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Polyakov loop in chiral quark models at finite temperature

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We describe how the inclusion of the gluonic Polyakov loop incorporates large gauge invariance and drastically modifies finite temperature calculations in chiral quark models after color neutral states are singled out. This generates an effective theory of quarks and Polyakov loops as basic degrees of freedom. We find a strong suppression of finite temperature effects in hadronic observables triggered by approximate triality conservation (Polyakov cooling), so that while the center symmetry breaking is exponentially small with the constituent quark mass, chiral symmetry restoration is exponentially small with the pion mass. To illustrate the point we compute some low-energy observables at finite temperature and show that the finite temperature corrections to the low energy coefficients are N_c suppressed due to color average of the Polyakov loop. Our analysis also shows how the phenomenology of chiral quark models at finite temperature can be made compatible with the expectations of chiral perturbation theory. The implications for the simultaneous center symmetry breaking/chiral symmetry restoration phase transition are discussed also.

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I. INTRODUCTION

The general belief that QCD undergoes a phase transition to a quark-gluon plasma phase at high temperature has triggered a lot of activity both on the theoretical and on the experimental side. The original argument put forward by Casher [1] suggesting that confinement implies dynamical chiral symmetry breaking and hence that the chiral and deconfinement phase transitions take place simultaneously at least at zero chemical potential has been pursued and so far confirmed in theoretical studies on the lattice [2]. This also agrees with the phenomenological determinations of the vacuum energy density in the bag model, with an energy density difference between the Wigner and Goldstone realizations of chiral symmetry. It has also been shown that in the large N_c limit with the temperature T kept fixed, if a chiral phase transition takes place it should be first order [3].

The coupling of QCD distinctive order parameters at finite temperature to hadronic properties has been the subject of much attention over the recent past [4–9] mainly in connection with theoretical expectations on the formation of quark-gluon plasma and the onset of deconfinement. Indeed, even if such a state of matter is produced in existing (Relativistic Heavy Ion Collider, Super Proton Synchrotron [10,11]) and future (Large Hadron Collider) facilities, the states which are detected are hadrons created in a hot environment. Thus, it makes sense to study the properties of hadrons in a medium which can undergo a confinement-deconfinement phase transition. For heavy masses, quarks become static sources and there is a general consensus that the order parameter can be taken to be the Polyakov loop or thermal Wilson line [12] where the breaking of the center symmetry signals the onset of deconfinement. Dynamical light quarks, however, break explicitly the center symmetry and no criterion for deconfinement has been established yet [13,14]. In QCD, there has been increasing interest in developing effective actions for the Polyakov loop as a confinement-deconfinement order parameter because of their relevance in describing the phase transition from above the critical temperature [15–18].

On the other hand, in a hot medium, one also expects that the spontaneously broken chiral symmetry is restored at some critical temperature. For this chiral phase transition the quark condensate is commonly adopted as the relevant order parameter. The melting of the chiral quark condensate has been observed on the lattice [2], is suggested by chiral perturbation theory extrapolations [19,20], and is numerically reproduced in chiral quark models before [21,22] and after inclusion of pion corrections [23] (for a review see e.g. Ref. [24]).

Where theory has most problems is precisely in the interesting intermediate temperature regime around the phase transition, because both the lightest Goldstone particles and the Polyakov loop degrees of freedom should play a role, if they coexist. Up to now it is uncertain how the corresponding states couple to each other from a fundamental QCD viewpoint, hence some modeling is required. Based on previous works [6,8,25], and to comply with chiral symmetry, it seems natural to couple chiral quark models and the Polyakov loop in a minimal way as an effective space-dependent color chemical potential. The work in Ref. [8] accounts for a crossover between the restoration of chiral symmetry and the spontaneous breaking of the center symmetry, reproducing qualitatively the features observed on the lattice simulations [26] and which find a natural explanation in terms of dimension two con-

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densates [27]. In this regard we want to argue below that the special role played by the gauge symmetry at finite temperature actually requires this coupling and elaborate on the consequences of it when the quantum gluon effects are considered.

The organization of the paper is as follows. We review some facts on large gauge symmetry at finite temperature in Sec. II which are put into the context of chiral quark models. Next, we address the problem suffered by chiral quark models at finite temperature in Sec. III where we argue that the origin of the difficulty is related to a defective treatment of the large gauge symmetry at finite temperature. Thus, to comply with gauge invariance at finite temperature one should at least couple the quarks to the A_0 gluon field. We do this in Sec. IV. This is equivalent to making the replacement

$$\partial_0 \to \partial_0 + iA_0, \tag{1.1}$$

which corresponds to an $\vec{x}$-dependent chemical potential coupling in the color fundamental representation. Obviously, this coupling introduces a color source into the problem for a fixed A_0 field. In order to project onto the color neutral states, we integrate over the A_0 field, in a gauge invariant manner. In Sec. V we describe the consequences of such a coupling and projection in chiral quark models for a variety of observables at the one quark loop approximation. Actually, as we will show, there is an accidental $\mathbb{Z}(N_c)$ symmetry in the model which generates a triality (super)selection rule at this level of approximation, from which a strong thermal suppression $\mathcal{O}(e^{-N_cM/T})$ follows in the quenched approximation. This puts some doubts on whether chiral quark models do predict a chiral phase transition at realistic temperatures as we advanced in previous communications [28,29]. Corrections beyond one quark loop are discussed in Sec. VI where we see that the suppression at low temperatures actually becomes $\mathcal{O}(e^{-m_\pi/T})$, very much along the expectations of chiral perturbation theory (ChPT) [19]. Gluonic corrections and local corrections in the Polyakov loop are analyzed also in this section. In view of our discussions we illustrate in Sec. VII the situation with schematic dynamical calculations involving quantum and local Polyakov loops in the unquenched theory as compared to lattice studies. In Sec. VIII we extend these calculations to the region around the phase transition. Finally, in Sec. IX, we summarize our points and draw our main conclusions.

II. GAUGE INVARIANCE OF CHIRAL QUARK MODELS AT FINITE TEMPERATURE AND THE POLYAKOV LOOP

In this section we review some relevant and naïvely very disparate concepts of gauge symmetry at finite temperature, Sec. II A, and the center symmetry in gluodynamics, Sec. II B, as well as the standard chiral quark models, Sec. II C, in order to fix our notation for the rest of the paper. Both subjects are well known on their own, although rarely discussed simultaneously, and the reader familiar with any of them may skip the corresponding subsections. Advancing the result of subsequent discussions made in latter sections, the basic Polyakov chiral quark model is first introduced in Sec. II D. The conflict between both large gauge symmetry and chiral quark models is discussed in Sec. III. The solution to the problem is elaborated in Sec. IV where the coupling of the Polyakov loop to chiral quark models is motivated.

A. Large gauge symmetry

One of the most striking features of a gauge theory like QCD at finite temperatures is the nonperturbative manifestation of the non-Abelian gauge symmetry. Indeed, in the Matsubara formalism of quantum field theory at finite temperature, the space-time becomes a topological cylinder: one introduces a compactified Euclidean imaginary time [30] and evaluates the path integral subjecting the fields to periodic or antiperiodic boundary conditions for bosons and fermions, respectively, in the imaginary time interval $\beta = 1/T$, where T is the temperature. We use the Euclidean notation $x_4 = ix_0$ and $A_4(\vec{x}, x_4) = iA_0(\vec{x}, x_0)$. Thus, only periodic gauge transformations, $g(\vec{x}, x_4) = g(\vec{x}, x_4 + \beta)$, are acceptable since the quark and gluon fields are stable under these transformations. In the Polyakov gauge $\partial_4 A_4 = 0$, with A_4 a diagonal traceless $N_c \times N_c$ matrix, one has for the gauge $\mathrm{SU}(N_c)$ group element

$$g(x_4) = \mathrm{diag}(e^{i2\pi x_4 n_j T}) \tag{2.1}$$

($\sum_{j=1}^{N_c} n_j = 0$) the following gauge transformation on the A_4 component of the gluon field

$$A_4 \to A_4 + 2\pi T \mathrm{diag}(n_j). \tag{2.2}$$

Thus, in this particular gauge, gauge invariance manifests as the periodicity in the A_4 gluon field. This property is crucial and prevents from the very beginning from the use of a perturbative expansion in the gluon field, A_4, at finite temperature. This large gauge symmetry[1] can be accounted for properly by considering the Polyakov loop or untraced Wilson line as an independent degree of freedom,

$$\Omega(x) = \mathcal{T} \exp i \int_{x_4}^{x_4+1/T} dx_4' A_4(\vec{x}, x_4'), \tag{2.3}$$

where $\mathcal{T}$ indicates the Euclidean temporal ordering operator and A_4 the gluon field. Under a general periodic gauge

[1]Technically speaking the transformations (2.1) may not be large in the topological sense (i.e., homotopically nontrivial). This depends on the topology of the spatial manifold as well as on the gauge group [31]. They are topologically large within the Polyakov gauge.

transformation one gets

$$\Omega(x) \to g(x)\Omega(x)g^{\dagger}(x). \tag{2.4}$$

In the Polyakov gauge, which we assume from now on, Ω becomes

$$\Omega(\vec{x}) = e^{iA_4(\vec{x})/T}, \tag{2.5}$$

and so it is invariant under the set of transformations (2.1). The failure of perturbation theory at finite temperature in a gauge theory has generated a lot of discussion in the past—mainly in connection with topological aspects, Chern-Simons terms, anomalies, etc. In the case of the topological Chern-Simons term radiatively induced by fermions in 2 + 1 dimensions [32] it was puzzling to find, in the perturbative treatment, that the Chern-Simons quantization condition [33] was violated at finite temperature [34,35]. It was subsequently shown that, within a nonperturbative treatment, no contradiction arises [36]. In [37,38] it was shown that a derivative expansion approach, suitably defined at finite temperature, was appropriate to deal with this problem. We will use this approach in the present work.

B. Center symmetry in gluodynamics

In pure gluodynamics at finite temperature one can use the center of the gauge group to extend the periodic transformations to aperiodic ones [39],

$$g\left(\vec{x}, \frac{1}{T}\right) = zg(\vec{x}, 0), \qquad z^{N_c} = 1, \tag{2.6}$$

so that z is an element of $\mathbb{Z}(N_c)$. An example of such a transformation (with $z = e^{i2\pi/N_c}$) in the Polyakov gauge is given by

$$g(x_4) = \mathrm{diag}(e^{i2\pi x_4 n_j T/N_c}), \qquad n_1 = 1 - N_c, \qquad n_{j\geq 2} = 1, \tag{2.7}$$

and the gauge transformation on the A_4 component of the gluon field is

$$A_4 \to A_4 + \frac{2\pi T}{N_c}\mathrm{diag}(n_j). \tag{2.8}$$

Under these transformations both gluonic action, measure and boundary conditions are invariant. The Polyakov loop, however, transforms as the fundamental representation of the $\mathbb{Z}(N_c)$ group, i.e. $\Omega \to z\Omega$, yielding $\langle\Omega\rangle = z\langle\Omega\rangle$ and hence $\langle\Omega\rangle = 0$. More generally, in the center symmetric or confining phase

$$\langle\Omega^n\rangle = 0 \quad \text{for } n \neq kN_c, \qquad k \in \mathbb{Z}. \tag{2.9}$$

Actually, this center symmetry is spontaneously broken *above* a critical temperature, $T_D \approx 270$ MeV for $N_c = 3$ [40]. The antiperiodic quark field boundary conditions are not preserved under nontrivial center transformations since $q(\vec{x}, 1/T) \to g(\vec{x}, 1/T)q(\vec{x}, 1/T) = -zg(\vec{x}, 0)q(\vec{x}, 0)$ instead of $-g(\vec{x}, 0)q(\vec{x}, 0)$. A direct consequence of such property is the vanishing of contributions to the quark bilinear of the form[2]

$$\langle\bar{q}(n/T)q(0)\rangle = 0 \quad \text{for } n \neq kN_c, \qquad k \in \mathbb{Z} \tag{2.10}$$

(in the confining phase) since under the large aperiodic transformations given by Eq. (2.6) $\bar{q}(n/T)q(0) \to z^{-n}\bar{q}(n/T)q(0)$. This generates an exact selection rule in quenched QCD. The center symmetry is explicitly broken by the presence of dynamical quarks and the choice of an order parameter for confinement is not obvious [41]. As a consequence the selection rule implied by Eq. (2.10) is no longer fulfilled. Nevertheless, such selection rule becomes relevant to chiral quark models in the large N_c limit and departures from it are found to be suppressed within chiral quark models in the large N_c limit at low temperatures, due to the spontaneous breaking of chiral symmetry which generates heavier constituent quarks from light current quarks.[3] This issue will be analyzed along this paper.

C. Chiral quark models at finite temperature

Chiral quark models have been used in the past to provide some semiquantitative understanding of hadronic features in the low energy domain. At zero temperature chiral quark models are invariant under *global* SU(N_c) transformations. There has always been the question of how the corresponding constituent quark fields transform under *local* color transformations or whether a physical gauge invariant definition can be attached to those fields [42]. If we assume that they transform in the same way as bare quarks, it seems unavoidable to couple gluons to the model in the standard way to maintain gauge invariance as done in previous works (see e.g. Refs. [43,44]). These gluon effects are treated within perturbation theory at $T = 0$. This approximation induces some subleading corrections in the calculation of color singlet states where the effects of confinement can be almost completely ignored for the low lying states [45]. This perturbative gluon dressing also complies with the interpretation that the whole quark model is defined at a low renormalization scale, from which QCD perturbative evolution to high energies processes can be successfully applied [46]. When going to

[2] In this formula $\langle\bar{q}(n/T)q(0)\rangle$ denotes contributions to the quark propagator including only paths which wind n times around the thermal cylinder. The average is for the quenched theory.

[3] We emphasize that our use of the approximate rule is in contrast to the so-called canonical ensemble description of QCD where, upon projection, triality is assumed to be exact even in the presence of dynamical quarks. See e.g. the discussion in [41].

finite temperature, chiral quark models predict already at the one-loop level a chiral phase transition [21,22] at realistic temperatures. However, even at low temperatures, single quark states are excited what is obviously not very realistic for it means that the hot environment is in fact a hot plasma of quarks. On the other hand, since the constituent quark mass is about a factor of 2 larger than the pion mass, pion loops dominate at low temperatures [23] (for a review see e.g. Ref. [24]), as expected from chiral perturbation theory [19,20]. In the present work we will deal with two chiral quark models, the Nambu-Jona-Lasinio (NJL) model [47–49], where quarks are characterized by a constant constituent mass in the propagator due to the spontaneous breaking of chiral symmetry, and the recently proposed spectral quark model (SQM) [50–53], where the notion of analytic confinement is explicitly verified. For completeness we review briefly the corresponding effective action below. One common and attractive feature of chiral quark models is that there is a one-to-one relation to the large N_c expansion and the saddle point approximation of a given path integral both at zero and at finite temperature.

1. The NJL model

The NJL Lagrangian as will be used in this paper reads in Minkowski space[4]

$$\mathcal{L}_{\rm NJL} = \bar{q}(i\not{\partial} - \hat{M}_0)q + \frac{G}{2}\sum_{a=0}^{N_f^2-1}((\bar{q}\lambda_a q)^2 + (\bar{q}\lambda_a i\gamma_5 q)^2), \tag{2.11}$$

where $q = (u, d, s, \ldots)$ represents a quark spinor with N_c colors and N_f flavors. The λ's are the Gell-Mann flavor matrices of the $\mathrm{U}(N_f)$ group and $\hat{M}_0 = \mathrm{diag}(m_u, m_d, m_s, \ldots)$ stands for the current quark mass matrix. In the limiting case of vanishing current quark masses, the classical NJL action is invariant under the global $\mathrm{U}(N_f)_R \otimes \mathrm{U}(N_f)_L$ group of transformations. Using the standard bosonization procedure [54], it is convenient to introduce auxiliary bosonic fields (S, P, V, A) so that after formally integrating out the quarks one gets the effective action[5]

$$\Gamma_{\rm NJL}[S, P] = -iN_c \,\mathrm{Tr}\log(i\mathbf{D}) - \frac{1}{4G}\int d^4x \,\mathrm{tr}_f(S^2 + P^2). \tag{2.12}$$

We use Tr for the full functional trace, tr_f for the trace in flavor space, and tr_c for the trace in color space. Here, the Dirac operator is given by

$$i\mathbf{D} = i\not{\partial} - \hat{M}_0 - (S + i\gamma_5 P). \tag{2.13}$$

The divergencies in Eq. (2.12) from the Dirac determinant can be regularized in a chiral gauge invariant manner by means of the Pauli-Villars method, although the issue of regularization is of little relevance at finite temperature [22] for $T \ll \Lambda$. This model is known not to confine and to produce a constituent quark mass $M \sim 300$ MeV due to the spontaneous breaking of chiral symmetry at zero temperature. The Goldstone bosons can be parameterized by taking

$$S + iP = \sqrt{U}\Sigma\sqrt{U} \tag{2.14}$$

with U a unitary matrix [see Eq. (2.19)] with $\Sigma^\dagger = \Sigma$, and one can use that $\Sigma = M + \phi$ with ϕ the scalar field fluctuation. The partition function for this model can be written as

$$Z_{\rm NJL} = \int DUD\Sigma e^{i\Gamma_{\rm NJL}[U,\Sigma]}. \tag{2.15}$$

By minimizing $\Gamma_{\rm NJL}$, one gets $S = M$, which generates the spontaneous breaking of chiral symmetry, and one obtains the gap equation

$$\frac{1}{G} = -i4N_c\sum_i c_i \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - M^2 - \Lambda_i^2}, \tag{2.16}$$

where the Pauli-Villars regularization has been used. The Pauli-Villars regulators fulfill $c_0 = 1$, $\Lambda_0 = 0$ and the conditions $\sum_i c_i = 0$, $\sum_i c_i \Lambda_i^2 = 0$, in order to render finite the logarithmic and quadratic divergencies, respectively. In practice it is common to take two cutoffs in the coincidence limit $\Lambda_1 \to \Lambda_2 = \Lambda$ and hence $\sum_i c_i f(\Lambda_i^2) = f(0) - f(\Lambda^2) + \Lambda^2 f'(\Lambda^2)$.

2. The SQM model

In the SQM the effective action reads

$$\Gamma_{\rm SQM}[U] = -iN_c \int d\omega \rho(\omega) \,\mathrm{Tr}\log(i\mathbf{D}), \tag{2.17}$$

where the Dirac operator is given by

$$i\mathbf{D} = i\not{\partial} - \omega U^{\gamma^5} - \hat{M}_0 \tag{2.18}$$

and $\rho(\omega)$ is the spectral function of a generalized Lehmann representation of the quark propagator with ω the spectral mass defined on a suitable contour of the complex plane [50–53]. The use of certain spectral conditions guarantees finiteness of the action. The matrix $U = u^2 = e^{i\sqrt{2}\Phi/f}$ (f is the pion weak decay constant in the chiral limit) is the flavor matrix representing the pseudoscalar octet of mesons in the nonlinear representation,

[4] We use Bjorken-Drell convention throughout the paper.

[5] Obviously at finite temperature the quark fields satisfy antiperiodic boundary conditions whereas the bosonized fields obey periodic boundary conditions.

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \tag{2.19}$$

A judicious choice of the spectral function based on vector meson dominance generates a quark propagator with no poles (analytic confinement). More details of the SQM at zero and finite temperature relevant for the present paper are further developed in Appendix A. The partition function for the SQM can be written as

$$Z_{\rm SQM} = \int DU e^{i\Gamma_{\rm SQM}[U]}. \tag{2.20}$$

D. The Polyakov chiral quark model

As we will show in Sec. III there is a conflict between large gauge invariance at finite temperature, reviewed in the previous Secs. II A and II B, and the standard chiral quark models presented in Sect. II C. The chiral quark model coupled to the Polyakov loop that will be motivated in Sec. IV and analyzed in the rest of this paper synthesizes the solution and corresponds to simply make the replacement

$$\partial_4 \to \partial_4 - iA_4 \tag{2.21}$$

in the Dirac operators, Eq. (2.13) and (2.18), and integrating further over the A_4 gluon field in a gauge invariant manner [55] yielding a generic partition function of the form

$$Z = \int DUD\Omega e^{i\Gamma_G[\Omega]} e^{i\Gamma_Q[U,\Omega]}, \tag{2.22}$$

where DU is the Haar measure of the chiral flavor group $\mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L$ and $D\Omega$ the Haar measure of the color group $\mathrm{SU}(N_c)$, Γ_G is the effective gluon action whereas Γ_Q stands for the quark effective action. If the gluonic measure is left out $A_4 = 0$ and $\Omega = 1$ we recover the original form of the corresponding chiral quark model, where there exists a one-to-one mapping between the loop expansion and the large N_c expansion both at zero and finite temperature. Equivalently one can make a saddle point approximation and corrections thereof. In the presence of the Polyakov loop such a correspondence does not hold, and we will proceed by a quark loop expansion, i.e. a saddle point approximation in the bosonic field U, keeping the integration on the Polyakov loop Ω. The work of Ref. [6] corresponds to make also a saddle point approximation in Ω. In Sec. V we stick to the one-loop approximation and keep the group integration. This is the minimal way to comply with center symmetry at low temperatures. Although in principle $\Omega(x)$ is a local variable, in what follows we will investigate the consequences of a spatially constant Polyakov loop. In this case the functional integration $D\Omega$ becomes a simple integration over the gauge group $d\Omega$. The issue of locality is reconsidered in Sec. VI C.

III. UNNATURALNESS OF CHIRAL QUARK MODELS AT FINITE TEMPERATURE

In this section we analyze the problem of chiral quark models at finite temperature, its interpretation in terms of thermal Boltzmann factors as well as the corresponding conflicts with chiral perturbation theory at finite temperature.

A. The problem

As already mentioned, chiral quark models at finite temperature have a problem since, even at low temperatures, excited states with any number of quarks are involved, whether they can form a color singlet or not. This is hardly a new observation, the surprising thing is that nothing has been done about it so far, attributing the failure to common diseases of the model, such as the lack of confinement. To illustrate this point in some more detail we will use a constituent quark model like the NJL model, where the quark propagator has a constant mass. To be specific, let us consider as an example the calculation of the quark condensate for a single flavor in constituent quark models with mass M. At finite temperature in the Matsubara formulation we have the standard rule

$$\int \frac{dk_0}{2\pi} F(\vec{k}, k_0) \to iT \sum_{n=-\infty}^{\infty} F(\vec{k}, i\omega_n) \tag{3.1}$$

with ω_n the fermionic Matsubara frequencies, $\omega_n = 2\pi T(n + 1/2)$. For the discussion in this and forthcoming sections it is convenient to elaborate this formula a bit further. Using Poisson's summation formula

$$\sum_{m=-\infty}^{\infty} F(m) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx F(x) e^{i2\pi xn} \tag{3.2}$$

one gets the rule

$$\int \frac{dk_0}{2\pi} F(\vec{k}, k_0) \to i \sum_{n=-\infty}^{\infty} (-1)^n \int \frac{dk_4}{2\pi} F(\vec{k}, ik_4) e^{ink_4/T}. \tag{3.3}$$

In terms of the Fourier transform, one obtains for a finite temperature fermionic propagator starting and ending at the same point,

$$\tilde{F}(x; x) \to \sum_{n=-\infty}^{\infty} (-1)^n \tilde{F}(\vec{x}, x_0 + in/T; \vec{x}, x_0). \tag{3.4}$$

Note that the zero temperature contribution corresponds to the term $n = 0$ in the sum. From a path integral point of view, the zero temperature term comes from contractile closed paths whereas thermal contributions come from closed paths which wind n times around the thermal cylinder. For fermions, each winding picks up a -1 factor.

For the (single flavor) condensate we get[6]

$$\begin{aligned}\langle\bar{q}q\rangle^* &= -iN_c \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{tr}_{\text{Dirac}} S(x) \Big|_{x_0=in/T} \\ &= -i4MN_c \sum_{n=-\infty}^{\infty} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}(-1)^n}{k^2-M^2} \Big|_{x_0=in/T} \\ &= \langle\bar{q}q\rangle - 2\frac{N_c M^2 T}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} K_1(nM/T). \end{aligned} \tag{3.5}$$

In writing the previous formula, finite cutoff corrections, appearing in the chiral quark models such as the NJL model at finite temperature, have been neglected. This is not a bad approximation provided the temperature is low enough $T \ll \Lambda$ (typically one has $\Lambda \approx 1$ GeV so even for $T \approx M \approx 300$ MeV the approximation works). At low temperatures we may use the asymptotic form of the Bessel function [56]

$$K_n(z) \sim e^{-z}\sqrt{\frac{\pi}{2z}} \tag{3.6}$$

to get for the leading contribution,

$$\langle\bar{q}q\rangle^* \sim \langle\bar{q}q\rangle + 4N_c\left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T}. \tag{3.7}$$

This means a rather flat dependence on temperature for $T \lesssim M$. [Numerically, the correction is about 1% for $T \approx 100$ MeV for $M = 300$ MeV and $\langle\bar{q}q\rangle \approx -(240\text{ MeV})^3$.] The strong attractive interaction which causes chiral dynamical symmetry breaking is reduced at finite temperature and the energy is decreased by a decreasing constituent quark mass M^*, eventually leading to a chiral phase transition [21,22]; the critical temperature is $T \approx 200$ MeV.[7] The coincidence of this number with lattice simulations has been considered a big success of chiral quark models and has triggered a lot of activity in the past (see e.g. Ref. [24] and references therein). We show below that this apparent success might be completely accidental, as it does not incorporate basic physical requirements.

B. Interpretation

An interpretation of the previous formula for the condensate is in terms of statistical Boltzmann factors. Using the definition of the quark propagator in coordinate space

$$S(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{\not{k} - M} = (i\not{\partial} + M)\Delta(x) \tag{3.8}$$

with

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2-M^2} = \frac{M^2}{4\pi^2 i} \frac{K_1(\sqrt{-M^2x^2})}{\sqrt{-M^2x^2}}, \tag{3.9}$$

at low temperature we get

$$S(\vec{x}, i/T) \sim e^{-M/T}. \tag{3.10}$$

Thus, for $\langle\bar{q}q\rangle^*$ and up to prefactors, we have the exponential suppression for a single quark propagator at low temperature. Using Eqs. (3.5) and (3.6), the quark condensate can be written in terms of Boltzmann factors with a mass formula $M_n = nM$ corresponding to any number of quark states.

One might object against the previous interpretation by arguing that these factors only reflect in the Euclidean coordinate space the pole of the propagator in Minkowski momentum space, and hence that they are a natural consequence of the lack of confinement. While the former statement is true, in the sense that singularities in Minkowski momentum space can be seen at large Euclidean coordinate values, the conclusion drawn from there is incorrect. As shown in Ref. [52] (see Appendix A) quark propagators with no poles but cuts can also produce a Boltzmann factor *without* prefactors, as it should be.[8] To the same level of approximation, i.e. one quark loop, in the SQM we get (see Appendix A for details)

$$\begin{aligned}\frac{\langle\bar{q}q\rangle^*}{\langle\bar{q}q\rangle} &= \tanh(M_S/4T) \\ &= 1 - 2e^{-M_S/2T} + 2e^{-M_S/T} + \cdots, \end{aligned} \tag{3.11}$$

where the "Boltzmann" constituent mass can be identified with half the scalar meson mass $M = M_S/2$.[9] These calculations illustrate our main point and can be extended to any observables which are color singlets in the zero temperature limit; quark model calculations at finite temperature in the one-loop approximation generate *all* possible quark states,

$$\mathcal{O}^* = \mathcal{O} + \mathcal{O}_q e^{-M/T} + \mathcal{O}_{qq} e^{-2M/T} + \cdots. \tag{3.12}$$

While there is no doubt that the leading term $\mathcal{O}_q$ has a Boltzmann factor corresponding to a single quark state, the term with mass $2M$ could in principle be equally a qq diquark state or a $\bar{q}q$ meson state. The latter possibility

[6] In what follows we use an asterisk as superscript for finite temperature quantities, i. e. $\mathcal{O}^* = \mathcal{O}_T$.

[7] The minimization can be written as the equation $\langle\bar{q}q\rangle^* \times (M^*)/M^* = \langle\bar{q}q\rangle(M)/M$, so one has to know the mass dependence of the condensate at zero temperature.

[8] Actually, the previous counter example shows that the lack of confinement has more to do with the presence of the exponential prefactors which are related to the available phase space.

[9] This relation together with the large N_c quark-hadron duality relation $M_S = M_V$ discussed in Ref. [53] yields $M = M_V/2 \sim 385$ MeV, a reasonable value.

should be discarded, however. At one loop a $\bar{q}q$ pair can only come from the quark line going upwards and then downwards in imaginary time propagation. Since such a path does not wind around the thermal cylinder it is already counted in the zero temperature term. The qq contribution, instead, corresponds to the single quark line looping twice around the thermal cylinder and is a proper thermal contribution. This is confirmed below. These Boltzmann factors control the whole physics, and temperature effects are sizeable for $T \approx M$.

C. Conflicts with ChPT

Our observation on the Boltzmann factor is rather puzzling because it seems hard to understand how it is possible to generate nonsinglet states by just increasing the temperature. The reason has to do with the fact that the condensate itself is not invariant under $\mathbb{Z}(N_c)$ transformations at finite temperature. For the example of the condensate we trivially obtain

$$\langle \bar{q}q \rangle^* = \sum_{n=-\infty}^{\infty} (-1)^n \langle \bar{q}(x_0) q(0) \rangle \Big|_{x_0 = in/T} \tag{3.13}$$

i.e., the condensate at finite temperature can be written as a coherent sum of nonlocal quark condensates at zero temperature. If we make a gauge transformation of the central type, we get

$$\langle \bar{q}q \rangle^* \to \sum_{n=-\infty}^{\infty} (-z)^n \langle \bar{q}(x_0) q(0) \rangle \Big|_{x_0 = in/T} \tag{3.14}$$

i.e., the condensate can be decomposed as a sum of irreducible representations of a given triality n. Thus, the state with Boltzmann factor $e^{-nM/T}$ is indeed a multiquark state.

This avoids the paradox, and suggests that in order to make a (centrally extended) gauge invariant definition of the condensate we could simply discard from the sum those terms which do not have zero triality, i.e. we would get

$$\langle \bar{q}q \rangle^* |_{\text{singlet}} = \sum_{n=-\infty}^{\infty} (-1)^{nN_c} \langle \bar{q}(x_0) q(0) \rangle \Big|_{x_0 = iN_c n/T}. \tag{3.15}$$

This would generate as a first thermal correction a term with a Boltzmann factor corresponding to mass $N_c M$ (a baryon) which is obviously very much suppressed. Since a quark loop generates a dependence proportional to N_c we would obtain a $N_c e^{-MN_c/T}$ dependence.

Another problem now comes from comparison with the expectations of chiral perturbation theory at finite temperature [19]. In the chiral limit, i.e., for $m_\pi \ll 2\pi T \ll 4\pi f_\pi$ the leading thermal corrections to the quark condensate for $N_f = 2$, for instance, are given by

$$\langle \bar{q}q \rangle^* |_{\text{ChPT}} = \langle \bar{q}q \rangle \left[1 - \frac{T^2}{8 f_\pi^2} - \frac{T^4}{384 f_\pi^4} + \cdots \right]. \tag{3.16}$$

Thus, the finite temperature correction is N_c suppressed as compared to the zero temperature value, since f_π^2 scales as N_c. This feature remains for finite pion mass and is generic to any thermal correction in ChPT; the dominant contribution comes from quantum pionic fluctuations and not from quark thermal excitations. Although the previous formula predicts a lowering of the quark condensate, it cannot describe the chiral phase transition since ChPT assumes from the start a nonvanishing chiral condensate. In this sense, the scaling behavior of the critical temperature with f_π and therefore with N_c suggested from direct extrapolation of the formula can only be regarded as an assumption.

At this point we should remind that the mechanism by which the chiral symmetry is restored at finite temperature in standard chiral quark models in the one quark loop approximation is quite different from the trend deduced from ChPT based mainly on pion loops. While in the first case it is due to populating states of the Dirac levels with the Fermi-Dirac thermal factor and a sudden decrease of the constituent quark mass gap $2M$, in ChPT the "phase transition" is merely due to large quark-antiquark excitations with the lightest pion quantum numbers with a fixed gap (otherwise ChPT method cannot be applied). These two pictures of the chiral symmetry restoration are not dual to each other; the N_c behavior of the critical temperature is different since in chiral quark models one has $T_c \sim M \sim N_c^0$ while in ChPT the extrapolated value of the "critical temperature" is $T_c \sim 2\sqrt{2} f_\pi \sim \sqrt{N_c}$. Quantum fluctuations have been included in chiral quark models at finite temperature [23] (for a review see e.g. Ref. [24]) and they are known to be $1/N_c$ suppressed. Actually, the subleading $1/N_c$ contribution reproduces the first term of ChPT, Eq. (3.16), thus largely dominating at low temperatures. Taking into account that ChPT by itself and more refined approaches incorporating meson resonance effects [57,58] provide similar values of the "critical temperature" quite close to the lattice predictions [2] for dynamical fermions and extrapolated to the chiral limit, one may wonder what is the meaning of the mean field quark chiral phase transition predicted in the past [21,22,24] and which has become a justification for chiral quark models at finite temperatures. These problems are also common to models where quarks and mesons are regarded as independent degrees of freedom.

We will see in the rest of the paper how a convenient N_c suppression of quark thermal corrections arises naturally when a color source, the Polyakov loop, is coupled to the chiral quark model and subsequent projection onto color neutral states is carried out. In this scenario one would have a large transition temperature $T_c \sim N_c M$ due to quarks, i.e. no symmetry restoration due to filling in the states above the Dirac levels in the absence of dynamical gluons and in the quenched approximation (Polyakov cooling). Nonperturbative gluonic corrections modify this picture; they do predict instead a critical temperature roughly equal the deconfinement phase transition, $T_c = T_D$. Finally, pion loops are protected from additional suppressions, so that the final result will be fully compatible with the ChPT behavior at low temperature.

IV. COUPLING THE POLYAKOV LOOP IN CHIRAL QUARK MODELS

A. General considerations

As we have said, one can formally maintain gauge invariance at zero temperature by coupling gluons to the model. In the spirit of the model these degrees of freedom should be treated within perturbation theory, since the constituent quarks carry some information on nonperturbative gluon effects (see e.g. Refs. [43,44] for explicit calculations in the low energy limit). At finite temperature the situation is radically different; a perturbative treatment of the A_0 component of gluon field manifestly breaks gauge invariance (namely, under large gauge transformations). The consequences of treating such a coupling nonperturbatively in the case of a constant A_0 field are straightforward and enlightening (see below for a discussion on the x-dependent case).

Actually, in a more general context, the Polyakov loop appears naturally in any finite temperature calculation in the presence of minimally coupled vector fields within a derivative expansion or a heat-kernel expansion approach. In this case, as shown in [31,59], a local one-loop quantity, such as the effective Lagrangian or an observable, takes the form

$$\mathcal{L}(x) = \sum_n \mathrm{tr}[f_n(\Omega(x))\mathcal{O}_n(x)], \tag{4.1}$$

where tr acts on all internal degrees of freedom, n labels all possible local gauge invariant operators $\mathcal{O}_n(x)$ (i.e. containing covariant derivatives), possibly with breaking of Lorentz symmetry down to rotational symmetry, and $f_n(\Omega(x))$ are temperature-dependent functions of the Polyakov loop which replace the numerical coefficients present in the zero temperature case. In this general context $\Omega(x)$ would be the local Polyakov loop of all minimally coupled fields.[10] In particular, a chemical potential would give a contribution $e^{\mu/T}$. Here we can see the necessity of the presence of Ω in (4.1): μ being a constant, it gives no contribution in the covariant derivative and so in $\mathcal{O}_n(x)$, therefore the chemical potential can only act through the presence of the Polyakov loop in the expression. This consideration also illustrates the breaking of gauge invariance in a perturbative treatment of Ω: $e^{\mu/T}$ depends periodically on the chemical potential, with period $2\pi i T$; this is a consequence of the coupling of μ to the integer quantized particle (or rather charge) number. Such periodicity would be spoiled in a perturbative treatment. Note that such periodicity is equivalent to one-valuedness of the functions f_n in (4.1).

[10] As noted below, in a model with vector mesons, there would be a corresponding flavor Polyakov loop. Such a contribution is expected to be much suppressed due to the large physical mass of the mesons.

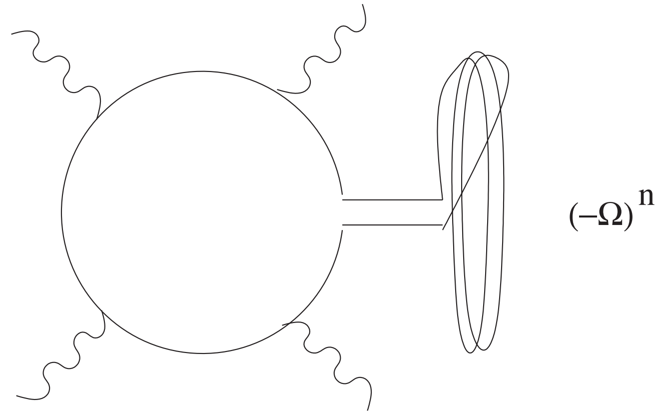

FIG. 1. Typical one quark loop diagram with a nontrivial Wilson line. For n windings around the U(1) compactified imaginary time the quarks get a topological factor Ω^n in addition to the Fermi-Dirac statistical factor $(-1)^n$. Wavy lines are external fields. The total contribution to the diagram is obtained by summing over all windings and tracing over color degrees of freedom.

B. Coupling the Polyakov loop

Coming back to chiral quark models with gluonic Polyakov loops, in fact, the analogy with the chemical potential has been invoked in a recent proposal of Fukushima [6],[11] which suggests coupling chiral quark models to the Polyakov loop at finite temperature in this way. Our own approach is similar, except that, as in (4.1), we consider a *local* Polyakov loop $\Omega(\vec{x})$ coupled to the quarks. This is what comes out of explicit one-loop calculations within a derivative expansion approach at finite temperature [59,62,63]. In those calculations there is a loop momenta integration at each given x, and the Polyakov loop appears minimally coupled, i.e., through the modified fermionic Matsubara frequencies,

$$\hat{\omega}_n = 2\pi T(n + 1/2 + \hat{\nu}), \tag{4.2}$$

which are shifted by the logarithm of the Polyakov loop

$$\Omega = e^{i2\pi\hat{\nu}}, \tag{4.3}$$

i.e. $\hat{\nu}(\vec{x}) = A_4(\vec{x})/(2\pi T)$. In our considerations, this is the only place where explicit dependence on color degrees of freedom appears, so it is useful to think of $\hat{\nu}$ as the corresponding eigenvalues. The effect of such a shift corresponds to change Eq. (3.4) into

[11] After our work was sent for publication Refs. [60,61] appeared, extending the results of Fukushima.

$$\tilde{F}(x;x) \to \sum_{n=-\infty}^{\infty} (-\Omega(\vec{x}))^n \tilde{F}(\vec{x}, x_0 + in/T; \vec{x}, x_0). \quad (4.4)$$

The interpretation of this formula can be visualized in Fig. 1; in a one quark loop with any number of external fields at finite temperature and with a nontrivial Polyakov line, the quarks pick up a phase (-1) due to Fermi-Dirac statistics, and a non-Abelian Aharonov-Bohm[12] factor Ω each time the quarks wind around the compactified Euclidean time. The total contribution to the diagram is obtained by summing over all windings and tracing over color degrees of freedom.

C. Dynamical Polyakov loop

The above prescription gives the contribution for a given gluon field configuration, of which we have only retained the Polyakov loop.[13] The next step is to integrate the gluons according to the QCD dynamics. This implies an average over the local Polyakov loop with some normalized weight $\rho(\Omega;\vec{x})d\Omega$. Here $d\Omega$ is the Haar measure of $SU(N_c)$ and $\rho(\Omega;\vec{x})$ the (temperature-dependent) probability distribution of $\Omega(\vec{x})$ over the gauge group. The emergence of the Haar measure of the integral representation of the Yang-Mills partition function was explicitly shown in Ref. [55]. Because of gauge invariance, $\rho(\Omega)$ will be invariant under similarity transformations, and hence it is really a function of the eigenvalues of Ω. In this section we will mainly remain within a quenched approximation and so the weight follows from a pure Yang-Mills dynamics, in particular, the weight will be $\vec{x}$ independent, as we do not consider external fields coupled to the gluons.[14] In Yang-Mills dynamics (in four dimensions and three colors) it its known to take place at first-order transition from a center symmetric phase to a broken symmetry or deconfining phase. Note that this is a rather peculiar phase transition where the symmetry is restored *below* the critical temperature, just the opposite as the standard case. Since the transition is discontinuous in observables such as the expectation value of the Polyakov loop, the probability distribution $\rho(\Omega)$ will also be discontinuous as a function of the temperature at the critical temperature. In the confining phase $\rho(\Omega)$ will be invariant under $\mathbb{Z}(N_c)$, $\rho(\Omega) = \rho(z\Omega)$. In the deconfining phase, such symmetry is spontaneously broken and one expects the Polyakov loop to concentrate around one of the elements of the center, at random. The coupling of dynamical quarks favors the perturbative value $\Omega = 1$ ($A_4 = 0$) as follows from computations of the effective potential at high temperature [64–66]. So in that phase we expect to have Ω concentrated near $\Omega = 1$, which would be equivalent to no Polyakov loop in the calculation.

Actually, one does not need the full distribution of Ω on $SU(N_c)$, but only the marginal distribution of eigenvalues. Denoting the Polyakov loop average by $\langle\rangle$, we have for a quark observable

$$\mathcal{L}(x) = \sum_n \langle \mathrm{tr}_c f_n(\Omega)\rangle \mathrm{tr}\mathcal{O}_n(x). \quad (4.5)$$

Consistently with gauge invariance, the functions $f_n(\Omega)$ are just ordinary functions $f_n(z)$ evaluated at $z = \Omega$ (e.g. e^{Ω}) hence, if $e^{i\phi_j}$, $j = 1, \ldots, N_c$ are the eigenvalues of Ω

$$\left\langle \frac{1}{N_c} \mathrm{tr}_c f(\Omega) \right\rangle = \int_{SU(N_c)} d\Omega \rho(\Omega) \frac{1}{N_c} \sum_{j=1}^{N_c} f(e^{i\phi_j}) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \hat{\rho}(\phi) f(e^{i\phi}) \quad (4.6)$$

with

$$\hat{\rho}(\phi) := \int_{SU(N_c)} d\Omega \rho(\Omega) \frac{1}{N_c} \sum_{j=1}^{N_c} 2\pi\delta(\phi - \phi_j). \quad (4.7)$$

Equivalently, all that it is needed is the set of momenta of the distribution, $\langle \mathrm{tr}_c(\Omega^n)\rangle$.

D. Group averaging

At sufficiently low temperature in the quenched theory we can go further on the analytical side, since the distribution of the Polyakov loop becomes just the Haar measure in this regime. As it will be discussed in Sec. VI B, this fact is justified with results based on strong coupling expansions and in one massive gluon loop approximation. Actually, from Eq. (6.18) we find that in observables such as the quark condensate, the effect of $\rho(\Omega)$ being different from unity is almost negligible for all temperatures below the transition, implying that a Haar measure distribution is an extremely good approximation in the confined phase. We elaborate further on gluonic corrections in Sec. VI B.

The corresponding density of eigenvalues of the $SU(N_c)$ group is given by [67,68]

$$\frac{1}{N_c!} 2\pi\delta\left(\sum_{i=1}^{N_c} \phi_i\right) \prod_{i<j}^{N_c} |e^{i\phi_i} - e^{i\phi_j}|^2 \prod_{i=1}^{N_c} \frac{d\phi_i}{2\pi}, \quad (4.8)$$

so $\hat{\rho}(\phi)$ of (4.7) is simply

$$\hat{\rho}(\phi) = 1 - \frac{2(-1)^{N_c}}{N_c} \cos(N_c\phi). \quad (4.9)$$

Using this result one can easily deduce the following useful

[12] This is an electric type of phase and not the standard magnetic one. The name is nonetheless appropriate since this electric phase was discussed first in the original AB paper.

[13] In addition, gluons appear also perturbatively through the covariant derivative. This will produce perturbative gluon exchange contributions as in the zero temperature case. We will not consider those in this work.

[14] In Secs. VI C and VII we will discuss some implications about local corrections in the Polyakov loop and unquenched results, respectively.

formulas for the average over the $SU(N_c)$ Haar measure

$$\langle \mathrm{tr}_c(-\Omega)^n\rangle_{SU(N_c)} = \begin{cases} N_c, & n=0 \qquad (4.10)\\ -1, & n=\pm N_c \qquad (4.11)\\ 0, & \text{otherwise} \qquad (4.12). \end{cases}$$

When this is inserted in, e.g., Eq. (4.5), one finds that the effect is not only to remove the triality breaking terms, as in Eq. (3.15), but additionally, the surviving thermal contributions are N_c suppressed as compared to the naïve expectation. This solves the second problem noted in Sec. III.

E. Polyakov cooling mechanism

In an insightful work, Fukushima [6] has modeled the coupling of the Polyakov loop to chiral quarks, with emphasis in the description of the deconfining and chiral phase transitions (or rather, crossovers). The fact that the critical temperatures for both transitions are nearly equal, according to lattice calculations [40], finds a natural explanation in that model. This follows from what we will call the *Polyakov cooling* mechanism, namely, the observation that, upon introduction of coupling with the Polyakov loop, any quark observable at temperature T (below T_D) roughly corresponds to the same observable in the theory without Polyakov loop but at a lower temperature, of the order of T/N_c, as already noted in [69]. This is a direct consequence of triality conservation. As discussed for Eqs. (3.14) and (3.15) at the end of Sec. III, Boltzmann weights $e^{-M/T}$ are suppressed in favor of $e^{-N_cM/T}$. An extreme example of cooling would come from considering an U(1) gauge theory in a confined phase in such a way that Ω is a completely random phase coupled to the quark. This would be equivalent to a uniform average of the variable $\hat{\nu}$ in Eq. (4.2) in the interval $[0, 1]$. Clearly, such an average completely removes the discretization of the Matsubara frequencies and gives back the continuum frequency of the zero temperature theory. The same extreme cooling would be obtained in an $U(N_c)$ gauge theory. In the $SU(N_c)$ case the average is not so effective since the phases corresponding to each of the N_c colors are not changed independently, owing to the restriction $\det\Omega = 1$. The cooling mechanism will be substantially modified in the unquenched theory, since sea quark loops allow the creation of thermal [i.e., with n different from zero in e.g. Eq. (3.4)] color singlet quark-antiquark pairs which propagate without any direct influence of the Polyakov loop.

The way Polyakov cooling brings the chiral and deconfining critical points to coincide is as follows. In the chiral theory without Polyakov loop, the critical temperature of the chiral transition is such that $T_\chi^{\Omega=1} < T_D$ yet $N_c T_\chi^{\Omega=1} > T_D$. Hence, in the theory with coupling to the Polyakov loop, one finds that for $T < T_D$ Polyakov cooling acts, $\langle\bar{q}q\rangle^*$ becomes roughly that of T/N_c which is below $T_\chi^{\Omega=1}$ and chiral symmetry is broken. On the other hand, for $T > T_D$, Polyakov cooling no longer acts and Ω quickly becomes unity, as in the theory without Polyakov loop at the same temperature; since T is above $T_\chi^{\Omega=1}$, chiral symmetry is restored. As a consequence the chiral transition is moved up and takes place at the same temperature as the deconfining transition, $T_\chi^{\langle\Omega\rangle} \approx T_D$. This result is consistent with [70] where it is shown that, at least in the large N_c limit, confinement implies chiral symmetry breaking.

We note a difference in our treatment of the Polyakov loop coupling and that in [6], namely, we use a local Polyakov loop subject to thermal and quantum fluctuations, as described by the distribution $\rho(\Omega;\vec{x})d\Omega$. This is in contrast with [6] where Ω is global and does not fluctuate. Instead Ω is determined through a mean field minimization procedure plus a specific choice of the allowed values (orbit) of Ω on the group manifold. In this way a model is obtained which is simple and at the same time can be used to address key issues of QCD at finite temperature. Nevertheless, let us argue why such an approach needs to be improved. At sufficiently low temperature the model in Ref. [6] for the gluon dynamics consists just of the invariant Haar measure on the gauge group, therefore any group element is just as probable as any other. If one takes some coordinates on the group manifold and makes a maximization of the resulting probability density, one is actually maximazing the Jacobian and the result will depend on the coordinates chosen. In the deconfined phase the local Polyakov loop is still subject to fluctuations (even in the thermodynamic limit). A different quantity is $\bar{\Omega}$, the spatial average of the local loop.[15] This is a global object by construction. Both quantities, $\Omega(x)$ and $\bar{\Omega}$, have the same expectation value, due to translational invariance, but $\bar{\Omega}$ does not fluctuate in the thermodynamic limit. The usual effective potential is so defined that its minimum gives the correct expectation value, and so $\bar{\Omega}$, but it does not give information on the fluctuations of $\Omega(x)$.

In the confining phase of the quenched theory triality is preserved, hence, after gluon average, Eq. (4.4) becomes

$$\tilde{F}(x;x) \to \sum_{n=-\infty}^{\infty} \langle(-\Omega(\vec{x}))^{nN_c}\rangle \tilde{F}(\vec{x}, x_0 + inN_c/T; \vec{x}, x_0), \tag{4.13}$$

which is the quenching invoked in Sec. III.

V. ONE QUARK LOOP RESULTS

The calculations outlined above in Sec. IV can be routinely applied to all observables. A more thorough and systematic study will be presented elsewhere. As an illustration we show here low temperature results (i.e. retaining only the Haar measure in the gluon averaging) for the quark condensate and the pion weak and electromagnetic anomalous decays for their relevance in chiral symmetry

[15] The quantity $\bar{\Omega}$ so defined does not lie on the group manifold, so some prescription should be devised to map it onto the group.

breaking, both for the NJL model as well as for the SQM at the one quark loop level. In Sec. VI A we discuss the structure of higher order corrections due to quark loops while in Sec. VI B dynamical gluonic effects are considered. Corrections beyond the quenched approximation will be explicitly computed in Sec. VII. In Ref. [71] we compute the full chiral Lagrangian at finite temperature at the one quark loop level.

A. Results for constituent quark models

To visualize the additional suppression, we apply the previous result to the calculation of the condensate at finite temperature. At the one-loop level we just make the substitution $N_c(-1)^n \to \mathrm{tr}_c\langle(-\Omega)^n\rangle$. We get

$$\langle\bar{q}q\rangle^* = -i4M \sum_{n=-\infty}^{\infty} \mathrm{tr}_c\langle(-\Omega)^n\rangle \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - M^2} \bigg|_{x_0 = in/T}. \tag{5.1}$$

This yields

$$\langle\bar{q}q\rangle^* = \langle\bar{q}q\rangle + \frac{2M^2T}{\pi^2 N_c} K_1(N_c M/T) + \cdots. \tag{5.2}$$

The dots indicate higher gluonic or sea quark effects. Because T is small we have further

$$\langle\bar{q}q\rangle^* \sim \langle\bar{q}q\rangle + 4\left(\frac{MT}{2\pi N_c}\right)^{3/2} e^{-N_c M/T}. \tag{5.3}$$

When compared to the ChPT result Eq. (3.16) we see that the N_c suppression of the constituent quark loop model is consistent with the expectations.

For the pion weak decay constant we obtain

$$f_\pi^{*2} = -i4M^2 \sum_{n=-\infty}^{\infty} \mathrm{tr}_c\langle(-\Omega)^n\rangle \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{[k^2 - M^2]^2} \bigg|_{x_0 = in/T} \tag{5.4}$$

yielding

$$\frac{f_\pi^{*2}}{f_\pi^2} = 1 - \frac{M^2}{\pi^2 f_\pi^2} K_0(N_c M/T) + \cdots. \tag{5.5}$$

The $\pi^0 \to \gamma\gamma$ amplitude is given by

$$F_{\pi\gamma\gamma}^* = i\frac{8M^2}{N_c f_\pi} \sum_{n=-\infty}^{\infty} \mathrm{tr}_c\langle(-\Omega)^n\rangle \times \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{[k^2 - M^2]^3} \bigg|_{x_0 = in/T}. \tag{5.6}$$

Using the value obtained at zero temperature, $F_{\pi\gamma\gamma} = 1/4\pi^2 f_\pi$, consistent with the anomaly, we get

$$\frac{F_{\pi\gamma\gamma}^*}{F_{\pi\gamma\gamma}} = 1 - \frac{2M}{T} K_1(N_c M/T) + \cdots. \tag{5.7}$$

This obviously complies again to the fact that the leading low temperature corrections should be encoded in pionic thermal excitations rather than quark excitations.

B. Spectral quark model

In the spectral quark model one averages with a given spectral function the previous result (3.11) and including the Polyakov loop average we get (see Appendix A for details)

$$\frac{\langle\bar{q}q\rangle^*}{\langle\bar{q}q\rangle} = 1 - \frac{2}{N_c} e^{-N_c M_S/2T} + \cdots. \tag{5.8}$$

For the pion weak decay constant we obtain

$$\frac{f_\pi^{*2}}{f_\pi^2} = 1 - \frac{1}{N_c}\left(2 + \frac{N_c M_V}{T}\right) e^{-N_c M_V/2T} + \cdots, \tag{5.9}$$

and the $\pi^0 \to \gamma\gamma$ amplitude is given by

$$\frac{F_{\pi\gamma\gamma}^*}{F_{\pi\gamma\gamma}} = 1 - \frac{1}{6N_c}\left[12 + \frac{6N_c M_V}{T} + \left(\frac{N_c M_V}{T}\right)^2\right] e^{-N_c M_V/2T} + \cdots. \tag{5.10}$$

VI. CORRECTIONS BEYOND ONE QUARK LOOP

In the previous sections we have restricted discussion to the one quark loop approximation for observables. This corresponds to the quenched approximation within the model, and to some extent this provides an oversimplified picture. In the present section we discuss the kind of corrections that we expect in this approximation.

A. Higher quark loop corrections

Going beyond the one quark loop approximation may require tedious calculations (see e.g. Refs. [23,24] for

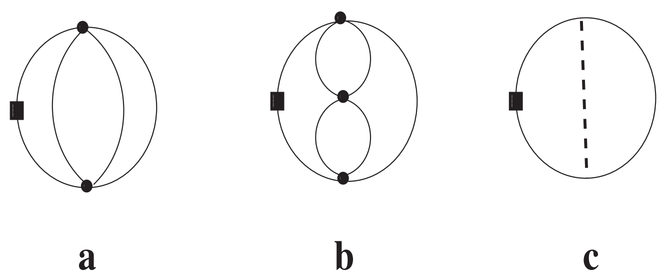

FIG. 2. Typical higher quark loop diagram for the quark condensate operator $\bar{q}q$. Quark lines with independent momenta may wind n times around the compactified Euclidean time, yielding a Fermi-Polyakov factor $(-\Omega)^n$. Triality conservation allows the internal quark-antiquark lines to wind with opposite signs only once, yielding an exponential suppression $e^{-2M/T}$ for diagram (a). A similar suppression occurs for diagram (b) if the quark-antiquark windings happen at any of the bubbles. Diagram (c) corresponds to summing up all intermediate states with the same quantum numbers and can be interpreted as a meson line.

explicit calculations in the standard NJL model with no Polyakov loop). However, some general features based on N_c counting rules at finite temperature can be deduced as follows. Let us, for instance, consider the three loop diagram of Fig. 2(a) contribution to the quark condensate in the NJL model in terms of quark propagators. Writing out for simplicity the Matsubara frequencies only we have

$$\text{Fig. 2(a)} = \sum_{w^{(1)},w^{(2)},w^{(3)}} S(w^{(1)}) \otimes S(w^{(1)}) \otimes S(w^{(2)}) \otimes S(w^{(3)}) \otimes S(w^{(1)} + w^{(3)} - w^{(2)}), \quad (6.1)$$

where $\otimes$ means tensor product in the Dirac and internal space sense. Using the Poisson's summation formula Eq. (3.2) and going to Euclidean time space we get

$$\begin{aligned}\text{Fig. 2(a)} = &\sum_{n_1,n_2,n_3} \langle \Omega^{n_1+n_2+n_3} \rangle \int_{-\infty}^{\infty} d\tau_1 d\tau_3 S(\tau_1) \\ &\otimes S(-\tau_1 - \tau_3 + n_1/T + n_3/T) \\ &\otimes S(-\tau_3 + n_2/T + n_3/T) \otimes S(\tau_3 - n_3/T) \\ &\sim \sum_{n_1,n_2,n_3} \langle \Omega^{n_1+n_2+n_3} \rangle e^{-M/T(|n_1|+|n_2|+|n_3|)}. \end{aligned} \quad (6.2)$$

For this diagram triality conservation implies $n_1 + n_2 + n_3 = kN_c$ and the minimum argument of the exponent corresponds to take $n_1 = n_2 = n_3 = 0$, which is the zero temperature contribution. The next thermal correction at low temperature is given by $n_1 = 0$, $n_2 = -n_3 = 1$ so the 3 loop diagram of Fig. 2(a) is suppressed by a thermal factor $e^{-2M/T}$, to be compared to the one quark loop suppression $e^{-N_c M/T}$. A similar thermal suppression is obtained by inserting the standard bubble summation which can be coupled to meson quantum numbers transforming the argument of the exponent $2M \to M_{\bar{q}q}$. Obviously, this contribution becomes most important for the lightest pion state. Actually, the quark-meson diagram in Fig. 2(b) looks like a two loop bosonized diagram as shown in Fig. 2(c). For such a bosonized diagram the previous argument becomes actually much simpler, since the number of loops equals the number of quark propagators. The pion polarization operator, proportional to the pion propagator, can then be taken at zero temperature, since the most important suppression comes from the quark lines not coupled to pion quantum numbers.

For a bosonized diagram with L quark loops we have to consider L-fold Matsubara generalization of the previous one quark loop correction Eq. (4.4). Actually, the analysis becomes simpler in coordinate space. Regardless of the total number of quark propagators we may choose to apply the Poisson's summation to L quark propagators. This can be seen by just using the formula

$$\sum_{n,m=-\infty}^{\infty} \int_0^{1/T} dx_4 F(x_4 + n/T + m/T) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx_4 F(x_4 + n/T) \quad (6.3)$$

and its multidimensional generalization both in the sum and in the integral sense. This effectively means that it is possible to remove as many Poisson summations as coordinate integrals appear in the expression. Using $L = I - (V - 1)$ and $4V = E + 2I$ we also have

$$\prod_{i=0}^{L} \int d^4 z_i G^{2L} \sum_{n_1,\ldots,n_L} \prod_{i=1}^{L} (-\Omega)^{n_i} S(\vec{x}_i, t_i + in_i/T). \quad (6.4)$$

Actually, this rule does not depend on the precise form of the quark interaction. At low temperatures, each quark line with an independent Poisson index generates a constituent quark mass suppression. Thus, the contribution to an observable can schematically be decomposed as follows:

$$\mathcal{O}^* = \sum_L \sum_{n_1,\ldots,n_L} \mathcal{O}_{n_1\ldots n_L} \langle \Omega^{n_1+\cdots+n_L} \rangle e^{-(|n_1|+\cdots+|n_L|)M/T}. \quad (6.5)$$

Triality conservation of the measure $\Omega \to z\Omega$ at this level implies

$$n_1 + \cdots + n_L = N_c k \quad (6.6)$$

with k an integer. The dominant term in the previous expansion is the one for which $n_1 = \ldots = n_L = 0$ with any arbitrary number of quark loops L and corresponds to the zero temperature contribution. One also sees that for $L = 1$ we only have contributions from $n_1 = kN_c$, which give the correction $e^{-N_c M/T}$, hence reproducing the results of Sec. V. According to Eq. (6.5), we can organize the thermal expansion at finite but low temperatures. The most important contributions come from minimizing $\sum_{i=1}^{L} |n_i|$ subjected to the triality constraint, Eq. (6.6). At finite T and for $N_c \geq 3$ the leading temperature-dependent contribution is given by $L \geq 2$ and $n_1 = -n_2 = 1$ with $n_3 = \cdots = n_L = 0$, which gives a factor $e^{-2M/T}$ and corresponds to a $\bar{q}q$ singlet meson state. This contribution has an additional $1/N_c$ power suppression, as compared to the zero temperature contribution. For $N_c = 3$ the next term in the expansion would correspond to $L \geq 3$ and $n_1 = n_2 = n_3 = 1$ and yields a finite temperature suppression $e^{-N_c M/T}$. For $N_c \geq 5$ we would instead get $L \geq 4$ and $n_1 = -n_2 = n_3 = n_4 = 1$ and $n_5 = \cdots n_L = 0$. Assuming $N_c = 3$ we have[16]

[16] In the case without Polyakov loop one would have $Z_{q^{N_q}(\bar{q}q)^{N_m}} \sim \frac{1}{N_c^{N_m}} e^{-(2N_m + N_q)M/T}$ instead. So the leading contributions are those corresponding to one quark state.

$$Z_{\bar{q}q} \sim \frac{1}{N_c} e^{-2M/T}, \tag{6.7}$$

$$Z_{qqq} \sim e^{-N_c M/T}, \tag{6.8}$$

$$Z_{qqq\bar{q}q} \sim \frac{1}{N_c} e^{-(2+N_c)M/T}, \tag{6.9}$$

$$\ldots, \tag{6.10}$$

$$Z_{(\bar{q}q)^{N_M}(qqq)^{N_B}} \sim \frac{1}{N_c^{N_M}} e^{-(2N_M+N_BN_c)M/T}. \tag{6.11}$$

Obviously, for $N_c = 3$ the meson loop contribution dominates over the baryon loop contribution. The previous argument ignores completely the quark binding effects so we should actually consider the relevant meson mass m; thus in summary one would get

$$\mathcal{O} = 1 + \sum_m \mathcal{O}_m \frac{1}{N_c} e^{-m/T} + \sum_B \mathcal{O}_B e^{-M_B/T} + \cdots. \tag{6.12}$$

This is how quark-hadron duality works at finite temperature in chiral quark models. As we see contributions of pion loops are the most important ones, even though they are $1/N_c$ suppressed. Higher meson states contribute next to the total observable at finite T. This is what one naïvely expects and it is rewarding to see that such a feature arises as a consequence of including the Polyakov loop into the chiral quark model and subsequent projection onto the gauge invariant color singlet sector.

Thus, at finite temperature there are the standard power-like $1/N_c e^{-2M/T}$ suppression for meson loops accompanied by an exponential suppression and a finite temperature exponential $e^{-N_c M/T}$ for baryon loops. Obviously, the most important contributions at large N_c or low T are those due to meson loops. We conclude from this discussion that thermal pion loops are protected.

The previous discussion has concentrated on quark observables. For an observable like the Polyakov loop one would have instead

$$\sum_L \sum_{n_1,\ldots,n_L} \mathcal{O}_{n_1\ldots n_L} \langle \Omega^{1+n_1+\cdots+n_L} \rangle e^{-(|n_1|+\cdots+|n_L|)M/T} \tag{6.13}$$

and

$$1 + n_1 + \cdots + n_L = N_c k. \tag{6.14}$$

The leading low temperature contribution (in this case there is no zero temperature term) is then of the type $n_1 = -1$, $n_2 = \cdots = n_L = 0$, corresponding to a single antiquark loop screening the charge of the test Polyakov loop. The leading term scales as $e^{-M/T}$ and is controlled by the constituent quark mass. Unlike the quark condensate case this behavior should remain unchanged by pionic loops.

B. Gluonic corrections

Up to now we have chosen to represent the full dynamical gluonic measure by a simple group integration. Unfortunately, we do not know at present any general argument supporting the idea that there is a low temperature exponential suppression of gluon degrees of freedom, leaving only the Haar measure as the only remnant of gluon dynamics. However, results based on strong coupling expansions [68,72] and in one massive gluon loop approximation [16,18] do provide such a suppression and indeed recent lattice findings confirm a striking universality in all group representations and favoring the simple group averaging dominance mechanism in gluodynamics below the phase transition [17]. More specifically, one finds both from lattice calculations [17] and from the group measure that

$$\langle \widehat{\mathrm{tr}}_c \hat{\Omega} \rangle = 0 \tag{6.15}$$

in the confining phase for the Polyakov loop in the adjoint representation. [In the group integration case, the previous formula follows from (7.6) below.] We stress that this result is not a consequence of triality preservation since $\hat{\Omega}$ is invariant under 't Hooft transformations. The previous equation is equivalent to $\langle |\mathrm{tr}_c \Omega|^2 \rangle = 1$. We note in passing that in the mean field approximation [6] $\langle |\mathrm{tr}_c \Omega|^2 \rangle$ vanishes instead, due to the absence of fluctuations.

We analyze now the two above mentioned models.

1. Strong coupling expansion

The gluon potential at the leading order result of the strong coupling expansion, for $N_c = 3$, is taken as [68,72]

$$-i\Gamma_G[\Omega] = V_{\mathrm{glue}}[\Omega] \cdot a^3/T = -2(d-1)\mathrm{e}^{-\sigma a/T} |\mathrm{tr}_c \Omega|^2 \tag{6.16}$$

with the string tension $\sigma = (425\ \mathrm{MeV})^2$. At the mean field level V_{glue} leads to a first-order phase transition with the critical coupling $2(d-1)\mathrm{e}^{-\sigma a/T_D} = 0.5153$. One can fix the deconfinement transition temperature as the empirical value $T_D = 270$ MeV by choosing $a^{-1} = 272$ MeV [6]. The corresponding mass is $m_G = \sigma a = 664$ MeV. At low temperatures we may expand the exponential in powers of the gluon action,

$$e^{i\Gamma_G} = 1 + i\Gamma_G - \tfrac{1}{2}\Gamma_G^2 + \cdots, \tag{6.17}$$

which introduces an exponential suppression for $e^{-m_G/T}$. For a treatment based on an average over the Polyakov loop, the normalized weight $\rho(\Omega)d\Omega$ suggested by the strong coupling expansion will be

$$\rho(\Omega) = N \exp(2(d-1)e^{-m_G/T} |\mathrm{tr}_c \Omega|^2), \tag{6.18}$$

where N is the normalization constant. Such distribution preserves exact triality. At low temperature, $\rho(\Omega)$ is close to unity and the distribution coincides with the Haar measure, hence Ω is completely random with equal probability to take any group value. At higher temperature $\rho(\Omega)$ tends to favor concentration of Ω near the central elements of the group, with equal probability.

This provides the following mass formula for the Boltzmann argument of the exponential (in the notation of Sec. VI A)

$$\mathcal{M} = nN_c M_q + mM_{\bar{q}q} + lm_G, \qquad (6.19)$$

which clearly shows that the leading thermal contribution at low temperatures is, again, provided by pion thermal loops, corresponding to $n = l = 0$ and $m = 1$ due to $N_c M_q \gg m_G \gg M_{\bar{q}q} = m_\pi$. Note that numerically, even the two pion contributions would be more important than gluonic corrections.

2. One massive gluon loop approximation

In a series of recent works [16,18], the equation of state has been deduced for a gas of massive gluons with a temperature-dependent mass in the presence of the Polyakov loop, reproducing the lattice data quite accurately above the deconfinement phase transition. The vacuum energy density reads

$$V_{\rm glue}[\Omega] = T \int \frac{d^3k}{(2\pi)^3} \widehat{\rm tr}_c \log[1 - e^{-\omega_k/T} \hat{\Omega}], \qquad (6.20)$$

where $\omega_k = \sqrt{k^2 + m_G^2}$, with m_G the gluons mass and $\hat{\Omega}$ and $\widehat{\rm tr}_c$ are the Polyakov loop and the color trace in the adjoint representation, respectively. This expression was discussed with a temperature-dependent mass in the deconfined phase given by plugging the Debye screening mass $m_G(T) = Tg(T)\sqrt{2}$, which at the phase transition, $T = T_c$ takes the value $m_G(T_c) = 1.2$–$1.3T_c$. It is worth noticing that, if one assumes a constant value for the gluon mass below the phase transition one gets at low temperatures

$$V_{\rm glue}[\Omega] = -T \sum_{n=1}^{\infty} \frac{1}{n} (|{\rm tr}_c \Omega^n|^2 - 1) \int \frac{d^3k}{(2\pi)^3} e^{-n\omega_k/T}, \qquad (6.21)$$

where the identity

$$\widehat{\rm tr}_c \hat{\Omega}^n = |{\rm tr}_c \Omega^n|^2 - 1 \qquad (6.22)$$

has been used. Using the asymptotic representation of the Bessel functions we see that, up to prefactors, a similar suppression of the sort described in the strong coupling limit, Sec. VI B 1, takes place.

C. Local corrections in the Polyakov loop

Up to now we have assumed a constant Ω field in space in our calculations. Quite generally, however, the Polyakov loop depends both on the Euclidean time and the space coordinate, as it comes out of explicit one-loop calculations within a derivative expansion approach at finite temperature [59,62,63]. In the Polyakov gauge, the temporal dependence becomes simple, but there is still an unknown space coordinate dependence. In such a case, the previous rules have to be modified, since Polyakov loop insertions carry finite momentum, and the result depends on the ordering of these insertions. If we still assume that the Polyakov loop is the only color source in the problem, we are naturally lead to consider Polyakov loop correlation functions. In the confining phase, we expect a cluster decomposition property to hold for any pair of variables. A convenient model to account for Polyakov loop correlations is

$$\langle {\rm tr}_c \Omega(\vec{x}) \, {\rm tr}_c \Omega^{-1}(\vec{y}) \rangle = e^{-\sigma|\vec{x}-\vec{y}|/T}, \qquad (6.23)$$

with σ the string tension. This includes the correct screening of the color charge at large distances due to confinement and is consistent with (7.7) for two Polyakov loops at the same point. Thus, very different values of the spatial coordinate are suppressed, and it makes sense to make a sort of local approximation within the correlation length and expand correlation functions in gradients in that limited region of space. Effectively, this corresponds to replacing the volume to a given confinement domain, by means of the rule

$$\frac{V}{T} = \frac{1}{T} \int d^3x \to \frac{1}{T} \int d^3x e^{-\sigma r/T} = \frac{8\pi T^2}{\sigma^3}. \qquad (6.24)$$

In Ref. [71] we will see explicitly that when computing the low energy chiral Lagrangian by expanding the effective action in derivatives of the meson fields, there appear also gradients of the Polyakov loop. Actually, since we couple the coordinate-dependent Polyakov loop effectively as a x-dependent color chemical potential, our approach resembles a non-Abelian generalization of the local density approximation of many body physics in nuclear physics and condensed matter systems, very much in the spirit of a density functional theory.

VII. RESULTS BEYOND THE QUENCHED APPROXIMATION AT LOW TEMPERATURES

A. General remarks

The full Polyakov chiral quark model is given in Sec. II D by Eq. (2.22). Therefore, any expectation value is defined as

$$\langle \mathcal{O} \rangle^* = \frac{1}{Z} \int DUD\Omega e^{i\Gamma_G[\Omega]} e^{i\Gamma_Q[U,\Omega]} \mathcal{O}, \qquad (7.1)$$

with $\Gamma_G[\Omega]$ given in (6.16) and $\Gamma_Q[U, \Omega]$ the quark contribution to the full action, given by (2.12) in the NJL model and (2.17) for the SQM case. In the latter model the full quark contribution coincides with the fermion determinant, while in the NJL model there is an additional term arising from the bosonization procedure,

$$e^{i\Gamma_Q[U,\Omega]} = \det(i\mathbf{D})_\Omega \exp\Big(-\frac{i}{4G}\int d^4x\, \mathrm{tr}_f(M-\hat{M}_0)^2\Big). \tag{7.2}$$

(Note that here we have included in **D** the color degrees of freedom.)

In this section we gather all our results to provide an estimate of the Polyakov loop expectation value at low temperatures as well as the quark condensate. This is particularly interesting since in the quenched approximation $\langle \mathrm{tr}_c\Omega\rangle = 0$, due to triality conservation. The fermion determinant does not conserve triality, but we show below that at low temperatures the violation is exponentially suppressed, so that it is still a good quantum number, and the Polyakov loop can be used as an order parameter for center symmetry in the same way as the chiral condensate provides a measure of chiral symmetry restoration away from the chiral limit.

In order to go beyond the quenched approximation, we will evaluate the fermion determinant in the presence of a slowly varying Polyakov loop following the techniques developed in our previous work [59]. According to our discussion of Sec. VI C of local corrections, such an approximation makes sense in a confining region where there are very strong correlations between Polyakov loops. In the presence of the Polyakov loop, the quark contribution can be generally written as

$$e^{i\Gamma_Q[U,\Omega]} = e^{i\int d^4x \mathcal{L}(x,\Omega)}, \tag{7.3}$$

where $\mathcal{L}$ is the chiral Lagrangian as a function of the Polyakov loop which will be computed at finite temperature in Ref. [71] in chiral quark models for nonvanishing meson fields. For our purposes here only the vacuum contribution with vanishing meson fields will be needed.

B. SQM model

In our case it is simpler to consider first the SQM. We have

$$e^{i\Gamma_Q[U,\Omega]} = \det(i\mathbf{D})_\Omega = e^{VB^*/T}, \tag{7.4}$$

where V is the three-dimensional volume and $-B^*$ the vacuum energy density at finite temperature in the presence of the Polyakov loop. The result for B^* is quite simple and is listed in (A19) in Appendix A. At low temperatures we may expand to get

$$e^{VB^*/T} = e^{VB/T}\Big[1 - \frac{VB}{T}e^{-M/T}\frac{1}{N_c}\mathrm{tr}_c(\Omega+\Omega^{-1}) + \cdots\Big] \tag{7.5}$$

with $M = M_V/2$ the constituent quark mass in the SQM and $-B$ is the vacuum energy density at zero temperature, $B = M_V^4 N_c N_f/192\pi^2 = (0.2\ \mathrm{GeV})^4$ for three flavors (see Appendix A). The calculation of observables requires the group integration formula [73],

$$\int d\Omega\, \Omega_{ij}\Omega^*_{kl} = \frac{1}{N_c}\delta_{ik}\delta_{jl}, \tag{7.6}$$

whence one gets for the constant Polyakov loop case

$$\int d\Omega\, \mathrm{tr}_c\Omega\, \mathrm{tr}_c\Omega^{-1} = 1. \tag{7.7}$$

Note that the effect of ignoring the Polyakov loop (i.e., setting $\Omega = 1$) promotes this result by two orders in N_c. In this model the average over pion fields is trivial since the vacuum energy density does not depend on U at the one quark loop level. Neglecting momentarily the gluonic corrections Γ_G, using the previous formulas and (7.1) we get the leading order result

$$L = \Big\langle \frac{1}{N_c}\mathrm{tr}_c\Omega\Big\rangle = -\frac{1}{N_c^2}\frac{BV}{T}e^{-M_V/2T}. \tag{7.8}$$

Note that at this order the contribution from the denominator is trivial. As expected, triality is not preserved due to the presence of dynamical quarks, but the relevant scale is the constituent quark mass. In addition, note that since B is proportional to N_c there is an extra $1/N_c$ suppression. So the Polyakov loop can be effectively used as an order parameter. Actually, our calculation suggests that a low temperature calculation of the Polyakov loop in full QCD might provide a method of extracting a gauge invariant constituent quark mass. Proceeding in a similar way from the expression of the quark condensate (A18) we get the leading order contribution

$$\frac{\langle\bar{q}q\rangle^*}{\langle\bar{q}q\rangle} = 1 + \frac{2BV}{N_c^2 T}e^{-(M_V+M_S)/2T} + \cdots. \tag{7.9}$$

It is noteworthy that the thermal correction scales as $1/N_c$ (B scales as N_c), as in the ChPT case. This again is not just a consequence of triality, but requires the proper integration over the Polyakov loop manifold. The presence of the (infinite) four-volume factor V/T has to do with our assumption on a constant Polyakov loop. As we have argued in Sec. VI C, one has indeed a local Polyakov loop and the volume should be replaced according to the rule in Eq. (6.24) by an effective confinement-domain volume.[17]

The first gluonic correction contributes in L as $e^{-(M_V+2m_G)/2T}$, and in the quark condensate as $e^{-(M_V+M_S+2m_G)/2T}$.

C. NJL model

The previous computation can also be considered within the NJL model. In this model the fermion determinant can be obtained by means of a derivative expansion [59,62]. The result will be presented in Ref. [71]. Retaining only the

[17] For the expectation value of a local observable $\mathcal{O}(\vec{x})$, points outside the volume V are not correlated and their contribution approximately cancels in numerator and denominator.

vacuum contribution, which coincides with the result given in Eq. (3) of [6], we have

$$\det(i\mathbf{D})_\Omega = \exp\left(i\int d^4x(\mathcal{L}_q(T=0) + \mathcal{L}_q(\Omega, T))\right), \tag{7.10}$$

where $\mathcal{L}_q(T=0)$ is the zero temperature contribution. At low temperature, the thermal correction reads

$$\mathcal{L}_q(\Omega, T) = N_f\sqrt{\frac{M^3T^5}{2\pi^3}}e^{-M/T}\,\mathrm{tr}_c(\Omega + \Omega^{-1}) + \cdots. \tag{7.11}$$

Using the volume rule $\int d^4x\mathcal{L}_q \to (V/T)\mathcal{L}_q$, expanding Eq. (7.10) in powers of $\mathcal{L}_q(\Omega, T)$, and considering the group integration formula as above, we get the leading order result[18]

$$L = \left\langle\frac{1}{N_c}\,\mathrm{tr}_c\Omega\right\rangle = \frac{N_f}{N_c}\frac{V}{T}\sqrt{\frac{M^3T^5}{2\pi^3}}e^{-M/T}. \tag{7.12}$$

[Since the NJL bosonization term in (7.2) cancels in the calculation of observables, it needs not be included in this calculation. Also the gluonic corrections have been omitted. Their effect is discussed below.]

For the quark condensate we take into account the result similar to Eq. (3.5) but replacing $(-1)^n$ with $(-\Omega)^n$, corresponding to the quark condensate for fixed Polyakov loop. Thus, including the leading fermion determinant contribution, using (7.6), and taking into account that $\langle\mathrm{tr}_c\Omega\rangle = \langle\mathrm{tr}_c\Omega^{-1}\rangle$, we get for the single flavor condensate

$$\langle\bar{q}q\rangle^* = \langle\bar{q}q\rangle + \frac{N_fV}{\pi^3}(MT)^3e^{-2M/T}. \tag{7.13}$$

Note that the N_f factor comes from the fermion determinant. As in the spectral quark model, the first gluonic correction contributes in L with $e^{-(M+m_G)/T}$, and in the quark condensate with $e^{-(2M+m_G)/T}$.

As we see, beyond the quenched approximation the Polyakov cooling persists although it is a bit less effective as in the quenched case, and for instance the temperature dependence of the low energy constants of the tree-level chiral effective Lagrangian becomes $L_i^* - L_i \sim_{\mathrm{Low}\,T} e^{-M_V/T}$ [71].

Finally, on top of this one must include higher quark loops, or equivalently mesonic excitations, from which the pions are the dominant ones. They yield exactly the results of ChPT [23] for the chiral condensate $\langle\bar{q}q\rangle$, and for the would-be Goldstone bosons, pions dominate at low temperatures. Thus, we see that when suitably coupled to chiral quark models, the Polyakov loop provides a quite natural explanation of results found long ago on purely hadronic grounds [19] as a direct consequence of the genuinely nonperturbative finite temperature gluonic effects. The expected leading correction effect on the Polyakov loop is also an additional exponential suppression $\mathcal{O}(e^{-m_\pi/T})$.

VIII. IMPLICATIONS FOR THE PHASE TRANSITION

The inclusion of the Polyakov loop has the consequence that one changes the one quark state Boltzmann factor $N_ce^{-M/T}$ into $\langle\mathrm{tr}_c\Omega\rangle$ at low temperatures. In the quenched approximation one has $\langle\mathrm{tr}_c\Omega\rangle = 0$, whereas the first nonvanishing contribution stemming from the Dirac sea behaves as $\langle\mathrm{tr}_c\Omega\rangle \sim e^{-M/T}$, due to the explicit breaking of the center symmetry induced by the fermion determinant. Likewise, for the quark condensate $\langle\bar{q}q\rangle$, the finite temperature correction changes $N_ce^{-M/T} \to e^{-2M/T}$ after the Polyakov loop integration is considered. Taking into account the large number of approximations and possible sources of corrections it is difficult to assess the accuracy of these Polyakov chiral quark models, in spite of the phenomenological success achieved in Refs. [6,60] within the mean field approach. Nevertheless, it is tempting to see how these results may be modified not only at low temperatures but also in the region around the phase transition when the proper quantum and local nature of the Polyakov loop is considered. This requires going beyond low temperature truncations like (7.12) and (7.13). Clearly, a proper description would demand a good knowledge of the Polyakov loop distribution as a function of the temperature. Unfortunately, such a distribution is poorly known and lattice simulations are not designed to extract it, since a subtle renormalization issue is involved [17,26]. As a first step to investigate the phase transition in the Polyakov chiral quark model beyond the mean field approximation we just take the strong coupling model for the gluonic action of (6.18). Because of the rather large exponential suppression, this ansatz has the virtue of reducing to the Haar measure in the low temperature regime, and as a consequence the vanishing of the adjoint Polyakov loop expectation value observed in lattice calculations [17] follows. In our view this is a compelling reason to go beyond the mean field by integrating over Polyakov loops. However, such a distribution preserves center symmetry and would not generate a phase transition *per se* in gluodynamics. This is unlike the mean field approximation where the action is minimized by center symmetry breaking configurations. As discussed before a side product of this approximation is to miss the fluctuations and also to introduce an explicit coordinate dependence in the gauge group. In our model the breaking of the center symmetry is attributed only to quarks. As we will see, this explicit breaking is rather large precisely due to the simultaneous

[18] Actually we find a negative value for the SQM and positive for the NJL model. While based on color-charge conjugation symmetry it can rigorously be shown that L must be real; no proof exists to our knowledge that $L > 0$ at any temperature, although lattice data [74] favor the positive case.

restoration of the chiral symmetry, since the constituent quark mass drops to zero. The qualitative agreement with lattice calculations in full QCD suggests that an important part of the physics has been retained by the model, leaving room for improvement in the Polyakov loop distribution.

We will present calculations only for the NJL model. In practice, we use (7.1), where the fermion determinant corresponds to Eq. (3) of [6] plus the volume rule (6.24). The Polyakov loop integration is carried out numerically. Because of gauge invariance the Polyakov loop dependence is through its eigenvalues, and thus one may use the marginal distribution of eigenvalues (4.8), which for $N_c = 3$ amounts to two independent integration variables. Full details are given in Appendix B.

In Fig. 3 we show the effect on both the chiral condensate $\langle \bar{q}q \rangle$ and Polyakov loop expectation value $L = \langle \mathrm{tr}_c \Omega \rangle / N_c$ within several schemes. In all cases we always minimize with respect to the quark mass and use $\rho(\Omega)$ in (6.18) for all temperatures. We compare the standard NJL model with no Polyakov loop with the mean field calculation of Ref. [6], which corresponds to minimize the vacuum energy as a function of the constituent mass and a given choice of the Polyakov loop matrix. We also compare with the result one obtains by integrating in the Polyakov loop instead and minimizing with respect to the quark mass afterwards. We work in these calculations with the NJL model with 2-flavor, $N_f = 2$, and consider for the current quark mass matrix $\hat{M}_0 = \mathrm{diag}(m_u, m_d)$ the isospin-symmetric limit with $m_u = m_d \equiv m_q = 5.5$ MeV. The zero temperature part of the effective action of Eq. (2.12) is regulated by the Pauli-Villars method, with the cutoff $\Lambda_{\mathrm{PV}} = 828$ MeV. The coupling is $G = 13.13\ \mathrm{GeV}^{-2}$, which is obtained from the gap Eq. (2.16), corresponding to a constituent quark mass $M = 300$ MeV. These parameters reproduce the empirical values of the pion weak decay constant and the quark condensate at zero temperature. Aspects of locality have been considered in the treatment of the NJL model with the integration in the Polyakov loop, by introducing the volume rule (6.24), where the string tension has been fixed to its zero temperature value $\sigma = (425\ \mathrm{MeV})^2$. Also displayed in the figure is the expectation value L in gluodynamics within the model of Eq. (6.16) in the mean field approximation, which leads to a first-order phase transition at $T_D = 270$ MeV. As we see, the net effect of the Polyakov loop integration is to displace the transition temperature to somewhat higher values. So, the method based on the integration provides an effective cooling at higher temperatures for fixed parameters. As we can see in Fig. 4, the crossover transitions for the chiral condensate $\langle \bar{q}q \rangle$ and for the Polyakov loop expectation value L coincide at the value $T_c \simeq 256$ MeV.

We have checked that a temperature dependence of the string tension may accommodate the unquenched lattice results [74], as we can see in Fig. 5. This provides a range of string tensions $\sigma = 0.181 \pm 0.085\ \mathrm{GeV}^2$ which somehow account for an estimate of the uncertainty in the present model. In Fig. 5 the error band associated to such an uncertainty reflects a critical temperature of about $T_D = 250 \pm 50$ MeV. This is compatible with the large rescaling advocated in Ref. [60]. At present, and taking into account the many possible sources of corrections to our calculations, we do not see how more accurate predictions could reliably be made in the context of Polyakov chiral quark models. Nevertheless, the semiquantitative success indicates that essential features for the center symmetry break-

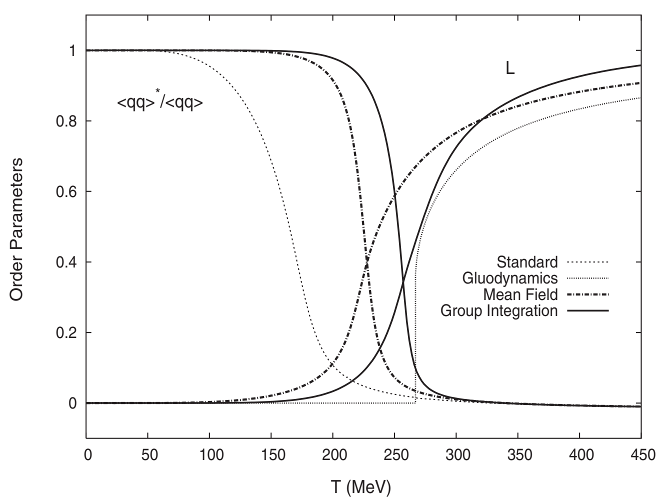

FIG. 3. Temperature dependence of the chiral condensate $\langle \bar{q}q \rangle$ and Polyakov loop expectation value $L = \langle \mathrm{tr}_c \Omega \rangle / N_c$ in relative units. The standard result for $\langle \bar{q}q \rangle^*$ corresponds to the pure NJL model uncoupled to the Polyakov loop. The result of L for gluodynamics within the strong coupling expansion is also displayed. We compare the mean field approach of Ref. [6] where the Polyakov loop is classical and coupled to the quarks, with the integration over the Polyakov loop Ω.

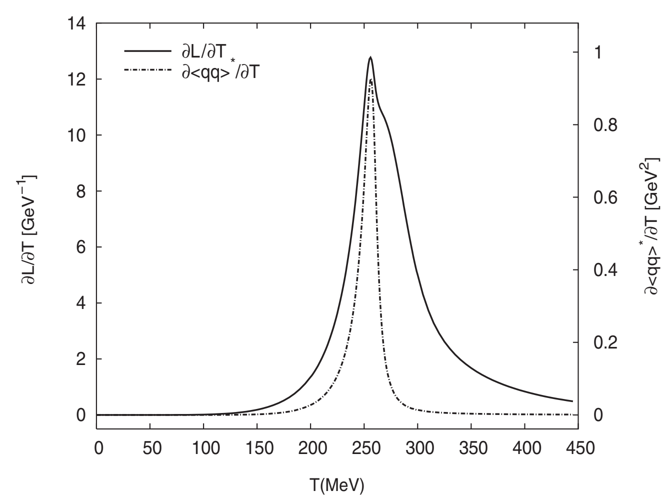

FIG. 4. Temperature dependence of $\partial \langle \bar{q}q \rangle^* / \partial T$ and $\partial L / \partial T$ in the NJL model when the integration over the Polyakov loop Ω is carried out.

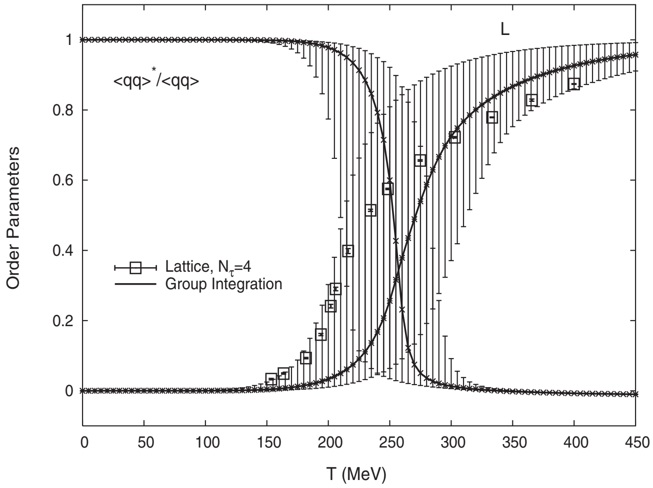

FIG. 5. Temperature dependence of the chiral condensate $\langle\bar{q}q\rangle$ and Polyakov loop expectation value $L = \langle \mathrm{tr}_c \Omega\rangle / N_c$ in relative units, in the NJL model when the integration over the Polyakov loop Ω is carried out. The error bands are associated to an uncertainty in the string tension of $\sigma = 0.181 \pm 0.085\ \mathrm{GeV}^2$. We compare with lattice data corresponding to 2-flavor QCD, taken from [74].

ing phase transition are encapsulated by these models, and further attempts along these lines should be striven. Nevertheless, the reader should be reminded that although the breaking of the center symmetry in this model is only attributed to the presence of quarks, one also has a contribution from gluons. In this regard let us mention that ignoring the exponentially suppressed gluon action (6.18) in the averaging has almost no effect below the phase transition and shifts up the transition temperature by about 30 MeV, a value within our error estimate. Given the importance of quarks in the phase transition one may wonder if the temperature-dependent volume enhances the breaking of the center symmetry. In fact, the volume at the transition temperature is roughly equal to gluon volume a^3 in (6.16). At low temperatures the exponential suppression dominates in the Polyakov loop expectation value where the volume appears as a harmless prefactor, see e.g. (7.12). The effect of replacing the temperature-dependent volume by a constant one can be seen in Fig. 8. Again changes are within our expected uncertainties.

As we have argued, the expectation value of the Polyakov loop is rather small at temperatures well below the phase transition. The difference between the mean field and the direct integration can be best quantified at the level of the fluctuations. While at the mean field level the probability of finding a given Polyakov loop would be a delta function, one expects a spreading of such probability due to quantum effects. For $N_c = 3$ the Polyakov loop contains two independent variables, which correspond to gluon fields in temperature units

$$\Omega = \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{-i(\phi_1+\phi_2)}). \tag{8.1}$$

The joint distribution $\rho(\phi_1, \phi_2)$ can be factorized as a product of the purely gluonic and the quark determinant contributions (see Appendix B)

$$\rho(\phi_1, \phi_2) = \rho_G(\phi_1, \phi_2)\rho_Q(\phi_1, \phi_2) \tag{8.2}$$

echoing the effective action displayed in Eq. (2.22) in Euclidean space. Note that $\rho(\phi_1, \phi_2)$ is not normalized to unity, instead its integral gives the full partition function (see Appendix B). As noted in Sec. IV C by gauge invariance the distribution is invariant under permutation of the three angles ϕ_1, ϕ_2 and $\phi_3 = -\phi_1 - \phi_2$. The use of such a symmetry is that the trace of any arbitrary function of the Polyakov loop $f(\Omega)$ (a one-body operator) can be averaged over the group by integrating out one angle, Eq. (4.6). Thus

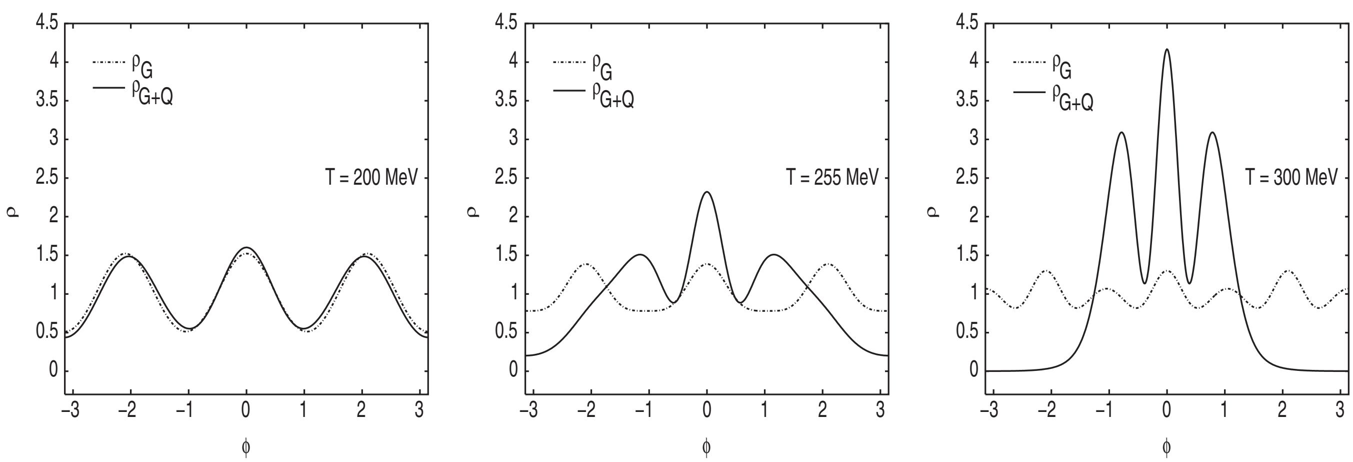

FIG. 6. Temperature dependence of the one-angle Polyakov loop distribution $\hat{\rho}(\phi)$ of (8.3) as a function of the angle. Dashed-dotted lines: quenched result (ρ_G is included, ρ_Q is not). Center symmetry is preserved. Solid lines: unquenched result (both factors ρ_G and ρ_Q are included) in the NJL model. Center symmetry is explicitly broken. Three temperatures nearby the transition (255 MeV) are considered. For convenience all distributions have been normalized to unity.

one obtains an equivalent one-body distribution as

$$\hat{\rho}(\phi) \propto \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi' \rho_G(\phi, \phi')\rho_Q(\phi, \phi'). \tag{8.3}$$

It is interesting to compare how this distribution evolves across the phase transition, and to look for the effects generated explicitly by the fermion determinant. In Fig. 6 we present such a comparison. Below the phase transition, and as already advanced in Sec. IV C, the weighting function presents three maxima at equidistant values, as required by the center symmetry. In this case the quark determinant plays a negligible role, although a tiny, indeed exponentially small, center symmetry breaking can be observed. As we see, there appears an interesting concentration of angles in the region around the origin as the phase transition takes place. The quarks are very effective suppressing contributions not near $\Omega = 1$. As a consequence, the lack of the spontaneous breaking of the center symmetry in (6.18) becomes not very relevant for temperatures above the transition.

A further trace of fluctuations can be seen by considering higher group representations of the Polyakov loop. In Fig. 7 we also show the expectation value of the Polyakov loop in the adjoint representation, $\langle \widehat{\mathrm{tr}}_c \hat{\Omega}\rangle/(N_c^2 - 1)$. According to the lattice results of the matrix model in Ref. [17] one has a vanishing expectation below the phase transition. As we have argued above, this feature is not preserved at the mean field level, where a nonvanishing value $-1/(N_c^2 - 1)$ is obtained instead [see Eq. (6.22) for the case $n = 1$]. Considering the Polyakov loop integration, as we do, complies with the lattice expectations and indicates that further developments should consider these constraints. The full fluctuation of the Polyakov loop, δ, is

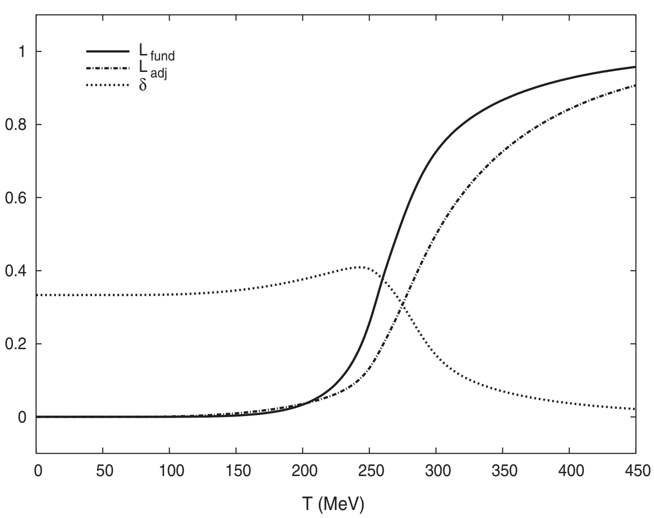

FIG. 7. Temperature dependence of the Polyakov loop expectation value in the fundamental, $\langle \mathrm{tr}_c \Omega\rangle/N_c$, and adjoint, $\langle \widehat{\mathrm{tr}}_c \hat{\Omega}\rangle/(N_c^2 - 1)$, representations and the total fluctuation δ of the Polyakov loop, in the NJL model when the integration over the Polyakov loop is carried out.

defined by

$$\delta^2 \equiv (\langle |\mathrm{tr}_c \Omega|^2\rangle - \langle \mathrm{tr}_c \Omega\rangle^2)/N_c^2$$
$$= (1 + \langle \widehat{\mathrm{tr}}_c \hat{\Omega}\rangle - \langle \mathrm{tr}_c \Omega\rangle^2)/N_c^2. \tag{8.4}$$

The fluctuation is also shown in Fig. 7. δ goes to zero in the large T regime, and this is compatible with the fact that the one-body distribution $\hat{\rho}(\phi)$ tends to concentrate near $\phi = 0$ as the temperature increases. In the second equality of Eq. (8.4) we have used the identity (6.22) with $n = 1$.

IX. CONCLUSIONS

In the present work we have discussed how the problem of conventional chiral quark models at finite temperature may be overcome by introducing the Polyakov loop. In order to maintain gauge invariance at finite temperature some nonperturbative explicit gluonic degrees of freedom must be kept. In practice, and in particular gauges such as the Polyakov gauge, the approach corresponds to treat the A_0 component of the gluon field as a color-dependent chemical potential in the quark propagator. This introduces, however, a color source which generates any possible color nonsinglet states, calling for a projection onto the physical color singlet states, or equivalently evaluating the path integral over the A_0 field in a gauge invariant fashion. As such, the average includes both the gluon action and the quark determinant. Models for the gluonic part have been discussed in light of pure gluodynamics results on the lattice. The net result is that, contrary to standard chiral quark model calculations at finite temperature, no single quark excitations are allowed in physical observables. More generally, the leading thermal corrections at the one-quark loop level start only at temperatures near the deconfinement transition. Given the fact that this strong suppression effect is triggered by a group averaging of Polyakov loops, we have named this effect Polyakov cooling of the quark excitations. Thus, and to a very good approximation, we do not expect any important finite temperature effect on quark observables below the deconfinement transition. In particular the chiral symmetry breaking transition cannot occur before the deconfinement transition. In such a situation the biggest change of observables such as the quark condensate should come from pseudoscalar loops at low temperatures and perhaps other higher meson resonances at intermediate temperatures. This is precisely what one expects from ChPT or unitarized approaches thereof which effectively include these loops on resonances. It is rewarding to see how in practice the apparent contradiction between chiral quark models and ChPT in the intermediate temperature region may be resolved by a judicious implementation of the Polyakov loop.

The extrapolation of these ideas to the phase transition is straightforward but more ingredients are needed. As an illustration we have investigated in a model the kind of effects one might expect from such a schematic Polyakov chiral quark model when both the quantum and local nature of the Polyakov loop are taken into account. Several interesting features arise out of such an investigation. At low temperatures the Polyakov loop is suppressed exponentially in the constituent quark mass suggesting that eventually more accurate lattice measurements might provide a method to extract the constituent quark mass in a gauge invariant fashion. According to our analysis, corrections to this leading behavior are provided by pion loops. It would be extremely helpful to find a general theoretical setup where these chiral corrections might be reliably computed. Moreover, we find that the explicit breaking of the center symmetry due to dynamical quarks at low temperature is $1/N_c$ suppressed. This is a direct consequence of averaging over gauge field configurations and confirms the current usage of the Polyakov loop as an order parameter in the unquenched case. In light of the present findings one might conjecture that in the large N_c limit the Polyakov loop becomes a true order parameter of full QCD.

Another feature we find is that the contribution of the gluon dynamics below the phase transition does not seem to be crucial. This is welcome since this is precisely the region where least information can be deduced from lattice simulations besides the known preservation of the center symmetry. Nevertheless, it would be rather interesting for our understanding of the low temperature gluon dynamics to compute directly from the lattice the Polyakov loop probability distribution. From our results we deduce that although the qualitative features observed in more simplified treatments are confirmed by calculations, one might expect large uncertainties in the determination of critical parameters, such as the critical temperature. Our estimate is $T_D = 250 \pm 50$ MeV for $N_f = 2$. Even given these large uncertainties, the very fact that a crossover between chiral symmetry restoration and center symmetry breaking takes place in the bulk part of the expected lattice QCD simulations with a minimal number of parameters is very encouraging and motivates that further studies along these lines should be pursued.

Finally, a more intriguing aspect regards what kind of model independent information could be inferred out of these models, where quarks and Polyakov loops are coupled, in the regime around the phase transition. For instance, the low temperature behavior of the chiral condensate can be described using chiral perturbation theory in terms of the zero temperature chiral condensate with no explicit reference to the underlying quark and gluonic degrees of freedom due to the dominance of pionic fluctuations. Given the fact that the Polyakov loop is a gauge invariant object which vanishes at zero temperature, it would be extremely helpful to isolate what physical states could equivalently describe such an observable and what specific zero temperature QCD operators drive its low temperature behavior.

ACKNOWLEDGMENTS

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APPENDIX A: THE SPECTRAL QUARK MODEL AT FINITE TEMPERATURE

Here we show how calculations for the spectral quark model introduced in Refs. [50–53] can be extended at finite temperature.

1. The spectral quark model at zero temperature

In the spectral quark model [50,51] the quark propagator is written in the generalized Lehmann form

$$S(k) = \int_C d\omega \frac{\rho(\omega)}{\not{k} - \omega} = \not{k} A(k^2) + B(k^2) = \frac{Z(k^2)}{\not{k} - M(k^2)}, \tag{A1}$$

where $\rho(\omega)$ is the, generally complex, spectral function and C denotes a contour in the complex ω plane. $M(k^2)$ is the self-energy and $Z(k^2)$ is the wave function renormalization. As discussed already in Ref. [51] the proper normalization and the conditions of finiteness of hadronic observables are achieved by requesting an infinite set of *spectral conditions* for the moments of the quark spectral function, $\rho(\omega)$, namely,

$$\rho_0 \equiv \int d\omega \rho(\omega) = 1, \tag{A2}$$

$$\rho_n \equiv \int d\omega \omega^n \rho(\omega) = 0, \quad \text{for } n = 1, 2, 3, \ldots \tag{A3}$$

Physical observables are proportional to the zeroth and the inverse moments,

$$\rho_{-n} \equiv \int d\omega \omega^{-n} \rho(\omega), \quad \text{for } n = 0, 1, 2, \ldots \tag{A4}$$

as well as to the *"log moments"*,

$$\rho_n' \equiv \int d\omega \log(\omega^2) \omega^n \rho(\omega), \quad \text{for } n = 1, 2, 3, \ldots \tag{A5}$$

Obviously, when an observable is proportional to the dimensionless $\rho_0 = 1$ the result does not depend explicitly on the regularization. The spectral conditions (A3) remove the dependence on a scale μ in (A5), thus guaranteeing the absence of any dimensional transmutation. No standard requirement of positivity for the spectral strength $\rho(\omega)$ is made. The spectral regularization is a physical regularization in the sense that it provides a high energy suppression in one quark loop amplitudes and will not be removed at the end of the calculation. Using the methods of Ref. [53] one finds at zero temperature $T = 0$, and trivial Polyakov loop $\Omega = 1$, and in the chiral limit $m_\pi = 0$, the following results for the pion weak decay constant f_π, the single flavor condensate $\langle \bar{q} q \rangle$, the energy-momentum tensor $\theta^{\mu\nu}$, and the anomalous $\pi^0 \to 2\gamma$ amplitude

$$f_\pi^2 = 4 i N_c \int \frac{d^4 k}{(2\pi)^4} [k^2 A(k^2)]', \tag{A6}$$

$$N_f \langle \bar{q} q \rangle = -4 i N_c N_f \int \frac{d^4 k}{(2\pi)^4} B(k^2), \tag{A7}$$

$$\langle \theta^{\mu\nu} \rangle = -4 i N_c N_f \int \frac{d^4 k}{(2\pi)^4} [k^\mu k^\nu A(k^2) - g^{\mu\nu}], \tag{A8}$$

$$F_{\pi\gamma\gamma} = i \frac{4}{N_c f_\pi} \int \frac{d^4 k}{(2\pi)^4} [k^2 A(k^2)]''. \tag{A9}$$

In these formulas, the prime indicates derivative with respect to k^2. The vacuum energy density is defined by $\epsilon_V = \frac{1}{4} \langle \theta^\mu_\mu \rangle = -B$ with B the bag constant. In the meson dominance version of the SQM one obtains for the even and odd components of the spectral function

$$\begin{aligned} \rho_V(\omega) &= \frac{1}{2} [\rho(\omega) + \rho(-\omega)] \\ &= \frac{1}{2\pi i} \frac{1}{\omega} \frac{1}{(1 - 4\omega^2/M_V^2)^{5/2}}, \end{aligned} \tag{A10}$$

$$\begin{aligned} \rho_S(\omega) &= \frac{1}{2} [\rho(\omega) - \rho(-\omega)] \\ &= \frac{1}{2\pi i} \frac{12 \rho_3'}{M_S^4 (1 - 4\omega^2/M_S^2)^{5/2}}. \end{aligned} \tag{A11}$$

(M_V and M_S are the vector and scalar meson masses, respectively) and hence

$$\begin{aligned} A(k^2) &= \frac{1}{k^2} \left[1 - \frac{1}{(1 - 4k^2/M_V^2)^{5/2}} \right], \\ B(k^2) &= \frac{48 \pi^2 \langle \bar{q} q \rangle}{M_S^4 N_c (1 - 4k^2/M_S^2)^{5/2}}. \end{aligned} \tag{A12}$$

Thus,

$$f_\pi^2 = \frac{M_V^2 N_c}{24 \pi^2}, \tag{A13}$$

$$\epsilon_V = -B = -\frac{M_V^4 N_f N_c}{192 \pi^2} = -\frac{N_f}{8} f_\pi^2 M_V^2. \tag{A14}$$

For three flavors, one has $B = (0.2\ \mathrm{GeV})^4$ for $M_V = 770$ MeV.

2. The spectral quark model at finite temperature and arbitrary Polyakov loop

We introduce finite temperature and arbitrary Polyakov loop by using the rule

$$\int \frac{dk_0}{2\pi} F(\vec{k}, k_0) \to i T \sum_{n=-\infty}^{\infty} F(\vec{k}, i \hat{\omega}_n), \tag{A15}$$

with $\hat{\omega}_n$ the fermionic Matsubara frequencies, $\hat{\omega}_n = 2\pi T(n + 1/2 + \hat{\nu})$, shifted by the logarithm of the Polyakov loop $\Omega = e^{i 2\pi \hat{\nu}}$. In the meson dominance model, we use the momentum space representation, evaluate first the three-dimensional integrals and finally the sums over the Matsubara frequencies. In practice all sums appearing are of the form

$$\begin{aligned} S_l(M, T) &= T \sum_{n=-\infty}^{\infty} \frac{1}{(M^2 + \hat{\omega}_n^2)^l} \\ &= \frac{1}{(l-1)!} \left(-\frac{d}{dM^2} \right)^{l-1} S_1(M, T). \end{aligned} \tag{A16}$$

The basic sum is given by

$$S_1(M, T) = \frac{1}{2M} \frac{\sinh(M/T)}{\cos(2\pi \hat{\nu}) + \cosh(M/T)}. \tag{A17}$$

Using the relations in [51] and the previous formulas, we get

$$\frac{\langle \bar{q} q \rangle^*}{\langle \bar{q} q \rangle} = \frac{1}{N_c} \mathrm{tr}_c \left[\frac{\sinh(M_S/2T)}{\cos(2\pi \hat{\nu}) + \cosh(M_S/2T)} \right] \tag{A18}$$

and, for the vacuum energy density

$$\frac{B^*}{B} = \frac{1}{N_c} \mathrm{tr}_c \left[\frac{\sinh(M_V/2T)}{\cos(2\pi \hat{\nu}) + \cosh(M_V/2T)} \right]. \tag{A19}$$

Note that the relative temperature dependence of the bag constant and the quark condensate are alike in the present model, if we also interchange the vector and scalar masses. The T and Ω dependence of f_π^2 is

$$\frac{f_\pi^{*2}}{f_\pi^2} = \frac{1}{N_c} \mathrm{tr}_c \left[\frac{T \sinh(M_V/T) - M_V - \cos(2\pi \hat{\nu})[M_V \cosh(M_V/2T) - 2T \sinh(M_V/2T)]}{2T(\cos(2\pi \hat{\nu}) + \cosh(M_V/2T))^2} \right]. \tag{A20}$$

To compute the group averages, we observe that for a sum of the form of Eq. (A17) we can make $z = e^{i\phi}$ and $t = e^{-M/T}$. Expanding in powers of t we get

$$f(z) = \frac{1/t - t}{z + 1/z + t + 1/t} = -1 + \sum_{n=0}^{\infty} (-t)^n (z^n + z^{-n}), \tag{A21}$$

and applying Eq. (4.10)-(4.12) for the average over the SU(N_c) Haar measure one has

$$\left\langle \frac{1}{N_c} \mathrm{tr}_c f(z) \right\rangle = 1 - \frac{2}{N_c} t^{N_c}. \tag{A22}$$

Thus, undoing the change of variables we get

$$\left\langle \frac{1}{N_c} \mathrm{tr}_c \frac{\sinh(M/T)}{\cos(2\pi\hat{\nu}) + \cosh(M/T)} \right\rangle = 1 - \frac{2}{N_c} e^{-N_c M/T}. \quad \text{(A23)}$$

Beyond the quenched approximation, integrals may be computed at low temperatures using Eq. (7.7) and generalizations thereof.

APPENDIX B: DETAILS ON NUMERICAL GROUP INTEGRATION

In this appendix we give details relative to the calculation presented in Sec. VIII for the Nambu-Jona-Lasinio model.

The chiral condensate is obtained from maximization of the partition function Z, Eq. (2.22), with respect to the constituent quark mass. To compute Z we need to carry out first the color group integration. To this end we consider the Polyakov gauge and parameterize the SU(3) Polyakov loop matrix as in Eq. (8.1). The expression is

$$Z = \int_{-\pi}^{\pi} \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \rho_G(\phi_1, \phi_2) \rho_Q(\phi_1, \phi_2), \quad \text{(B1)}$$

where we have separated a gluonic distribution

$$D\Omega e^{i\Gamma_G[\Omega]} = \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \rho_G(\phi_1, \phi_2) \quad \text{(B2)}$$

and a fermionic one

$$e^{i\Gamma_Q[\Omega]} = \rho_Q(\phi_1, \phi_2). \quad \text{(B3)}$$

Note that in $\Gamma_Q[\Omega]$ we do not consider any dependence in the mesonic U fields, because we only retain the vacuum contribution. ρ_G contains the Haar measure associated with the SU(3) group integration, as well as the gluonic corrections given in Eq. (6.16), i.e.

$$\rho_G(\phi_1, \phi_2) = \tfrac{1}{6}(27 - |\mathrm{tr}_c\Omega|^4 + 8\,\mathrm{Re}((\mathrm{tr}_c\Omega)^3) - 18|\mathrm{tr}_c\Omega|^2) \times \exp(2(d-1)e^{-\sigma a/T}|\mathrm{tr}_c\Omega|^2), \quad \text{(B4)}$$

and

$$\mathrm{tr}_c(\Omega) = e^{i\phi_1} + e^{i\phi_2} + e^{-i(\phi_1+\phi_2)}. \quad \text{(B5)}$$

The quark distribution $\rho_Q(\phi_1, \phi_2)$ follows from the vacuum contribution in Eqs. (7.2) and (7.10). To obtain $\Gamma_Q[\Omega]$ we use, passing over to Euclidean space,

$$i\Gamma_Q[\Omega] = -\frac{V}{T}(\mathcal{L}_q(T=0) + \mathcal{L}_q(T,\Omega) + \frac{1}{4G}\mathrm{tr}_f(M - \hat{M}_0)^2), \quad \text{(B6)}$$

where the correlation volume V is given in (6.24). Moreover,

$$\mathcal{L}_q(T=0) = -2N_c N_f \int \frac{d^3k}{(2\pi)^3} E_k, \quad \text{(B7)}$$

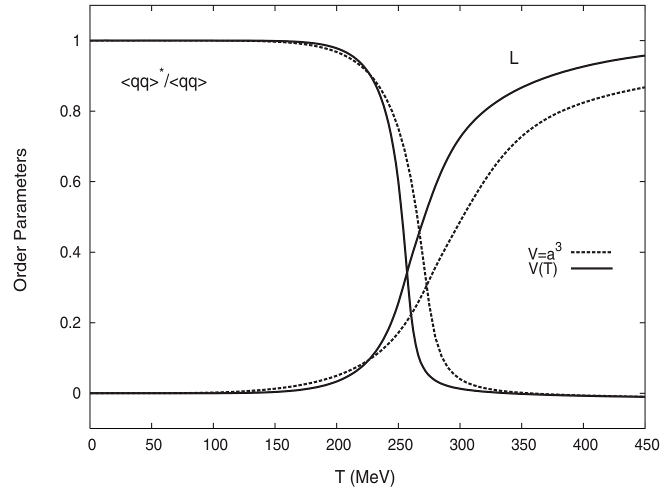

FIG. 8. Comparison of observables computed using a constant correlation volume a^3, $a^{-1} = 272$ MeV (dashed line) versus the temperature-dependent volume V of Eq. (6.24) (solid line).

$$\mathcal{L}_q(T,\Omega) = -2TN_f \int \frac{d^3k}{(2\pi)^3} (\mathrm{tr}_c \log[1 + \Omega e^{-E_k/T}] + \mathrm{tr}_c \log[1 + \Omega^\dagger e^{-E_k/T}]). \quad \text{(B8)}$$

Here $E_k = \sqrt{\mathbf{k}^2 + M^2}$ is the energy of quasiquarks and the constituent quark mass is $M = m_q - G\langle\bar{q}q\rangle$. For the zero temperature term we use the Pauli-Villars regularization. After momentum integration we get

$$\mathcal{L}_q(T=0) = -\frac{N_c N_f}{(4\pi)^2} \sum_i c_i (\Lambda_i^2 + M^2)^2 \log(\Lambda_i^2 + M^2). \quad \text{(B9)}$$

For the temperature-dependent part, after momentum integration and expanding the logarithm function, we obtain

$$\mathcal{L}_q(T,\Omega) = \frac{N_f}{\pi^2}(MT)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} K_2(nM/T) \times (\mathrm{tr}_c(\Omega^n) + \mathrm{tr}_c(\Omega^{-n})). \quad \text{(B10)}$$

In numerical calculations we check for convergence in these sums. The expectation value of any observable is computed as in Eq. (7.1). For any general function $f(\Omega)$, the group averaging can be advantageously evaluated using Eq. (4.6), where the one-body distribution is given by Eq. (8.3). It proves convenient to evaluate the double integral as an iterated one, since with the exception of the adjoint Polyakov loop, observables depend on one angle only. A comparison of observables calculated using a constant correlation volume a^3, $a^{-1} = 272$ MeV [6] and the temperature-dependent volume V of Eq. (6.24) is shown in Fig. 8.

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