19) are not apparent at all from the form of the supertube angular momenta, and only become apparent after expressing the N_I using (5.8). When $Q = 1$ the angular momenta are interchanged under the exchange of n_I and f_I. Hence, a solution with $Q = 1$ has two interpretations: for small n_I it is a supertube of dipole charges n_I in the $\tilde{\phi}_1$ plane, and for small f_I it is a supertube of dipole charges f_I in the $\tilde{\phi}_2$ plane. This feature—the existence of one supergravity solution with two different brane interpretations—is present in all the other systems that contain branes wrapped on topologically trivial cycles [33,42,43] and might be the key to finding the microscopic description of our bubbling supertube geometries.

When the size of one bubble is much smaller than the size of the other, the harmonic functions that give the bubbling solution (5.2), (5.3), and (5.4) become approximately equal to the supertube harmonic functions (5.20).

If we work in the gauge with $k_0^I = 0$, one can easily see that (5.2) combined with (5.7) reproduces the correct identification of the dipole charges with the flux integrals. From (5.3) one finds

$$\bar{N}_I = \frac{2}{Q} C_{IJK}(k_a^J k_a^K - k_b^J k_b^K). \tag{5.27}$$

This again agrees with (5.7) and (5.8) in the gauge $k_0^I = 0$ after using the fact that the total M2-brane charge of the solution is

$$N_I = \bar{N}_I + \frac{1}{2}C^{IJK} n_J n_K$$
$$= \frac{2}{Q}C_{IJK}((k_a^J k_a^K - k_b^J k_b^K) + Q(k_a^J - k_b^J)(k_a^K - k_b^K))$$
$$= \frac{2}{Q}C_{IJK}(k_a^J + k_b^J)((Q+1)k_a^K + (Q-1)k_b^K). \tag{5.28}$$

Finally, the harmonic function M in (5.4) reproduces that of the naive supertube after identifying

$$J_T = \frac{4}{3Q^2}C_{IJK}(k_a^I k_a^J k_a^K + k_b^I k_b^J k_b^K)$$
$$= \frac{1}{2}C^{IJK}(f_I f_J n_K - n_I n_J f_K)$$
$$+ \frac{1}{24}\left(3 + \frac{4}{Q} + \frac{1}{Q^2}\right)C^{IJK} n_I n_J n_K \tag{5.29}$$

and remembering that the solutions agree in the large Q or small n limit.

Hence, the bubbling solution is identical to the naive solutions at distances much larger than the size of the bubble. Moreover, in the limit when $Q \to \infty$ or in the limit when $n_I \to 0$ with fixed N_I, the bubble that is nucleated to resolve the singularity of the three-charge supertube becomes very small, and the resolved solution becomes virtually indistinguishable from the naive solution. This confirms the intuition coming from the discussion of geometric transitions in Sec. II, and from the similar phenomenon observed in [33].

The fact that for small dipole charges the singularity resolution is local provides very strong support to the expectation that there exist similar bubbling supertube solutions that correspond to supertubes of arbitrary shape and arbitrary charge densities.

E. Bubbling supertubes in Taub-NUT

In order to compare our singularity resolution mechanism to that of the zero-entropy four-dimensional black hole [22], we need to put the bubbling supertube in Taub-NUT. This solution resolves the singularity of the zero-entropy black ring in Taub-NUT [25–27].

The construction of bubbling supertubes in Taub-NUT is almost identical to the construction in asymptotically Euclidean space. To simplify the algebra we make use of the gauge freedom (3.18) to set the three k_0^I to zero. The asymptotically Taub-NUT base is obtained by modifying the harmonic function V to

$$V = h + \frac{1}{r} - \frac{Q}{r_a} + \frac{Q}{r_b}, \tag{5.30}$$

where we do not fix h to one in order to make the interpolation between the asymptotically $\mathbb{R}^4$ and the asymptotically Taub-NUT solutions easier. The functions K^I and L_I are the same as before [see (5.2) and (5.3)]. The coefficients m_j in M are given by the requirement that μ be regular at the three centers (4.18), and they are also not changed. However, because of the changed asymptotics there is no longer a requirement that μ vanish at infinity, and so m_0 is a free parameter. Hence,

$$M = m_0 + \frac{1}{2}C_{IJK}\left(\frac{k_0^I k_0^J k_0^K}{r} + \frac{k_a^I k_a^J k_a^K}{Q^2 r_a} + \frac{k_b^I k_b^J k_b^K}{Q^2 r_b}\right). \quad (5.31)$$

The requirement that μ vanishes at the three centers gives three equations, which are now independent:

$$\frac{1}{aQ^2}\prod_{I=1}^{3}\left(\frac{1}{2}(Q-1)n_I - Qf_I\right) - \frac{1}{bQ^2}\prod_{I=1}^{3}\left(\frac{1}{2}(Q+1)n_I - Qf_I\right) - 16m_0 = 0, \quad (5.32)$$

$$\frac{1}{(b-a)Q}\prod_{I=1}^{3}n_I - \left(h + \frac{1}{b}\right)\frac{1}{Q^2}\prod_{I=1}^{3}\left(\frac{1}{2}(Q+1)n_I - Qf_I\right) - 16m_0 + \sum_{I=1}^{3}\left(\frac{1}{2}(Q+1)n_I - Qf_I\right) = 0. \quad (5.33)$$

$$\frac{1}{(b-a)Q}\prod_{I=1}^{3}n_I - \left(h + \frac{1}{a}\right)\frac{1}{Q^2}\prod_{I=1}^{3}\left(\frac{1}{2}(Q-1)n_I - Qf_I\right) - 16m_0 Q + \sum_{I=1}^{3}\left(\frac{1}{2}(Q-1)n_I - Qf_I\right) = 0. \quad (5.34)$$

The compatibility of these equations determines m_0 to be

$$m_0 = \frac{1}{32}hC^{IJK}(f_I f_J n_K - n_I n_J f_K) + \frac{1}{384}\left(3 + \frac{1}{Q^2}\right)hC^{IJK}n_I n_J n_K - \frac{1}{4}\sum_{I=1}^{3}n_I. \quad (5.35)$$

When $h = 0$ the value of m_0 becomes $-\frac{1}{4}\sum_{I=1}^{3}n_I$, and one recovers the solution of the previous subsection in the gauge $k_0^I = 0$.

One can solve these equations in the large Q or in the small n_I limit, in which the separation between a and b is much smaller than their distance from the origin. Setting $a = b$ in Eq. (5.32), one obtains

$$h + \frac{1}{a} \approx h_0 \equiv \frac{16\sum_{I=1}^{3}n_I}{2C^{IJK}(f_I f_J n_K - n_I n_J f_K) + (3 + \frac{1}{Q^2})\frac{1}{6}C^{IJK}n_I n_J n_K}. \quad (5.36)$$

In the large Q limit, this equation reproduces the correct radius of the zero-entropy black ring in Taub-NUT [27].

To reach the four-dimensional black hole limit, one needs to move the bubbling supertube away from the Taub-NUT center, to very large a and b, keeping the fluxes n_I and f_I fixed.

To do this, one adjusts h until it reaches h_0. In this limit one has $m_0 = 0$, and (5.32) is trivially satisfied. The distance between a and b is obtained from either of the remaining two equations:

$$b - a \approx \frac{Q\prod_{I=1}^{3}n_I}{h_0\prod_{I=1}^{3}(\frac{1}{2}(Q-1)n_I - Qf_I) - Q^2\sum_{I=1}^{3}(\frac{1}{2}(Q-1)n_I - Qf_I)}, \quad (5.37)$$

where h_0 is defined in (5.36).

Thus, by putting the supertube in Taub-NUT we have related the singularity resolution mechanism of the zero-entropy black ring (the nucleation of two oppositely charged GH centers) to the singularity resolution of the zero-entropy four-dimensional black hole (the splitting of the branes that form the black hole into two stacks at a finite radius from each other [22]). Similar solutions have also been analyzed from the point of view of four-dimensional supergravity in [44–46].

We see therefore that the interpolation between Taub-NUT and $\mathbb{R}^4$ is not only a good tool to obtain the microscopic description of black rings and black holes [27,47], but is also useful in understanding their singularity resolution.

VI. CONCLUSIONS

We have constructed smooth geometries that resolve the zero-entropy singularity of BPS black rings. The $U(1) \times U(1)$ invariant geometries must have a Gibbons-Hawking base space with several centers of positive and negative charge. Despite the fact that the signature of the base changes, the full geometries are regular.

These geometries stem naturally from implementing the mechanism of geometric transitions to the supertube. The physics is closely related to that of other systems containing branes wrapped on topologically trivial cycles [33]. The fact that our solutions reproduce in one limit the geometries found by Mathur and collaborators [5–7], and in another limit they reduce to the ground states of the four-dimensional black hole found in [22], is a nontrivial confirmation that these geometries are indeed the correct black ring/supertube ground states.

The BPS black rings have two microscopic interpretations: one in terms of the D1-D5-P CFT [18], and another in terms of a four-dimensional black hole CFT [18,19,27]. Hence, our solutions should similarly be thought of as *both* microstates of the D1-D5-P system, and as microstates of the four-dimensional black hole CFT that describes the black ring [18,27]. To establish the four-dimensional black hole microstates that are dual to our solutions, it is best perhaps to put the bubbling supertubes in Taub-NUT, and go to the limit when they become four-dimensional configurations. However, to establish their interpretation in the D1-D5-P CFT is somewhat nontrivial. One possibility is to start from the D1-D5-P microscopic description of the BPS black ring [18], and to explore the zero-entropy limit. However, this might not be so straightforward, since the naive supertube and resolved geometries only agree in the very large Q limit. Another option is to use the fact that when the two positively charged centers coincide these solutions reproduce those of [5–7], which do have a D1-D5-P microscopic interpretation.

Our analysis also indicates that the most generic bound state with a GH base is determined by a gas of positive and negative centers, with fluxes threading the many nontrivial two-cycles of the base, and no localized brane charges. This proposal has similar features to the foam described in [48], but in [48] the foam was restricted to the compactification space, whereas here the foam naturally lives in the macroscopic space-time and defines the interior structure of a black hole. Another interesting exploration of a similar type of space-time foam from the point of view of a dual boundary theory has appeared in [49].

Our results suggest quite a number of very interesting consequences and suggestions for future work. First, we have only considered geometries with a Gibbons-Hawking base, because such geometries are easy to find [13,17], and appear in the $U(1) \times U(1)$ invariant background. However, the most general smooth solution will have a base given by an asymptotically $\mathbb{R}^4$ hyper-Kähler manifold whose signature can change from $(+, +, +, +)$ to $(-, -, -, -)$. We expect these solutions to give regular geometries,[9] provided that there are nonzero dipole fluxes.

[9]It is also worth stressing that it is the possibility of signature change that enables us to avoid the extremely restrictive conditions, familiar to relativists, on the existence of four-dimensional, asymptotically Euclidean metrics. By allowing the signature of the base to change we have found a large number, and conjectured an even larger number of base spaces that should give smooth three-charge geometries.

The classification of the generalized hyper-Kähler manifolds that we use here is far from known. However, one may not need the metrics to do interesting physics. We are proposing that the black hole microstates are described by a foam of nontrivial S^2's in a four-dimensional base. One might be able to do some statistical analysis of such a foam, perhaps using toric geometry, to see if one can describe the macroscopic, bulk ''state functions'' of the black hole. It is also interesting to investigate the transitions between different geometries, nucleation of GH points, instantons for such transitions, and probabilities of transition. Presumably nucleation is easy for small fluxes and small GH charges, but there should be some kind of correspondence limit in which large, classical bubbling supertubes, which involve only two GH points with very large Q's, should be relatively stable and not decay into a foam.

It is interesting to note that ideas of space-time foam have made regular appearances in the discussion of quantum gravity, see, for example [50]. In a very general sense, what we are proposing here is in a similar spirit to the ideas of [50]: Space-time on small scales becomes a topological foam. Here, however, we have managed to find it as a limit of supersymmetric D-brane physics, and with this comes a great deal more computational control of the problem. It is also important to note that the same physical ideas that led to the idea that space-time becomes foamy near the Planck scale also come into play here. At the Planck scale, even in empty space, there are virtual black holes, and we are proposing that their microstates be described by foams of two-spheres that will be hyper-Kähler only for BPS black holes. Consistency would therefore suggest that these virtual black holes should really be virtual fluctuations in bubbling hyper-Kähler geometries. Therefore, even empty space should be thought of in terms of some generalization of the foamy geometries considered here. Obviously our geometric description will break down at the Planck scale, but the picture is still rather interesting, and it is certainly supported by the fact that large bubbles are needed to resolve singularities in macroscopic supertubes and black rings.

There are evidently many things to be tested and lots of interesting things that might be done, but we believe that we have made important progress by resolving the supertube singularity and thereby giving a semiclassical description of black-hole microstates that may also give new insight into the structure of space-time on very small scales.

ACKNOWLEDGMENTS

We would like to thank Eric Gimon, Jan Gutowski, Clifford Johnson, Per Kraus, Jim Liu, Don Marolf, Harvey Reall and Radu Roiban or interesting discussions. The work of I. B. is supported in part by the NSF Grant No. PHY-00-99590. The work of N. W. is supported in part by the DOE Grant No. DE-FG03-84ER-40168.

[1] O. Lunin and S. D. Mathur, Nucl. Phys. **B623**, 342 (2002).
[2] S. D. Mathur, A. Saxena, and Y. K. Srivastava, Nucl. Phys. **B680**, 415 (2004).
[3] S. D. Mathur, Fortschr. Phys. **53**, 793 (2005).
[4] O. Lunin, J. High Energy Phys. 04 (2004) 054.
[5] S. Giusto, S. D. Mathur, and A. Saxena, Nucl. Phys. **B701**, 357 (2004).
[6] S. Giusto, S. D. Mathur, and A. Saxena, Nucl. Phys. **B710**, 425 (2005).
[7] S. Giusto and S. D. Mathur, Nucl. Phys. **B729**, 203 (2005).
[8] V. Jejjala, O. Madden, S. F. Ross, and G. Titchener, Phys. Rev. D **71**, 124030 (2005).
[9] I. Bena and P. Kraus, Phys. Rev. D **70**, 046003 (2004).
[10] D. Mateos and P. K. Townsend, Phys. Rev. Lett. **87**, 011602 (2001).
[11] I. Bena and N. P. Warner, hep-th/0408106.
[12] J. B. Gutowski and H. S. Reall, J. High Energy Phys. 04 (2004) 048.
[13] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, Classical Quantum Gravity **20**, 4587 (2003).
[14] I. Bena, Phys. Rev. D **70**, 105018 (2004).
[15] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, Phys. Rev. Lett. **93**, 211302 (2004).
[16] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, Phys. Rev. D **71**, 024033 (2005).
[17] J. P. Gauntlett and J. B. Gutowski, Phys. Rev. D **71**, 045002 (2005).
[18] I. Bena and P. Kraus, J. High Energy Phys. 12 (2004) 070.
[19] M. Cyrier, M. Guica, D. Mateos, and A. Strominger, Phys. Rev. Lett. **94**, 191601 (2005).
[20] I. Bena, C. W. Wang, and N. P. Warner, J. High Energy Phys. 03 (2006) 015.
[21] G. T. Horowitz and H. S. Reall, Classical Quantum Gravity **22**, 1289 (2005).
[22] I. Bena and P. Kraus, Phys. Rev. D **72**, 025007 (2005).
[23] J. P. Gauntlett and J. B. Gutowski, Phys. Rev. D **71**, 025013 (2005).
[24] P. Kraus and F. Larsen, Phys. Rev. D **72**, 024010 (2005).
[25] D. Gaiotto, A. Strominger, and X. Yin, J. High Energy Phys. 02 (2006) 023.
[26] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, J. High Energy Phys. 08 (2005) 042.
[27] I. Bena, P. Kraus, and N. P. Warner, Phys. Rev. D **72**, 084019 (2005).
[28] D. Marolf and A. Virmani, J. High Energy Phys. 11 (2005) 026.
[29] F. Larsen, J. High Energy Phys. 10 (2005) 100.
[30] R. Gopakumar and C. Vafa, Adv. Theor. Math. Phys. **3**, 1415 (1999).
[31] I. R. Klebanov and M. J. Strassler, J. High Energy Phys. 08 (2000) 052.
[32] C. Vafa, J. Math. Phys. (N.Y.) **42**, 2798 (2001).
[33] H. Lin, O. Lunin, and J. Maldacena, J. High Energy Phys. 10 (2004) 025.
[34] M. Cvetic and F. Larsen, Nucl. Phys. **B531**, 239 (1998).
[35] O. Lunin and S. D. Mathur, Nucl. Phys. **B610**, 49 (2001).
[36] O. Lunin, J. Maldacena, and L. Maoz, hep-th/0212210.
[37] P. Berglund, E. G. Gimon, and T. S. Levi, hep-th/0505167.
[38] I. Bena, Phys. Rev. D **62**, 126006 (2000).
[39] C. N. Pope and N. P. Warner, J. High Energy Phys. 04 (2004) 011.
[40] I. Bena and N. P. Warner, J. High Energy Phys. 12 (2004) 021.
[41] G. W. Gibbons and P. J. Ruback, Commun. Math. Phys. **115**, 267 (1988).
[42] J. Polchinski and M. J. Strassler, hep-th/0003136.
[43] I. Bena and D. J. Smith, Phys. Rev. D **71**, 025005 (2005).
[44] F. Denef, J. High Energy Phys. 08 (2000) 050.
[45] B. Bates and F. Denef, hep-th/0304094.
[46] F. Denef, J. High Energy Phys. 10 (2002) 023.
[47] D. Gaiotto, A. Strominger, and X. Yin, J. High Energy Phys. 02 (2006) 024.
[48] A. Iqbal, N. Nekrasov, A. Okounkov, and C. Vafa, hep-th/0312022.
[49] V. Balasubramanian, V. Jejjala, and J. Simon, Int. J. Mod. Phys. D **14**, 2181 (2005); V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, J. High Energy Phys. 12 (2005) 006.
[50] S. W. Hawking, Nucl. Phys. **B144**, 349 (1978).