

Hamiltonian reduction and supersymmetric mechanics with Dirac monopoleStefano Bellucci,¹ Armen Nersessian,^{2,3} and Armen Yeranyan²¹*INFN-Laboratori Nazionali di Frascati, P.O. Box 13, I-00044, Italy*²*Yerevan State University, Alex Manoogian St., 1, Yerevan, 375025, Armenia*³*Artsakh State University, Stepanakert & Yerevan Physics Institute, Yerevan, Armenia*

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We apply the technique of Hamiltonian reduction for the construction of three-dimensional $\mathcal{N} = 4$ supersymmetric mechanics specified by the presence of a Dirac monopole. For this purpose we take the conventional $\mathcal{N} = 4$ supersymmetric mechanics on the four-dimensional conformally-flat spaces and perform its Hamiltonian reduction to three-dimensional system. We formulate the final system in the canonical coordinates, and present, in these terms, the explicit expressions of the Hamiltonian and supercharges. We show that, besides a magnetic monopole field, the resulting system is specified by the presence of a spin-orbit coupling term. A comparison with previous work is also carried out.

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I. INTRODUCTION

The Hamiltonian reduction appears as an effective procedure for studying the qualitative properties of classical systems. Also, it is one of the most powerful methods for the construction of nontrivial integrable systems in classical mechanics. In fact, all known integrable models of classical mechanics, including multiparticle ones, could be obtained by an appropriate Hamiltonian reduction from higher-dimensional trivial integrable mechanical systems (free-particle and oscillator) [1]. A specific, particular case of the Hamiltonian reduction is the reduction of four-dimensional mechanical systems by the Hamiltonian action of the $U(1)$ group, which yields the three-dimensional mechanical systems specified by the presence of Dirac monopole. The best known application of this procedure is the construction of the so-called MIC-Kepler system (which is the generalization of the three-dimensional Coulomb problem specified by the presence of Dirac monopole [2]) from the four-dimensional oscillator [3]. In a similar way the generalization of the MIC-Kepler system on the three-dimensional hyperboloid has been constructed from the oscillator on the four-dimensional sphere and hyperboloid (this system has been suggested in [4]) [5] and from the oscillator on two-dimensional complex projective space $\mathbb{C}P^2$ and Lobachevsky space \mathcal{L}_2 [6]. Notice that the appearance in the reduced system of the Dirac monopole is the result of this specific reduction procedure, and it has no any direct relation with the structure of the initial Hamiltonian. Particularly, one can apply this reduction procedure to the supersymmetric Hamiltonian systems, and, reducing the number of its bosonic variables, to obtain the three-dimensional supersymmetric Hamiltonian system with Dirac monopole. Such an approach to the construction of three-dimensional supersymmetric mechanics looks quite attractive. This is because standard (and powerful) approaches to the construction of the systems with extended supersymmetries are based on the superfield technique. The latter is related

with the complex structures, and, as a consequence, the configuration spaces of the corresponding supersymmetric mechanics are Kähler or quaternionic spaces. For example, the $\mathcal{N} = 4$ supersymmetric mechanics constructed by the use of chiral superfields, has $(n|2n)_{\mathbb{C}}$ -dimensional configurational superspace, underlied by the the Kähler manifold (see, e.g. [7]), and the $\mathcal{N} = 8$ supersymmetric mechanics constructed by the use of chiral superfields, has $((n|4n)_{\mathbb{C}}$ -dimensional configurational superspace, underlied by the special Kähler manifold [8]. The $\mathcal{N} = 4$ supersymmetric mechanics constructed by the use of the so-called “root” supermultiplets [9] possesses a $(2n, 2n)_{\mathbb{C}}$ -dimensional configuration space, which is conformally-flat for $n = 1$ [10]. There exists the model of $\mathcal{N} = 8$ supersymmetric mechanics with $(2n, 4n)_{\mathbb{C}}$ -dimensional configuration space, which is also conformally-flat for $n = 1$ [11] (the $n > 1$ case has been suggested in [12]). Also, one can increase the number of supersymmetries, passing from Kähler spaces to hyper-Kähler ones, without expanding the number of fermionic degrees of freedom [13,14]. Although supersymmetric mechanics with a Dirac monopole is known in the literature (see, e.g. [15–17] and references therein), they where found, in some sense, occasionally. While the regular superfield approach to supersymmetric mechanics does not give the way to incorporate in the system the interaction with external gauge fields without breaking supersymmetry. Probably, the only exceptions are the three-dimensional $\mathcal{N} = 4$ superconformal mechanics [18] and the two-dimensional $\mathcal{N} = 4, 8$ supersymmetric mechanics constructed within the “non-linear chiral superfield” approach [19]. However, it is unclear how to construct nontrivial higher-dimensional analogs of these systems. The Hamiltonian reduction could be useful also in this subject, i.e. in the construction of the even-dimensional supersymmetric mechanics interacting with gauge fields.

The idea to construct supersymmetric mechanics by the Hamiltonian reduction is, in some sense, part of the physics

folklore. Explicitly it was written down, e.g., in [20], and exemplified there by the concrete example of one-dimensional $\mathcal{N} = 4$ supersymmetric mechanics, constructed by the reduction of the two-dimensional one based on a chiral superfield (this reduction was performed for the first time in [21]). Naturally, there is no interaction with nontrivial gauge fields in this system. Another example is the five-dimensional supersymmetric mechanics constructed in [22]. Let us also mention the old paper [23], where the complex projective superspaces were constructed as reduced phase spaces of super-Hamiltonian systems. The Hamiltonian reduction seems to be a natural procedure for the construction of supersymmetric mechanics including the interaction with external gauge fields from higher-dimensional supersymmetric systems (without external gauge fields) constructed within the superfield approach.

In the present paper we demonstrate this fact on the simple case of the reduction of the $(2|2)_{\mathbb{C}}$ -dimensional $\mathcal{N} = 4$ supersymmetric mechanics with conformally-flat configuration space to the three-dimensional system. In some sense, the content of the presented paper can be considered the Hamiltonian counterpart of an earlier work [10]. There it was considered the *Lagrangian* reduction of the four-dimensional $\mathcal{N} = 4$ supersymmetric mechanics mechanics constructed by the use of the root supermultiplet to the three- and two-dimensional systems. It was also observed there that the resulting two-dimensional system coincides with that constructed by the use of the nonlinear chiral multiplet. The appearance of the Dirac monopole field has been detected in the three-dimensional system, and that of the constant magnetic field was seen in the two-dimensional one.

However, the present paper contains some new features. Our resulting system is formulated purely in three-dimensional terms and canonical Poisson brackets, so that passing to supersymmetric *quantum* mechanics is straightforward here. This formulation allows us to clarify the nature of the resulting system. Particularly, we indicate the appearance of the spin-orbit coupling term there. Even when the configuration space of the reduced system is the Euclidean space, and in the absence of a magnetic monopole field, the Hamiltonian of the system contains nonzero fermionic terms. Also, in contrast with [10], we get the supercharges of the reduced system as well, and find that they possess a quite unusual structure, which seems not to be predictable from current intuition. Finally, in our consideration the odd coordinates of the reduced system are singular in the coordinate origin only, in contrast with [10], where they are singular in the ‘‘Dirac string’’, i.e. on a semiaxis. As a consequence, in the previous consideration, the reduced supercharges and the Poisson brackets are also singular on the Dirac string, whereas in the present picture all the ingredients have singularities on the coordinate origin only.

Before going into details, let us briefly present our procedure of Hamiltonian reduction. Let the initial phase superspace be parameterized by the local complex coordinates $(z^\alpha; \pi_\alpha; \eta^\alpha)$, $\alpha, \beta = 1, 2$. For the Hamiltonian reduction by the action of the constant of motion J_0 , we should find another set of coordinates $(x_i, u; p_i; \xi^\alpha)$, where

$$\{J_0, p_i\} = \{J_0, x_i\} = \{J_0, \xi^\alpha\} = 0, \quad i, j, k = 1, 2, 3. \quad (1)$$

The latter coordinate u , necessarily has a nonzero Poisson bracket with \mathcal{J} (because the Poisson brackets are non-degenerate) $\{u, J_0\} \neq 0$. Then, we immediately get that in these coordinates the Hamiltonian is independent of u

$$\{J_0, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial u} \cdot \{u, J_0\} \neq 0, \Rightarrow \mathcal{H} = \mathcal{H}(J_0, x_i, p_i, \xi^\alpha). \quad (2)$$

From the Jacobi identity we get that all Poisson brackets for the phase superspace coordinates (x_i, p_i, ξ^α) are also independent on u . Since J_0 is a constant of motion, we can fix its value $J_0 = c$, and describe the system in terms of the local coordinates (x_i, p_i, ξ^α) only. In this way we shall reduce the initial super-Hamiltonian system with a $(8|8)_{\mathbb{R}}$ -dimensional phase superspace to the system with a $(6|8)$ -dimensional one. Geometrically, this Hamiltonian reduction means that we fix, in the phase superspace, the $(7|8)_{\mathbb{R}}$ -dimensional level surface \mathcal{M}_γ by the $J_0 = c$, and then factorize it by the action of a vector field $\{J_0\}$, which is tangential to \mathcal{M}_γ . The resulting space $\mathcal{M}_r = \mathcal{M}_\gamma / \{J_0\}$ is a phase superspace of the reduced system.

II. FLAT CASE

Firstly, we consider the simplest case of a $\mathcal{N} = 4$ supersymmetric free particle with *four* fermionic degrees of freedom, moving on $\mathbb{R}^4 = \mathbb{C}^2$ equipped with a Euclidean metrics $ds^2 = dz^\alpha d\bar{z}^\alpha$.

This system can be conveniently described in terms of a $(4|2)_{\mathbb{C}}$ -dimensional phase superspace equipped with the canonical Poisson bracket. The latter is defined by the following nonzero relations (and their complex-conjugates):

$$\{\pi_\alpha, z^\beta\} = \delta_\alpha^\beta, \quad \{\bar{\eta}^\beta, \eta^\alpha\} = \delta^{\beta\alpha}. \quad (3)$$

In order to construct the system with $\mathcal{N} = 4$ superalgebra

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\delta_{\alpha\bar{\beta}}\mathcal{H}, \quad \{Q_\alpha, Q_\beta\} = 0, \quad (4)$$

we choose the following supercharges:

$$Q_1 = \pi_1 \eta^1 + \bar{\pi}_2 \bar{\eta}^2, \quad Q_2 = \pi_2 \eta^1 - \bar{\pi}_1 \bar{\eta}^2, \quad (5)$$

which obey the second equation in (4). Then, ‘‘squaring’’ these supercharges (with respect to the Poisson bracket) we get the $\mathcal{N} = 4$ supersymmetric Hamiltonian

$$\mathcal{H} = \bar{\pi}_\alpha \pi_\alpha / 2. \quad (6)$$

Let us notice that the supercharges look quite simple in the quaternionic notation

$$\begin{aligned} \mathcal{Q} &= Q_1 - jQ_2 = \boldsymbol{\pi}\boldsymbol{\eta}, & \boldsymbol{\pi} &= \pi_1 - j\pi_2, \\ \boldsymbol{\eta} &= \eta^1 + \eta^2 j. \end{aligned} \quad (7)$$

Clearly, the free-particle Hamiltonian and the supercharges are invariant under $U(1)$ rotations

$$\delta z^\alpha = \iota z^\alpha, \quad \delta \eta^\alpha = \iota \eta^\alpha, \quad \delta \pi_\alpha = -\iota \pi_\alpha, \quad (8)$$

given by the generator

$$\begin{aligned} J_0 &= \iota(z\pi - \bar{z}\bar{\pi}) + \iota\eta\bar{\eta}: \\ \{J_0, Q_\alpha\} &= \{J_0, \bar{Q}_\alpha\} = \{J_0, \mathcal{H}\} = 0. \end{aligned} \quad (9)$$

Hence, performing the Hamiltonian reduction by the action of J_0 , we shall get three-dimensional $\mathcal{N} = 4$ supersymmetric mechanics.

Also, we can define the generators of $SU(2)$ rotations acting separately on bosonic and fermionic variables, which also commute with the generator J_0

$$\begin{aligned} \mathcal{J}_i &= \frac{\iota(z\hat{\sigma}_i\pi - \bar{\pi}\hat{\sigma}_i\bar{z})}{2}: & \{J_0, \mathcal{J}_i\} &= 0, \\ \{\mathcal{J}_i, \mathcal{J}_j\} &= -\varepsilon_{ijk}\mathcal{J}_k, \end{aligned} \quad (10)$$

$$\delta_i z^\alpha = \frac{\iota}{2}(z\hat{\sigma}_i)^\alpha, \quad \delta_i \pi_\alpha = -\frac{\iota}{2}(\hat{\sigma}_i\pi)_\alpha, \quad \delta_i \eta^\alpha = 0, \quad (11)$$

and

$$\begin{aligned} R_i &= \frac{\iota}{2}\eta\sigma_i\bar{\eta}: & \{J_0, R_i\} &= 0, & \{R_i, R_j\} &= -\varepsilon_{ijk}R_k, \\ \delta_i z^\alpha &= \delta_i \pi_\alpha = 0, & \delta_i \eta^\alpha &= \frac{\iota}{2}(\eta\hat{\sigma}_i)^\alpha, \end{aligned} \quad (12)$$

where $\hat{\sigma}_i$ denote Pauli matrices. Notice that these $SU(2)$ generators commute with the Hamiltonian

$$\{\mathcal{J}_i, \mathcal{H}\} = 0, \quad \{R_i, \mathcal{H}\} = 0 \quad (13)$$

but do not commute with the supercharges

$$\begin{aligned} \{\mathcal{J}_i, Q_\alpha\} &= -\frac{\iota}{2}(\hat{\sigma}_i Q)_\alpha, & \{R_1 - \iota R_2, Q_\alpha\} &= -\iota\varepsilon_{\alpha\beta}\bar{Q}_\beta, \\ \{R_1 + \iota R_2, Q_\alpha\} &= 0, & \{R_3, Q_\alpha\} &= \frac{\iota}{2}Q_\alpha. \end{aligned} \quad (14)$$

Hence, performing their reduction by the Hamiltonian action of J_0 , we shall get the three-dimensional generators of $SU(2)$ rotations of $\mathcal{N} = 4$ supersymmetric mechanics which form, with the supercharges, a nontrivial superalgebra.

Now, let us perform the Hamiltonian reduction. For this purpose we should fix the $(7|4)_{\mathbb{R}}$ -dimensional level surface of the J_0 generator

$$J_0 = 2s, \quad (15)$$

and then factorize it by the $U(1)$ -group action given by the

tangent vector field $\{J_0, \cdot\}$. The resulting $(6|4)_{\mathbb{R}}$ -dimensional phase superspace could be parameterized by the following functions:

$$p_i = \frac{z\hat{\sigma}_i\pi + \bar{\pi}\hat{\sigma}_i\bar{z}}{2z\bar{z}}, \quad x_i = z\sigma_i\bar{z}, \quad (16)$$

and

$$\xi^1 = \frac{z^1\bar{\eta}^1 + \bar{z}^2\eta^2}{|z|}, \quad \xi^2 = \frac{z^2\bar{\eta}^1 - \bar{z}^1\eta^2}{|z|}, \quad (17)$$

which are clearly $U(1)$ -invariant

$$\{J_0, p_i\} = \{J_0, x_i\} = \{J_0, \xi^\alpha\} = 0. \quad (18)$$

The reduced bosonic coordinates (16) are exactly the same, which appear in the Kustaanheimo-Stiefel transformation of the four-dimensional bosonic systems. These coordinates could be supplemented with various choices of odd coordinates. Perhaps, the present choice of odd coordinates looks more clear in quaternionic terms

$$\boldsymbol{\xi} = \xi^1 + \xi^2 j = \frac{\bar{\boldsymbol{\eta}}\mathbf{z}}{|z|}, \quad \mathbf{z} = z^1 + z^2 j. \quad (19)$$

Calculating the Poisson brackets among these functions and restricting them on the level surface (15), we shall get the Poisson brackets on the reduced phase space

$$\begin{aligned} \{p_i, x_j\} &= \delta_{ij}, & \{p_i, p_j\} &= \varepsilon_{ijk}\left(s + \frac{\iota}{2r}\xi\hat{x}\bar{\xi}\right)\frac{x_k}{r^3}, \\ \{p_i, \xi^\alpha\} &= -\frac{\iota}{2r^2}\varepsilon_{ijk}x_j(\xi\hat{\sigma}_k)^\alpha, & \{\xi^\alpha, \bar{\xi}^\beta\} &= \delta^{\alpha\beta}. \end{aligned} \quad (20)$$

The reduced Hamiltonian and supercharges look as follows:

$$\begin{aligned} \mathcal{H}^{\text{red}} &= r\left[\frac{p_i^2}{2} + \frac{s^2}{2r^2} + s\frac{(\iota\xi\hat{x}\bar{\xi})}{2r^3} + \frac{(\iota\xi\bar{\xi})^2}{8r^2}\right], \\ Q_\alpha &= \sqrt{r}\left[p_r\bar{\xi}^\alpha - \iota\frac{(\hat{\mathcal{J}}\bar{\xi})_\alpha}{r}\right], \end{aligned} \quad (21)$$

while the reduced constants of motion (10) and (12) take the form

$$\begin{aligned} \mathcal{J}_i &= \varepsilon_{ijk}x_j p_k + \left(s + \frac{\iota}{2r}\xi\hat{x}\bar{\xi}\right)\frac{x_i}{r}, \\ R_+ = R_1 + \iota R_2 &= -\iota\xi\bar{\xi}_1\bar{\xi}_2, & R_3 &= -\frac{\iota\xi\bar{\xi}}{2}. \end{aligned} \quad (22)$$

Here and further below we use, for any A_i , the notation $\hat{A} \equiv A_i\hat{\sigma}_i$ and $A_r = A_i x_i / r$. The Poisson brackets of these generators with the coordinates of the reduced phase space look as follows:

$$\begin{aligned} \{\mathcal{J}_i, p_j\} &= -\varepsilon_{ijk}p_k, & \{\mathcal{J}_i, x_j\} &= -\varepsilon_{ijk}x_k, \\ \{\mathcal{J}_i, \xi^\alpha\} &= \frac{\iota}{2}(\xi\hat{\sigma}_i)^\alpha, \end{aligned} \quad (23)$$

$$\begin{aligned} \{R_i, x_j\} = \{R_i, p_j\} = 0, \quad \{R_+, \xi^\alpha\} = -i\varepsilon_{\alpha\beta}\bar{\xi}^\beta, \\ \{R_-, \xi^\alpha\} = 0, \quad \{R_3, \xi^\alpha\} = -\frac{l}{2}\xi^\alpha. \end{aligned} \quad (24)$$

In the given form the Hamiltonian has a canonical structure (in the sense that it is quadratic on momenta), but the Poisson brackets are noncanonical. For a better understanding of the structure of the system it is convenient, by a redefinition of momenta, to transform the Poisson bracket to the canonical (in the absence of Dirac monopole) one

$$P_i = p_i + \frac{l}{2r^2} \varepsilon_{ijk} x_j (\xi \hat{\sigma}_k \bar{\xi}): \quad (25)$$

$$\begin{aligned} \{P_i, x_j\} = \delta_{ij}, \quad \{P_i, P_j\} = s\varepsilon_{ijk} \frac{x_k}{r^3}, \\ \{P_i, \xi^\alpha\} = 0, \quad \{\xi^\alpha, \bar{\xi}^\beta\} = \delta^{\alpha\beta}. \end{aligned} \quad (26)$$

In these terms, the $so(3)$ generators \mathcal{J}_i take the form

$$\mathcal{J}_i \equiv J_i + \frac{l}{2} (\xi \hat{\sigma}_i \bar{\xi}), \quad J_i \equiv \varepsilon_{ijk} x_j P_k + s \frac{x_i}{r}, \quad (27)$$

and the Hamiltonian looks as follows:

$$\mathcal{H}^{\text{red}} = r \left[\frac{P_i^2}{2} + \frac{s^2}{2r^2} + \frac{(\xi \hat{J} \bar{\xi})}{2r^2} - \frac{(\xi \bar{\xi})^2}{8r^2} \right]. \quad (28)$$

The supercharges read

$$\begin{aligned} Q_\alpha = \sqrt{r} \left[P_r \bar{\xi}^\alpha - l \frac{(\hat{J} \bar{\xi})_\alpha}{r} \right] \\ = \sqrt{r} \left[\left(P_r + \frac{3l(\xi \bar{\xi})}{2r} \right) \bar{\xi}^\alpha - l \frac{(\hat{J} \bar{\xi})_\alpha}{r} \right]. \end{aligned} \quad (29)$$

Thus, we got the three-dimensional $\mathcal{N} = 4$ supersymmetric mechanics, specified by the presence of Dirac monopole. Its Hamiltonian, in contrast with the supercharges, looks quite simple. But, actually, this model is quite specific, since its configuration space is nonconstant, namely, it is equipped with the metric $ds^2 = (d\mathbf{r})^2/r$. Actually, it is not only a nonconstant space, but it has a conic singularity at the origin of the coordinates. However, one can construct, in a similar manner, the $\mathcal{N} = 4$ supersymmetric mechanics on a three-dimensional euclidean space, as well as on the generic three-dimensional conformally-flat spaces. For this purpose we should choose, as the initial system, the four-dimensional supersymmetric mechanics on conformally-flat spaces.

III. CONFORMALLY-FLAT CASE

Let us consider the reduction of the $\mathcal{N} = 4$ supersymmetric mechanics which lives on a four-dimensional space equipped with the conformally-flat metric

$$ds^2 = G(z, \bar{z}) dz^\alpha d\bar{z}^\alpha. \quad (30)$$

The $\mathcal{N} = 4$ supersymmetric mechanics is defined by the following Hamiltonian and supercharges:

$$\begin{aligned} Q_1 = \frac{1}{\sqrt{G}} (\Pi_1 \eta^1 + \bar{\Pi}_2 \bar{\eta}^2), \\ Q_2 = \frac{1}{\sqrt{G}} (\Pi_2 \eta^1 - \bar{\Pi}_1 \bar{\eta}^2), \end{aligned} \quad (31)$$

$$\begin{aligned} \mathcal{H} = \frac{(\Pi_\alpha - (\bar{\eta} \bar{\partial} \log G) \bar{\eta}^\alpha) (\bar{\Pi}_\alpha - (\eta \partial \log G) \eta^\alpha)}{2G} \\ - \frac{\partial_\alpha \bar{\partial}_\alpha G}{4G} (\iota \eta \bar{\eta})^2. \end{aligned} \quad (32)$$

Here

$$\Pi_\alpha \equiv \pi_\alpha + \iota \frac{\partial_\alpha \log G}{2} (\iota \eta \bar{\eta}).$$

In order to have the possibility to perform the Hamiltonian reduction by the generator J_0 (9), the metric (30) should be invariant under the transformation (8). This means that the conformal factor $g(z, \bar{z})$ has to depend solely on $U(1)$ invariant functions x_i , which are given by the expression (16): $G = G(x_i)$.

Repeating the Hamiltonian reduction procedure performed in the previous Section, we shall get the three-dimensional supersymmetric mechanics whose configuration space is equipped with the metric

$$ds^2 = g dx_i dx_i, \quad g \equiv \frac{G}{r}. \quad (33)$$

The connection components and scalar curvature of this metric look as follows:

$$\begin{aligned} \Gamma_{jk}^i = \Gamma_j \delta_{ik} + \Gamma_k \delta_{ij} - \Gamma_i \delta_{jk}, \\ \mathcal{R} = -\frac{4\partial_i \Gamma_i + 2\Gamma_i \Gamma_i}{g}, \quad \Gamma_i \equiv \frac{\partial_i g}{2g}. \end{aligned} \quad (34)$$

The reduced supercharges are given by the following expressions:

$$Q_\alpha = \frac{1}{\sqrt{g}} \left[\left(P_r + \iota \left(\Gamma_r + \frac{2}{r} \right) \Lambda_0 - \frac{(\vec{r} \times \vec{\Gamma}) \cdot \vec{\Lambda}}{r} \right) \bar{\xi}^\alpha - \iota \frac{(\hat{J} \bar{\xi})_\alpha}{r} \right]. \quad (35)$$

Here and in the following we use the notation and identities

$$\begin{aligned} \Lambda_0 = (\iota \xi \bar{\xi}), \quad \Lambda_i = (\iota \xi \hat{\sigma}_i \bar{\xi}): \quad \Lambda_i \Lambda_j = -\delta_{ij} \Lambda_0^2, \\ \Lambda_i (\hat{\sigma}_j \bar{\xi})_\alpha = (-\delta_{ij} \Lambda_0 + \iota \varepsilon_{ijk} \Lambda_k) \bar{\xi}^\alpha. \end{aligned} \quad (36)$$

The reduced Hamiltonian looks as follows:

$$\begin{aligned} \mathcal{H} = \frac{\vec{p}^2}{2g} + \frac{s^2}{2gr^2} + \frac{\vec{\Lambda} \cdot \vec{\mathcal{V}}}{g} - \left(\text{div} \vec{\Gamma} - \vec{\Gamma}^2 + \frac{2\Gamma_r}{r} + \frac{2}{r^2} \right) \\ \times \frac{\Lambda_0^2}{2g} \end{aligned} \quad (37)$$

where

$$\vec{\mathcal{V}}(\vec{P}, \vec{r}, s) = \frac{\vec{J}}{r^2} + \frac{2(\vec{J} \cdot \vec{\Gamma})\vec{r}}{r} + \vec{\Gamma} \times \vec{P} - \frac{s}{r}\vec{\Gamma} \quad (38)$$

For the $so(3)$ invariant metric, $g = g(r)$, the Hamiltonian takes a quite simple form

$$\mathcal{H} = \frac{\vec{P}^2}{2g} + \frac{s^2}{2gr^2} + \left(\Gamma + \frac{1}{r}\right)\frac{\vec{J} \cdot \vec{\Lambda}}{2rg} - \left(\Gamma' - \Gamma^2 + \frac{4\Gamma}{r} + \frac{2}{r^2}\right)\frac{\Lambda_0^2}{2g} \quad (39)$$

where $\Gamma \equiv \Gamma_r = d \log g(r)/(2dr)$, $\Gamma' = d\Gamma/dr$.

The above-obtained, explicit expressions for the supercharges (35) and the Hamiltonian (37) are the main results of our paper. Since the system is formulated in canonical coordinates, it could be immediately considered at the quantum mechanical level. For this purpose we should replace the Grassman coordinates ξ^α by the four-dimensional Euclidean gamma-matrices $\hat{\gamma}^\alpha = (\hat{\gamma}^\alpha + i\hat{\gamma}^{\alpha+2})/\sqrt{2}$, and the momenta variables P_i by the momenta operators $\hat{P}_i = -i\partial_i + sA_i(x)$ (where A_i is the potential of the Dirac monopole).

Let us draw the reader's attention to the presence of the spin-orbit coupling term $\vec{J} \cdot \vec{\Lambda}$ and the vanishing of the explicit dependence of the Hamiltonian from the monopole number s in the $so(3)$ symmetric case. Even in the Euclidean space ($g = 1$, $\Gamma = 0$), and in the absence of a magnetic monopole field ($s = 0$), the Hamiltonian has nonzero fermionic terms. Hence, these terms could be interpreted as an interaction energy of the neutral particle spin with the external field. Notice also, that the angular part of the constructed system is the two-dimensional mechanics obtained by the Lagrangian reduction of the initial system in [10].

IV. COMPARISON WITH PREVIOUS WORK

As we mentioned in the Introduction, the construction presented in this work is the Hamiltonian counterpart of the reduction performed in [10], but with a different choice of odd coordinates. Let us show which Poisson brackets arise in the original construction. Namely, let us choose, instead of (17), the following odd coordinates:

$$\xi^\alpha = \frac{\bar{z}^1}{|z^1|} \eta^\alpha \quad (40)$$

as it was suggested in [24].

In these terms the reduced Poisson brackets are defined by the relations

$$\begin{aligned} \{p_i, x_j\} &= \delta_{ij}, & \{p_i, p_j\} &= s\varepsilon_{ijk} \frac{x_k}{r^3} - \iota R_{ij\alpha\beta} \xi^\alpha \bar{\xi}^\beta, \\ \{p_i, \xi^\alpha\} &= \Gamma_{i\beta}^\alpha \xi^\beta, & \{\bar{\xi}^\beta, \xi^\alpha\} &= \delta^{\beta\alpha}. \end{aligned} \quad (41)$$

Here

$$\Gamma_{i\beta}^\alpha = \frac{\iota}{2} A_i \delta_{\alpha\beta}^i, \quad R_{ij\alpha\beta} = \frac{1}{2} F_{ij} \delta_{\alpha\beta}^i, \quad A_i = -\frac{\varepsilon_{ij3} x_j}{r(r+x_3)}, \quad (42)$$

i.e. F_{ij} and A_i are, respectively, the strength and the vector potential of the magnetic field of the Dirac monopole.

In contrast with (17), the functions (40) are singular in the line $z^1 = 0$, and, in terms of the reduced space, on the semiaxis $x_3 = -r$, i.e. on the ‘‘Dirac string’’. Thus, in order to cover the whole space, we should introduce another set of odd coordinates, $\bar{\xi}^\alpha = \frac{\bar{z}^2}{|z^2|} \eta^\alpha$, which are regular on the line $z^1 = 0$, but singular on $z^2 = 0$. These two sets of local coordinates are related as follows: $\bar{\xi}^\alpha = \exp(i\gamma)\xi^\alpha$, where $\exp(i\gamma) = \frac{z^1|z^2|}{z^2|z^1|}$, $\gamma \in [0, 4\pi)$. Upon this choice of odd coordinates, the vector potential A_i appearing in the Poisson brackets looks as follows: $A_i = \frac{\varepsilon_{ij3} x_j}{r(r-x_3)}$.

We could get the (twisted) canonical Poisson brackets (26) by the following redefinition of momenta:

$$P_i = p_i + \frac{1}{2} A_i(x) \Lambda_0. \quad (43)$$

In these terms the Hamiltonian is again given by the expression (21), i.e. it is free from singularities on the Dirac string. However, the supercharges remain singular. Moreover, on the intersection of the (super)charts the (two sets of) reduced supercharges are not equal to each other, but differ in the phase factor, which has no impact on the Hamiltonian. In other words, the supercharges are not scalar functions in this picture.

An important remark is that even for $s = 0$, i.e. in the absence of Dirac monopole, this singularity appears in the reduced Poisson bracket. Nevertheless, this choice of coordinates is appropriate if we reduce ourselves to the two-dimensional system, as it was done in [10]. Indeed, this reduction assumes the choice of the bosonic coordinate $w = z^1/z^2$; hence, the resulting system has the topology of sphere $S^2 = \mathbb{C}P^1$ and, consequently, it is covered by two charts. Also, the odd coordinates (40) are quite convenient, when we reduce to three dimensions the $\mathcal{N} = 4$ supersymmetric mechanics on Kähler spaces. We are planning to present these systems elsewhere.

V. CONCLUSION

In this paper we performed the Hamiltonian reduction of the simplest, $(2|2)_{\mathbb{C}}$ -dimensional $\mathcal{N} = 4$ supersymmetric mechanics with flat and conformally-flat configuration spaces to the $(3|4)_{\mathbb{R}}$ dimensional ones with flat and conformally-flat phase configuration spaces. We formulated the system in canonical coordinates, so that it could be immediately considered at the quantum mechanical level. Let us mention the appearance, in the reduced system, of the Dirac monopole magnetic field, and of a specific spin-orbit interaction term mixing the momenta and Grassmann variables. Further reduction of this system to three dimensions should yield a system where the spin-

orbit coupling term still appears, but the Dirac monopole field is transformed into some nonsingular magnetic field (including, as a particular case, the constant magnetic field). Hence, the constructed system could have an application in condensed matter physics. For example, one can hope that it will be useful in the study of the spin-Hall effect, which was observed experimentally very recently [25]. This phenomenon has been proposed to occur, as a result of the spin-orbit coupling term of the electron in the initial Hamiltonian. We recall that the classical Hall effect arises physically from a velocity dependent force, such as the Lorentz force, whereas another velocity dependent force in condensed matter systems is the SO coupling force [26,27]. Thus, in finite-size electron systems the presence of some kind of spin-Hall effect can be due to the interplay between the spin-orbit coupling (generating a kind of Lorentz force) and the edge of the device [28–32], analogously to what happens in the Hall effect.

We have found that the constructed system has a quite unusual structure of supercharges, and it is free of singularities (except for the one in the coordinate origin). It seems that the constructed system is the generalization of the the quantum mechanics suggested in [16] to curved spaces. We have restricted ourselves to the reduction of $\mathcal{N} = 4$ supersymmetric mechanics, though, in a completely similar way, one can construct the three-dimensional $\mathcal{N} = 8$ supersymmetric mechanics, reducing the four-dimensional system suggested in [11]. Also, one can apply the same technique to the $\mathcal{N} = 4, 8$ four-dimensional supersymmetric mechanics on Kähler spaces. In particular, in this way one can construct the $\mathcal{N} = 4$ supersymmetric (repulsive) MIC-Kepler system, performing the Hamiltonian reduction of the $\mathcal{N} = 4$ supersymmetric particle on the Taub-NUT space with negative mass.

The connection between the corresponding bosonic systems has been established in [33].

It also appears that the procedure of Hamiltonian reduction could explain the freedom in the fermion-boson coupling observed in two-dimensional systems with nonlinear chiral multiplet [34]. These works are currently in progress and will be published elsewhere.

Another perspective development is in the construction of the five-dimensional supersymmetric mechanics specified by the presence of a $SU(2)$ Yang monopole (instanton) from the eight-dimensional supersymmetric systems (without monopoles). For this purpose one should perform the Hamiltonian reduction by the $SU(2)$ group action, related with the second Hopf map. The bosonic counterpart of this procedure is widely known in classical and quantum mechanics. For example, by means of such a reduction procedure the five-dimensional Coulomb problem with $SU(2)$ Yang monopole has been constructed from the eight-dimensional oscillator in [35]. Extending this procedure to the supersymmetric system on the eight-dimensional conformally-flat case, one can construct the expected five-dimensional supersymmetric system.

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