

Infrared freezing of Euclidean QCD observables

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(Received 3 May 2006; published 18 September 2006)

We consider the leading one-chain term in a skeleton expansion for QCD observables and show that for energies $Q^2 > \Lambda^2$, where $Q^2 = \Lambda^2$ is the Landau pole of the coupling, the skeleton expansion result is equivalent to the standard Borel integral representation, with ambiguities related to infrared (IR) renormalons. For $Q^2 < \Lambda^2$ the skeleton expansion result is equivalent to a previously proposed modified Borel representation where the ambiguities are connected with ultraviolet (UV) renormalons. We investigate the Q^2 -dependence of the perturbative corrections to the Adler- D function, the GLS sum rule and the polarized and unpolarized Bjorken sum rules. In all these cases the one-chain result changes sign in the vicinity of $Q^2 = \Lambda^2$, and then exhibits freezing behavior, vanishing at $Q^2 = 0$. Finiteness at $Q^2 = \Lambda^2$ implies specific relations between the residues of IR and UV renormalon singularities in the Borel plane. These relations, only one of which has previously been noted (though it remained unexplained), are shown to follow from the continuity of the characteristic function in the skeleton expansion. By considering the compensation of nonperturbative and perturbative ambiguities we are led to a result for the Q^2 -dependence of these observables at all Q^2 , in which there is a single undetermined nonperturbative parameter, and which involves the skeleton expansion characteristic function. The observables freeze to zero in the infrared. We briefly consider the freezing behavior of the Minkowskian $R_{e^+e^-}$ ratio.

DOI: [10.1103/PhysRevD.74.065012](https://doi.org/10.1103/PhysRevD.74.065012)

PACS numbers: 12.38.-t, 11.15.Pg, 11.55.Hx, 12.38.Lg

I. INTRODUCTION

Thanks to asymptotic freedom, fixed-order QCD perturbation theory can potentially provide accurate approximations to physical observables at suitably large energy scales, Q^2 . Such a perturbative description necessarily breaks down below the Landau singularity at $Q^2 = \Lambda^2$, and the infrared behavior unavoidably involves nonperturbative effects. In fact nonperturbative information is needed even to make sense of perturbation theory, since higher perturbative coefficients exhibit factorial growth, and the perturbation series is not convergent. Using a Borel integral to represent the resummed perturbation series, the Borel integral is ambiguous due to singularities on the integration contour along the positive real semiaxis in the Borel plane, so-called infrared (IR) renormalons. These ambiguities are structurally the same as terms in the operator product expansion (OPE) in powers of Λ^2/Q^2 . OPE ambiguities and Borel representation ambiguities can compensate each other, allowing the perturbative Borel and nonperturbative OPE components to be separately well-defined once a regulation of the Borel integral, such as principal value (PV), has been chosen [1]. For $Q^2 < \Lambda^2$, however, the Borel representation which is correlated with terms in the OPE breaks down. In a recent paper Ref. [2], which focussed on the infrared freezing of the Minkowskian $R_{e^+e^-}$ ratio, it was suggested that below $Q^2 = \Lambda^2$ one should use a modified Borel representation whose ambiguities come from singularities lying on the integration contour along the negative real semiaxis, so-

called ultraviolet (UV) renormalons. This Borel representation has ambiguities which are structurally the same as a modified expansion in powers of Q^2/Λ^2 , and once regulated both components can remain defined in the infrared. This change of Borel representation has been claimed not to be physically motivated in Ref. [3], where different conclusions about infrared behavior are reached. In this paper we shall show that if we postulate a QCD skeleton expansion [4,5], then the leading one-chain term reproduces the standard Borel representation for $Q^2 > \Lambda^2$, and the proposed modified Borel representation for $Q^2 < \Lambda^2$.

We consider the infrared behavior of the one-chain result for some Euclidean QCD observables. We shall concentrate on the Adler- D function, the GLS sum rule and the polarized and unpolarized Bjorken sum rules [6,7]. The skeleton expansion result automatically freezes to zero as $Q^2 \rightarrow 0$. For the observables we consider, the freezing to zero occurs after the Borel resummed perturbative corrections to the parton model result change sign in the vicinity of $Q^2 = \Lambda^2$. Individual renormalon contributions to the Borel integral diverge at $Q^2 = \Lambda^2$, but we find that when all of the renormalons are summed over, one obtains a finite result. This finiteness requires relations between the residues of infrared and ultraviolet renormalons. Only one of these relations has previously been noted [8], and we show that they arise from the continuity of the characteristic function in the skeleton expansion. Considering the compensation of perturbative and OPE ambiguities alluded to above, we are led to an expression for the Q^2 -dependence of the observable written in terms of the characteristic function, and containing a single undetermined nonperturbative parameter. This result freezes to zero in the infrared. Existing discussions of infrared freez-

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ing behavior have largely focused on the Analytic Perturbation Theory (APT) approach [9]. In this formalism one expands observables in a basis of functions which have smooth infrared behavior. For Euclidean observables the unphysical Landau singularity in the coupling is cancelled by a powerlike correction. In contrast in our discussion finiteness and continuity emerge thanks to a subtle interplay between UV and IR renormalons.

The plan of the paper is as follows. In Sec. II we shall introduce the QCD skeleton expansion, and show that the one-chain leading term is equivalent to the standard Borel representation for $Q^2 > \Lambda^2$, and to the modified representation for $Q^2 < \Lambda^2$. We discuss what can be learnt about the infrared freezing of observables. In Sec. III we describe the Borel plane renormalon structure for our chosen Euclidean observables, and we show that finiteness at $Q^2 = \Lambda^2$ only holds if there are cancellations between the residues of IR and UV renormalons, the cancellations rely on a previously unknown relation between the IR and UV residues. We write down a result for the Q^2 dependence of the resummed observables in terms of Exponential Integral (Ei) functions, and plot the infrared freezing behavior to zero noted above. In Sec. IV we consider the skeleton expansion for the Adler D function, and give an expression for the characteristic function of the leading one-chain term. Making a power series expansion, and changing variables, we explicitly obtain the Borel representations, and relate the IR and UV renormalon residues to the power series coefficients of the characteristic function. Continuity of the characteristic function is shown to underwrite the relations between UV and IR renormalon residues noted above. In Sec. V we derive the result for Q^2 -dependence including nonperturbative effects mentioned above. In Sec. VI we briefly consider Minkowskian observables, specifically $R_{e^+e^-}$, and modify some of the conclusions of Ref. [2] in the light of the criticisms of Ref. [3]. Section VII contains a discussion and our conclusions.

II. QCD SKELETON EXPANSION AND BOREL REPRESENTATIONS

Consider a generic Euclidean QCD observable $\mathcal{D}(Q^2)$ having the perturbative expansion

$$\mathcal{D}_{PT}(Q^2) = a(Q^2) + \sum_{n>0} d_n a^{n+1}(Q^2). \quad (1)$$

Here $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ is the renormalized coupling. Throughout this paper we will use the one-loop approximation for the coupling,

$$a(Q^2) = \frac{2}{b \ln(Q^2/\Lambda^2)}, \quad (2)$$

where $b = (33 - 2N_f)/6$ is the leading beta-function coefficient in SU(3) QCD with N_f active quark flavours. $Q^2 \equiv -q^2 > 0$ is the single spacelike energy scale. As

$Q^2 \rightarrow \infty$ asymptotic freedom ensures that $\mathcal{D}(Q^2) \rightarrow 0$. Our interest is in the infrared limit $Q^2 \rightarrow 0$, and the infrared behavior of $\mathcal{D}(Q^2)$. Specifically, is it possible that freezing to a finite infrared limit $\mathcal{D}(0)$ occurs? This is an intrinsically nonperturbative question which cannot be answered by perturbation theory alone. One has in addition the nonperturbative contribution arising from the operator product expansion (OPE),

$$\mathcal{D}_{NP}(Q^2) = \sum_n c_n \left(\frac{\Lambda^2}{Q^2}\right)^n. \quad (3)$$

The freezing limit, if any, of $\mathcal{D}(Q^2) = \mathcal{D}_{PT}(Q^2) + \mathcal{D}_{NP}(Q^2)$, depends on the behavior of *both* components as $Q^2 \rightarrow 0$. Perturbative freezing will not arise from fixed-order perturbation theory, one needs an all-orders resummation of Eq. (1). Unfortunately our exact information about the higher-order coefficients is limited, at best, to calculations of d_1 and d_2 , higher-orders are unknown. All-orders information is only available in the large- N_f limit where one expands each d_n as

$$d_n = d_n^{[n]} N_f^n + d_n^{[n-1]} N_f^{n-1} + \dots + d_n^{[0]}. \quad (4)$$

The leading large- N_f coefficient $d_n^{[n]}$ can be computed exactly to all-orders since it arises from a restricted set of Feynman diagrams in which a chain of n fermion bubbles (shown in Fig. 1) is inserted in a basic skeleton diagram [10,11]. In principle one can consider more than one chain and construct a QED skeleton expansion [12]. In QCD one can replace N_f by $(33/2 - 3b)$, and obtain an expansion in powers of b ,

$$d_n = d_n^{(n)} b^n + d_n^{(n-1)} b^{n-1} + \dots + d_n^{(0)}. \quad (5)$$

The leading- b term $d_n^{(L)} \equiv d_n^{(n)} b^n$ can then be used to approximate d_n [8,13,14] and an all-orders resummation of these terms performed to obtain $\mathcal{D}_{PT}^{(L)}(Q^2)$. Use of the one-loop form of the coupling in Eq. (2) ensures that this resummed result is RS-independent.

The leading term of the skeleton expansion arises from integrating over the momentum k flowing through the chain of bubbles [4,5,15].

$$\mathcal{D}_{PT}^{(L)}(Q^2) = \int_0^\infty dt \omega(t) a(e^C t Q^2). \quad (6)$$

Here $t \equiv k^2/Q^2$, and $\omega(t)$ is the so-called characteristic function of the observable. The constant C depends on the subtraction procedure used to renormalize the bubble. Standard \overline{MS} subtraction corresponds to $C = -5/3$.



FIG. 1. A chain of fermion bubbles with momentum k running through them.

From now on we shall assume $C = 0$ which corresponds to the so-called V -scheme, \overline{MS} subtraction with renormalization scale $\mu^2 = e^{-5/3} Q^2$. Λ in Eq. (2) will refer to that in the V -scheme. The characteristic function satisfies the normalization condition

$$\int_0^\infty dt \omega(t) = 1, \quad (7)$$

which ensures the leading $a(Q^2)$ coefficient of unity assumed in Eq. (1). The form of $\omega(t)$ changes at $t = 1$, and the range of integration splits into an IR and a UV part

$$\mathcal{D}_{PT}^{(L)}(Q^2) = \int_0^1 dt \omega_{IR}(t) a(tQ^2) + \int_1^\infty dt \omega_{UV}(t) a(tQ^2), \quad (8)$$

the IR part corresponding to $k^2 < Q^2$, and the UV part to $k^2 > Q^2$. By making a change of variable one can transform the leading skeleton term into a Borel representation. For $Q^2 > \Lambda^2$ one has the standard Borel representation (we shall explicitly write down the required changes of variable in Sec. IV),

$$\mathcal{D}_{PT}^{(L)}(Q^2) = \int_0^\infty dz e^{-z/a(Q^2)} B[\mathcal{D}_{PT}^{(L)}](z). \quad (9)$$

Here $B[\mathcal{D}](z)$ is the Borel transform, defined by,

$$B[\mathcal{D}_{PT}^{(L)}](z) = \sum_{n=0}^\infty \frac{z^n d_n^{(L)}}{n!}. \quad (10)$$

$B[\mathcal{D}_{PT}^{(L)}](z)$ contains singularities along the real z -axis. In the large- b approximation these are single and double poles at positions $z = z_n$ and $z = -z_n$, with $z_n \equiv 2n/b$, $n = 1, 2, 3 \dots$. The singularities on the positive real semiaxis are referred to as *infrared* renormalons, IR_n , and those on the negative real semiaxis as *ultraviolet* renormalons, UV_n . The IR_n renormalons cause the Borel representation to be ambiguous since they lie on the integration contour along the positive real z -axis. The difference between routing the contour above or below the singularity yields an ambiguity

$$\Delta \mathcal{D}_{PT}^{(L)} \sim \left(\frac{\Lambda^2}{Q^2} \right)^n, \quad (11)$$

which has the same form as a term in the OPE in Eq. (3), so that OPE ambiguities associated with the $(\Lambda^2/Q^2)^n$ OPE term in $\mathcal{D}_{NP}(Q^2)$ can potentially cancel against the IR_n renormalon ambiguity allowing each component separately to be well defined [1]. In practice we shall choose to take a Principal Value (PV) definition of the integral. The IR part of the t integration in Eq. (8) produces the IR renormalon part of the Borel representation, and needs to be PV regulated. The second UV component produces the UV renormalons and does not require regulation. As we shall see in the next section the standard Borel representation of Eq. (9) for Euclidean quantities diverges like

$\ln a(Q^2)$ at $Q^2 = \Lambda^2$ for each *individual* IR_n or UV_n renormalon contribution. When the full set is resummed, however, the $\ln a$ divergence is cancelled and a finite result is found. We shall explore this further in Secs. III and IV.

For $Q^2 < \Lambda^2$, $a(Q^2) < 0$, and the representation of Eq. (9) is invalid. The key point is that the change of variable from t to z is proportional to $a(Q^2)$, and so if $a(Q^2)$ changes sign the limits of integration in z change sign, yielding the modified Borel representation

$$\mathcal{D}_{PT}^{(L)}(Q^2) = \int_0^{-\infty} dz e^{-z/a(Q^2)} B[\mathcal{D}_{PT}^{(L)}](z). \quad (12)$$

This is the modified Borel representation proposed in Ref. [2] where it was motivated as a standard Borel representation corresponding to an expansion in $|a(Q^2)| = -a(Q^2)$, since by changing variables one can write Eq. (12) as

$$\mathcal{D}_{PT}^{(L)}(Q^2) = - \int_0^\infty dz e^{-z/|a(Q^2)|} B[\mathcal{D}_{PT}^{(L)}](-z). \quad (13)$$

So we see that the one-chain skeleton contribution of Eq. (6) is equivalent to the standard Borel representation of Eq. (9) for $Q^2 > \Lambda^2$, and to the modified representation of Eq. (12) for $Q^2 < \Lambda^2$. Note that when we substitute Eq. (10) into the Borel representation of Eq. (13), then it reproduces the correct form of the perturbative expansion in Eq. (1), for negative a . The modified Borel representation now has a contour of integration along the *negative* real semiaxis, and so it is rendered ambiguous by the ultraviolet UV_n renormalon singularities. Correspondingly the IR component of Eq. (8) is now well-defined and it is now the UV component which requires regulation. The ambiguity from routing the contour is now

$$\Delta \mathcal{D}_{PT}^{(L)}(Q^2) \sim \left(\frac{Q^2}{\Lambda^2} \right)^n. \quad (14)$$

It was suggested in Ref. [2] that the usual OPE of Eq. (3) breaks down for $Q^2 < \Lambda^2$, as does the associated PT Borel representation of Eq. (9), and should be recast and replaced by a modified expansion in powers of Q^2/Λ^2 ,

$$\mathcal{D}_{NP}(Q^2) = \sum_n \tilde{C}_n \left(\frac{Q^2}{\Lambda^2} \right)^n. \quad (15)$$

The n th term in this expansion has then structurally the same form as the ambiguity associated with the UV_n renormalon contribution. It was further suggested in Ref. [2] that a \tilde{C}_0 term independent of Q^2 could arise from rearrangement of the standard OPE. This was motivated by a simple toy example. In fact in the one-chain approximation no such term arises and both PT and NP components freeze to zero. The terms in Eq. (15) are then in one-to-one correspondence with the UV_n renormalon ambiguities. From its definition, the QCD skeleton expansion implies $\mathcal{D}_{PT}(0) = 0$ in the $Q^2 \rightarrow 0$ limit. For the one-

chain term in Eq. (6) this simply follows because as $Q^2 \rightarrow 0$ the integrand vanishes everywhere in the range of integration, since $a(tQ^2) \rightarrow 0$ for any given t . Higher multiple chain terms will contain products of the form $a(t_1 Q^2)a(t_2 Q^2) \dots$ in the integrand and will similarly vanish. This then implies that in the infrared limit $\mathcal{D}_{NP}(Q^2)$ behaves as

$$\mathcal{D}_{NP}(Q^2) \approx k \left(\frac{Q^2}{\Lambda^2} \right)^{n_0}, \quad (16)$$

where UV_{n_0} is the UV renormalon singularity nearest to the origin in the Borel plane. We should note that the modified Borel representation, its infrared behavior and its connection with UV renormalons, has also been discussed in Ref. [16]. In Appendix B of that paper the infrared freezing of the Adler function $D(Q^2)$ was discussed and it was concluded that from general arguments of nonperturbative spontaneous chiral symmetry breaking in the limit of a large number of colors, N_c , one expected that as $Q^2 \rightarrow 0$, $D(Q^2) \rightarrow 0$ like

$$D(Q^2) \sim \frac{Q^2}{M^2}, \quad (17)$$

where M is the mass of a one-meson state, these states remaining massive in the chiral limit. A similar result is obtained in Ref. [17]. Since UV_1 is the singularity nearest the origin for the Adler function, $n_0 = 1$, and the freezing expectation is indeed consistent with Eq. (16). Notice that strictly the leading behavior as $Q^2 \rightarrow 0$ is the logarithmic freezing to zero of $a(Q^2)$ contributed by the PT component. It is the nonperturbative effects which reflect the UV renormalon structure.

III. Q^2 -DEPENDENCE OF THE EUCLIDEAN OBSERVABLES

We begin by defining the three Euclidean observables we shall consider. The QCD vacuum polarization function, $\Pi(Q^2)$, is the correlator of two vector currents in the Euclidean region,

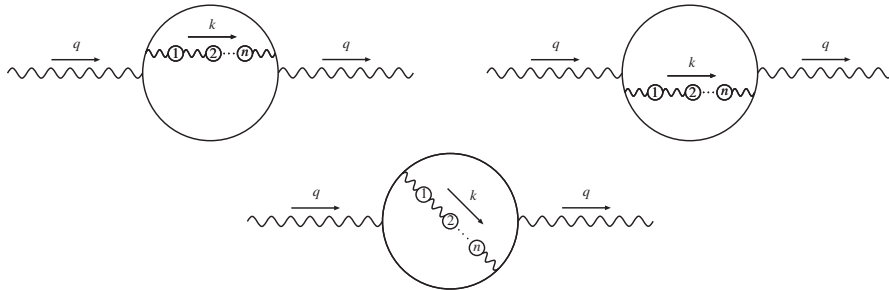


FIG. 2. Leading large- N_f contributions to the vacuum polarization function at n th order in perturbation theory.

$$(q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(Q^2) = 16\pi^2 i \int d^4x e^{iq \cdot x} \times \langle 0 | T [j_\mu(x) j_\nu(0)] | 0 \rangle, \quad (18)$$

The leading- N_f component of $\Pi(Q^2)$ can be calculated from the diagrams in Fig. 2. The Adler function, $D(Q^2)$, is then defined via the logarithmic derivative of $\Pi(Q^2)$

$$D(Q^2) = -\frac{3}{4} Q^2 \frac{d}{dQ^2} \Pi(Q^2). \quad (19)$$

This can be split into the parton model result and QCD corrections, $\mathcal{D}(Q^2)$,

$$D(Q^2) = N_c \sum_f Q_f^2 \left(1 + \frac{3}{4} C_F \mathcal{D}(Q^2) \right), \quad (20)$$

where N_c is the number of colors, $C_F = \frac{(N_c^2 - 1)}{2N_c}$, and Q_f is the charge of quark flavour f . Here $\mathcal{D}(Q^2) = \mathcal{D}_{PT}(Q^2) + \mathcal{D}_{NP}(Q^2)$, with the two components defined as in Eqs. (1) and (3). The polarized Bjorken (pBj) [18] and GLS [19] sum rules are defined as

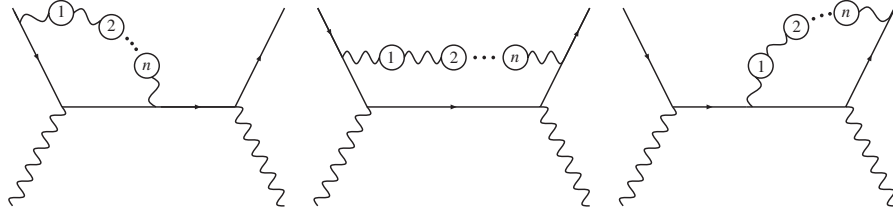
$$\begin{aligned} K_{\text{pBj}} &\equiv \int_0^1 g_1^{ep-en}(x, Q^2) dx \\ &= \frac{1}{3} \left| \frac{g_A}{g_V} \right| \left(1 - \frac{3}{4} C_F \mathcal{K}(Q^2) \right), \end{aligned} \quad (21)$$

$$K_{\text{GLS}} \equiv \frac{1}{6} \int_0^1 F_3^{\bar{\nu}p+\nu p}(x, Q^2) dx = \left(1 - \frac{3}{4} C_F \mathcal{K}(Q^2) \right). \quad (22)$$

$\mathcal{K}(Q^2)$ being the QCD corrections to the parton model result, again split into PT and NP components as for $\mathcal{D}(Q^2)$. We have neglected contributions due to ‘‘light-by-light’’ diagrams—which when omitted render the perturbative corrections to K_{GLS} and K_{pBj} identical. Finally, the unpolarized Bjorken sum rule (uBj) [20] is defined as

$$U_{\text{uBj}} \equiv \int_0^1 F_1^{\bar{\nu}p-\nu p}(x, Q^2) dx = \left(1 - \frac{1}{2} C_F \mathcal{U}(Q^2) \right). \quad (23)$$

The QCD corrections to the parton model result are again split into PT and NP components. The leading- N_f contributions to these three sum rules can be calculated from the


 FIG. 3. Leading large- N_f contributions to the DIS sum rules of Eqs. (21)–(23) at n th order in perturbation theory.

diagrams in Fig. 3. These large- N_f results can be used to compute leading- b all-orders resummations for these observables, $\mathcal{D}_{PT}^{(L)}(Q^2)$, $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$, as described in Sec. II. The Borel transform of $\mathcal{D}_{PT}^{(L)}(Q^2)$ is well-known and can be found in Ref. [8],

$$B[\mathcal{D}_{PT}^{(L)}](z) = \sum_{n=1}^{\infty} \frac{A_0(n) - A_1(n)z_n}{(1 + \frac{z}{z_n})^2} + \frac{A_1(n)z_n}{(1 + \frac{z}{z_n})} + \sum_{n=1}^{\infty} \frac{B_0(n) + B_1(n)z_n}{(1 - \frac{z}{z_n})^2} - \frac{B_1(n)z_n}{(1 - \frac{z}{z_n})}. \quad (24)$$

Here

$$\begin{aligned} A_0(n) &= \frac{8}{3} \frac{(-1)^{n+1}(3n^2 + 6n + 2)}{n^2(n+1)^2(n+2)^2}, \\ A_1(n) &= \frac{8}{3} \frac{b(-1)^{n+1}(n + \frac{3}{2})}{n^2(n+1)^2(n+2)^2}, \\ B_0(1) &= 0, \quad B_0(2) = 1, \\ B_0(n) &= -A_0(-n) \quad n \geq 3 \\ B_1(1) &= 0, \quad B_1(2) = -\frac{b}{4}, \\ B_1(n) &= -A_1(-n) \quad n \geq 3 \end{aligned} \quad (25)$$

These definitions coincide with Ref. [8], except for $B_1(2) = -\frac{b}{4}$. The purpose of the slight change of definition is to make more explicit the single and double pole structure. The Borel transforms of $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$

can be found in Refs. [6,7], respectively. They have a much simpler structure than that of the Adler- D function since they arise from insertion of the chain of bubbles into a tree-level diagram, rather than into a quark loop, as shown in Fig. 3. There are only a finite number of single poles and no double poles. Consequently we can write out their Borel transforms explicitly

$$B[\mathcal{K}_{PT}^{(L)}](z) = \frac{4/9}{(1 + \frac{z}{z_1})} - \frac{1/18}{(1 + \frac{z}{z_2})} + \frac{8/9}{(1 - \frac{z}{z_1})} - \frac{5/18}{(1 - \frac{z}{z_2})}. \quad (26)$$

and

$$B[\mathcal{U}_{PT}^{(L)}](z) = \frac{1/6}{(1 + \frac{z}{z_2})} + \frac{4/3}{(1 - \frac{z}{z_1})} - \frac{1/2}{(1 - \frac{z}{z_2})}. \quad (27)$$

As noted in Refs. [7,8] the leading- b approximations for the NLO and NNLO coefficients for these observables are in reasonable agreement with the known exact coefficients.

We can now evaluate the Borel integral of Eq. (9) to obtain $\mathcal{D}_{PT}^{(L)}(Q^2)$, $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$. Using the integrals

$$\int_0^{\infty} dz \frac{e^{-z/a}}{(1 + z/z_n)} = -z_n e^{z_n/a} \text{Ei}(-z_n/a), \quad (28)$$

$$\int_0^{\infty} dz \frac{e^{-z/a}}{(1 + z/z_n)^2} = z_n \left[1 + \frac{z_n}{a} e^{z_n/a} \text{Ei}(-z_n/a) \right] \quad (29)$$

the following resummed expressions are obtained,

$$\begin{aligned} \mathcal{D}_{PT}^{(L)}(Q^2) &= \sum_{n=1}^{\infty} z_n \left\{ e^{z_n/a(Q^2)} \text{Ei}\left(-\frac{z_n}{a(Q^2)}\right) \left[\frac{z_n}{a(Q^2)} (A_0(n) - z_n A_1(n)) - z_n A_1(n) \right] + (A_0(n) - z_n A_1(n)) \right\} \\ &+ \sum_{n=1}^{\infty} z_n \left\{ e^{-z_n/a(Q^2)} \text{Ei}\left(\frac{z_n}{a(Q^2)}\right) \left[\frac{z_n}{a(Q^2)} (B_0(n) + z_n B_1(n)) - z_n B_1(n) \right] - (B_0(n) + z_n B_1(n)) \right\}, \end{aligned} \quad (30)$$

$$\begin{aligned} \mathcal{K}_{PT}^{(L)}(Q^2) &= \frac{1}{9b} \left[-8e^{z_1/a(Q^2)} \text{Ei}\left(-\frac{z_1}{a(Q^2)}\right) + 2e^{z_2/a(Q^2)} \text{Ei}\left(-\frac{z_2}{a(Q^2)}\right) + 16e^{-z_1/a(Q^2)} \text{Ei}\left(\frac{z_1}{a(Q^2)}\right) \right. \\ &\left. - 10e^{-z_2/a(Q^2)} \text{Ei}\left(\frac{z_2}{a(Q^2)}\right) \right], \end{aligned} \quad (31)$$

$$\mathcal{U}_{PT}^{(L)}(Q^2) = \frac{1}{3b} \left[8e^{-z_1/a(Q^2)} \text{Ei}\left(\frac{z_1}{a(Q^2)}\right) - 6e^{-z_2/a(Q^2)} \text{Ei}\left(\frac{z_2}{a(Q^2)}\right) - 2e^{z_2/a(Q^2)} \text{Ei}\left(-\frac{z_2}{a(Q^2)}\right) \right]. \quad (32)$$

Where $\text{Ei}(x)$ is the exponential integral function defined (for $x < 0$) as

$$\text{Ei}(x) \equiv - \int_{-x}^{\infty} dt \frac{e^{-t}}{t}, \quad (33)$$

and for $x > 0$ by taking the PV of the integral. It has the expansion

$$\text{Ei}(x) = \ln|x| + \gamma_E + \mathcal{O}(x), \quad (34)$$

for small x , where $\gamma_E = 0.57721\dots$ is the Euler constant.

A crucial point is that the above expressions for the Q^2 -dependence apply at *all* values of Q^2 . For $Q^2 < \Lambda^2$ the modified Borel representation, written as an ordinary Borel representation for an expansion in powers of $|a|$, as in Eq. (13), corresponds to changing $a(Q^2) \rightarrow -a(Q^2)$, $z_n \rightarrow -z_n$, and adding an overall minus sign in Eqs. (30)–(32). One can easily see that these equations are invariant under these changes. In Eq. (30) one needs to change $A_1 \rightarrow -A_1$ and $B_1 \rightarrow -B_1$, since they contain a hidden z_n factor in their definitions, also in Eqs. (31) and (32), the prefactor proportional to $1/b$ also needs to change sign since it has been factorized from z_1, z_2 . The $\text{Ei}(z_n/a(Q^2))$ functions exhibit a logarithmic divergence as their argument goes to zero, and so it would appear that one does not obtain a finite result at $Q^2 = \Lambda^2$. Using Eq. (34), one has,

$$\begin{aligned} \text{Ei}\left[\frac{2n}{ba(Q^2)}\right] &= \text{Ei}[n \log(Q^2/\Lambda^2)] \quad \simeq \text{Ei}\left[n\left(\frac{Q^2}{\Lambda^2} - 1\right)\right] \\ &\simeq \gamma_E + \ln\left[n\left(\frac{Q^2}{\Lambda^2} - 1\right)\right], \end{aligned} \quad (35)$$

for $\Lambda^2 \approx Q^2$. Note that the only terms in Eqs. (30)–(32) which could possibly contribute to the divergence are $e^{\pm z_n/a} \text{Ei}(\mp z_n/a)$ terms and, as can be seen from Eqs. (28) and (29), these are generated exclusively by the single pole terms in the Borel transform. The double pole terms only generate finite contributions at $Q^2 = \Lambda^2$.

Using Eq. (35) we obtain the $Q^2 \rightarrow \Lambda^2$ limit of $\mathcal{D}_{PT}^{(L)}(Q^2)$

$$\begin{aligned} \mathcal{D}_{PT}^{(L)}(Q^2) &= - \sum_{n=1}^{\infty} z_n^2 [A_1(n) + B_1(n)] \ln\left[n\left(\frac{Q^2}{\Lambda^2} - 1\right)\right] \\ &\quad + \sum_{n=1}^{\infty} [z_n^2(1 + \gamma_E)(-A_1(n) - B_1(n)) \\ &\quad + z_n(A_0(n) - B_0(n))] + \mathcal{O}\left(\frac{Q^2}{\Lambda^2} - 1\right). \end{aligned} \quad (36)$$

So the coefficient of the divergent log term in $\mathcal{D}_{PT}^{(L)}(Q^2)$ is,

$$- \sum_{n=1}^{\infty} z_n^2 [A_1(n) + B_1(n)], \quad (37)$$

and for $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$ the equivalent coefficients are $(-8 + 2 + 16 - 10 = 0)$ and $(8 - 6 - 2 = 0)$, respectively. Cancellation clearly occurs in the cases of $\mathcal{K}_{PT}^{(L)}(Q^2)$

and $\mathcal{U}_{PT}^{(L)}(Q^2)$ and in the case of $\mathcal{D}_{PT}^{(L)}(Q^2)$ the previously unnoticed relation

$$z_{n+3}^2 B_1(n+3) = -z_n^2 A_1(n), \quad (38)$$

ensures that $\mathcal{D}_{PT}^{(L)}(\Lambda^2)$ is finite,

$$\sum_{n=1}^{\infty} z_n^2 [A_1(n) + B_1(n)] = 0. \quad (39)$$

A similar relation

$$A_0(n) = -B_0(n+2), \quad (40)$$

was noted in [8]. We shall show in the next section that the relations of Eqs. (38) and (40) are underwritten by the continuity of the skeleton expansion characteristic function $\omega_{\Pi}(t)$ and its first derivative at $t = 1$. The form of the perturbative corrections, $\mathcal{D}_{PT}^{(L)}(Q^2)$, $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$, are shown in Fig. 4.

Although we have shown that when summed to infinity Eq. (30) is finite at $Q^2 = \Lambda^2$, we obviously can only plot the expression including a finite number of terms in the n sum. The expression can remain finite, however, if we sum the UV renormalons to finite $n = N$ and the IR renormalons to $n = N + 3$. In this case the relation of Eq. (38) will ensure that the divergent terms cancel. We took $N = 50$ and assumed $N_f = 0$ quark flavours, avoiding the need to match at quark flavour thresholds, since we are only interested here in the form of the freezing behavior, not in a phenomenological analysis.

The plots in Fig. 4 demonstrate two important points about the Euclidean quantities we are considering. Firstly the finite behavior at $Q^2 = \Lambda^2$, and secondly that the Borel resummed perturbative corrections to the parton model result change sign just below or above this point. For \mathcal{D} these corrections become negative but crucially the full observable $D(Q^2)$ remains positive at all values of Q^2 . They then freeze to zero as noted in Sec. II.

The relation of Eq. (38) simplifies the expression for the finite part of Eq. (36), it becomes

$$\begin{aligned} \mathcal{D}_{PT}^{(L)}(Q^2 = \Lambda^2) &= \sum_{n=1}^{\infty} z_n [A_0(n) - B_0(n)] \\ &\quad - \sum_{n=1}^{\infty} z_n^2 [A_1(n) + B_1(n)] \ln n \\ &\simeq \frac{0.679938}{b}. \end{aligned} \quad (41)$$

The values $\mathcal{K}_{PT}^{(L)}(Q^2 = \Lambda^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2 = \Lambda^2)$ are given by a formula identical to Eq. (41), but using values of $A_{0,1}(n)$ and $B_{0,1}(n)$ appropriate to \mathcal{K} and \mathcal{U} . Although we have not given these values explicitly, they are of a much simpler form than in the case of \mathcal{D} , and they can easily be deduced by comparing Eqs. (26) and (27) with Eq. (24). From this we obtain,

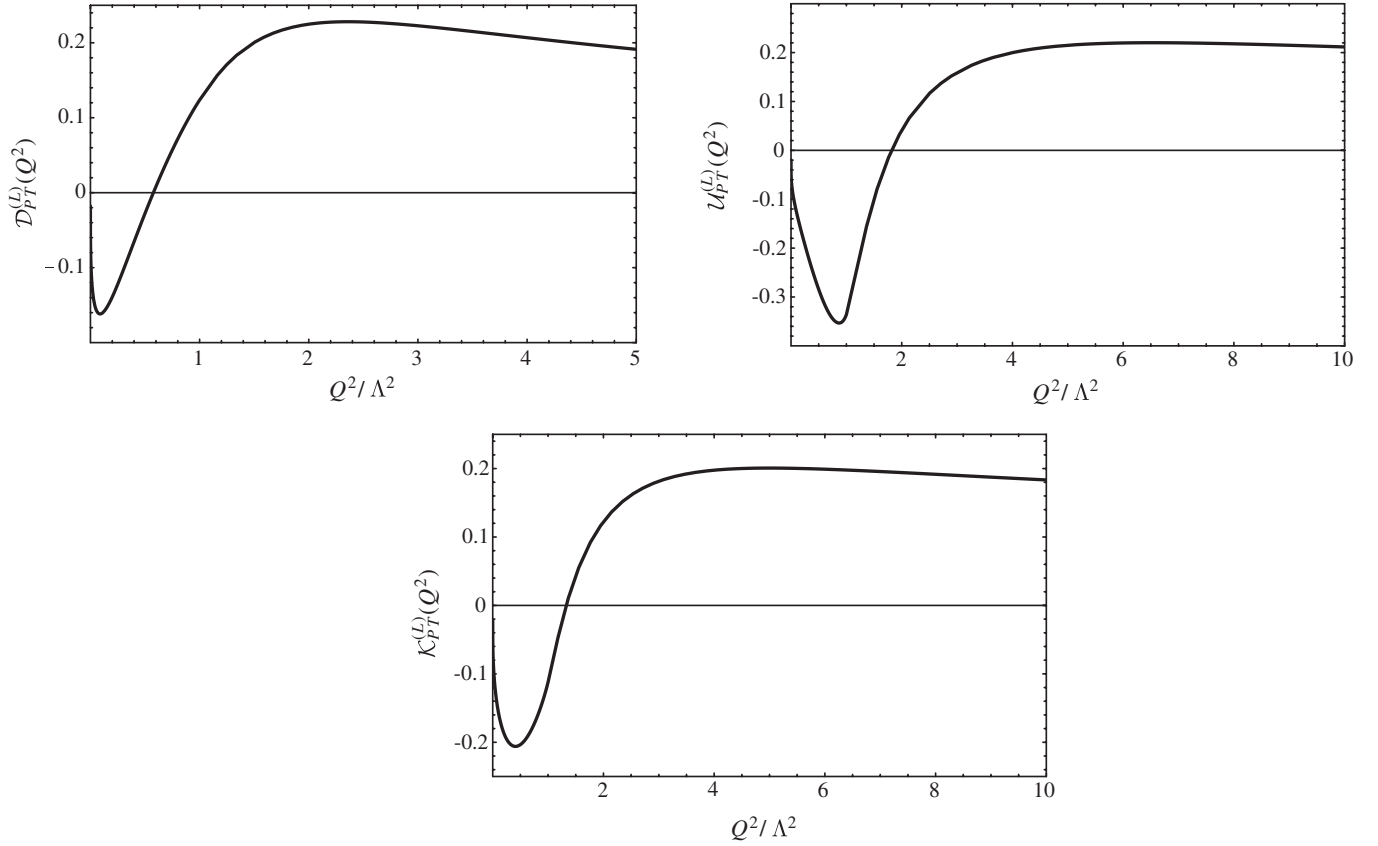


FIG. 4. Q^2 -dependence of the *perturbative* corrections to the observables in Eqs. (21)–(23), resummed to all orders in the leading- b approximation.

$$\begin{aligned} \mathcal{K}_{PT}^{(L)}(Q^2 = \Lambda^2) &= -\frac{8}{9b} \ln 2, \\ \mathcal{U}_{PT}^{(L)}(Q^2 = \Lambda^2) &= -\frac{8}{3b} \ln 2. \end{aligned} \quad (42)$$

IV. SKELETON EXPANSION AND BOREL REPRESENTATIONS FOR THE ADLER FUNCTION

We begin with the one-chain skeleton expansion result for the vacuum polarization function $\Pi(Q^2)$ defined in Eq. (18),

$$\Pi(Q^2) = \int_0^\infty dt \omega_\Pi(t) a(tQ^2), \quad (43)$$

where the characteristic function $\omega_\Pi(t)$ is given by

$$\omega_\Pi(t) = -\frac{4}{3} \begin{cases} t\Xi(t) & t \leq 1 \leftrightarrow \text{IR} \\ \frac{1}{t}\Xi\left(\frac{1}{t}\right) & t \geq 1 \leftrightarrow \text{UV} \end{cases} \quad (44)$$

It can be obtained from the classic QED work of Ref. [21] by simply including appropriate color factors.¹ In this language it is related to the Bethe-Salpeter kernel for the

¹The origin of the minus sign in Eq. (18) is the difference between the definitions of Π given in Eq. (18) and Ref. [21].

scattering of light-by-light, and is the first term in a well-defined QED skeleton expansion [12]. The diagrams relevant to the kernel are shown in Fig. 5. It is easy to see how, by connecting the ends of the fermion bubble chain in Fig. 1 to the momentum k external propagators in Fig. 5, one can reproduce the topology of the diagrams in Fig. 2. The existence of the QCD skeleton expansion is more problematic [15]. $\Xi(t)$ is given by [21]

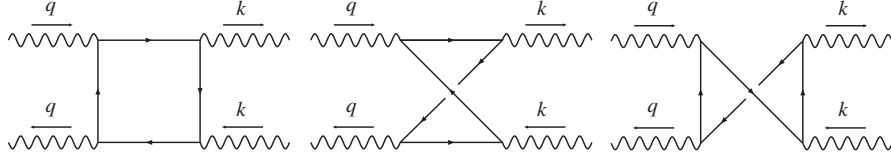
$$\begin{aligned} \Xi(t) &\equiv \frac{4}{3t} \left\{ 1 - \ln t + \left(\frac{5}{2} - \frac{3}{2} \ln t \right) t \right. \\ &\quad \left. + \frac{(1+t)^2}{t} [L_2(-t) + \ln t \ln(1+t)] \right\}, \end{aligned} \quad (45)$$

where $L_2(x)$ is the dilogarithmic function.

$$L_2(x) = -\int_0^x dy \frac{\ln(1-y)}{y}. \quad (46)$$

Though we define $\omega_\Pi(t)$ separately in the IR and UV domains, the two regions are related by the conformal symmetry $t \leftrightarrow \frac{1}{t}$.

The Adler- D function, related to $\Pi(Q^2)$ through Eq. (19), will have the one-chain skeleton expansion term with characteristic function $\omega_{\mathcal{D}}(t)$,


 FIG. 5. Light-by-light scattering diagrams, used to calculate $\omega_{\Pi}(t)$.

$$\mathcal{D}_{pT}^{(L)}(Q^2) = \int_0^\infty dt \omega_{\mathcal{D}}(t) a(tQ^2). \quad (47)$$

$\omega_{\mathcal{D}}(t)$ is obtained from $\omega_{\Pi}(t)$ by performing the differentiation of Eq. (19) on Eq. (43) and then performing integration by parts on the resulting expression.

$$\begin{aligned} \mathcal{D}_{pT}^{(L)}(Q^2) &= -\frac{3}{4} Q^2 \frac{d}{dQ^2} \int_0^\infty dt \omega_{\Pi}(t) t \left(\frac{a(tQ^2)}{t} \right) \\ &= +\frac{3}{2b} Q^2 \frac{d}{dQ^2} \int_0^\infty dt \frac{d}{dt} [\omega_{\Pi}(t) t] \ln[a(tQ^2)] \\ &= -\frac{3}{4} \int_0^\infty dt \left[\omega_{\Pi}(t) + t \frac{d}{dt} \omega_{\Pi}(t) \right] a(tQ^2). \end{aligned} \quad (48)$$

The transformation from Π to \mathcal{D} therefore induces a transformation in $\omega_{\Pi}(t)$ of

$$\begin{aligned} \Pi(Q^2) &\rightarrow Q^2 \frac{d}{dQ^2} \Pi(Q^2) = -\frac{4}{3} \mathcal{D}(Q^2) \Rightarrow \omega_{\Pi}(t) \\ &\rightarrow \omega_{\Pi}(t) + t \frac{d}{dt} \omega_{\Pi}(t) = -\frac{4}{3} \omega_{\mathcal{D}}(t). \end{aligned} \quad (49)$$

This transformation spoils the conformal symmetry present in $\omega_{\Pi}(t)$. Indeed the expressions for $\omega_{\mathcal{D}}(t)$ in the UV and IR regions are slightly more complicated.

$$\omega_{\mathcal{D}}^{IR}(t) = \frac{8}{3} \left\{ \left(\frac{7}{4} - \text{Lnt} \right) t + (1+t) [L_2(-t) + \text{Lnt} \ln(1+t)] \right\} \quad (50)$$

$$\omega_{\mathcal{D}}^{UV}(t) = \frac{8}{3} \left\{ 1 + \text{Lnt} + \left(\frac{3}{4} + \frac{1}{2} \text{Lnt} \right) \frac{1}{t} + (1+t) [L_2(-t^{-1}) - \text{Lnt} \ln(1+t^{-1})] \right\} \quad (51)$$

However, a partial symmetry remains in $\omega_{\mathcal{D}}(t)$ and this will be elucidated upon in the following discussion. We shall now convert the skeleton expansion form into the Borel representations of Eqs. (9) and (12) by making a

change of variables. To achieve this it is necessary to write $\omega_{\Pi}(t)$ as an expansion in powers of t . This yields expressions in both the IR and UV regions comprising an expansion plus an expansion times a logarithm.

$$\omega_{\Pi}^{IR}(t) = -\frac{4}{3} \left(\sum_{n=1}^{\infty} \xi_n t^n + \text{Lnt} \sum_{n=2}^{\infty} \hat{\xi}_n t^n \right). \quad (52)$$

The conformal symmetry expressed in Eq. (44) means that the UV part can also be written in terms of the coefficients ξ_n and $\hat{\xi}_n$

$$\omega_{\Pi}^{UV}(t) = -\frac{4}{3} \left(\sum_{n=1}^{\infty} \xi_n t^{-n} - \text{Lnt} \sum_{n=2}^{\infty} \hat{\xi}_n t^{-n} \right). \quad (53)$$

From Eq. (45), ξ_n and $\hat{\xi}_n$ are found to be

$$\begin{aligned} \xi_{n>1} &= \frac{4}{3} \frac{(2-6n^2)(-1)^n}{(n-1)^2 n^2 (n+1)^2}, \\ \hat{\xi}_{n>1} &= \frac{4}{3} \frac{2(-1)^n}{(n-1)n(n+1)} \\ \xi_1 &= 1 \quad \hat{\xi}_1 = 0 \end{aligned} \quad (54)$$

Performing the transformation in Eq. (49) allows us to write $\omega_{\mathcal{D}}(t)$ as a similar expansion

$$\omega_{\mathcal{D}}^{IR}(t) = \sum_{n=1}^{\infty} [\xi_n(1+n) + \hat{\xi}_n] t^n + \text{Lnt} \sum_{n=2}^{\infty} \hat{\xi}_n (n+1) t^n \quad (55)$$

$$\omega_{\mathcal{D}}^{UV}(t) = \sum_{n=1}^{\infty} [\xi_n(1-n) - \hat{\xi}_n] t^{-n} + \text{Lnt} \sum_{n=2}^{\infty} \hat{\xi}_n (n-1) t^{-n} \quad (56)$$

Using the expansions of Eqs. (55) and (56) we can now represent $\mathcal{D}_{pT}^{(L)}(Q^2)$ in terms of a Borel integral. We take $\mathcal{D}_{pT}^{(L)}(Q^2)$ expressed in terms of $\omega_{\mathcal{D}}(t)$ and then split the integral into IR and UV regions

$$\begin{aligned}
 \mathcal{D}_{PT}^{(L)}(Q^2) &= \int_0^\infty dt \omega_{\mathcal{D}}(t) a(tQ^2) = \sum_{k=0}^\infty a(Q^2) \int_0^1 dt \omega_{\mathcal{D}}^{IR}(t) \left(-\frac{ba(Q^2)}{2} \ln t\right)^k + \sum_{k=0}^\infty a(Q^2) \int_1^\infty dt \omega_{\mathcal{D}}^{UV}(t) \left(-\frac{ba(Q^2)}{2} \ln t\right)^k \\
 &= a(Q^2) \sum_{k=0}^\infty \left(-\frac{ba(Q^2)}{2}\right)^k \left[\int_0^1 dt \left(\sum_{n=1}^\infty [\xi_n(1+n) + \hat{\xi}_n](t)^n + \ln t \sum_{n=2}^\infty \hat{\xi}_n(n+1)(t)^n \right) (\ln t)^k \right. \\
 &\quad \left. + \int_1^\infty dt \left(\sum_{n=1}^\infty [\xi_n(1-n) - \hat{\xi}_n](t)^{-n} + \ln t \sum_{n=2}^\infty \hat{\xi}_n(n-1)(t)^{-n} \right) (\ln t)^k \right] \quad (57)
 \end{aligned}$$

Where we have used

$$a(xy) = a(y) \sum_{k=0}^\infty \left(-\frac{ba(y)}{2} \ln x\right)^k. \quad (58)$$

We note that $[\xi_n(1-n) - \hat{\xi}_n] = 0$ for $n = 1$, which allows us to omit this term from the above sum. This

expression may be transformed into a Borel integral of the form of Eq. (9) by changes of variables and integration by parts. We use the change of variables $z = -a(Q^2) \times (n+1) \ln t$ and $z = a(Q^2)(n-1) \ln t$ for IR and UV parts, respectively. Integration by parts is necessary for the integrals with an extra $\ln t$ term. For $Q^2 > \Lambda^2$, $a(Q^2) > 0$, we then obtain the standard Borel representation, of Eq. (9)

$$\begin{aligned}
 \mathcal{D}_{PT}^{(L)}(Q^2) &= \int_0^\infty dz e^{-z/a(Q^2)} \left[\sum_{n=1}^\infty \frac{[\xi_n(1+n) + \hat{\xi}_n]}{n+1} \frac{1}{1 - \frac{bz}{2(n+1)}} - \sum_{n=2}^\infty \frac{\hat{\xi}_n(n+1)}{(n+1)^2} \frac{1}{\left(1 - \frac{bz}{2(n+1)}\right)^2} \right] \\
 &\quad + \int_0^\infty dz e^{-z/a(Q^2)} \left[\sum_{n=2}^\infty \frac{[\xi_n(1-n) - \hat{\xi}_n]}{n-1} \frac{1}{1 + \frac{bz}{2(n-1)}} + \sum_{n=2}^\infty \frac{\hat{\xi}_n(n-1)}{(n-1)^2} \frac{1}{\left(1 + \frac{bz}{2(n-1)}\right)^2} \right], \quad (59)
 \end{aligned}$$

and for $Q^2 < \Lambda^2$, $a(Q^2) < 0$, we obtain the modified Borel representation of Eq. (12), in which the upper limit in z is $-\infty$. Having obtained the Borel transform we can now make contact with Eq. (24) and this allows us to make the identifications

$$\frac{\xi_n(1+n) + \hat{\xi}_n}{n+1} = -B_1(n+1)z_{n+1} \quad n \geq 1 \quad (60)$$

$$\frac{\xi_n(1-n) - \hat{\xi}_n}{n-1} = A_1(n-1)z_{n-1} \quad n \geq 2 \quad (61)$$

for the single pole residues and

$$-\frac{\hat{\xi}_n(n+1)}{(n+1)^2} = B_0(n+1) + B_1(n+1)z_{n+1} \quad n \geq 2 \quad (62)$$

$$\frac{\hat{\xi}_n(n-1)}{(n-1)^2} = A_0(n-1) - A_1(n-1)z_{n-1} \quad n \geq 2 \quad (63)$$

for the double pole residues. Substituting the form of ξ_n and $\hat{\xi}_n$ given by Eq. (54), and comparison with Eq. (25), verifies the above equations.

Equations. (60)–(63) can be used to rewrite the $\omega_{\mathcal{D}}^{IR}(t)$ and $\omega_{\mathcal{D}}^{UV}$ expansions of Eqs. (55) and (56) in terms of the $A_0(n)$, $A_1(n)$, and $B_0(n)$, $B_1(n)$ renormalon residues. One finds

$$\begin{aligned}
 \omega_{\mathcal{D}}^{IR}(t) &= \frac{b}{2} \sum_{n=1}^\infty -z_{n+1}^2 B_1(n+1)t^n - \ln t \sum_{n=2}^\infty (n+1)^2 \\
 &\quad \times [B_0(n+1) + z_{n+1} B_1(n+1)]t^n \quad (64)
 \end{aligned}$$

$$\begin{aligned}
 \omega_{\mathcal{D}}^{UV}(t) &= \frac{b}{2} \sum_{n=1}^\infty z_{n-1}^2 A_1(n-1)t^{-n} + \ln t \sum_{n=2}^\infty (n-1)^2 \\
 &\quad \times [A_0(n-1) - z_{n-1} A_1(n-1)]t^{-n}. \quad (65)
 \end{aligned}$$

The discontinuity at $t = 1$ is then found to be

$$\omega_{\mathcal{D}}^{UV}(1) - \omega_{\mathcal{D}}^{IR}(1) = \frac{b}{2} \sum_{n=1}^\infty z_n^2 [A_1(n) + B_1(n)], \quad (66)$$

which vanishes using Eq. (39). In the language of ξ_n and $\hat{\xi}_n$ coefficients, Eq. (66) is equivalent to

$$-2 \sum_{n=1}^\infty (n\xi_n + \hat{\xi}_n) = 0. \quad (67)$$

So the relation between UV and IR renormalon residues of Eq. (38), which guarantees finiteness at $Q^2 = \Lambda^2$, ensures that the characteristic function $\omega_{\mathcal{D}}(t)$ is continuous at $t = 1$.

For the first derivative at $t = 1$ one finds the discontinuity

$$\begin{aligned}
 \frac{d\omega_{\mathcal{D}}}{IR} dt \Big|_{t=1} - \frac{d\omega_{\mathcal{D}}}{UV} dt \Big|_{t=1} &= b \sum_{n=1}^{\infty} z_n^2 [A_1(n) + B_1(n)] \\
 &\quad - \frac{b^2}{4} \sum_{n=1}^{\infty} z_n^2 [A_0(n) \\
 &\quad + B_0(n)] + \frac{b^2}{2} \\
 &\quad \times \sum_{n=1}^{\infty} z_n^3 [A_1(n) - B_1(n)],
 \end{aligned} \tag{68}$$

Equation (39), which ensures that $\mathcal{D}_{PT}^{(L)}(\Lambda^2)$ is finite, means that the first line of this expression vanishes. The sum of the remaining terms also vanishes, ensuring continuity of the first derivative of $\omega_{\mathcal{D}}(t)$. This also ensures that the $\mathcal{D}_{PT}^{(L)'}(\Lambda^2)$ is finite (the prime denoting the first derivative $d/d \ln Q$). Indeed, the required relation corresponding to the vanishing of the coefficient of the potentially divergent $\ln a$ term in $\mathcal{D}_{PT}^{(L)'}(\Lambda^2)$ is,

$$\sum_{n=1}^{\infty} [2z_n^3(A_1(n) - B_1(n)) - z_n^2(A_0(n) + B_0(n))] = 0. \tag{69}$$

So finiteness of the first derivative of $\mathcal{D}^{(L)}(\Lambda)$ at $Q = \Lambda$, corresponds to continuity of the first derivative of $\omega(t)$ at $t = 1$. Furthermore, Eq. (69) written in terms of ξ_n and $\hat{\xi}_n$ is simply Eq. (67), with an extra factor of -2 . Consequently, the continuity of $\omega_{\mathcal{D}}(t)$ and its first derivative stem from a single relation, Eq. (67). The second and third derivatives are also continuous at $t = 1$, and their discontinuities involve additional new structures built from combinations of the $A_{0,1}$ and $B_{0,1}$. To ensure finiteness of $\mathcal{D}_{PT}^{(L)''}(Q^2)$ at $Q^2 = \Lambda^2$, one requires the relation

$$\sum_{n=1}^{\infty} [3z_n^4(A_1(n) + B_1(n)) - 2z_n^3(A_0(n) - B_0(n))] = 0. \tag{70}$$

For finiteness of $\mathcal{D}^{(L)'''}(Q^2)$ at $Q^2 = \Lambda^2$ one requires the relation

$$\sum_{n=1}^{\infty} [4z_n^5(A_1(n) - B_1(n)) - 3z_n^4(A_0(n) + B_0(n))] = 0. \tag{71}$$

Equations (70) and (71) are also required in order for the second and third derivatives of $\omega_{\mathcal{D}}(t)$ to be continuous at $t = 1$, furthermore, they can both be derived from the following relation

$$\sum_{n=1}^{\infty} (n^3 \xi_n + 3n^2 \hat{\xi}_n) = 0. \tag{72}$$

The fourth and higher derivatives of $\omega_{\mathcal{D}}(t)$ are discontinuous at $t = 1$ as noted in [4].

V. SKELETON EXPANSION AND THE NP COMPONENT

In this section we wish to consider more carefully the compensation of ambiguities between renormalons and the OPE. The regular OPE is a sum over the contributions of condensates with different mass dimensions. In the case of the Adler function the dimension four gluon condensate is the leading contribution,

$$G_0(a(Q^2)) = \frac{1}{Q^4} \langle 0 | GG | 0 \rangle C_{GG}(a(Q^2)), \tag{73}$$

where $C_{GG}(a(Q^2))$ is the Wilson coefficient. In general the n th term in the OPE expansion of Eq. (3) will have the coefficient

$$C_n(a(Q^2)) = C_n[a(Q^2)]^{\delta_n} (1 + O(a)). \tag{74}$$

The exponent δ_n corresponding to the anomalous dimension of the condensate operator concerned. Nonlogarithmic UV divergences [22] lead to an ambiguous imaginary part in the coefficient so that $C_n = C_n^{(R)} \pm iC_n^{(I)}$. If one considers an IR_n renormalon singularity in the Borel plane to be of the form $K_n/(1 - z/z_n)^{\gamma_n}$ then one finds an ambiguous imaginary part arising of the form

$$\text{Im}[\mathcal{D}_{PT}] = \pm K_n \frac{\pi z_n^{\gamma_n}}{\Gamma(\gamma_n)} e^{-z_n/a(Q^2)} a^{1-\gamma_n} [1 + O(a)]. \tag{75}$$

Here the \pm ambiguity comes from routing the contour above or below the real z -axis in the Borel plane. This is structurally the same as the ambiguous OPE term in Eq. (74), and if $C_n^{(I)} = K_n \pi z_n^{\gamma_n} / \Gamma(\gamma_n)$ and $\delta_n = 1 - \gamma_n$, then the PT Borel and NP OPE ambiguities can cancel against each other [23]. Taking a PV of the Borel integral corresponds to averaging over the \pm possibilities. For $Q^2 < \Lambda^2$ the modified expansion of Eq. (15) will have an n th coefficient of the form

$$\tilde{C}_n(a(Q^2)) = \tilde{C}_n[a(Q^2)]^{\tilde{\delta}_n} (1 + O(a)). \tag{76}$$

Now the exponent $\tilde{\delta}_n$ is related to the anomalous dimension of dimension 6, four-fermion operators associated with UV renormalons [24], IR divergences associated with these render the imaginary part ambiguous, and $\tilde{C}_n = \tilde{C}_n^{(R)} \pm i\tilde{C}_n^{(I)}$. The modified Borel representation of Eq. (12) has ambiguities arising from UV renormalons. Assuming that the UV_n singularity is of the form $\tilde{K}_n/(1 + z/z_n)^{\tilde{\gamma}_n}$ one finds

$$\text{Im}[\mathcal{D}_{PT}] = \pm \tilde{K}_n \frac{\pi z_n^{\tilde{\gamma}_n}}{\Gamma(\tilde{\gamma}_n)} e^{z_n/a(Q^2)} a^{1-\tilde{\gamma}_n} [1 + O(a)]. \tag{77}$$

This is structurally the same as the ambiguity in the modified NP expansion coefficient in Eq. (76), and if $\tilde{C}_n^{(I)} = \tilde{K}_n \pi z_n^{\tilde{\gamma}_n} / \Gamma(\tilde{\gamma}_n)$ and $\tilde{\delta}_n = 1 - \tilde{\gamma}_n$, the ambiguities can be cancelled.

In the one-chain (leading- b) approximation the renormalons are single or double poles corresponding to $\gamma = 1$ or $\gamma = 2$, and correspondingly the ambiguous imaginary parts in Eqs. (75) and (77) contain factors of $a^{1-\gamma}$ which are 1 or $1/a$, respectively. For the Adler function $\text{Im}[\mathcal{D}_{PT}^{(L)}]$ is obtained by making the change $\text{Ei} \rightarrow \text{Ei} \pm i\pi$ in the first line of Eq. (30) for $Q^2 > \Lambda^2$, and in the second line for $Q^2 < \Lambda^2$. For continuity of the Im part at $Q^2 = \Lambda^2$ one needs to choose the sign of $i\pi$ oppositely in the two regions. One then finds for $Q^2 > \Lambda^2$

$$\begin{aligned} \text{Im}[\mathcal{D}_{PT}^{(L)}(Q^2)] = & \pm i\pi \left[\sum_{n=1}^{\infty} B_1(n+1) z_{n+1}^2 \left(\frac{\Lambda^2}{Q^2}\right)^{(n+1)} \right. \\ & - \frac{1}{a(Q^2)} \sum_{n=2}^{\infty} z_{n+1}^2 [B_0(n+1) \\ & \left. + z_{n+1} B_1(n+1)] \left(\frac{\Lambda^2}{Q^2}\right)^{n+1} \right]. \end{aligned} \quad (78)$$

Correspondingly, for $Q^2 < \Lambda^2$ one finds

$$\begin{aligned} \text{Im}[\mathcal{D}_{PT}^{(L)}(Q^2)] = & \mp i\pi \left[\sum_{n=2}^{\infty} A_1(n-1) z_{n-1}^2 \left(\frac{Q^2}{\Lambda^2}\right)^{n-1} \right. \\ & - \frac{1}{a(Q^2)} \sum_{n=2}^{\infty} z_{n-1}^2 [A_0(n-1) \\ & \left. - z_{n-1} A_1(n-1)] \left(\frac{Q^2}{\Lambda^2}\right)^{n-1} \right]. \end{aligned} \quad (79)$$

Comparing these expressions with Eqs. (64) and (65) one then finds that the imaginary part may be written directly in terms of the characteristic function $\omega_{\mathcal{D}}(t)$,

$$\begin{aligned} \text{Im}[\mathcal{D}_{PT}^{(L)}(Q^2)] = & \pm \frac{2\pi}{b} \frac{\Lambda^2}{Q^2} \omega_D^{IR} \left(\frac{\Lambda^2}{Q^2}\right) \quad (Q^2 > \Lambda^2) \\ \text{Im}[\mathcal{D}_{PT}^{(L)}(Q^2)] = & \pm \frac{2\pi}{b} \frac{\Lambda^2}{Q^2} \omega_D^{UV} \left(\frac{\Lambda^2}{Q^2}\right) \quad (Q^2 < \Lambda^2) \end{aligned} \quad (80)$$

Continuity at $Q^2 = \Lambda^2$ then follows from continuity of $\omega(t)$ at $t = 1$. The $C_n^{(R)}$, and $\tilde{C}_n^{(R)}$ coefficients of the OPE and the modified NP expansion are in principle independent of the imaginary part, but continuity at $Q^2 = \Lambda^2$ is dependent upon relations between the $A_{0,1}$ and $B_{0,1}$ residues, such as Eqs. (38) and (40), and the more complicated structures of Eqs. (69)–(71), needed for finiteness of the Q^2

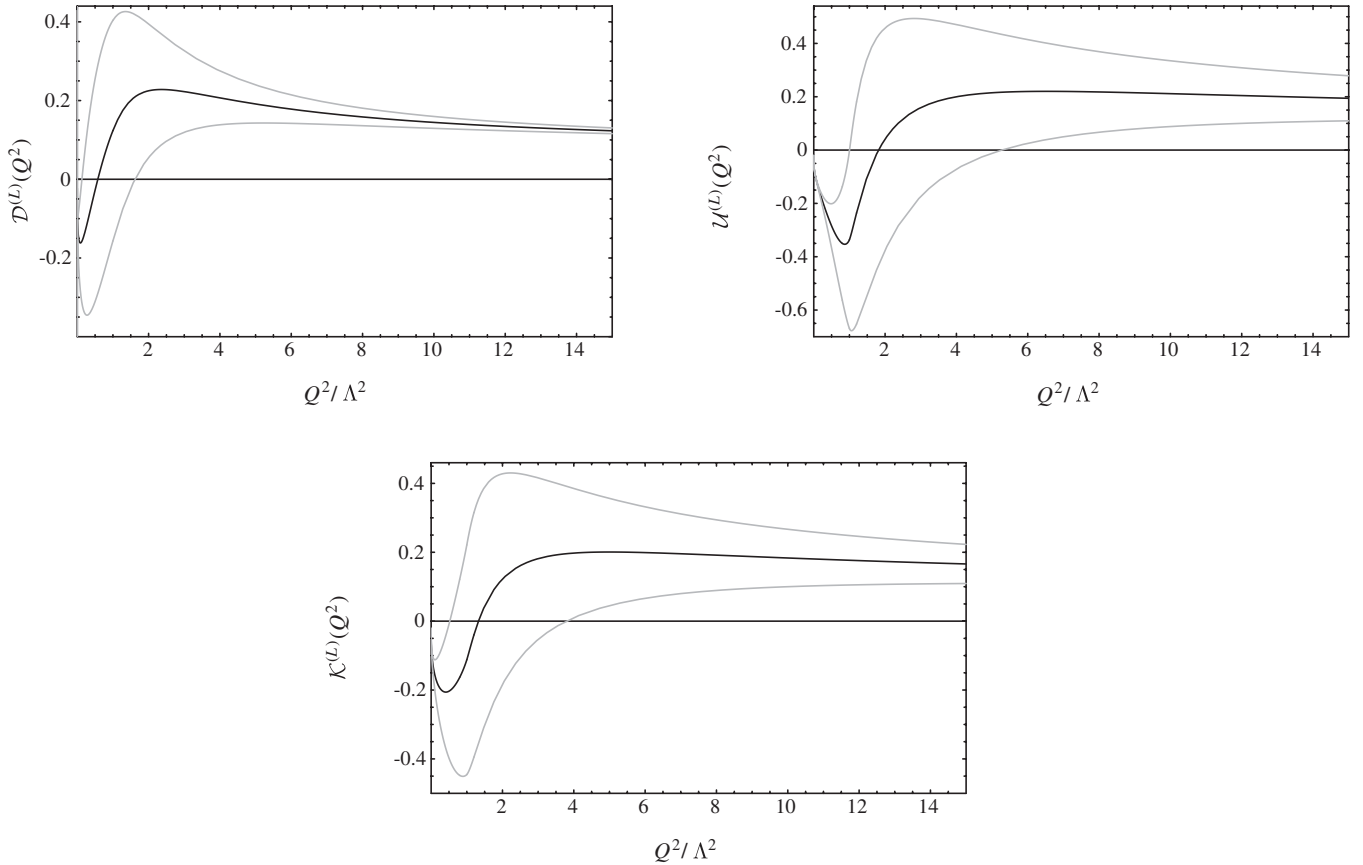


FIG. 6. The bold curves corresponding to $\kappa = 0$ are the perturbative corrections to the observables in Eqs. (21)–(23), as in Fig. 4. The upper and lower curves correspond to the overall result including NP contributions with the choice $\kappa = 1$ and $\kappa = -1$, respectively.

derivatives. Although not strictly necessary for continuity, this continuity follows naturally if we write

$$\mathcal{D}_{NP}^{(L)}(Q^2) = \left(\kappa \pm \frac{2\pi i}{b} \right) \int_0^{\Lambda^2/Q^2} dt \left(\omega_{\mathcal{D}}(t) + t \frac{d\omega_{\mathcal{D}}(t)}{dt} \right). \quad (81)$$

Here κ is an undetermined overall real, nonperturbative factor. The t integration here reproduces the expressions of Eq. (80) in the two Q^2 regions. If the PT component is PV regulated one averages over the \pm possibilities, and combining Eq. (81) with Eq. (47) for $\mathcal{D}_{PT}^{(L)}(Q^2)$ one can write down a result for $\mathcal{D}^{(L)}(Q^2)$ for all values of Q^2 ,

$$\mathcal{D}^{(L)}(Q^2) = \int_0^\infty dt \left[\omega_{\mathcal{D}}(t) a(tQ^2) + \kappa \left(\omega_{\mathcal{D}}(t) + t \frac{d\omega_{\mathcal{D}}(t)}{dt} \right) \times \theta(\Lambda^2 - tQ^2) \right]. \quad (82)$$

The Q^2 evolution is fixed by the nonperturbative constant κ , and by Λ . The infrared limit is $\mathcal{D}^{(L)}(0) = 0$, we have already noted that $\mathcal{D}_{PT}^{(L)}(0) = 0$, the NP component also freezes to zero since on integrating the second term one finds an IR limit of $\omega_{\mathcal{D}}^{IR}(1) - \omega_{\mathcal{D}}^{UV}(1) = 0$, from continuity of the characteristic function at $t = 1$. The same expression holds for the other Euclidean observables $\mathcal{K}_{PT}^{(L)}(Q^2)$ and $\mathcal{U}_{PT}^{(L)}(Q^2)$ on replacing $\omega_{\mathcal{D}}(t)$ by $\omega_{\mathcal{K}}(t)$ and $\omega_{\mathcal{U}}(t)$, respectively. We plot in Fig. 6 the overall result for $\mathcal{D}^{(L)}(Q^2)$, $\mathcal{K}^{(L)}(Q^2)$ and $\mathcal{U}^{(L)}(Q^2)$ for the choices $\kappa = 0$, $\kappa = 1$ and $\kappa = -1$. For the DIS sum rules $\omega_{\mathcal{K}}(t)$, $\omega_{\mathcal{U}}(t)$ and their first derivatives are continuous at $t = 1$. In the case of $\mathcal{U}_{NP}^{(L)}(Q^2)$ there are a total of three nonperturbative terms, and hence the continuity of the characteristic function and its first derivative fixes the form of the function up to an overall constant factor. Thus Eq. (82) does indeed hold for $\mathcal{U}^{(L)}(Q^2)$ without conjecturing the form of Eq. (81).

VI. INFRARED FREEZING BEHAVIOR OF $R_{e^+e^-}$

We turn in this section to a consideration of freezing behavior of the Minkowskian quantity $R_{e^+e^-}$ which was discussed in Ref. [2]. This treatment was criticized in Ref. [3], which argued that in fact there is an unphysical divergence in the infrared limit. We wish to address these criticisms. $R_{e^+e^-}(s)$ will be defined by Eq. (20) with the perturbative corrections $\mathcal{D}(Q^2)$ replaced by $\mathcal{R}(s)$, \sqrt{s} here is the e^+e^- c.m. energy. $\mathcal{R}(s)$ is related to $\mathcal{D}(-s)$ by analytical continuation from Euclidean to Minkowskian. One may write the dispersion relation

$$\mathcal{R}(s) = \frac{1}{2\pi i} \int_{-s-i\epsilon}^{-s+i\epsilon} dt \frac{\mathcal{D}(t)}{t}. \quad (83)$$

If $\mathcal{D}(t)$ is represented by a Borel representation as in Eq. (9) one arrives at

$$\mathcal{R}_{PT}^{(L)}(s) = \int_0^\infty dz e^{-z/a(s)} \frac{\sin(\pi b z/2)}{\pi b z/2} B[\mathcal{D}_{PT}^{(L)}](z). \quad (84)$$

There is now an extra oscillatory factor of $\sin(\pi b z/2)/(\pi b z/2)$ arising from the analytical continuation. In consequence each *individual* IR or UV renormalization contribution at $Q^2 = \Lambda^2$ will be finite, and the cancellation of Eq. (39) is not required. One can also analytically continue the one-chain skeleton expansion result for $\mathcal{D}_{PT}^{(L)}(Q^2)$ to obtain

$$\mathcal{R}_{PT}^{(L)}(s) = \frac{2}{\pi b} \int_0^\infty dt \omega_{\mathcal{D}}(t) \arctan\left(\frac{\pi b a(t s)}{2}\right). \quad (85)$$

Here the principal branch of \arctan is assumed so it lies in the interval $[-\pi/2, +\pi/2]$, and $\arctan(0) = 0$. This form is equivalent to the Borel representation of Eq. (84) for $s > \Lambda^2$, and to the modified Borel representation for $s < \Lambda^2$. Notice that the choice of principal branch is crucial if the PV Borel sum is to be continuous at $s = \Lambda^2$. The result freezes to the IR limit $\mathcal{R}_{PT}^{(L)}(0) = 0$, since $\arctan(0) = 0$ on the principal branch. This freezing limit differs from that found in the APT approach [9], where a freezing to an IR limit of $2/b$ occurs. This freezing limit was also erroneously claimed in Ref. [2], but then the PV Borel sum is discontinuous. In Ref. [3] unphysical singularities in the region $-\Lambda^2 < s < 0$ lead to extra terms and they find

$$\begin{aligned} \mathcal{R}_{PT}^{(L)}(s) = & \frac{2}{\pi b} \int_0^\infty dt \omega_{\mathcal{D}}(t) \arctan\left(\frac{\pi b a(t s)}{2}\right) \\ & + \frac{2}{b} \int_0^{\Lambda^2/s} dt \omega_{\mathcal{D}}(t) + \frac{2}{b} \int_{-\Lambda^2/s}^0 dt \omega_{\mathcal{D}}^{IR}(t). \end{aligned} \quad (86)$$

These extra terms may be treated as contributions to $\mathcal{R}_{NP}^{(L)}(Q^2)$. The final term leads to an infrared divergence as $s \rightarrow 0$, and has an expansion of the same form as the OPE. Notice, however, that the Minkowskian OPE for $\mathcal{R}(Q^2)$ is pathological and contains delta-functions $\delta(s)$ and their derivatives [25]. It is only when a smearing procedure in Q^2 is used [26] that it makes sense. In contrast for Euclidean quantities the regular OPE is potentially well-defined, and no smearing is required.

We will now consider the evaluation of the PV Borel integral for $\mathcal{R}_{PT}^{(L)}$, and correct the erroneous statements made in Ref. [2] noted above. This can be expressed in terms of generalized exponential integral functions $\text{Ei}(n, w)$, defined for $\text{Re} w > 0$ by

$$\text{Ei}(n, w) = \int_1^\infty dt \frac{e^{-wt}}{t^n}. \quad (87)$$

One also needs the integral

$$\int_0^\infty dz e^{-z/a} \frac{\sin(\pi b z/2)}{z} = \arctan\left(\frac{\pi b a}{2}\right). \quad (88)$$

Here the principal branch of the \arctan is again assumed. Care needs to be taken when $\text{Re} w < 0$. With the standard continuation one arrives at a function analytic everywhere in the cut complex w -plane, except at $w = 0$, and with a branch cut running along the negative real semiaxis.

Explicitly [27]

$$\text{Ei}(n, w) = \frac{(-w)^{n-1}}{(n-1)!} \left[-\ln w - \gamma_E + \sum_{m=1}^{n-1} \frac{1}{m} \right] - \sum_{\substack{m=0 \\ m \neq n-1}} \frac{(-w)^m}{(m-n+1)m!}. \quad (89)$$

The $\ln w$ contributes the branch cut along the negative real semiaxis. To obtain the PV of the Borel integral one needs to compensate for the discontinuity across the branch cut and make the replacement $\text{Ei}(n, w) \rightarrow \text{Ei}(n, w) + i\pi \text{sign}(\text{Im}w)$. One then finds that the IR renormalon contributions to $\mathcal{R}_{PT}^{(L)}(s)$ can be written in terms of the functions

$$\begin{aligned} \phi_-(p, q) &\equiv e^{-z_q/a(s)} (-1)^q \text{Im}[Ei(p, -z_q/a(s) \\ &\quad - i\pi b z_q/2)] \\ &\quad - \frac{e^{-z_q/a(s)} (-1)^q z_q^{p-1}}{(p-1)!} \pi \text{Re}[(z_q/a(s) \\ &\quad + i\pi b/2)^{p-1}] \theta(s - \Lambda^2). \end{aligned} \quad (90)$$

The UV renormalon contributions can be written in terms of the functions

$$\begin{aligned} \phi_+(p, q) &\equiv e^{z_q/a(s)} (-1)^q \text{Im}[Ei(p, z_q/a(s) + i\pi b z_q/2)] \\ &\quad + \frac{e^{z_q/a(s)} (-1)^q z_q^{p-1}}{(p-1)!} \pi \text{Re}[(-z_q/a(s) \\ &\quad - i\pi b/2)^{p-1}] \theta(\Lambda^2 - s). \end{aligned} \quad (91)$$

The PV regulated $\mathcal{R}_{PT}^{(L)}(s)$ is then given for *all* values of s by

$$\begin{aligned} \mathcal{R}_{PT}^{(L)}(s) &= \mathcal{R}_{PT}^{(L)}(s)|_{UV} + \mathcal{R}_{PT}^{(L)}(s)|_{IR} \\ &= \frac{2}{\pi b} \arctan\left(\frac{\pi b a(s)}{2}\right) + \frac{2}{\pi b} \sum_{j=1}^{\infty} (A_0(j) \phi_+(1, j) \\ &\quad + (A_0(j) - A_1(j) z_j) \phi_+(2, j)) \\ &\quad + \frac{2B_0(2)}{\pi b} \phi_-(1, 2) + \frac{2}{\pi b} \sum_{j=3}^{\infty} (B_0(j) \phi_-(1, j) \\ &\quad + (B_0(j) + B_1(j) z_j) \phi_-(2, j)). \end{aligned} \quad (92)$$

Note that the presence of the θ -functions is crucial in Eqs. (90) and (91). The terms they multiply are the extra contributions necessary to obtain the PV when $\text{Re}w < 0$. For $s > \Lambda^2$ the second contribution is required for the IR renormalon contribution, but for $s < \Lambda^2$ it must be switched off, otherwise the Borel integral will not be correctly evaluated. With $a(s) < 0$ for $s < \Lambda^2$, $\text{Re}w < 0$ occurs for the UV renormalon contributions and the extra term must be switched on to obtain a PV regulation of the UV component. Leaving out the θ -function in Eq. (90) would cause an unphysical divergence in the infrared, and leaving it out in Eq. (91) would cause asymptotic freedom to fail in the ultraviolet. If the PV is correctly evaluated with \arctan remaining on the principal branch for $s^2 < \Lambda^2$

then one obtains $\mathcal{R}_{PT}^{(L)}(0) = (2/\pi b) \arctan(0) = 0$. Notice that at first sight the PV result appears to be discontinuous at $s = \Lambda^2$, as the θ -function contributions switch over. However the discontinuity is given by the $\phi_{\pm}(1, j)$ terms, and one finds, upon summing them, a discontinuity

$$\frac{2}{b} \sum_{j=1}^{\infty} [B_0(j) (-1)^j + A_0(j) (-1)^j] = \frac{2}{b} B_0(2) = \frac{2}{b}. \quad (93)$$

Here the relation of Eq. (40) ensures pairwise cancellations of terms, and $B_0(2) = 1$ is left over. If we remain on the principal branch, however, the \arctan term has an equal discontinuity of opposite sign, since $(2/\pi b) \arctan(+\infty) = 1/b$, whereas $(2/\pi b) \arctan(-\infty) = -1/b$, and overall there is continuity at $s = \Lambda^2$. In Ref. [2] it was wrongly claimed that the PV result is discontinuous at $s = \Lambda^2$, and instead it was suggested to use a regulation where one throws away the second terms in Eqs. (90) and (91). These terms are of the form $(\Lambda^2/s)^q$, and $(s/\Lambda^2)^q$, respectively, and so they can simply be absorbed into the regular OPE and its modified form.

We finally discuss the ambiguous $\text{Im}[\mathcal{R}_{PT}^{(L)}(s)]$. This may be straightforwardly evaluated as

$$\begin{aligned} \text{Im}[\mathcal{R}_{PT}^{(L)}(s)] &= \pm i\pi \sum_{n=1}^{\infty} [B_0(n) + B_1(n) z_n] \\ &\quad \times z_n (-1)^n \left(\frac{\Lambda^2}{s}\right)^n \quad (s > \Lambda^2) \\ \text{Im}[\mathcal{R}_{PT}^{(L)}(s)] &= \mp i\pi \sum_{n=1}^{\infty} [A_0(n) - A_1(n) z_n] \\ &\quad \times z_n (-1)^n \left(\frac{\Lambda^2}{s}\right)^n \quad (s < \Lambda^2). \end{aligned} \quad (94)$$

If one defines $\omega_{\mathcal{D}}(t) \equiv \omega_{\mathcal{D}}^{(1)}(t) + \text{Int} \omega_{\mathcal{D}}^{(2)}(t)$, the split being into the single and double pole renormalon contributions, then comparing with Eqs. (64) and (65) one finds

$$\begin{aligned} \text{Im}[\mathcal{R}_{PT}^{(L)}(s)] &= \pm \frac{2\pi}{b} \frac{\Lambda^2}{s} \omega_{\mathcal{D}}^{(2)IR} \left(\frac{-\Lambda^2}{s}\right) \quad (s > \Lambda^2) \\ \text{Im}[\mathcal{R}_{PT}^{(L)}(s)] &= \pm \frac{2\pi}{b} \frac{\Lambda^2}{s} \omega_{\mathcal{D}}^{(2)UV} \left(\frac{-\Lambda^2}{s}\right) \quad (s < \Lambda^2). \end{aligned} \quad (95)$$

Notice that only the double poles contribute since the $\sin(\pi b z/2)/(\pi b z/2)$ analytical continuation term in Eq. (83) contains zeros at $z = \pm z_n$ which nullify the single pole contributions. Whilst the characteristic function $\omega_{\mathcal{D}}(t)$ is continuous at $t = 1$, the $\omega_{\mathcal{D}}^{(2)}(t)$ function is *discontinuous* at $t = -1$. The discontinuity is $\pm 2/b$ and arises from the same sum in Eq. (93) which gives an apparent discontinuity in the PV $\mathcal{R}_{PT}^{(L)}(s)$ component, although in the PT case this is cancelled by the \arctan term. Thus defined in this way $\text{Im}[\mathcal{R}_{PT}^{(L)}(s)]$ is discontinuous at $s = \Lambda^2$. It would seem that the proper way to proceed is rather to use the dispersion relation of

Eq. (83) to analytically continue into the Minkowskian region the expression for $\mathcal{D}^{(L)}Q^2$ arrived at in Eq. (82). Unfortunately the one-chain skeleton expansion form for $\mathcal{D}(Q^2)$ is hard to consistently analytically continue, which was a key motivation for the alternative inverse Mellin representation introduced in Ref. [28]. We shall defer further discussion of the more subtle issue of Minkowskian freezing until a later work.

VII. DISCUSSION AND CONCLUSIONS

We have shown in this paper that in the approximation of the one-chain QCD skeleton expansion (leading- b approximation), the perturbative corrections to the parton model result for Euclidean observables undergo a smooth freezing to an infrared limit of zero. We explicitly studied the Adler function, GLS sum rule and polarized and unpolarized Bjorken DIS sum rules as explicit examples, and found that they changed sign in the vicinity of $Q^2 = \Lambda^2$, and then froze to zero at $Q^2 = 0$. Continuity and finiteness at $Q^2 = \Lambda^2$ follow from continuity of the characteristic function $\omega(t)$, and its derivatives at $t = 1$. The one-chain term is equivalent to the standard Borel representation of Eq. (9) for $Q^2 > \Lambda^2$, and to the modified Borel representation of Eq. (12), previously proposed in Ref. [2], for $Q^2 < \Lambda^2$. For the Adler function we established a dictionary between the residues of the IR and UV renormalon singularities, and the series expansion coefficients of $\omega(t)$. Continuity of $\omega_{\mathcal{D}}(t)$ and its first three derivatives at $t = 1$ implies relations between the residues of IR and UV renormalon singularities: Eqs. (38), (40), and (69)–(71). IR renormalons for $Q^2 > \Lambda^2$ lie on the contour of integration in the Borel representation, and similarly UV renormalons lie on the contour of integration in the modified Borel representation for $Q^2 < \Lambda^2$. In both cases these singularities lead to an ambiguous imaginary part in $\mathcal{D}_{PT}^{(L)}(Q^2)$, which can be cancelled against an ambiguous imaginary part in the coefficients of the nonperturbative terms, in the two Q^2 regions. The ambiguous imaginary part may be written directly in terms of the characteristic function, as in Eq. (80), and is continuous at $Q^2 = \Lambda^2$. If the real parts of the condensates are to result in a $\mathcal{D}_{NP}^{(L)}(Q^2)$ which is continuous at $Q^2 = \Lambda^2$ this suggests that one should write these in terms of the characteristic function as well, which led us to conjecture Eq. (81) in which there is a real overall nonperturbative factor κ which is undetermined and observable-dependent. All of these properties and results hold in general for Euclidean observables for which a one-chain result of the form of Eq. (6) can be written down. As pointed out in Ref. [4] this is not possible for Minkowskian observables such as $R_{e^+e^-}$, and in this case the question of freezing is more delicate. There is no characteristic function $\omega_{\mathcal{R}}(t)$ to underwrite the smooth transition through $s = \Lambda^2$ from UV to IR. Indeed the Minkowskian gluon condensate OPE contribution is proportional to $\delta'(s)$ [25], so

without a smearing procedure it will give an apparent infrared divergence as $s \rightarrow 0$. We corrected some erroneous statements about the continuity of $\mathcal{R}_{PT}^{(L)}(s)$ at $s = \Lambda^2$ made in [2]. The issue of Minkowskian freezing is interesting and requires further investigation.

An interesting feature of the skeleton expansion representation of Eq. (6) concerns the definition of $a(Q^2)$ for $Q^2 < \Lambda^2$. In this region the result of Eq. (2) is not in fact the solution of the RG equation, but is an analytical continuation of the $Q^2 > \Lambda^2$ result. However notice that in evaluating $\mathcal{D}_{PT}^{(L)}(Q^2)$ for $Q^2 > \Lambda^2$ one is integrating over the region $t < \Lambda^2/Q^2$ where the analytically continued $a(Q^2)$ is required. Also notice that QED skeleton expansion results can be obtained simply by interchanging UV and IR renormalons, and the limits $Q^2 \rightarrow \infty$ and $Q^2 \rightarrow 0$. The implication is that QCD in the IR energy region is QED-like, and conversely that QED in the UV energy region beyond the Landau ghost is QCD-like. Our result of Eq. (82) is closely related to successful models for power corrections based on isolating the IR renormalon ambiguity, such as [29], and to the power correction model of [30,31]. The latter postulates an infrared finite running coupling, and uses a dispersive approach. The infrared limit of the coupling $\bar{\alpha}_0$ is a *universal* parameter in this picture, whereas κ in our approach is expected to be observable-dependent. A more sophisticated discussion of power corrections to DIS sum rules has recently appeared in [32]. Notice that continuity at $Q^2 = \Lambda^2$ is the key constraint leading us to suggest Eq. (81), which is arguably not a model for power corrections but the actual form of the NP component in the one-chain (leading- b) approximation. In future work we intend to report on fits of κ and Λ to experimental data on the Adler function (e.g., the analysis of [33]), and DIS sum rules. It would also be interesting to compare our results for the Q^2 -dependence of the polarized and unpolarized Bjorken sum rules in the light of the relations between them noted in Ref. [34]. Whilst the freezing is straightforward to analyze in the leading- b , one-chain approximation, it is much harder to analyze at the two-chain level, where the anomalous dimensions for the operators will enter. The IR \leftrightarrow UV conformal relations between renormalon residues and condensate coefficients would appear to be the key ingredient in the freezing picture, and there is hope that they can continue to hold at higher orders in the QCD skeleton expansion, if indeed such an expansion can be consistently formulated [15]. There is clearly much still to investigate.

ACKNOWLEDGMENTS

We would like to thank Irinel Caprini, Jan Fischer, Andrei Kataev and Dimitri Shirkov for extremely useful discussions, which have helped us to improve on an earlier version of this paper. P.M.B. gratefully acknowledges support from PPARC UK.

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