Non-Abelian vortices of higher winding numbers

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We make a detailed study of the moduli space of winding number two ($k = 2$) axially symmetric vortices (or equivalently, of coaxial composite of two fundamental vortices), occurring in $U(2)$ gauge theory with two flavors in the Higgs phase, recently discussed by Hashimoto and Tong and by Auzzi, Shifman, and Yung. We find that it is a weighted projective space $W\mathbf{C}P^2_{(2,1,1)} \simeq \mathbf{C}P^2/\mathbf{Z}_2$. This manifold contains an A_1-type ($\mathbf{Z}_2$) orbifold singularity even though the full moduli space including the relative position moduli is smooth. The $SU(2)$ transformation properties of such vortices are studied. Our results are then generalized to $U(N)$ gauge theory with N flavors, where the internal moduli space of $k = 2$ axially symmetric vortices is found to be a weighted Grassmannian manifold. It contains singularities along a submanifold.

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I. INTRODUCTION

Vortices have played important roles in various areas of fundamental physics since their discovery [1,2]. A particularly interesting type of vortices are the ones possessing an exact, continuous non-Abelian flux moduli (called non-Abelian vortices below), found recently [3,4]. A motivation for studying such non-Abelian vortices is that a monopole is confined in the Higgs phase by related vortices [5–10], so that such systems provide a dual model of color confinement of truly non-Abelian kind. Another motivation could come from the interest in the physics of cosmic strings. Many papers on non-Abelian vortices appeared lately [11–22], where the discussion often encompasses the context of more general soliton physics, involving domain walls, monopoles and vortices, or composite thereof.

The moduli space of non-Abelian vortices was obtained in certain D-brane configuration in string theory [3] as well as in a field theory framework [15]. In particular, the moduli subspace of $k = 2$ axially symmetric vortices was studied by two groups: Hashimoto and Tong (HT) [14] and Auzzi, Shifman, and Yung (ASY) [16]. The former concluded that it is $\mathbf{C}P^2$ by using the brane construction [3] whereas the latter found $\mathbf{C}P^2/\mathbf{Z}_2$ by using a field theoretical construction. This discrepancy is crucial when one discusses the reconnection of vortices because the latter contains an orbifold singularity. The study of non-Abelian vortices of higher winding numbers (or equivalently, of composite vortices) can be important in the understanding of confinement mechanism, or in the detailed model study of cosmic string interactions. Motivated by these considerations, we study in this paper the moduli space of $k = 2$ axially symmetric non-Abelian vortices of $U(N)$ gauge theories, by using the method of moduli matrix [15], which was originally introduced in the study of domain walls in [21] (see [22] for a review).

This paper is organized as follows. In Sec. II we briefly review the non-Abelian vortices and their moduli matrix description in the context of most frequently discussed models: $U(N)$ gauge theory with N flavors of scalar quarks, and discuss, in the case of $U(2)$ model, the transformation properties of the fundamental $k = 1$ vortices. The moduli space and the transformation properties of $k = 2$ coaxial vortices are studied in Sec. III, where we reproduce the results by Hashimoto and Tong, and by Auzzi, Shifman and Yung, and resolve the apparent discrepancy between their results. We generalize our results on $k = 2$ coaxial vortices to analogous vortices of $U(N)$ theories in Sec. IV. In Appendices A and B we present detailed comparison between our results and those by Hashimoto and Tong, and by Auzzi, Shifman, and Yung. In Appendix C, we show the relation between the Kähler quotient construction and the moduli matrix approach. In Appendix D we give a simple method to construct the moduli matrix for vortices of

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higher winding number as a product of those for fundamental vortices.

II. NON-ABELIAN VORTICES

A. Vortex equations

Our model is an $U(N)_{\rm G}$ gauge theory coupled with N Higgs fields in the fundamental representation, denoted by an N by N complex (color-flavor) matrix H. The Lagrangian is given by

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \mathcal{D}_\mu H \mathcal{D}^\mu H^\dagger - \lambda(c\mathbf{1}_N - HH^\dagger)^2\right], \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\nu + i[W_\mu, W_\nu]$ and $\mathcal{D}_\mu H = (\partial_\mu + iW_\mu)H$. g is the $U(N)_{\rm G}$ gauge coupling and λ is a scalar quartic coupling. In this article we shall restrict ourselves to the critical case, $\lambda = \frac{g^2}{4}$ (BPS limit): in this case the model can be regarded as the bosonic sector of a supersymmetric gauge theory. In the supersymmetric context the constant c is the Fayet-Iliopoulos parameter. In the following we set $c > 0$ to ensure stable vortex configurations. This model has an $SU(N)_{\rm F}$ flavor symmetry acting on H from the right while the $U(N)_{\rm G}$ gauge symmetry acts on H from the left. The vacuum of this model, determined by $HH^\dagger = c\mathbf{1}_N$, is unique up to a gauge transformation and is in the Higgs phase. The $U(N)_{\rm G}$ gauge symmetry is completely broken. This vacuum preserves a global unbroken symmetry $SU(N)_{\rm G+F}$ (the color-flavor locking phase).

This system admits the Abrikosov-Nielsen-Olesen (ANO) type of vortices [1,2] which saturate Bogomol'nyi bound. The equations of motion reduce to the first order non-Abelian vortex equations [3,4,15,22]:

$$(\mathcal{D}_1 + i\mathcal{D}_2)H = 0, \qquad F_{12} + \frac{g^2}{2}(c\mathbf{1}_N - HH^\dagger) = 0. \tag{2.2}$$

It turns out that the matter equation can be solved by

$$H = S^{-1}(z,\bar{z})H_0(z), \quad W_1 + iW_2 = -2iS^{-1}(z,\bar{z})\bar{\partial}_z S(z,\bar{z}). \tag{2.3}$$

where the elements of the N by N moduli matrix $H_0(z)$ are holomorphic functions of the complex coordinate $z \equiv x^1 + ix^2$ [11,15,22], and S is an N by N matrix invertible over the whole z-plane. For any given $H_0(z)$, S is uniquely determined up to a gauge transformation by the second equation in Eq. (2.2). The physical fields H and W are obtained by plugging the solution S back into Eq. (2.3). Each element of the matrix $H_0(z)$ must be a polynomial of z in order to satisfy the boundary condition, $\det H_0(z) = \mathcal{O}(z^k)$, k being the vortex (winding) number.

A great advantage of the method lies in the fact that all the integration constants of the BPS equations Eq. (2.2)—moduli parameters—are encoded in the moduli matrix $H_0(z)$ as various coefficients of the polynomials, justifying its name [15,22]. The zeros $\{z_i\}$ of

$$\det H_0(z) \propto \prod_i (z - z_i)$$

can be interpreted as the positions of the component vortices, when they are sufficiently far apart from each other; vice versa, when they overlap significantly, z_i's have no clear physical meaning as the center of each component vortex: they are just part of the moduli parameters, characterizing the shape and color-flavor orientation of the vortex under consideration.

Notice that the rank of H gets reduced by one at vortex positions z_i when all the vortices are separated, $z_i \neq z_j$ for $i \neq j$. A constant vector defined by

$$H|_{z=z_i}\vec{\phi}_i = 0 \tag{2.4}$$

is associated with each component vortex at $z = z_i$. An overall constant of $\vec{\phi}_i$ cannot be determined from Eq. (2.4) so we should introduce an equivalence relation "$\sim$," given by

$$\vec{\phi}_i \sim \lambda\vec{\phi}_i, \quad \text{with} \quad \lambda \in \mathbf{C}^*. \tag{2.5}$$

Thus, each vector $\vec{\phi}_i$ takes a value in the projective space $\mathbf{C}P^{N-1} = SU(N)/[SU(N-1)\times U(1)]$. This space can be understood as a space parameterized by Nambu-Goldstone modes associated with the symmetry breaking,

$$SU(N)_{\rm G+F} \to U(1)\times SU(N-1), \tag{2.6}$$

caused by the presence of a vortex [3,4,10,15,22]. We call ϕ_i the orientational vector.

The solutions Eq. (2.3) are invariant under

$$(H_0, S) \to (V(z)H_0, V(z)S) \tag{2.7}$$

with $V(z) \in GL(N, \mathbf{C})$ being holomorphic with respect to z. We call this V-transformation or V-equivalence relation. The moduli space of the vortex equations Eq. (2.2) is obtained as the quotient space $\mathcal{M}_{\rm total} = \{H_0(z)\}/GL(N,\mathbf{C})$. This space is infinite dimensional and can be decomposed into topological sectors according to the vortex number k. The k-th topological sector $\mathcal{M}_{N,k}$, the moduli space of k vortices, is determined by the condition that $\det H_0(z)$ is of order z^k:

$$\mathcal{M}_{N,k} \simeq \{H_0(z) \mid \det H_0(z) = \mathcal{O}(z^k)\}/\{V(z)\}. \tag{2.8}$$

B. Fundamental ($k = 1$) vortices

Let us first discuss a single non-Abelian vortex in $U(2)$ gauge theory. The condition on the moduli matrix H_0 is $\det H_0 = \mathcal{O}(z)$. Modulo V-equivalence relation Eq. (2.7), the moduli matrix can be brought to one of the following two forms [10]:

$$H_0^{(1,0)}(z) = \begin{pmatrix} z - z_0 & 0 \\ -b' & 1 \end{pmatrix}, \qquad H_0^{(0,1)}(z) = \begin{pmatrix} 1 & -b \\ 0 & z - z_0 \end{pmatrix} \tag{2.9}$$

with b, b' and z_0 complex parameters. Here z_0 gives the position moduli whereas b and b' give the orientational moduli as we see below. The two matrices in Eq. (2.9) describe the same single vortex configuration but in two different patches of the moduli space. Let us denote them $\mathcal{U}^{(1,0)} = \{z_0, b'\}$ and $\mathcal{U}^{(0,1)} = \{z_0, b\}$. The transition function between these patches is given, except for the point $b' = 0$ in the patch $\mathcal{U}^{(1,0)}$ and $b = 0$ in $\mathcal{U}^{(0,1)}$, by the V-transformation Eq. (2.7) of the form [10]

$$V = \begin{pmatrix} 0 & -1/b' \\ b' & z - z_0 \end{pmatrix} \in GL(2, \mathbf{C}). \tag{2.10}$$

This yields the transition function

$$b = \frac{1}{b'}, \qquad (b, b' \neq 0). \tag{2.11}$$

b' and b are seen to be the two patches of a $\mathbf{C}P^1$, leading to the conclusion that the moduli space of the single non-Abelian vortex is

$$\mathcal{M}_{N=2,k=1} \simeq \mathbf{C} \times \mathbf{C}P^1, \tag{2.12}$$

where the first factor $\mathbf{C}$ corresponds to the position z_0 of the vortex.

The same conclusion can be reached from the orientation vector, defined by $H_0(z = z_0)\vec{\phi} = 0$: $\vec{\phi}$ is given by

$$\vec{\phi} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} b \\ 1 \end{pmatrix}. \tag{2.13}$$

We see that the components of $\vec{\phi}$ are the homogeneous coordinates of $\mathbf{C}P^1$; b, b' are the inhomogeneous coordinates.[1]

The individual vortex breaks the color-flavor diagonal symmetry $SU(2)_{\mathrm{G+F}}$, so that it transforms nontrivially under it. The transformation property of the vortex moduli parameters can be conveniently studied by the $SU(2)_{\mathrm{F}}$ flavor transformations on the moduli matrix, as the color transformations acting from the left can be regarded as a V transformation. The flavor symmetry acts on H_0 as $H_0 \to H_0 U$ with $U \in SU(2)_{\mathrm{F}}$. A general $SU(2)$ matrix

$$U = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, \tag{2.14}$$

with $u, v \in \mathbf{C}$ satisfying $|u|^2 + |v|^2 = 1$, acts for instance on $H_0^{(0,1)}$ in Eq. (2.9) as

$$H_0^{(0,1)}(z) \to H_0^{(0,1)}(z)U = \begin{pmatrix} u + v^* b & -u^* b + v \\ -v^*(z - z_0) & u^*(z - z_0) \end{pmatrix}. \tag{2.15}$$

The right hand side should be pulled back to the form $H_0^{(0,1)}$ in Eq. (2.9) by using an appropriate V-transformation Eq. (2.7). This can be achieved by

$$V_U H_0^{(0,1)}(z) U = \begin{pmatrix} 1 & -\frac{u^* b - v}{v^* b + u} \\ 0 & z - z_0 \end{pmatrix}, \qquad V_U = \begin{pmatrix} (u + v^* b)^{-1} & 0 \\ v^*(z - z_0) & u + v^* b \end{pmatrix} \in GL(2, \mathbf{C}). \tag{2.16}$$

The $SU(2)_{\mathrm{G+F}}$ transformation law of b is then

$$b \to \frac{u^* b - v}{v^* b + u}, \tag{2.17}$$

which is the standard $SU(2)$ transformation law of the inhomogeneous coordinate of $\mathbf{C}P^1$.[2]

In terms of the orientational vector $\vec{\phi}$ in Eq. (2.13), the transformation law Eq. (2.17) can be derived more straightforwardly. According to the definition Eq. (2.4), $\vec{\phi}$ is transformed in the fundamental representation of $SU(2)_{\mathrm{F}}$:

$$\vec{\phi} \to U^\dagger \vec{\phi}, \qquad U \in SU(2)_{\mathrm{F}}. \tag{2.18}$$

III. $k = 2$ VORTICES IN $U(2)$ GAUGE THEORY

A. Moduli space of $k = 2$ vortices

Configurations of $k = 2$ vortices at arbitrary positions are given by the moduli matrix whose determinant has degree two, $\det H_0 = \mathcal{O}(z^2)$. By using V-transformations Eq. (2.7) the moduli matrix satisfying this condition can be brought into one of the following three forms [15,22]

$$\begin{aligned} H_0^{(2,0)} &= \begin{pmatrix} z^2 - \alpha' z - \beta' & 0 \\ -a' z - b' & 1 \end{pmatrix}, \\ H_0^{(1,1)} &= \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z - \tilde{\phi} \end{pmatrix}, \\ H_0^{(0,2)} &= \begin{pmatrix} 1 & -az - b \\ 0 & z^2 - \alpha z - \beta \end{pmatrix}. \end{aligned} \tag{3.1}$$

These define the three patches $\mathcal{U}^{(2,0)} = \{a', b', \alpha', \beta'\}$, $\mathcal{U}^{(1,1)} = \{\phi, \tilde{\phi}, \eta, \tilde{\eta}\}$, $\mathcal{U}^{(0,2)} = \{a, b, \alpha, \beta\}$ of the moduli space $\mathcal{M}_{N=2,k=2}$. The transition from $\mathcal{U}^{(1,1)}$ to $\mathcal{U}^{(0,2)}$ is given via the V-transformation

$$V = \begin{pmatrix} 0 & -1/\tilde{\eta} \\ \tilde{\eta} & z - \phi \end{pmatrix}:$$

[1] Similarly, in the case of $U(N)$ gauge theory, the components of $\vec{\phi}$ correspond to the homogeneous coordinates of $\mathbf{C}P^{N-1}$.

[2] The coordinate b is invariant (more precisely the orientational vector receives a global phase) under the $U(1)$ transformations generated by $\hat{\mathbf{n}} \cdot \vec{\sigma}/2$, $\hat{\mathbf{n}} = \frac{1}{(1+|b|^2)}(2\Re b, 2\Im b, |b|^2 - 1)$. This implies the coset structure $\mathbf{C}P^1 \simeq SU(2)/U(1)$.

$$a = \frac{1}{\tilde{\eta}}, \qquad b = -\frac{\tilde{\phi}}{\tilde{\eta}}, \qquad \alpha = \phi + \tilde{\phi}, \qquad \beta = \eta\tilde{\eta} - \phi\tilde{\phi}. \tag{3.2}$$

Similarly the transition from $\mathcal{U}^{(2,0)}$ to $\mathcal{U}^{(0,2)}$ is given by

$$V = \begin{pmatrix} \frac{-a'^2}{a'^2\beta' - a'b'\alpha' - b'^2} & \frac{-a'z + a'\alpha' + b'}{a'^2\beta' - a'b'\alpha' - b'^2} \\ a'z + b' & z^2 - \alpha' z - \beta' \end{pmatrix},$$

which yields

$$a = \frac{a'}{a'^2\beta' - a'b'\alpha' - b'^2}, \qquad b = -\frac{b' + a'\alpha'}{a'^2\beta' - a'b'\alpha' - b'^2}, \qquad \alpha = \alpha', \qquad \beta = \beta'. \tag{3.3}$$

Finally those between $\mathcal{U}^{(1,1)}$ and $\mathcal{U}^{(2,0)}$ are given by the composition of the transformations Eqs. (3.2) and (3.3). Let us now discuss the moduli space of $k = 2$ vortices separately for the cases where the two vortex centers are (1) distinct ($z_1 \neq z_2$), and (2) coincident ($z_1 = z_2$).

(1) At the vortex positions z_i the orientational vectors are determined by Eq. (2.4). The orientational vectors $\vec{\phi}_i$ ($i = 1, 2$) are then obtained by

$$\vec{\phi}_i \sim \begin{pmatrix} az_i + b \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_i - \tilde{\phi} \\ \tilde{\eta} \end{pmatrix} \sim \begin{pmatrix} \eta \\ z_i - \phi \end{pmatrix} \sim \begin{pmatrix} 1 \\ a'z_i + b' \end{pmatrix}. \tag{3.4}$$

The two ($i = 1, 2$) parameters defined by $b_i \equiv az_i + b$ (or $b'_i \equiv a'z_i + b'$) parameterize the two different $\mathbf{C}P^1$'s separately. Conversely if the two vortices are separated $z_1 \neq z_2$, then moduli parameters a, b are described by b_1, b_2 with positions of vortices z_1, z_2 as

$$a = \frac{b_1 - b_2}{z_1 - z_2}, \qquad b = \frac{b_2 z_1 - b_1 z_2}{z_1 - z_2}, \qquad \alpha = z_1 + z_2, \qquad \beta = -z_1 z_2 \tag{3.5}$$

(and similar relations for the primed variables). Thus in the case of separated vortices $\{b_1, b_2, z_1, z_2\}$ ($\{b'_1, b'_2, z_1, z_2\}$) can be taken as appropriate coordinates of the moduli space, instead of $\{a, b, \alpha, \beta\}$ ($\{a', b', \alpha', \beta'\}$). The transition functions are also obtained by applying the equivalence relation Eq. (2.5) to Eq. (3.4), for instance,

$$b_i = \frac{1}{b'_i} \qquad (b_i, b'_i \neq 0). \tag{3.6}$$

It can be shown that these are equivalent to Eqs. (3.3) by use of Eq. (3.5) and analogous relation for the primed parameters. The coordinates in $\mathcal{U}^{(1,1)}$ are also the orientational moduli. If we take $\{b_1, b'_2, z_1, z_2\}$ as a set of independent moduli and substitute Eq. (3.5) and $b_2 = 1/b'_2$ to Eq. (3.2), then we obtain, for $b_1 b'_2 \neq 1$

$$\phi = \frac{z_2 - b_1 b'_2 z_1}{1 - b_1 b'_2}, \qquad \eta = \frac{z_1 - z_2}{1 - b_1 b'_2} b_1, \qquad \tilde{\phi} = \frac{z_1 - b_1 b'_2 z_2}{1 - b_1 b'_2}, \qquad \tilde{\eta} = -\frac{z_1 - z_2}{1 - b_1 b'_2} b'_2. \tag{3.7}$$

It can be seen that the representation Eq. (3.5) implies that $\mathcal{U}^{(0,2)}$ and $\mathcal{U}^{(2,0)}$ are suitable for describing the situation when two orientational moduli are parallel or nearby. On the other hand, Eq. (3.7) implies that $\mathcal{U}^{(1,1)}$ is suitable to describe the situation when orientational moduli are orthogonal or close to such a situation, while not adequate for describing a parallel set. Therefore, the moduli space for two separated vortices are completely described by the positions and the two orientational moduli b_1, b_2, (b'_1, b'_2): the moduli space for the composite vortices in this case is given by [14,15]

$$\mathcal{M}^{\text{separated}}_{k=2,N=2} \simeq (\mathbf{C} \times \mathbf{C}P^1)^2/\mathfrak{S}_2, \tag{3.8}$$

where $\mathfrak{S}_2$ permutes the centers and orientations of the two vortices.

(2) We now focus on coincident (coaxial) vortices ($z_1 = z_2$), with the moduli space denoted by

$$\tilde{\mathcal{M}}_{N=2,k=2} \equiv \mathcal{M}_{N=2,k=2}|_{z_1=z_2}. \tag{3.9}$$

As an overall translational moduli is trivial, we set $z_1 = z_2 = 0$ without loss of generality. According to Eqs. (3.5) and (3.7), all points in the moduli space tend to the origin of $\mathcal{U}^{(1,1)}$ in the limit of $z_2 \to z_1$, as long as b_1 and b_2 take different values. A more careful treatment is needed in this case. In terms of the moduli matrix, the condition of coincidence is given by $\det H_0(z) = z^2$. We have

$$\{\alpha = 0, \beta = 0\}, \qquad \{\tilde{\phi} = -\phi, \phi\tilde{\phi} - \eta\tilde{\eta} = 0\}, \quad \text{and} \quad \{\alpha' = 0, \beta' = 0\}, \tag{3.10}$$

in $\mathcal{U}^{(2,0)}$, $\mathcal{U}^{(1,1)}$, and $\mathcal{U}^{(0,2)}$, respectively. $\tilde{\mathcal{M}}_{N=2,k=2}$ is covered by the reduced patches $\tilde{\mathcal{U}}^{(2,0)}$, $\tilde{\mathcal{U}}^{(1,1)}$, and $\tilde{\mathcal{U}}^{(0,2)}$, defined by the moduli matrices

$$H_0^{(2,0)} = \begin{pmatrix} z^2 & 0 \\ -a'z - b' & 1 \end{pmatrix}, \qquad H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z + \phi \end{pmatrix}, \qquad H_0^{(0,2)} = \begin{pmatrix} 1 & -az - b \\ 0 & z^2 \end{pmatrix}. \tag{3.11}$$

The following constraint exists among the coordinates in $\tilde{\mathcal{U}}^{(1,1)}$:

$$\phi^2 + \eta\tilde{\eta} = 0. \tag{3.12}$$

The transition functions between $\tilde{\mathcal{U}}^{(0,2)}$ and $\tilde{\mathcal{U}}^{(1,1)}$ are given by

$$a = \frac{1}{\tilde{\eta}}, \qquad b = \frac{\phi}{\tilde{\eta}} = -\frac{\eta}{\phi}, \tag{3.13}$$

and those between $\tilde{\mathcal{U}}^{(0,2)}$ and $\tilde{\mathcal{U}}^{(2,0)}$ by

$$a = -\frac{a'}{b'^2}, \qquad b = \frac{1}{b'}. \tag{3.14}$$

All the patches defined by Eq. (3.11) are parameterized by two independent complex parameters. The reduced patches $\tilde{\mathcal{U}}^{(2,0)}$ and $\tilde{\mathcal{U}}^{(0,2)}$ are locally isomorphic to $\mathbf{C}^2$ with a, b or a', b' being good coordinates. However, $\tilde{\mathcal{U}}^{(1,1)}$ suffers from the constraint Eq. (3.12) which gives the A_1-type ($\mathbf{Z}_2$) orbifold singularity at the origin and therefore $\tilde{\mathcal{U}}^{(1,1)} \simeq \mathbf{C}^2/\mathbf{Z}_2$ locally (See Eq. (3.17), below). Note that the moduli matrix $H_0(z)$ is proportional to the unit matrix at the singularity: $H_0(z) = z\mathbf{1}_2$. This implies that configurations of the physical fields (H and F_{12}) are also proportional to the unit matrix where the global symmetry $SU(2)_{\mathrm{G+F}}$ is fully recovered at that singularity. The full gauge symmetry is also recovered at the core of coincident vortices.

Remark: A brief comment on the orientational vectors. We could extract a part of moduli in the moduli matrix as the orientational vector at $z = 0$, as in the case of separated vortices discussed above:

$$\vec{\phi} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} \eta \\ -\phi \end{pmatrix} \sim \begin{pmatrix} \phi \\ \tilde{\eta} \end{pmatrix} \sim \begin{pmatrix} b \\ 1 \end{pmatrix}. \tag{3.15}$$

From the identification $\vec{\phi} \sim \lambda\vec{\phi}(\lambda \in \mathbf{C}^*)$ with the transition functions given in Eqs. (3.13) and (3.14), we find that the orientational moduli again parameterizes $\mathbf{C}P^1$. However, the orientational vectors in Eq. (3.15) are not sufficient to pick up all the moduli parameters in the moduli matrix H_0. For instance a is lost in the $\tilde{\mathcal{U}}^{(0,2)}$ patch. It is even ill-defined at the singular point, as $H_0^{(1,1)}(z = 0) = 0$.

To clarify the whole structure of the space $\tilde{\mathcal{M}}_{N=2,k=2}$, let us define new coordinates, solving the constraint Eq. (3.12)

$$XY \equiv -\phi, \qquad X^2 \equiv \eta, \qquad Y^2 \equiv -\tilde{\eta}. \tag{3.16}$$

This clarifies the structure of the singularity at the origin. The coordinates (X, Y) describe the patch $\tilde{\mathcal{U}}^{(1,1)}$ correctly *modulo* $\mathbf{Z}_2$ identification

$$(X, Y) \sim (-X, -Y). \tag{3.17}$$

Using the transition functions Eq. (3.13) and (3.14), the three local domains are patched together as in Table I. In terms of the new coordinates (X, Y), the orientational vector defined at $z = 0$ is given by

$$\vec{\phi} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} b \\ 1 \end{pmatrix} \tag{3.18}$$

with $\vec{\phi} \sim \lambda\vec{\phi}(\lambda \in \mathbf{C}^*)$. This equivalence relation recovers the transition functions between b, b' and (X, Y) in the Table I. These are coordinates on the $\mathbf{C}P^1$ as was mentioned above. But this $\mathbf{C}P^1$ is only a subspace of the moduli space $\tilde{\mathcal{M}}_{N=2,k=2}$.

The full space $\tilde{\mathcal{M}}_{N=2,k=2}$ can be made visible by attaching the remaining parameters a, a' to $\mathbf{C}P^1$. We arrange the moduli parameters in the three patches $\tilde{\mathcal{U}}^{(2,0)}$, $\tilde{\mathcal{U}}^{(1,1)}$ and $\tilde{\mathcal{U}}^{(0,2)}$ as

$$\begin{pmatrix} a' \\ 1 \\ b' \end{pmatrix} \sim \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} \sim \begin{pmatrix} -a \\ b \\ 1 \end{pmatrix}, \tag{3.19}$$

respectively, with the equivalence relation "$\sim$," *defined* by

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} \sim \begin{pmatrix} \lambda^2\phi_0 \\ \lambda\phi_1 \\ \lambda\phi_2 \end{pmatrix} \quad \text{with} \quad \lambda \in \mathbf{C}^*. \tag{3.20}$$

All the transition functions in Table I are then nicely reproduced. The equivalence relation Eq. (3.20) defines a *weighted complex projective space* with the weights (2, 1, 1). We thus conclude that the moduli space for the coincident (coaxial) $k = 2$ non-Abelian vortices is a weighted projective space,

$$\tilde{\mathcal{M}}_{N=2,k=2} \simeq W\mathbf{C}P^2_{(2,1,1)}. \tag{3.21}$$

While the complex projective spaces with common weights, $\mathbf{C}P^n$, are smooth, weighted projective spaces have singularities. In fact, we have shown that $\tilde{\mathcal{U}}^{(1,1)} \simeq \mathbf{C}^2/\mathbf{Z}_2$, and it has a conical singularity at the origin by $(1, X, Y) \sim (1, -X, -Y)$, whose existence was first pointed out by ASY [16]. The origin of the conical singularity can be seen clearly from the equivalence relation Eq. (3.20). As mentioned above the transition functions in Table I are reproduced via the equivalence relation Eq. (3.20). In fact, one finds that $\lambda = \frac{1}{X}$ gives $(\lambda^2, \lambda X, \lambda Y) = (a', 1, b')$ and $\lambda = \frac{1}{Y}$ gives $(\lambda^2, \lambda X, \lambda Y) = (-a, b, 1)$. Note that λ in the equivalence relation Eq. (3.20) is completely fixed in the patches $\tilde{\mathcal{U}}^{(2,0)}$ and $\tilde{\mathcal{U}}^{(0,2)}$ given in Eq. (3.19). However, in the middle patch $(1, X, Y)$ we still have a freedom $\lambda = -1$ which leaves the first component 1 untouched, but changes $(1, X, Y) \to (1, -X, -Y)$.

The relation between our result and that in [16] becomes clear by defining $\xi^2 \equiv \phi_0$ ($\xi = \pm\sqrt{\phi_0}$). Now the parameters (ξ, ϕ_1, ϕ_2) have a common weight λ, so they can be regarded as the homogeneous coordinates of $\mathbf{C}P^2$. But one must identify $\xi \sim -\xi$ clearly, and this leads to the $\mathbf{Z}_2$ quotient $(\xi, \phi_1, \phi_2) \sim (\xi, -\phi_1, -\phi_2)$. Therefore our moduli space can also be rewritten as

$$\tilde{\mathcal{M}}_{N=2,k=2} \simeq \mathbf{C}P^2/\mathbf{Z}_2 \tag{3.22}$$

reproducing the result of [16]. Such a $\mathbf{Z}_2$ equivalence, however, does not change the homotopy of $\mathcal{M}_{N=2,k=2}$: it

TABLE I. Transition functions between the three patches $\tilde{\mathcal{U}}^{(2,0)}$, $\tilde{\mathcal{U}}^{(1,1)}$ and $\tilde{\mathcal{U}}^{(0,2)}$.

	(a, b)	(a', b')	(X, Y)
$(a, b) =$	$\cdots$	$(-a'/b'^2, 1/b')$	$(-1/Y^2, X/Y)$
$(a', b') =$	$(-a/b^2, 1/b)$	$\cdots$	$(1/X^2, Y/X)$
$(X, Y) =$	$(\pm ib/\sqrt{a}, \pm i1/\sqrt{a})$	$(\pm 1/\sqrt{a'}, \pm b'/\sqrt{a'})$	$\cdots$

remains $\mathbf{C}P^2$ [14]. This is analogous to an $(x, y) \sim (-x, -y)$ equivalence relation (with real x, y) introduced in one local coordinate system of $\mathbf{C}P^1$ (a sphere), which leads to a sphere with two conic singularities (a rugby ball, or a lemon) instead of the original smooth sphere.[3] See Appendices A and B for more details.

B. $SU(2)$ transformation law of coaxial $k = 2$ vortices

The complex projective space $\mathbf{C}P^2 \simeq \frac{SU(3)}{SU(2)\times U(1)}$ with the Fubini-Study metric has an $SU(3)$ isometry. On the other hand, the weighted projective space $W\mathbf{C}P^2_{(2,1,1)}$ can have an $SU(2)$ isometry at most due to the difference of the weights Eq. (3.20). This matches with the fact that we have only $SU(2)_{\rm G+F}$ symmetry acting on the moduli space. In this subsection we investigate the $SU(2)_{\rm G+F}$ transformation laws of the moduli for the coaxial two vortices, as was done for the fundamental vortex in Sec. II B.

Let us start with the patch $\tilde{\mathcal{U}}^{(0,2)}$, with the moduli matrix

$$H_0^{(0,2)} = \begin{pmatrix} 1 & -az - b \\ 0 & z^2 \end{pmatrix}. \tag{3.23}$$

An $SU(2)$ matrix U like Eq. (2.14) acts on the above H_0 from the right,

$$H_0^{(0,2)} \to H_0^{(0,2)} U = \begin{pmatrix} v^* az + u + v^* b & -u^* az + v - u^* b \\ -v^* z^2 & u^* z^2 \end{pmatrix}. \tag{3.24}$$

A V-transformation $V = V_1 V_2$, where

$$V_1 = \begin{pmatrix} -\frac{v^* a}{(u+v^* b)^2} & 0 \\ 0 & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} z - \frac{u+v^* b}{v^* a} & a \\ v^* z^2 & v^* az + u + v^* b \end{pmatrix} \tag{3.25}$$

brings the result back to the upper-right triangle form,

$$H_0^{(0,2)} \to H_0^{(0,2)} U \sim V H_0^{(0,2)} U = \begin{pmatrix} 1 & -\frac{a}{(u+v^* b)^2} z + \frac{v - u^* b}{u + v^* b} \\ 0 & z^2 \end{pmatrix}. \tag{3.26}$$

The $SU(2)$ transformation laws of the parameters a, b are then

$$a \to \frac{a}{(v^* b + u)^2}, \qquad b \to \frac{u^* b - v}{v^* b + u}. \tag{3.27}$$

As in the previous section, b can be regarded as an inhomogeneous coordinate of $\mathbf{C}P^1$; in fact, b is invariant under a $U(1)$ subgroup (see footnote 2) and this means that b parameterizes $\mathbf{C}P^1 \simeq \frac{SU(2)}{U(1)}$. On the other hand, the transformation law of the parameter a can be rewritten as

$$a \to \left[\frac{d}{db}\left(\frac{u^* b - v}{v^* b + u}\right)\right] a, \tag{3.28}$$

showing that the parameter a is a tangent vector on the base space $\mathbf{C}P^1$ parameterized by b. This is very natural. First recall the situation for separated vortices. The moduli parameters are extracted from their positions and their orientations, defined at the vortex centers. However, once the vortices overlap exactly ($z_i \to z_j$), the positions and the orientations only do not have enough information. When $l(\leq k)$ vortices are coincident, we need $1, 2, \cdots, l-1$ derivatives at the coincident point in order to extract all the information. They define how vortices approach each other ($b_i \to b_j$) [15].

Some $SU(2)$ action sends the points in the patch $\tilde{\mathcal{U}}^{(0,2)}$ to where a better description is in the patch $\tilde{\mathcal{U}}^{(2,0)}$, and vice versa. Compare Eq. (3.27) with $u = 0$, $v = i$, with Eq. (3.14). This shows indeed that

$$\tilde{\mathcal{U}}^{(0,2)} \cup \tilde{\mathcal{U}}^{(2,0)} \simeq T\mathbf{C}P^1. \tag{3.29}$$

Next consider the patch $\tilde{\mathcal{U}}^{(1,1)}$ with

$$H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z + \phi \end{pmatrix}, \qquad \phi^2 + \eta\tilde{\eta} = 0. \tag{3.30}$$

It is convenient to rewrite this as

$$H_0^{(1,1)} = z\mathbf{1}_2 - \vec{X}\cdot\vec{\sigma}, \tag{3.31}$$

where $\vec{\sigma}$ are the Pauli matrices and

$$\phi \equiv X_3, \qquad \eta \equiv X_1 - iX_2, \qquad \tilde{\eta} \equiv X_1 + iX_2. \tag{3.32}$$

X_1, X_2, X_3 are then complex coordinates with a constraint $X_1^2 + X_2^2 + X_3^2 = 0$. To keep the form Eq. (3.31) under $SU(2)_{\rm F}$ transformation, we perform the V-transformation Eq. (2.7) with $V = U^\dagger$: $H_0^{(1,1)} \to U^\dagger H_0^{(1,1)} U$. Equivalently, we study the transformation property of the vortex under

[3] For instance, it is easily seen that $\mathcal{M}_{N=2,k=2} \simeq \mathbf{C}P^2/\mathbf{Z}_2$ remains simply connected. The higher homotopy groups cannot change by a discrete fibration [23].

$SU(2)_{\rm G+F}$. We find

$$\vec{X}\cdot\vec{\sigma}\to U^{\dagger}(\vec{X}\cdot\vec{\sigma})U, \tag{3.33}$$

that is, the vector $\vec{X}$ transforms as an adjoint (triplet) representation, except at $\vec{X}=0$. This last point—singular point of $W\mathbf{C}P^2_{(2,1,1)}$—or the origin of the patch $\tilde{\mathcal{U}}^{(1,1)}$, is a fixed point of $SU(2)$ (a singlet). Note also that the transition functions between the patches $\tilde{\mathcal{U}}^{(0,2)}$ and $\tilde{\mathcal{U}}^{(1,1)}$ are given by

$$X_3=\frac{b}{a},\qquad X_1-iX_2=-\frac{b^2}{a},\qquad X_1+iX_2=\frac{1}{a}. \tag{3.34}$$

The patch $\mathcal{U}^{(1,1)}$ does not cover points at "infinity," namely, the subspace defined by $a=0$ in the patch $\mathcal{U}^{(0,2)}$. That submanifold is nothing but $\mathbf{C}P^1$ parameterized by b which is an edge of $W\mathbf{C}P^2_{(2,1,1)}$. See Fig. 1. One can verify that the transformation law for a, b in Eq. (3.27) and that for ϕ, η, $\tilde{\eta}$ in Eq. (3.33) are consistent through the transition function Eq. (3.34). These results confirm those in [16].

IV. $k=2$ VORTICES IN $U(N)$ GAUGE THEORY

In this section the composition of two non-Abelian vortices in a $U(N)$ gauge theory is systematically investigated. Up to now we made use of the direct form of the moduli matrix $H_0(z)$ for studying the moduli space structure. Another method for studying the latter will be developed and used to determine the moduli space below.

A. The case of $U(N)$

Let $\mathbf{Z}$ and Ψ be k by k and N by k constant complex matrices, respectively. We consider the $GL(k,\mathbf{C})$ action defined by

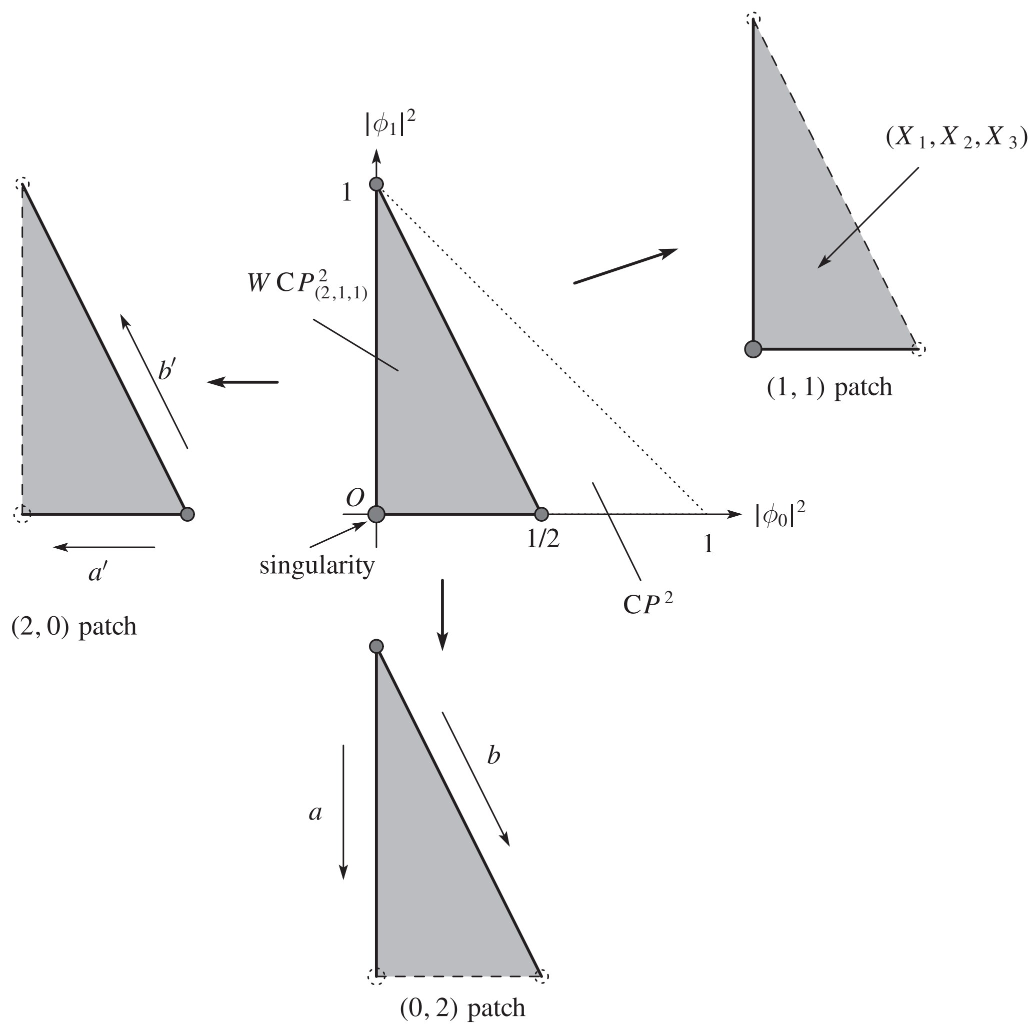

FIG. 1. Toric diagram of $W\mathbf{C}P^2_{(2,1,1)}$ and their three patches $\tilde{\mathcal{U}}^{(2,0)}$, $\tilde{\mathcal{U}}^{(1,1)}$ and $\tilde{\mathcal{U}}^{(0,2)}$. The diagram is drawn under a gauge fixing condition (called the D term constraint) $\sum_{a=0}^{2} q^a|\phi_a|^2=1$ where $U(1)^{\mathbf{C}}$ charges are $q_a=(2,1,1)$ for $W\mathbf{C}P^2_{(2,1,1)}$ while $q_a=(1,1,1)$ for the ordinary $\mathbf{C}P^2$. The triangle with the broken line and O (without singularity) denotes the ordinary $\mathbf{C}P^2$.

$$\mathbf{Z} \to \mathcal{V}\mathbf{Z}\mathcal{V}^{-1}, \qquad \Psi \to \Psi\mathcal{V}^{-1}, \qquad \mathcal{V} \in GL(k, \mathbf{C}). \tag{4.1}$$

It was shown in [22] that the moduli space $\mathcal{M}_{N,k}$ of k vortices can be written as the Kähler quotient [24] defined by

$$\mathcal{M}_{N,k} \simeq \{\mathbf{Z}, \Psi\}//GL(k, \mathbf{C}), \tag{4.2}$$

where $GL(k, \mathbf{C})$ action is free on these matrices.[4] The moduli space $\mathcal{M}_{N,k}$ given by the moduli matrix $H_0(z)$ in Eq. (2.8) and hence by the complex Kähler quotient in Eq. (4.2) is identical to that obtained by use of the D-brane construction by Hanany-Tong [3]. The concrete correspondence between them is obtained by fixing the imaginary part of $GL(k, \mathbf{C})$ in Eq. (4.2) by the moment map $[\mathbf{Z}^\dagger, \mathbf{Z}] + \Psi^\dagger\Psi$:

$$\mathcal{M}_{N,k} \simeq \{(\mathbf{Z}, \Psi)|[\mathbf{Z}^\dagger, \mathbf{Z}] + \Psi^\dagger\Psi \propto \mathbf{1}_k\}/U(k), \tag{4.4}$$

where $\mathbf{Z}$ and Ψ are again in the adjoint and fundamental representations of $U(k)$ group, respectively.

We now use the Kähler quotient construction to generalize the discussion to general N. The authors in [14,16] used the expression Eq. (4.4) but Eq. (4.2) is easier to deal with. Let us discuss the moduli space of coaxial vortices in terms of Eq. (4.2). A subspace of the moduli space $\mathcal{M}_{N,k}$ for coincident vortices at the origin of the x^1-x^2 plane is given by putting the constraint $\det(z - \mathbf{Z}) = z^k$, that is,

$$\mathrm{Tr}\,(\mathbf{Z}^n) = 0, \quad \text{for} \quad n = 1, 2, \cdots k. \tag{4.5}$$

To understand the subspace clearly we need to solve the above constraint by taking appropriate coordinates with $k^2 - k$ complex parameters.

In the case of $k = 2$, $U(N)$ vortices, the constraints Eq. (4.5) are equivalent to the constraint Eq. (3.12) for the $N = 2$ case. $\mathbf{Z}$ can be solved in this case as

$$\mathbf{Z} = \epsilon v v^T, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.6}$$

with v a column two-vector with complex components. The fact that the above form of $\mathbf{Z}$ transforms as in the adjoint representation under $SL(2, \mathbf{C}) \subset GL(2, \mathbf{C})$ means that v is a fundamental representation of $SL(2, \mathbf{C})$ since $\mathbf{2}$ and $\mathbf{2}^*$ is equivalent in the $k = 2$ case. Let us define a complex 2 by $N + 1$ matrix by

$$M = (\Psi^T \quad v). \tag{4.7}$$

The $GL(2, \mathbf{C}) = SL(2, \mathbf{C}) \times \mathbf{C}^*$ action with elements $\mathcal{S} \in SL(2, \mathbf{C})$ and $\lambda \in \mathbf{C}^*$ read

$$M \to \mathcal{S}M, \qquad (\Psi^T \quad v) \to (\lambda\Psi^T \quad v). \tag{4.8}$$

The quotient by $GL(2, \mathbf{C})$ results in a kind of complex Grassmannian manifold whose $\mathbf{C}^*$ action has weights $(\underbrace{1, \cdots, 1}_{N}, 0)$. The moduli subspace of coincident two vortices in $U(N)$ gauge theory is therefore found to be a weighted Grassmannian manifold,

$$\mathcal{M}_{N,k=2}|_{\text{coincident}} \simeq WGr_{N+1,2}^{(1,\cdots,1,0)}. \tag{4.9}$$

We note again that the elements $\mathcal{S} = -\mathbf{1}_2$ and $\lambda = -1$ acting as

$$(\Psi^T \quad v) \to (\Psi^T \quad -v) \tag{4.10}$$

is precisely the $\mathbf{Z}_2$ action which gives orbifold singularities. Note that, although the ordinary complex Grassmannian manifold $Gr_{N+1,2} \simeq \frac{SU(N+1)}{SU(N-1)\times SU(2)\times U(1)}$ naturally enjoys an $SU(N+1)$ isometry, the weighted Grassmannian manifold $WGr_{N+1,2}^{(1,\cdots,1,0)}$ can have an $SU(N)$ isometry at most, due to the difference of $U(1)^{\mathbf{C}}$ charges. This is consistent with the existence of the $SU(N)_{\mathrm{G+F}}$ symmetry acting on the moduli space in the $U(N)$ case. In cases of $N > 2$ the orbifold singularities are not isolated points but form a submanifold given by $v = 0$, which is the ordinary complex Grassmannian manifold $Gr_{N,2} \subset WGr_{N+1,2}^{(1,\cdots,1,0)}$ reflecting the $SU(N)_{\mathrm{G+F}}$ symmetry.

In Appendix C, beside giving the general procedure to pass from the moduli matrix $H_0(z)$ to the Kähler quotient construction, we directly show a one-to-one correspondence among the patches of the moduli matrix for the $U(N)$, $k = 2$ vortices and those of the weighted Grassmannian manifold $WGr_{N+1,2}^{(1,\cdots,1,0)}$ and verify that the transition functions of the latter perfectly match the ones obtained from $H_0(z)$. We enforce this way the above result based on general grounds (specifically the equivalence of moduli matrix and Kähler quotient approach discovered in [22]).

B. The case of $U(2)$ revisited

As an illustration consider again the case of $U(2)$ theory. 2 by 2 matrices $\mathbf{Z}$ and Ψ correspond to the moduli space of $k = 2$ non-Abelian vortices in the $U(2)$ gauge theory with the equivalence relation Eq. (4.2). The double coaxial vortices are described by $\det(z\mathbf{1}_2 - \mathbf{Z}) = z^2$. This can be rewritten as $\mathrm{Tr}\mathbf{Z} = \mathrm{Tr}\mathbf{Z}^2 = 0$. These conditions are easily solved and we find that these vortices are described by the following two 2 by 2 matrices $\mathbf{Z}$ and Ψ:

$$\mathbf{Z} = \epsilon v v^T = \begin{pmatrix} v_1 v_2 & v_2^2 \\ -v_1^2 & -v_1 v_2 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}, \tag{4.11}$$

---

[4] The relation between the moduli matrix $H_0(z)$ and the two matrices $(\mathbf{Z}, \Psi)$ is given by the following ADHM-like equation

$$\nabla^\dagger L = 0, \qquad \det(z - \mathbf{Z}) = \det H_0(z), \tag{4.3}$$

where $L^\dagger \equiv (H_0(z), \mathbf{J}(z))$ and $\nabla^\dagger \equiv (-\Psi^\dagger, \bar{z} - \mathbf{Z}^\dagger)$. Here $\mathbf{J}(z)$ is N by k matrix whose elements are holomorphic function of z. $(\mathbf{Z}, \Psi)$ and $\mathbf{J}(z)$ can be uniquely determined from a given $H_0(z)$ [22].

where $v^T = (v_1, v_2)$ and

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These obey the equivalence relation $GL(2, \mathbf{C})$ given in Eq. (C14). At this stage the matter would become simple if we consider v rather than $\mathbf{Z}$. Since $\mathbf{Z}$ is in the adjoint representation of $GL(2, \mathbf{C})$, v is in the fundamental representation of $SL(2, \mathbf{C})$. Notice that v is not charged under the overall $U(1)^{\mathbf{C}} \subset GL(2, \mathbf{C})$.

It is natural to define $k(=2)$ by $N+1(=2+1)$ matrix

$$M = (\Psi^T, v) = \begin{pmatrix} \psi_{11} & \psi_{21} & v_1 \\ \psi_{12} & \psi_{22} & v_2 \end{pmatrix}. \tag{4.12}$$

This matrix M transforms under $GL(2, \mathbf{C}) = U(1)^{\mathbf{C}} \times SL(2, \mathbf{C})$ as follows

$$M = (\Psi^T, v) \sim (\mathcal{S}\lambda\Psi^T, \mathcal{S}v), \tag{4.13}$$

where $\mathcal{S} \in SL(2, \mathbf{C})$ and $\lambda \in U(1)^{\mathbf{C}}$. If the vector v had a charge 1 under $U(1)^{\mathbf{C}}$ (that is, $v \to \lambda v$), the above identification would correspond to the complex Grassmannian $Gr_{3,2} \simeq M/GL(2, \mathbf{C})$ which is same as $\mathbf{C}P^2$. But since v is not charged under $U(1)^{\mathbf{C}}$, the manifold is not a Grassmannian. Equation (4.13) is an example of a weighted Grassmannian manifold and we denote it by $WGr_{3,2}^{(1,1,0)}$. Here the numbers $(1, 1, 0)$ denote the $U(1)^{\mathbf{C}}$-charges (the weights) of columns of M.

We choose appropriate $GL(2, \mathbf{C})$ matrices to obtain various patches on the moduli space for the composite vortices. Let us define the 2 by 2 minors $M_{[ij]}$ and their determinants as

$$M_{[ij]} = \begin{pmatrix} M_{i1} & M_{j1} \\ M_{i2} & M_{j2} \end{pmatrix}, \qquad \tau_{ij} = \det M_{[ij]}. \tag{4.14}$$

There are 3 minors $M_{[12]}$, $M_{[23]}$, and $M_{[13]}$. Using the $GL(2, \mathbf{C})$, one of them can be brought to identity. So one has 3 patches as follows.

(i) $M_{[23]} = \mathbf{1}_2$ patch. First act

$$\mathcal{S} = \begin{pmatrix} \tau_{23} & 0 \\ 0 & 1 \end{pmatrix} M_{[23]}^{-1}$$

to the matrix M in Eq. (4.12), and after that by $\lambda = \tau_{23}^{-1}$:

$$M \to \begin{pmatrix} \tau_{13} & \tau_{23} & 0 \\ -\frac{\tau_{12}}{\tau_{23}} & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \frac{\tau_{13}}{\tau_{23}} & 1 & 0 \\ -\frac{\tau_{12}}{\tau_{23}^2} & 0 & 1 \end{pmatrix}. \tag{4.15}$$

(ii) $M_{[12]} = \mathbf{1}_2$ patch. First one acts $\mathcal{S} = (\tau_{12})^{1/2} M_{[12]}^{-1}$ to the matrix M in Eq. (4.12), and after that then by $\lambda = (\tau_{12})^{-1/2}$:

$$M \to \begin{pmatrix} \sqrt{\tau_{12}} & 0 & \frac{-\tau_{23}}{\sqrt{\tau_{12}}} \\ 0 & \sqrt{\tau_{12}} & \frac{\tau_{13}}{\sqrt{\tau_{12}}} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{-\tau_{23}}{\sqrt{\tau_{12}}} \\ 0 & 1 & \frac{\tau_{13}}{\sqrt{\tau_{12}}} \end{pmatrix}. \tag{4.16}$$

Notice that the element $\mathcal{S} = -\mathbf{1}_2$ with $\lambda = -1$, has not been fixed, thus one has a $\mathbf{Z}_2$ symmetry in this patch

$$\begin{pmatrix} -\frac{\tau_{23}}{\sqrt{\tau_{12}}} \\ \frac{\tau_{13}}{\sqrt{\tau_{12}}} \end{pmatrix} \sim \begin{pmatrix} \frac{\tau_{23}}{\sqrt{\tau_{12}}} \\ -\frac{\tau_{13}}{\sqrt{\tau_{12}}} \end{pmatrix}. \tag{4.17}$$

(iii) $M_{[13]} = \mathbf{1}_2$ patch. Act first by

$$\mathcal{S} = \begin{pmatrix} \tau_{13} & 0 \\ 0 & 1 \end{pmatrix} M_{[13]}^{-1}$$

to the matrix M in Eq. (4.12), and then by $\lambda = \tau_{13}^{-1}$:

$$M \to \begin{pmatrix} \tau_{13} & \tau_{23} & 0 \\ 0 & \frac{\tau_{12}}{\tau_{13}} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{\tau_{23}}{\tau_{13}} & 0 \\ 0 & \frac{\tau_{12}}{\tau_{13}^2} & 1 \end{pmatrix}. \tag{4.18}$$

The corresponding matrices $\begin{pmatrix} \Psi \\ \mathbf{Z} \end{pmatrix}$ for the above three patches are summarized as follows

$$\begin{pmatrix} \frac{\tau_{13}}{\tau_{23}} & \frac{-\tau_{12}}{\tau_{23}^2} \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-\tau_{23}\tau_{13}}{\tau_{12}} & \frac{\tau_{13}^2}{\tau_{12}} \\ \frac{-\tau_{23}^2}{\tau_{12}} & \frac{\tau_{23}\tau_{13}}{\tau_{12}} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \frac{\tau_{23}}{\tau_{13}} & \frac{\tau_{12}}{\tau_{13}^2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.19}$$

The leftmost one corresponds to the $M_{[23]} = \mathbf{1}_2$ patch, the middle to the $M_{[12]} = \mathbf{1}_2$ and the rightmost one to the $M_{[13]} = \mathbf{1}_2$ patch. Clearly, these should be identified with the matrices in Eq. (C18) which were obtained from the moduli matrices $H_0^{(2,0)}(z)$, $H_0^{(1,1)}(z)$ and $H_0^{(0,2)}(z)$ given in Eq. (3.1) through the relation Eq. (C4). Therefore, $M_{[23]} = \mathbf{1}_2$ patch and $\tilde{\mathcal{U}}^{(0,2)}$ patch, $M_{[12]} = \mathbf{1}_2$ patch and $\tilde{\mathcal{U}}^{(1,1)}$ patch, and $M_{[13]} = \mathbf{1}_2$ patch and $\tilde{\mathcal{U}}^{(2,0)}$ patch. The concrete identification is

$$\begin{pmatrix} \frac{\tau_{12}}{\tau_{13}^2} \\ \frac{\tau_{23}}{\tau_{13}} \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}, \qquad \begin{pmatrix} \frac{\tau_{13}}{\sqrt{\tau_{12}}} \\ \frac{\tau_{23}}{\sqrt{\tau_{12}}} \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \begin{pmatrix} \frac{\tau_{12}}{\tau_{23}^2} \\ \frac{\tau_{13}}{\tau_{23}} \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}. \tag{4.20}$$

Now we are ready to understand a little mysterious relation Eq. (3.20) which gave us the weighted complex projective space $W\mathbf{C}P^2_{(2,1,1)}$. Ordinary complex Grassmannian $Gr_{3,2}$ is known to be equivalent to $\mathbf{C}P^2$ and the weighted cases are quite analogous. Because all the parameters in Eq. (4.19) are functions of the determinants of the minors $M_{[12]}$, $M_{[23]}$ and $M_{[13]}$, it would be natural to consider that the manifold is naturally parameterized by them. In particular the origin of the weighted equivalence relation Eq. (3.20) becomes clear since τ_{ij} are invariant under $SL(2, \mathbf{C})$ while transforming under $U(1)^{\mathbf{C}}$ as

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{pmatrix} \sim \begin{pmatrix} \lambda^2\tau_{12} \\ \lambda\tau_{13} \\ \lambda\tau_{23} \end{pmatrix}. \tag{4.21}$$

This is nothing but Eq. (3.20). The $\tilde{\mathcal{U}}^{(0,2)}$ patch is obtained by fixing with $\lambda = \tau_{12}^{-1/2}$, the $\tilde{\mathcal{U}}^{(1,1)}$ patch by $\lambda = \tau_{23}^{-1/2}$ and the $\tilde{\mathcal{U}}^{(2,0)}$ patch by $\lambda = \tau_{13}^{-1/2}$. Thus

$$\tilde{\mathcal{M}}_{N=2,k=2} \simeq W\mathbf{C}P^2_{(2,1,1)} \simeq WGr^{(1,1,0)}_{3,2}. \tag{4.22}$$

C. The case of $U(3)$

The moduli space of the coaxial $k = 2$ vortices in the $U(3)$ gauge theory is the weighted complex Grassmannian $\tilde{\mathcal{M}}_{N=3,k=2} \simeq WGr^{(1,1,1,0)}_{4,2}$ as shown already for general N. The weighted Grassmannian is covered by ${}_4C_2 = 6$ patches as the ordinary Grassmannian $Gr_{4,2}$. The patches are obtained as follows. Let us begin with 2 by 4 matrix M

$$M = (\Psi^T, v) = \begin{pmatrix} \psi_{11} & \psi_{21} & \psi_{31} & v_1 \\ \psi_{12} & \psi_{22} & \psi_{32} & v_2 \end{pmatrix} \tag{4.23}$$

with an $GL(2, \mathbf{C}) = SL(2, \mathbf{C}) \times U(1)^{\mathbf{C}}$ weighted equivalence relation

$$M = (\Psi^T, v) \sim (\mathcal{S}\lambda\Psi^T, \mathcal{S}v) \tag{4.24}$$

with $\mathcal{S} \in SL(2, \mathbf{C})$ and $\lambda \in U(1)^{\mathbf{C}}$. The matrix M has 6 minor matrices $M_{[ij]}$ whose size are 2 by 2 as given in Eq. (4.14). By using the $GL(2, \mathbf{C})$, we can bring one of the 6 minors to the unit matrix.

(i) $M_{[a4]} = \mathbf{1}_2$ patches ($a = 1, 2, 3$): In order to obtain $M_{[a4]} = \mathbf{1}_2$ patch, one must perform

$$\mathcal{S}_a = \begin{pmatrix} \tau_{a4} & 0 \\ 1 & 0 \end{pmatrix} M^{-1}_{[a4]} \in SL(2, \mathbf{C})$$

and $\lambda_a = \tau_{a4}^{-1} \in U(1)^{\mathbf{C}}$. For example, $M_{[14]} = \mathbf{1}_2$ patch is obtained as follows:

$$M \to \begin{pmatrix} \tau_{14} & \tau_{24} & \tau_{34} & 0 \\ 0 & \frac{\tau_{12}}{\tau_{14}} & \frac{\tau_{13}}{\tau_{14}} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{\tau_{24}}{\tau_{14}} & \frac{\tau_{34}}{\tau_{14}} & 0 \\ 0 & \frac{\tau_{12}}{\tau_{14}^2} & \frac{\tau_{13}}{\tau_{14}^2} & 1 \end{pmatrix}. \tag{4.25}$$

Similarly one finds the remaining $M_{[24]} = \mathbf{1}_2$ and $M_{[34]} = \mathbf{1}_2$ patches

$$\begin{pmatrix} \frac{\tau_{14}}{\tau_{24}} & 1 & \frac{\tau_{34}}{\tau_{24}} & 0 \\ -\frac{\tau_{12}}{\tau_{24}^2} & 0 & \frac{\tau_{23}}{\tau_{24}^2} & 1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{\tau_{14}}{\tau_{34}} & \frac{\tau_{24}}{\tau_{34}} & 1 & 0 \\ -\frac{\tau_{13}}{\tau_{34}^2} & -\frac{\tau_{23}}{\tau_{34}^2} & 0 & 1 \end{pmatrix}, \tag{4.26}$$

respectively.

(ii) $M_{[ab]} = \mathbf{1}_2$ patches ($a, b = 1, 2, 3,\ a < b$): In order to get a $M_{[ab]} = \mathbf{1}_2$ patch, one must make a transformation with $\mathcal{S}_{ab} = (\tau_{ab})^{1/2} M^{-1}_{[ab]} \in SL(2, \mathbf{C})$ and $\lambda_{ab} = (\tau_{ab})^{-1/2} \in U(1)^{\mathbf{C}}$. For example, $M_{[12]} = \mathbf{1}_2$ patch is obtained as follows:

$$M \to \begin{pmatrix} \sqrt{\tau_{12}} & 0 & \frac{-\tau_{23}}{\sqrt{\tau_{12}}} & \frac{-\tau_{24}}{\sqrt{\tau_{12}}} \\ 0 & \sqrt{\tau_{12}} & \frac{\tau_{13}}{\sqrt{\tau_{12}}} & \frac{\tau_{14}}{\sqrt{\tau_{12}}} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{-\tau_{23}}{\tau_{12}} & \frac{-\tau_{24}}{\sqrt{\tau_{12}}} \\ 0 & 1 & \frac{\tau_{13}}{\tau_{12}} & \frac{\tau_{14}}{\sqrt{\tau_{12}}} \end{pmatrix}. \tag{4.27}$$

The patches $M_{[13]} = \mathbf{1}_2$ and $M_{[23]} = \mathbf{1}_2$ can be obtained similarly

$$\begin{pmatrix} 1 & \frac{\tau_{23}}{\tau_{13}} & 0 & -\frac{\tau_{34}}{\sqrt{\tau_{13}}} \\ 0 & \frac{\tau_{12}}{\tau_{13}} & 1 & \frac{\tau_{14}}{\sqrt{\tau_{13}}} \end{pmatrix}, \quad \begin{pmatrix} \frac{\tau_{13}}{\tau_{23}} & 1 & 0 & -\frac{\tau_{34}}{\sqrt{\tau_{23}}} \\ -\frac{\tau_{12}}{\tau_{23}} & 0 & 1 & \frac{\tau_{24}}{\sqrt{\tau_{23}}} \end{pmatrix}. \tag{4.28}$$

Notice that $\mathcal{S} = -\mathbf{1}_2$, $\lambda = -1$ have not used up, so one has a $\mathbf{Z}_2$ symmetry in these patches.

τ_{ab} ($a, b = 1, 2, 3, 4$ with $a < b$) can be seen as natural coordinates for the manifold $WGr^{(1,1,1,0)}_{4,2}$. $\lambda\mathcal{S} \in GL(2, \mathbf{C})$ equivalence relation is expressed on these coordinates as

$$\begin{pmatrix} \tau_{12} \\ \tau_{23} \\ \tau_{13} \\ \tau_{14} \\ \tau_{24} \\ \tau_{34} \end{pmatrix} \sim \begin{pmatrix} \lambda^2\tau_{12} \\ \lambda^2\tau_{23} \\ \lambda^2\tau_{13} \\ \lambda\tau_{14} \\ \lambda\tau_{24} \\ \lambda\tau_{34} \end{pmatrix}. \tag{4.29}$$

This weighted equivalence relation for 6 complex parameters defines the weighted complex projective space $W\mathbf{C}P^5_{(2,2,2,1,1,1)}$ whose complex dimension is 5. The $W\mathbf{C}P^5_{(2,2,2,1,1,1)}$ is a bigger manifold than the $WGr^{(1,1,1,0)}_{4,2}$ whose complex dimension is 4. The embedding relation $WGr^{(1,1,1,0)}_{4,2} \subset W\mathbf{C}P^5_{(2,2,2,1,1,1)}$ is the same as that for the ordinary Grassmannian to the ordinary complex projective space through the so-called Plücker relation

$$\tau_{12}\tau_{34} - \tau_{13}\tau_{24} + \tau_{14}\tau_{23} = 0. \tag{4.30}$$

The $\mathbf{Z}_2$ symmetry on the $W\mathbf{C}P^5_{(2,2,2,1,1,1)}$ is now realized as the action of $\lambda = -1$ which acts as

$$\begin{pmatrix} \tau_{12} \\ \tau_{23} \\ \tau_{13} \\ \tau_{14} \\ \tau_{24} \\ \tau_{34} \end{pmatrix} \sim \begin{pmatrix} \tau_{12} \\ \tau_{23} \\ \tau_{13} \\ -\tau_{14} \\ -\tau_{24} \\ -\tau_{34} \end{pmatrix}. \tag{4.31}$$

The patches $M_{[a4]} = \mathbf{1}_2$ ($a = 1, 2, 3$) can be obtained fixing the $U(1)^{\mathbf{C}}$ ambiguity by choosing $\lambda = (\tau_{a4})^{-1}$ while the patches $M_{[ab]} = \mathbf{1}_2$ ($a, b = 1, 2, 3$ with $a < b$) are obtained by choosing $\lambda = (\tau_{ab})^{-\frac{1}{2}}$. Note that one can easily confirm that the whole of the orbifold singularities of the $\mathbf{Z}_2$ action form a submanifold $\mathbf{C}P^2 \simeq Gr_{3,2} \subset WGr_{4,2}^{(1,1,1,0)}$, since the condition $\tau_{14} = \tau_{24} = \tau_{34} = 0$ for the singularity solves the Plücker condition Eq. (4.30) and the unconstrained parameters $(\tau_{12}, \tau_{23}, \tau_{13})$ have the ordinary equivalence relation as that of $\mathbf{C}P^2$ with $\lambda' = \lambda^2$ in Eq. (4.29).

For completeness we list the moduli matrices in the case of $k = 2$ and $N = 3$. The six patches $\tilde{\mathcal{U}}^{(2,0,0)}$, $\tilde{\mathcal{U}}^{(0,2,0)}$, $\tilde{\mathcal{U}}^{(0,0,2)}$, $\tilde{\mathcal{U}}^{(1,1,0)}$, $\tilde{\mathcal{U}}^{(0,1,1)}$ and $\tilde{\mathcal{U}}^{(1,0,1)}$ of the $k = 2$ coaxial vortices in the $U(3)$ gauge theory are given by

$$H_0^{(0,0,2)}(z) = \begin{pmatrix} 1 & 0 & -a_1 z - b_1 \\ 0 & 1 & -a_2 z - b_2 \\ 0 & 0 & z^2 \end{pmatrix}, \quad H_0^{(1,1,0)}(z) = \begin{pmatrix} z + XY & -X^2 & 0 \\ Y^2 & z - XY & 0 \\ -\gamma & -\chi & 1 \end{pmatrix}, \tag{4.32}$$

$$H_0^{(0,2,0)}(z) = \begin{pmatrix} 1 & -a'_1 z - b'_1 & 0 \\ 0 & z^2 & 0 \\ 0 & -a'_2 z - b'_2 & 1 \end{pmatrix}, \quad H_0^{(1,0,1)}(z) = \begin{pmatrix} z + X'Y' & 0 & -X'^2 \\ -\gamma' & 1 & -\chi' \\ Y'^2 & 0 & z - X'Y' \end{pmatrix}, \tag{4.33}$$

$$H_0^{(2,0,0)}(z) = \begin{pmatrix} z^2 & 0 & 0 \\ -a''_1 z - b''_1 & 1 & 0 \\ -a''_2 z - b''_2 & 0 & 1 \end{pmatrix}, \quad H_0^{(0,1,1)}(z) = \begin{pmatrix} 1 & -\gamma'' & -\chi'' \\ 0 & z + X''Y'' & -X''^2 \\ 0 & Y''^2 & z - X''Y'' \end{pmatrix}, \tag{4.34}$$

with $(X, Y) \sim (-X, -Y), (X', Y') \sim (-X', -Y')$ and $(X'', Y'') \sim (-X'', -Y'')$. These identifications lead to the orbifold singularities along $\mathbf{C}P^2$, as we mentioned, which is parameterized by three patches $(\gamma, \chi), (\gamma', \chi')$ and (γ'', χ''). The determinant of each of these matrices is equal to z^2 corresponding to the fact that these describe double vortices one sitting on the other, at the origin of the z plane. The transition functions and other details are given in Appendix C.

V. CONCLUSION

In this paper we have studied and determined the structure of the moduli space of certain composite non-Abelian vortices, appearing in $U(N)$ gauge theories in the Higgs phase. The moduli subspace of two coaxial vortices (or equivalently, axially symmetric $k = 2$ vortices) in the $U(N)$ gauge theories with N flavors, is found to be a weighted Grassmannian manifold, Eq. (4.9). In the case of $U(2)$ gauge theory, it reduces to a weighted projective space $W\mathbf{C}P^2_{(2,1,1)} \simeq \mathbf{C}P^2/\mathbf{Z}_2(\mathbf{C}P^2$ homotopically), in agreement with the known results [14,16]. This space contains a $\mathbf{Z}_2$ orbifold (conic) singularity at the origin of the (1,1) patch. In the case of $U(N)$ gauge theory, it contains singularities along $Gr_{N,2}$.

The presence of this kind of orbifold singularities is a general feature of weighted Grassmannian manifold. This fact implies the necessity to reconsider the reconnection of non-Abelian vortices. So far this issue has been studied considering the moduli space of $k = 2$ coaxial vortices smooth everywhere [14]. We claim that this is not the case and that we need to analyze the metric on the $k = 2$ vortices moduli space to address the problem.

An interesting question is how our results are generalized in the case of semilocal non-Abelian vortices [20,22]. Extension of the results of this paper to the semilocal cases will be discussed elsewhere.

It would be interesting also to extend our study to vortices of different kind, such as those appearing in $SO(N)$ theories [17].

The implications of our results on the properties of non-Abelian *monopoles* appearing in related systems, will be discussed in a separate paper.

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APPENDIX A: RELATION TO ASY ANSATZ

The ansatz in [16] is formulated in terms of two independent unit vectors, $\vec{n}_1$ and $\vec{n}_2$. Using a global color-flavor rotation the two vectors can be rotated into the following form:

$$\vec{n}_1 = (0, 0, 1), \qquad \vec{n}_2 = (\sin\alpha, 0, \cos\alpha), \tag{A1}$$

where α is the relative angle between $\vec{n}_1$ and $\vec{n}_2$. Now it is straightforward to derive the moduli matrix H_0 that corresponds to this particular choice of parameters, as was done in [16]:

$$H_0(z, \alpha) = \begin{pmatrix} -\cos\frac{\alpha}{2} z^2 & \sin\frac{\alpha}{2} z \\ -\sin\frac{\alpha}{2} z & -\cos\frac{\alpha}{2} \end{pmatrix}. \tag{A2}$$

This matrix can be put into an upper-right triangular form

$$H_0(z,\alpha) = \begin{pmatrix} z & \cot\frac{\alpha}{2} \\ 0 & z \end{pmatrix}, \tag{A3}$$

by a V transformation.

The ASY vortices with generic orientation vectors $\vec{n}_1$ and $\vec{n}_2$ can be found simply by an overall $SO(3)$ rotation of the above. To find the moduli matrix representation of the general ASY ansatz, we must go to the system in which

$$\vec{n}_1 = (-\sin\alpha_1\cos\beta_1, \sin\alpha_1\sin\beta_1, \cos\alpha_1). \tag{A4}$$

To obtain such a general ASY solution, parameterized with four angular coordinates:

$$H_0(z,\alpha,\beta,\alpha_1,\beta_1), \tag{A5}$$

where (α_1, β_1) represent the orientation $\mathbf{n}_1$ while the angles (α, β) stand for the orientation of the vector $\mathbf{n}_2$ relative to $\mathbf{n}_1$, we rotate the moduli matrix Eq. (A3) with a global rotation matrix:

$$U = \exp\left(\frac{i}{2}\eta_1\tau_3\right)\exp\left(\frac{i}{2}\alpha_1\tau_2\right)\exp\left(\frac{i}{2}\beta_1\tau_3\right) = \begin{pmatrix} e^{i/2(\beta_1+\eta_1)}\cos\frac{\alpha_1}{2} & -e^{-i/2(\beta_1-\eta_1)}\sin\frac{\alpha_1}{2} \\ e^{i/2(\beta_1-\eta_1)}\sin\frac{\alpha_1}{2} & e^{-i/2(\beta_1+\eta_1)}\cos\frac{\alpha_1}{2} \end{pmatrix}, \tag{A6}$$

where $(\eta_1, \alpha_1, \beta_1)$ are the Euler angles. After the rotation

$$H_0(z,\alpha,\eta_1,\alpha_1,\beta_1) = H_0(z,\alpha)U(\eta_1,\alpha_1,\beta_1), \tag{A7}$$

we put the result into the upper-right triangular form, $H_0(z,\alpha,\eta_1,\alpha_1,\beta_1) \sim VH_0(z,\alpha,\eta_1,\alpha_1,\beta_1)$, to get

$$H_0(z,\alpha,\eta_1,\alpha_1,\beta_1) = \begin{pmatrix} 1 & -e^{-i\beta_1}\cot\frac{\alpha_1}{2} - ze^{-i(\beta_1-\eta_1)}\csc^2\frac{\alpha_1}{2}\tan\frac{\alpha}{2} \\ 0 & z^2 \end{pmatrix}. \tag{A8}$$

The V transformation needed is

$$V = \begin{pmatrix} e^{-i/2(\beta_1-\eta_1)}\csc\frac{\alpha_1}{2}\tan\frac{\alpha}{2} & -e^{-i/2(\beta_1-3\eta_1)}\cot\frac{\alpha_1}{2}\csc\frac{\alpha_1}{2}\tan\frac{\alpha}{2} \\ -ze^{-i/2(\beta_1-\eta_1)}\sin\frac{\alpha_1}{2}\tan\frac{\alpha}{2} & ze^{i/2\beta_1}\cos\frac{\alpha_1}{2} + e^{i/2(\beta_1-\eta_1)}\sin\frac{\alpha_1}{2} \end{pmatrix}. \tag{A9}$$

With an arbitrary choice for the origin of the β angle we can identify $\beta = \eta_1$. Thus the ASY ansatz has the moduli matrix representation, with

$$b = e^{-i\beta_1}\cot\frac{\alpha_1}{2}, \qquad a = e^{-i(\beta_1-\beta)}\csc^2\frac{\alpha_1}{2}\tan\frac{\alpha}{2}. \tag{A10}$$

Note that $a \to \infty$ as $\alpha \to \pi$ and we get the singlet point of the moduli space; while $a \to 0$ as $\alpha \to 0$ and we get a "doublet" transforming as a $k = 1$ vortex with the orientation vector $\mathbf{n}_1$, in accord with ASY and with our results.

APPENDIX B: RELATION TO HT-ASY ANALYSIS

The Kähler quotient construction (D-brane construction by Hanany-Tong) for the moduli space of k vortices in $U(N)$ gauge theory coupled with N Higgs fields is given by the two matrices Z and ψ. Here Z is k by k matrix and ψ is N by k matrix which satisfy the following constraint

$$[Z^\dagger, Z] + \psi^\dagger\psi = \mathbf{1}_2, \tag{B1}$$

and are divided by the $U(k)$ symmetry

$$Z \to \tilde{\mathcal{V}} Z \tilde{\mathcal{V}}^\dagger, \quad \psi \to \psi\tilde{\mathcal{V}}^\dagger, \tag{B2}$$

with $\tilde{\mathcal{V}} \in U(k)$. The eigenvalues of the matrix Z are thought of as the positions of the vortices. After performing an appropriate transformation $U(k)$, we can always bring the matrix Z into a triangle matrix. As a concrete example, let us consider composing two vortices at the origin in the $U(2)$ model. The matrices are of the form in that gauge:

$$Z = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} \vec{A} & \vec{B} \end{pmatrix}, \tag{B3}$$

with $w \in \mathbf{C}$ and two complex vectors $\vec{A}^T = (A_1, A_2)$ and $\vec{B}^T = (B_1, B_2)$. Note that we have not fixed two $U(1)$ symmetries $e^{i\theta_1} \in U(1)_1$ and $e^{i\theta_2} \in U(1)_2$ which do not change the triangle form of the matrix Z:

$$\tilde{\mathcal{V}} = \begin{pmatrix} e^{i(\theta_1+\theta_2)} & 0 \\ 0 & e^{i(\theta_1-\theta_2)} \end{pmatrix}. \tag{B4}$$

The charges of $\vec{A}$, $\vec{B}$, and w under those $U(1)$'s are summarized in the following Table II. We have to fix these $U(1)$'s to realize the moduli space of the composing two vortices. To this end, we first absorb the phase of w as $w = |w|$ by use of $U(1)_2$. Then we have only $U(1)_1$ as unfixed gauge symmetry. Plugging Eq. (B3) into Eq. (B1), we obtain three constraints

$$|\vec{A}|^2 - |w|^2 = 1, \qquad |\vec{B}|^2 + |w|^2 = 1, \qquad \vec{A}^\dagger \cdot \vec{B} = 0. \tag{B5}$$

From the middle constraint of Eq. (B5), the vector $\vec{B}$ can be written as the following form

TABLE II. The charges under $U(1)_1$ and $U(1)_2$.

	$U(1)_1$	$U(1)_2$
$\vec{A}$	-1	-1
$\vec{B}$	-1	1
w	0	2

$$\vec{B} = \sqrt{1-|w|^2}\begin{pmatrix} -e^{-i\beta_2}\sin\xi \\ e^{-i\beta_1}\cos\xi \end{pmatrix}. \tag{B6}$$

Then from the first and the last constraints in Eq. (B5), the remaining vector $\vec{A}$ can be expressed as the following form

$$\vec{A} = \sqrt{1+|w|^2}e^{i\alpha}\epsilon\frac{\vec{B}^*}{|\vec{B}|} = \sqrt{1+|w|^2}e^{i\alpha}\begin{pmatrix} e^{i\beta_1}\cos\xi \\ e^{i\beta_2}\sin\xi \end{pmatrix} \tag{B7}$$

with the antisymmetric tensor ϵ ($\epsilon^{12}=1$). Let us fix the remaining $U(1)_1$ gauge symmetry by choosing $\theta_1 = \frac{\alpha}{2}$. Finally, we get the following form

$$Z = \begin{pmatrix} 0 & |w| \\ 0 & 0 \end{pmatrix},$$
$$\psi = \begin{pmatrix} \sqrt{1+|w|^2}e^{i\gamma_1}\cos\xi & -\sqrt{1-|w|^2}e^{-i\gamma_2}\sin\xi \\ \sqrt{1+|w|^2}e^{i\gamma_2}\sin\xi & \sqrt{1-|w|^2}e^{-i\gamma_1}\cos\xi \end{pmatrix}, \tag{B8}$$

where $\gamma_1 = \frac{\alpha}{2} + \beta_1$ and $\gamma_2 = \frac{\alpha}{2} + \beta_2$. At this stage, we have fixed both $U(1)_1$ and $U(1)_2$. But we have to be careful because $U(1)_2$ has not been completely fixed yet. In fact, $\mathbf{Z}_2$ transformation by $\theta_2 = \pi$ is unfixed since w has charge 2 under $U(1)_2$ gauge symmetry ($w = |w| \to +|w|$). Therefore we need to take $\mathbf{Z}_2$ identification into account

$$\mathbf{Z}_2\colon \psi \to -\psi. \tag{B9}$$

Now we have completely fixed the $U(2)$ gauge symmetry and have reached a patch of the Kähler quotient in which $\mathbf{Z}_2$ symmetry is equipped. This patch should be identified with our matrices $\mathbf{Z}^{(1,1)}$ and $\Psi^{(1,1)}$ in the $\tilde{\mathcal{U}}^{(1,1)}$ patch given in Eq. (C18) in which the $\mathbf{Z}_2$ symmetry also exist. Our matrices $\{\mathbf{Z}^{(1,1)}, \Psi^{(1,1)}\}$ and matrices $\{Z, \psi\}$ in Eq. (B8) are transformed by $GL(2,\mathbf{C})$ transformation

$$\mathcal{V} = \psi \in GL(2,\mathbf{C})\colon \mathbf{Z}^{(1,1)} = \psi Z \psi^{-1}, \quad \Psi^{(1,1)} = \psi\psi^{-1} = \mathbf{1}_2. \tag{B10}$$

More concretely, this can be written as

$$\begin{pmatrix} -XY & X^2 \\ -Y^2 & XY \end{pmatrix} = \frac{|w|\sqrt{1+|w|^2}}{\sqrt{1-|w|^2}} \times \begin{pmatrix} -e^{i(\gamma_1+\gamma_2)}\sin\xi\cos\xi & e^{2i\gamma_1}\cos^2\xi \\ -e^{2i\gamma_1}\sin^2\xi & e^{i(\gamma_1+\gamma_2)}\sin\xi\cos\xi \end{pmatrix}. \tag{B11}$$

Thus we can find the relation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \sqrt{\frac{|w|\sqrt{1+|w|^2}}{\sqrt{1-|w|^2}}}\begin{pmatrix} e^{i\gamma_1}\cos\xi \\ e^{i\gamma_2}\sin\xi \end{pmatrix} = \sqrt{\frac{|w|}{\sqrt{1-|w|^4}}}\vec{A}. \tag{B12}$$

Thus we conclude that our $\mathbf{Z}_2$ symmetry $(X, Y) \to -(X, Y)$ and the other $\mathbf{Z}_2$ symmetry ($\vec{A} \to -\vec{A}$) in Eq. (B9) are completely equivalent.

To close completely the gap, here are the explicit relations among the HT parameters (w, $\gamma_{1,2}$, ξ) and those of ASY construction (α, β, α_1, β_1) discussed in the Appendix A:

$$\tan^{-1/2}\frac{\alpha}{2} = \sqrt{\frac{|w|\sqrt{1+|w|^2}}{\sqrt{1-|w|^2}}}; \tag{B13}$$

$$\gamma_1 = \frac{\beta_1 - \beta + \pi}{2}, \qquad \gamma_2 = -\frac{\beta_1 + \beta - \pi}{2}, \qquad \xi = \frac{\alpha_1}{2}. \tag{B14}$$

APPENDIX C: MATRIX REPRESENTATION

As was already emphasized, all the moduli parameters are contained in the moduli matrix. The positions of k vortices are given by the zeros $\{z_i\}$ of the determinant $P(z) = \det H_0(z)$ of the moduli matrix: $P(z = z_i) = 0$, and the orientations $\{\vec{\phi}_i\}$ are given by $H_0(z = z_i)\vec{\phi}_i = \vec{0}$, a null vector at the vortex positions in Eq. (2.4). This is a nice feature of our approach, as long as all the vortices are separated. However, it does not give us a good picture when the vortex axes overlap, as we have seen already.

In this Appendix we will explain a systematic method to extract moduli parameters from the moduli matrix. A general introduction to this method was given in [22].

1. The case of $U(2)$

Let us first extend the orientational vector $\vec{\phi}_i$ which is the *constant* vector in Eq. (2.4) to a vector $\vec{\phi}_i(z)$ whose elements are not constants but *holomorphic* polynomials of z of order $O(z^{k-1})$:

$$H_0(z)\vec{\phi}_i(z) = J_i(z)P(z) \equiv 0, \qquad \mathrm{Mod}[P(z)]. \tag{C1}$$

for some holomorphic $J_i(z)$. This extended definition of the orientational vector reduces to Eq. (2.4) when we set $z = z_i$, since $P(z = z_i) = 0$. The number of the linearly independent vectors $\vec{\phi}_i(z)$ is the same as the degree of the polynomial $P(z)$, so that index i runs from 1 to 2 for the $k = 2$ vortices.

Introduce an N by $k(=2)$ holomorphic matrix $\Phi(z)$ from $\vec{\phi}_i(z)$ as

$$\Phi(z) = (\vec{\phi}_1(z), \vec{\phi}_2(z)). \tag{C2}$$

Namely, $\Phi(z)$ satisfies the relation

$$H_0(z)\Phi(z) \equiv 0, \qquad \mathrm{Mod}[P(z)], \qquad P(z) = \det H_0(z). \tag{C3}$$

One can construct two constant matrices $\mathbf{Z}$ which is a $k(=2)$ by $k(=2)$ matrix and Ψ which is a N by $k(=2)$ matrix from $\Phi(z)$ as follows.

$$z\Phi(z) = \Phi(z)\mathbf{Z} + \Psi P(z). \tag{C4}$$

For example, we can choose the following matrix satisfying Eq. (C3) with the moduli matrix $H_0^{(0,2)}(z)$ in Eq. (3.1)

$$\Phi^{(0,2)}(z) = \begin{pmatrix} bz - b\alpha + a\beta & az + b \\ z - \alpha & 1 \end{pmatrix}. \tag{C5}$$

Here, $\vec{\phi}_2(z)^{\mathrm{T}} = (az + b, 1)$ is a straightforward solution for Eq. (C1) since $\phi_2(z_i)$ $(i = 1, 2)$ are just the two orientational vectors given by Eq. (3.4) and $\vec{\phi}_1(z)$ is given by $(z - \alpha)\vec{\phi}_2(z)$ with modulo $P(z) = z^2 - \alpha z - \beta$. According to the prescription given in Eq. (C4), two matrices $\mathbf{Z}^{(0,2)}$ and $\Psi^{(0,2)}$ can be constructed as follows:

$$\begin{aligned} z\Phi^{(0,2)}(z) &= \begin{pmatrix} bz^2 - b\alpha z + a\beta z & az^2 + bz \\ z^2 - \alpha z & z \end{pmatrix} = \begin{pmatrix} b(z^2 - P(z)) - b\alpha z + a\beta z & a(z^2 - P(z)) + bz \\ (z^2 - P(z)) - \alpha z & z \end{pmatrix} + P(z)\begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \beta(az + b) & (a\alpha + b)z + a\beta \\ \beta & z \end{pmatrix} + P(z)\begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix} = \Phi^{(0,2)}(z)\begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix} + P(z)\begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{C6}$$

Thus we obtain $\mathbf{Z}^{(0,2)}$ and $\Psi^{(0,2)}$ corresponding to $H_0^{(0,2)}(z)$:

$$\mathbf{Z}^{(0,2)} = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}, \qquad \Psi^{(0,2)} = \begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix}. \tag{C7}$$

Let us turn our attention to another patch $H_0^{(1,1)}(z)$, Eq. (3.1). The corresponding orientational matrix $\Phi^{(1,1)}(z)$ is

$$\Phi^{(1,1)}(z) = \begin{pmatrix} z - \tilde{\phi} & \eta \\ \tilde{\eta} & z - \phi \end{pmatrix}. \tag{C8}$$

One can verify that $H_0^{(1,1)}(z)\Phi(z) \equiv \mathbf{0}$ with modulo $P(z) = (z - \phi)(z - \tilde{\phi}) - \eta\tilde{\eta}$. Again two matrices $\mathbf{Z}$ and Ψ satisfying Eq. (C4) can be found:

$$\begin{aligned} z\Phi^{(1,1)}(z) &= \begin{pmatrix} z^2 - \tilde{\phi}z & \eta z \\ \tilde{\eta}z & z^2 - \phi z \end{pmatrix} = \begin{pmatrix} (z^2 - P(z)) - \tilde{\phi}z + P(z) & \eta z \\ \tilde{\eta}z & (z^2 - P(z)) - \phi z + P(z) \end{pmatrix} \\ &= \begin{pmatrix} \phi z - \phi\tilde{\phi} + \eta\tilde{\eta} & \eta z \\ \tilde{\eta}z & \tilde{\phi}z - \phi\tilde{\phi} + \eta\tilde{\eta} \end{pmatrix} + P(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Phi^{(1,1)}(z)\begin{pmatrix} \phi & \eta \\ \tilde{\eta} & \tilde{\phi} \end{pmatrix} + P(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{C9}$$

$\mathbf{Z}^{(1,1)}$ and $\Psi^{(1,1)}$,

$$\mathbf{Z}^{(1,1)} = \begin{pmatrix} \phi & \eta \\ \tilde{\eta} & \tilde{\phi} \end{pmatrix}, \qquad \Psi^{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{C10}$$

have the same information as the moduli matrix $H_0^{(1,1)}(z)$. Finally, from the orientational matrix

$$\Phi^{(2,0)}(z) = \begin{pmatrix} z - \alpha' & 1 \\ b'z - b'\alpha' + a'\beta' & a'z + b' \end{pmatrix}. \tag{C11}$$

for the last patch $H_0^{(2,0)}(z)$ in Eq. (3.1), one gets

$$\mathbf{Z}^{(2,0)} = \begin{pmatrix} 0 & 1 \\ \beta' & \alpha' \end{pmatrix}, \qquad \Psi^{(2,0)} = \begin{pmatrix} 1 & 0 \\ b' & a' \end{pmatrix}. \tag{C12}$$

Summarizing,

$$\begin{pmatrix} \Psi_{[N\times k]} \\ \mathbf{Z}_{[k\times k]} \end{pmatrix} = \begin{pmatrix} b & a \\ 1 & 0 \\ 0 & 1 \\ \beta & \alpha \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \phi & \eta \\ \tilde{\eta} & \tilde{\phi} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ b' & a' \\ 0 & 1 \\ \beta & \alpha \end{pmatrix}. \tag{C13}$$

As was shown before, the moduli matrix $H_0^{(0,2)}(z)$, $H_0^{(1,1)}(z)$ and $H_0^{(2,0)}(z)$ are connected by V-equivalence relation ($H_0(z) \sim V(z)H_0(z)$ with $V(z) \in GL(N = 2, \mathbf{C})$). This leads to the transition functions between moduli parameters, see Eqs. (3.2) and (3.3). From the view point of the matrices $\mathbf{Z}$ and Ψ, the transition functions between them are given by the $GL(k = 2, \mathbf{C})$ equivalence relation

$$\mathbf{Z} \sim \mathcal{V}\mathbf{Z}\mathcal{V}^{-1}, \qquad \Psi \sim \Psi\mathcal{V}^{-1}, \tag{C14}$$

with $\mathcal{V} \in GL(k = 2, \mathbf{C})$. This $\mathcal{V}$-equivalence relation comes from ambiguity in the definition of $\Phi(z)$ in Eq. (C3). In fact, $\Phi'(z) = \Phi(z)\mathcal{V}$ satisfies the same relation as Eq. (C3), so that $\Phi(z)$ and $\Phi(z)\mathcal{V}$ must be identified. Since the two matrices $\mathbf{Z}$ and Ψ are obtained from $z\Phi(z) = \Phi(z)\mathbf{Z} + P(z)\Psi$, we reach the $\mathcal{V}$-equivalence relation, Eq. (C14).

As an example, let us reproduce the transition function from the $\mathcal{U}^{(1,1)}$ patch to the $\mathcal{U}^{(0,2)}$ patch, Eq. (3.2). The transition matrix from $(\Psi^{(1,1)}, \mathbf{Z}^{(1,1)})$ to $(\Psi^{(0,2)}, \mathbf{Z}^{(0,2)})$ is

$$\mathcal{V}^{-1} = \begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix}.$$

One can easily show that the transition function Eq. (3.2) is the equivalent to $\mathbf{Z}^{(0,2)} = \mathcal{V}\mathbf{Z}^{(1,1)}\mathcal{V}^{-1}$:

$$\begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} \tilde{\phi} + b\tilde{\eta} & a\tilde{\eta} \\ \frac{\eta + b\phi - b(\tilde{\phi}+b\tilde{\eta})}{a} & \phi - b\tilde{\eta} \end{pmatrix}. \tag{C15}$$

$\mathbf{Z}$ and Ψ have a simple physical meaning. First note that the relation

$$\det(z\mathbf{1} - \mathbf{Z}) = P(z) \tag{C16}$$

with $P(z) = \det H_0(z)$. This means that zeros z_i of $P(z)$, namely, the position of the vortices, can be obtained as eigenvalues of the matrix $\mathbf{Z}$. As an example, let us diagonalize the matrix

$$\mathbf{Z}^{(0,2)} \to \mathcal{V}\mathbf{Z}^{(0,2)}\mathcal{V}^{-1} = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$$

by

$$\mathcal{V}^{-1} = \begin{pmatrix} 1 & 1 \\ z_1 & z_2 \end{pmatrix}$$

with $\alpha = z_1 + z_2$, $\beta = -z_1 z_2$. At the same time the other matrix $\Psi^{(0,2)}$ is transformed as follows

$$\Psi^{(0,2)} \to \Psi^{(0,2)}\mathcal{V}^{-1} = \begin{pmatrix} az_1 + b & az_2 + b \\ 1 & 1 \end{pmatrix}. \tag{C17}$$

The column vectors

$$\vec{\phi}_1 = \begin{pmatrix} az_1 + b \\ 1 \end{pmatrix}$$

and

$$\vec{\phi}_2 = \begin{pmatrix} az_2 + b \\ 1 \end{pmatrix}$$

are nothing but the orientational vectors given in Eq. (3.4) which are defined at the vortex positions. We conclude that the eigenvalues of the matrix $\mathbf{Z}$ are the positions (of the center) of the vortices while Ψ is related to the "orientation" of the vortices defined there.

When the two vortex centers coincide, the matrices reduce to the following form:

$$\begin{pmatrix} \Psi \\ \mathbf{Z} \end{pmatrix} = \begin{pmatrix} b & a \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -XY & X^2 \\ -Y^2 & XY \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ b' & a' \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{C18}$$

2. The case of $U(3)$

Let us next check the correspondence between the result obtained above in terms of the matrices $\mathbf{Z}$ and Ψ and the result from the viewpoint of the moduli matrix $H_0(z)$, for $U(3)$. There are six patches $\tilde{\mathcal{U}}^{(2,0,0)}$, $\tilde{\mathcal{U}}^{(0,2,0)}$, $\tilde{\mathcal{U}}^{(0,0,2)}$, $\tilde{\mathcal{U}}^{(1,1,0)}$, $\tilde{\mathcal{U}}^{(0,1,1)}$ and $\tilde{\mathcal{U}}^{(1,0,1)}$ for $k = 2$ coaxial vortices in the $U(3)$ gauge theory. These are given by

$$H_0^{(0,0,2)}(z) = \begin{pmatrix} 1 & 0 & -a_1 z - b_1 \\ 0 & 1 & -a_2 z - b_2 \\ 0 & 0 & z^2 \end{pmatrix}, \quad H_0^{(1,1,0)}(z) = \begin{pmatrix} z + XY & -X^2 & 0 \\ Y^2 & z - XY & 0 \\ -\gamma & -\chi & 1 \end{pmatrix}, \tag{C19}$$

$$H_0^{(0,2,0)}(z) = \begin{pmatrix} 1 & -a_1' z - b_1' & 0 \\ 0 & z^2 & 0 \\ 0 & -a_2' z - b_2' & 1 \end{pmatrix}, \quad H_0^{(1,0,1)}(z) = \begin{pmatrix} z + X'Y' & 0 & -X'^2 \\ -\gamma' & 1 & -\chi' \\ Y'^2 & 0 & z - X'Y' \end{pmatrix}, \tag{C20}$$

$$H_0^{(2,0,0)}(z) = \begin{pmatrix} z^2 & 0 & 0 \\ -a_1'' z - b_1'' & 1 & 0 \\ -a_2'' z - b_2'' & 0 & 1 \end{pmatrix}, \quad H_0^{(0,1,1)}(z) = \begin{pmatrix} 1 & -\gamma'' & -\chi'' \\ 0 & z + X''Y'' & -X''^2 \\ 0 & Y''^2 & z - X''Y'' \end{pmatrix} \tag{C21}$$

with identifications $(X, Y) \sim (-X, -Y)$, $(X', Y') \sim (-X', -Y')$ and $(X'', Y'') \sim (-X'', -Y'')$. The determinant of these matrices is equal to z^2, meaning that two vortices are sitting on the origin of z plane.

The corresponding matrices $\mathbf{Z}$ which is $2(=k)$ by $2(=k)$ matrix and Ψ which is $3(=N)$ by $2(=k)$ matrix for these moduli matrices can be constructed through the relation

$$z\Phi(z) = \Phi(z)\mathbf{Z} + P(z)\Psi \tag{C22}$$

with $P(z) = z^2$. Here $\Phi(z)$ is the orientational matrix defined by

$$H_0(z)\Phi(z) \equiv 0 \tag{C23}$$

with modulo $P(z) = z^2$. We start with the patch $H^{(1,1,0)}(z)$. The orientational matrix Φ can be chosen as follows

$$\Phi^{(1,1,0)}(z) = \begin{pmatrix} z - XY & X^2 \\ -Y^2 & z + XY \\ \gamma z - Y(\gamma X + \chi Y) & \chi z - X(\gamma X + \chi Y) \end{pmatrix}. \tag{C24}$$

It can be verified that this $\Phi^{(1,1,0)}(z)$ actually satisfies the equation Eq. (C20). According to the equation Eq. (C19), the matrices $\mathbf{Z}$ and Ψ can be found as follows:

$$z\Phi^{(1,1,0)}(z) = \Phi^{(1,1,0)}(z)\begin{pmatrix} -XY & X^2 \\ -Y^2 & XY \end{pmatrix} + z^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \gamma & \chi \end{pmatrix} \tag{C25}$$

so

$$\mathbf{Z}^{(1,1,0)} = \begin{pmatrix} -XY & X^2 \\ -Y^2 & XY \end{pmatrix} = \epsilon \begin{pmatrix} -Y \\ X \end{pmatrix} (-Y \quad X), \qquad \Psi^{(1,1,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \gamma & \chi \end{pmatrix}, \tag{C26}$$

with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the 2 by 4 matrix M given in Eq. (4.7) is of the form

$$M^{(1,1,0)} = \begin{pmatrix} 1 & 0 & \gamma & -Y \\ 0 & 1 & \chi & X \end{pmatrix}. \tag{C27}$$

Similarly other matrices corresponding to $H_0^{(1,0,1)}(z)$ and $H_0^{(0,1,1)}(z)$ can be found:

$$M^{(1,0,1)} = \begin{pmatrix} 1 & \gamma' & 0 & -Y' \\ 0 & \chi' & 1 & X' \end{pmatrix}, \qquad M^{(0,1,1)} = \begin{pmatrix} \gamma'' & 1 & 0 & -Y'' \\ \chi'' & 0 & 1 & X'' \end{pmatrix}. \tag{C28}$$

Let us next move to the other patch $H^{(0,0,2)}(z)$. The corresponding orientational matrix Φ is given by

$$\Phi^{(0,0,2)}(z) = \begin{pmatrix} b_1 z & a_1 z + b_1 \\ b_2 z & a_2 z + b_2 \\ z & 1 \end{pmatrix}. \tag{C29}$$

One can verify that this $\Phi^{(0,0,2)}(z)$ actually satisfies the equation Eq. (C20). According to the equation Eq. (C19), we find the matrices $\mathbf{Z}$ and Ψ as follows

$$z\Phi^{(0,0,2)}(z) = \Phi^{(0,0,2)}(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} b_1 & a_1 \\ b_2 & a_2 \\ 1 & 0 \end{pmatrix}, \tag{C30}$$

$$\mathbf{Z}^{(0,0,2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Psi^{(0,0,2)} = \begin{pmatrix} b_1 & a_1 \\ b_2 & a_2 \\ 1 & 0 \end{pmatrix} \Rightarrow M^{(0,0,2)} = \begin{pmatrix} b_1 & b_2 & 1 & 0 \\ a_1 & a_2 & 0 & 1 \end{pmatrix}. \tag{C31}$$

Similarly,

$$M^{(0,2,0)} = \begin{pmatrix} b_1' & 1 & b_2' & 0 \\ a_1' & 0 & a_2' & 1 \end{pmatrix}, \qquad M^{(2,0,0)} = \begin{pmatrix} 1 & b_1'' & b_2'' & 0 \\ 0 & a_1'' & a_2'' & 1 \end{pmatrix}. \tag{C32}$$

These are summarized as follows

$$\begin{pmatrix} \tau_{12} \\ \tau_{23} \\ \tau_{13} \\ \tau_{14} \\ \tau_{24} \\ \tau_{34} \end{pmatrix} \sim \begin{pmatrix} 1 \\ -\gamma \\ \chi \\ X \\ Y \\ \gamma X + \chi Y \end{pmatrix} \sim \begin{pmatrix} \chi' \\ \gamma' \\ 1 \\ X' \\ \gamma' X' + \chi' Y' \\ Y' \end{pmatrix} \sim \begin{pmatrix} -\chi'' \\ 1 \\ \gamma'' \\ \gamma'' X'' + \chi'' Y'' \\ X'' \\ Y'' \end{pmatrix} \sim \begin{pmatrix} b_1 a_2 - b_2 a_1 \\ -a_2 \\ -a_1 \\ b_1 \\ b_2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -a_1' \\ a_2' \\ b_1' a_2' - b_2' a_1' \\ b_1' \\ 1 \\ b_2' \end{pmatrix} \sim \begin{pmatrix} a_1'' \\ b_1'' a_2'' - b_2'' a_1'' \\ a_2'' \\ 1 \\ b_1'' \\ b_2'' \end{pmatrix}. \tag{C33}$$

The transition functions between these can be easily found via the weighted equivalence relation Eq. (4.29). For example, the transition function from $M^{(1,1,0)}$ to $M^{(1,0,1)}$ is given by $\lambda = \chi^{-1/2}$:

$$\chi' = \frac{1}{\chi}, \qquad \gamma' = -\frac{\gamma}{\chi}, \qquad X' = \frac{X}{\chi^{\frac{1}{2}}}, \qquad Y' = \frac{\gamma X + \chi Y}{\chi^{\frac{1}{2}}}. \tag{C34}$$

Similarly, the transition function from $M^{(1,1,0)}$ to $M^{(0,2,0)}$ is given by $\lambda = Y^{-1}$:

$$a_1' = -\frac{1}{Y^2}, \qquad a_2' = -\frac{\gamma}{Y^2}, \qquad b_1' = \frac{X}{Y}, \qquad b_2' = \frac{\gamma X + \chi Y}{Y}. \tag{C35}$$

All other transition functions can be obtained in an analogous way.

3. The case of $U(N)$

In the general $U(N)$ case there are always two kind of patches for the moduli matrix $H_0(z)$: N patches with a z^2 factor on the diagonal, which we denote $\tilde{\mathcal{U}}^{(i)}$ (i indicates the position of z^2 on the diagonal) and $N(N-1)/2$ patches with 2 diagonal elements of the form $z - c$, which we denote $\tilde{\mathcal{U}}^{(j,k)}$, $j < k$ (j, k indicate the position of the nontrivial diagonal elements). More explicitly

$$H_0^{(i)}(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -a_1^{(i)}z - b_1^{(i)} & 0 & \cdots \\ 0 & 1 & & & \vdots & & \\ \vdots & & \ddots & & -a_{i-1}^{(i)}z - b_{i-1}^{(i)} & & \\ \vdots & & & & z^2 & & \\ \vdots & & & & -a_{i+1}^{(i)}z - b_{i+1}^{(i)} & & \\ \vdots & & & & \vdots & & \end{pmatrix}, \tag{C36}$$

$$H_0^{(j,k)}(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\gamma_1^{(jk)} & 0 & \cdots & 0 & -\chi_1^{(jk)} & 0 & \cdots \\ 0 & 1 & & & \vdots & & & & \vdots & & \\ \vdots & & \ddots & & -\gamma_{j-1}^{(jk)} & & & & -\chi_{j-1}^{(jk)} & & \\ \vdots & & & & z + X^{(jk)}Y^{(jk)} & & & & -(X^{(jk)})^2 & & \\ \vdots & & & & -\gamma_{j+1}^{(jk)} & & & & -\chi_{j+1}^{(jk)} & & \\ \vdots & & & & \vdots & & & & \vdots & & \\ \vdots & & & & -\gamma_{k-1}^{(jk)} & & & & -\chi_{k-1}^{(jk)} & & \\ \vdots & & & & (Y^{(jk)})^2 & & & & z - X^{(jk)}Y^{(jk)} & & \\ \vdots & & & & -\gamma_{k+1}^{(jk)} & & & & -\chi_{k+1}^{(jk)} & & \\ \vdots & & & & \vdots & & & & \vdots & & \end{pmatrix}. \tag{C37}$$

It is possible to find the transition functions among the moduli in the different patches using appropriate V-transformations. It turns out that only a subset of transition functions is needed, then the others can be recovered using composition and inversion of the known ones. In particular we need

(i) $\tilde{\mathcal{U}}^{(i)} \to \tilde{\mathcal{U}}^{(j)}$

$$a_k^{(j)} = \frac{a_k^{(i)}b_j^{(i)} - a_j^{(i)}b_k^{(i)}}{(b_j^{(i)})^2}, \qquad k \neq i; \qquad a_i^{(j)} = -\frac{a_j^{(i)}}{(b_j^{(i)})^2}, \tag{C38}$$

$$b_k^{(j)} = \frac{b_k^{(i)}}{b_j^{(i)}}, \qquad k \neq i; \qquad b_i^{(j)} = \frac{1}{b_j^{(i)}}. \tag{C39}$$

(ii) $\tilde{\mathcal{U}}^{(i)} \to \tilde{\mathcal{U}}^{(k,i)}$

$$X^{(ki)} = \pm i\frac{b_k^{(i)}}{\sqrt{a_k^{(i)}}}; \qquad Y^{(ki)} = \pm\frac{1}{\sqrt{a_k^{(i)}}}, \tag{C40}$$

$$\gamma_l^{(ki)} = \frac{a_l^{(i)}}{a_k^{(i)}}; \qquad \chi_l^{(ki)} = b_l^{(i)} - \frac{a_l^{(i)}b_k^{(i)}}{a_k^{(i)}}. \tag{C41}$$

(iii) $\tilde{\mathcal{U}}^{(i)} \to \tilde{\mathcal{U}}^{(j,k)}$, $j, k \neq i$

$$X^{(jk)} = \pm\frac{b_j^{(i)}}{\sqrt{d_{jk}^{(i)}}}; \qquad Y^{(jk)} = \pm\frac{b_k^{(i)}}{\sqrt{d_{jk}^{(i)}}}, \tag{C42}$$

$$\gamma_l^{(jk)} = \frac{a_k^{(i)}b_l^{(i)} - a_l^{(i)}b_k^{(i)}}{\sqrt{-d_{jk}^{(i)}}}, \quad l \neq i; \quad \gamma_i^{(jk)} = \frac{a_k^{(i)}}{\sqrt{-d_{jk}^{(i)}}}, \tag{C43}$$

$$\chi_l^{(jk)} = \frac{a_j^{(i)}b_l^{(i)} - a_l^{(i)}b_j^{(i)}}{\sqrt{d_{jk}^{(i)}}}, \qquad l \neq i; \qquad \chi_i^{(jk)} = \frac{a_j^{(i)}}{\sqrt{d_{jk}^{(i)}}}, \tag{C44}$$

with $d_{jk}^{(i)} \equiv -a_j^{(i)}b_k^{(i)} + a_k^{(i)}b_j^{(i)}$.

On the other hand, given the general form of $H_0(z)$, Eq. (C36) and (C37), the moduli can always be collected into the 2 by $N+1$ matrix $M = (\Psi^T, \upsilon)$ with the usual procedure ($\mathbf{Z} = \epsilon\upsilon\upsilon^T$). If we denote $M^{(i)}$, $M^{(jk)}$ in the patches $\tilde{\mathcal{U}}^{(i)}$, $\tilde{\mathcal{U}}^{(j,k)}$ respectively we get

$$M^{(i)} = \begin{pmatrix} b_1^{(i)} & \cdots & b_{i-1}^{(i)} & 1 & b_{i+1}^{(i)} & \cdots & b_N^{(i)} & 0 \\ a_1^{(i)} & \cdots & a_{i-1}^{(i)} & 0 & a_{i+1}^{(i)} & \cdots & a_N^{(i)} & 1 \end{pmatrix}, \tag{C45}$$

$$M^{(jk)} = \begin{pmatrix} \gamma_1^{(jk)} & \cdots & \gamma_{j-1}^{(jk)} & 1 & \gamma_{j+1}^{(jk)} & \cdots & \gamma_{k-1}^{(jk)} & 0 & \gamma_{k+1}^{(jk)} & \cdots & \gamma_N^{(jk)} & -Y^{(jk)} \\ \chi_1^{(jk)} & \cdots & \chi_{j-1}^{(jk)} & 0 & \chi_{j+1}^{(jk)} & \cdots & \chi_{k-1}^{(jk)} & 1 & \chi_{k+1}^{(jk)} & \cdots & \chi_N^{(jk)} & X^{(jk)} \end{pmatrix}. \tag{C46}$$

The matrix M together with the weighted $GL(2, \mathbf{C})$, Eq. (4.8), defines the weighted Grassmannian manifold $WGr_{N+1,2}^{(1,\ldots,1,0)}$ and the $M^{(i)}$, $M^{(jk)}$ represent the standard covering of this space. One can pass from one patch to another by appropriate weighted $GL(2, \mathbf{C})$ transformation and so deduce the transition functions, which turn out to be the same of the moduli matrix representation, as expected. In particular it is possible to check that the transition functions listed above, which generate all the others, perfectly match with the corresponding ones of the $WGr_{N+1,2}^{(1,\ldots,1,0)}$.

We have thus explicitly pointed out that the moduli space of $k = 2$ vortices given by the moduli matrix $H_0(z)$ is indeed a weighted Grassmannian manifold $WGr_{N+1,2}^{(1,\ldots,1,0)}$. This enforces the general considerations coming from the established equivalence between the moduli matrix and the Kähler quotient construction [22].

APPENDIX D: PRODUCT OF MODULI MATRICES

Within the moduli matrix formalism, it is easy to construct vortices of higher winding number: the latter can be constructed from the moduli matrices of lower winding number as simple products. For instance, consider two fundamental vortices, and

$$H_0^{(1,0)} \times H_0^{(1,0)\prime} = \begin{pmatrix} z - z_0 & 0 \\ -b_0 & 1 \end{pmatrix}\begin{pmatrix} z - z_0' & 0 \\ -b_0' & 1 \end{pmatrix} = \begin{pmatrix} (z - z_0)(z - z_0') & 0 \\ -b_0 z + b_0 z_0' - b_0' & 1 \end{pmatrix}. \tag{D1}$$

Analogously for the product of two $(0, 1)$ vortices

$$H_0^{(0,1)} \times H_0^{(0,1)\prime} = \begin{pmatrix} 1 & -a_0 \\ 0 & z - z_0 \end{pmatrix}\begin{pmatrix} 1 & -a_0' \\ 0 & z - z_0' \end{pmatrix} = \begin{pmatrix} 1 & -a_0 z + a_0 z_0' - a_0' \\ 0 & (z - z_0)(z - z_0') \end{pmatrix}. \tag{D2}$$

By comparing these with $H_0^{(0,2)}$ or $H_0^{(2,0)}$ in Eq. (3.1), one finds

$$a = a_0, \qquad b = a_0 z_0' - a_0', \qquad \alpha = z_0 + z_0', \qquad \beta = -z_0 z_0'; \tag{D3}$$

$$a' = -b_0, \qquad b' = b_0 z_0' - b_0', \qquad \alpha = z_0 + z_0', \qquad \beta = -z_0 z_0'. \tag{D4}$$

Finally, for the product vortex of the type $(0, 1)$ times $(1, 0)$,

$$H_0^{(1,0)} \times H_0^{(0,1)\prime} = \begin{pmatrix} z - z_0 & 0 \\ -b_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a_0' \\ 0 & z - z_0' \end{pmatrix} = \begin{pmatrix} z - z_0 & -a_0'(z - z_0) \\ -b_0 & z - z_0' + b_0 a_0' \end{pmatrix}. \tag{D5}$$

Bringing it to the standard form by a $V(z)$ transformation

$$V = \begin{pmatrix} 1 & a_0' \\ 0 & 1 \end{pmatrix}, \tag{D6}$$

one has

$$H_0^{(1,1)} \sim H_0^{(1,0)} \times H_0^{(0,1)\prime} \simeq \begin{pmatrix} z - z_0 - b_0 a_0' & a_0'(z_0 - z_0') + a_0'^2 b_0 \\ -b_0 & z - z_0' + b_0 a_0' \end{pmatrix}. \tag{D7}$$

This has the same form as the middle form of Eq. (3.1), by identification

$$\eta = -a_0'(z_0 - z_0') - a_0'^2 b_0, \qquad \tilde{\eta} = b_0, \qquad \phi = z_0 + b_0 a_0', \qquad \tilde{\phi} = z_0' - b_0 a_0'. \tag{D8}$$

Note that

$$\phi + \tilde{\phi} = z_0 + z_0' = \alpha, \qquad \eta\tilde{\eta} - \phi\tilde{\phi} = -z_0 z_0' = \beta. \tag{D9}$$

in accord with the relations Eq. (3.2).

In fact, the transition function Eq. (3.3) between the sets (a, b, α, β) and $(\phi, \tilde{\phi}, \eta, \tilde{\eta})$ (patches ((0, 2) and (1, 1)) is simply a consequence of the transition function for the minimum vortex

$$b_0 = \frac{1}{a_0}, \tag{D10}$$

through the composition rule, Eqs. (D3) and (D8).

Analogously, to find the relation between the $(2, 0)$ and $(1, 1)$ patches, we first write $H_0^{(1,1)}$ as

$$H_0^{(1,0)} \times H_0^{(0,1)\prime} = \begin{pmatrix} 1 & -a_0 \\ 0 & z - z_0 \end{pmatrix}\begin{pmatrix} z - z_0' & 0 \\ -b_0' & 1 \end{pmatrix} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z - \tilde{\phi} \end{pmatrix}. \tag{D11}$$

The relation

$$a' = \frac{1}{\eta}; \qquad b' = -\frac{\phi}{\eta}; \qquad \alpha = \phi + \tilde{\phi}; \qquad \beta = \eta\tilde{\eta} - \phi\tilde{\phi}, \tag{D12}$$

follows then easily from the transition rule $b_0 = 1/a_0$ between $(0, 1)$ and $(1, 0)$ patches. Finally the relation Eq. (3.3) follows by composing Eq. (3.2) and (D12).

In the case of coaxis vortices, one gets, by eliminating the center-of mass position and the relative position (by setting $z_0 = z_0' = 0$),

$$H_0^{(1,1)} \sim \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z + \phi \end{pmatrix}, \qquad \phi^2 + \eta\tilde{\eta} = 0. \quad \text{(D13)}$$

Note that in this construction ($H_0^{(1,1)} \sim H_0^{(0,1)} \times H_0^{(0,1)}$), the constraint $\phi^2 + \eta\tilde{\eta} = 0$ is automatically satisfied once we set $z_0 = z_0' = 0$ due to the identification Eq. (D8).

These discussions simply show that the moduli matrices have a natural property under the product. Thus

$$H_0^{(m,n)} \sim H_0^{(m_1,n_1)} \times H_0^{(m_2,n_2)}, \qquad m_1 + m_2 = m; \quad n_1 + n_2 = n. \quad \text{(D14)}$$

The product moduli is simply

$$\mathcal{M}_1 \times \mathcal{M}_2, \quad \text{(D15)}$$

as long as no constraints (such as the coincident axes) are imposed. The transition functions between the "neighboring" patches (say $(m+1, n)$ and $(m, n+1)$) can be always reduced to the simple relation between $H_0^{(1,0)}$ and $H_0^{(0,1)}$.

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