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Classification of cosmological milestones

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In this paper causal geodesic completeness of Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models is analyzed in terms of generalized power expansions of the scale factor in coordinate time. The strength of the found singularities is discussed following the usual definitions due to Tipler and Królak. It is shown that while classical cosmological models are both timelike and lightlike geodesically incomplete, certain observationally allowed models which have been proposed recently are lightlike geodesically complete.

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I. INTRODUCTION

In the last few years the evidence that the Universe contains a large proportion of some not ordinary stuff which makes it expand acceleratedly is getting stronger grounds (see [1] for a recent review). Because of such considerations, quite a few names have been added to the original list of contents of the Universe along its different epochs. Among those, quintessence and phantom energy are the most popular, and, particularly the latter, has triggered a feverish activity in many directions, one of them being the investigation of the unusual geometric properties of the cosmological models they lead to. Phantom universes have rather awkward singularities [2], which go into the basket of "cosmological milestones" [3] along with other geometric curiosities such as bounces or extremality events appearing in other sorts of universes.

Observationally, phantom universes seem to be preferred over geometrywise dull lambda-cold dark matter (LCDM) cosmologies [4], thus making legitimate the sort of questions addressed here about the fate of phantom universes. We are going to perform an innovative analysis of those models in conjunction with all the other FLRW models in the literature, which will bring some surprises to build on the atypicality of phantom cosmologies. In addition, our analysis opens new paths for the exploration of other sort of milestones, such as sudden singularities which have received considerable attention recently.

We have borrowed the denomination "cosmological milestones" from [3], where FLRW cosmological models were analyzed in terms of a generalized power expansion of the scale factor. The appearance of polynomial scalar curvature singularities and derivative curvature singularities, together with the satisfaction of energy conditions were shown to depend most generally on just the first three terms of the expansion.

Clearly, there are many interesting geometrical features which are elusive to studies of that sort. Since the usual definitions of singularities are related not only to the properties of the curvature tensor but also to the existence of causal geodesics that cannot be extended to arbitrary values of their proper time (geodesic incompleteness) [5] or even of general causal curves with the same property (b-incompleteness) [6,7], it is of interest to analyze singularities in general FLRW cosmologies within this framework, as it was done in [8] for sudden singularities [9]. It is relevant to do so because causal geodesics describe the trajectories of observers subject just to gravitational forces. Note that curvature is a static concept, in the sense that it only reflects what happens at each event, whereas features derived from tracking the observer along its trajectory are more dynamical, and somewhat more enlightening. Thus, our study covers key issues that were overlooked in recent related classifications [3,10]

We begin therefore in Sec. II by arranging geodesic equations for FLRW cosmological models in a suitable fashion for integration in terms of a generalized power expansion in coordinate time. We proceed then in Sec. III to analyze the behavior of lightlike geodesics in these models, which sets the foundations allowing to check whether the singularities that are found are strong or not according to the usual definitions reviewed and refined in Sec. IV. Timelike geodesics are integrated in Sec. V and the strength of their singularities is dealt with in Sec. VI. The paper ends with a discussion of the results in Sec. VII. Special remarks are done throughout the paper regarding observationally allowed/favored phantom cosmologies, because there is a peculiar class of such cosmologies which persist to stand out of the crowd of all phantom models, as long as their geometrical properties are concerned.

II. GEODESICS IN FLRW COSMOLOGICAL MODELS

We consider spacetimes endowed with a FLRW metric of the form

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$$ds^2 = -dt^2 + a^2(t)\{f^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\},$$
$$f^2(r) = \frac{1}{1-kr^2}, \qquad k = 0, \pm 1. \tag{1}$$

As in [3], we assume that at a coordinate time t_0, a singular event or cosmological milestone comes up in such spacetime. To allow our results to be most general, we just require the scale factor $a(t)$ to have a generalized Puiseux expansion around the event at t_0,

$$a(t) = c_0|t-t_0|^{\eta_0} + c_1|t-t_1|^{\eta_1} + \cdots, \tag{2}$$

where the exponents η_i are real and ordered,

$$\eta_0 < \eta_1 < \cdots$$

This framework covers every proposal of the FLRW cosmological model in the literature.

At first order, a model admitting such expansion behaves as a power-law model of exponent η_0, which in the case of a flat universe, $k = 0$, would correspond to a linear equation of state for the cosmological fluid

$$p = w\rho, \qquad w = -1 + 2/3\eta_0.$$

In order to have a positive expansion factor, we require c_0 to be positive.

Depending on the value of η_0, several types of cosmological milestones arise [3]:

(i) $\eta_0 > 0$: the scale factor vanishes at t_0 and generically we have a big bang or big crunch singularity.

(ii) $\eta_0 = 0$: the scale factor is finite at t_0. If $a(t)$ is analytical, the event at t_0 is regular. Otherwise a weak or sudden singularity appears [9].

(iii) $\eta_0 < 0$: the scale factor diverges at t_0 and a big rip singularity appears.

Since the singular event at t_0 is approached from just one side (the past for a big bang, the future for a big crunch singularity), there is usually no need to consider absolute values in the expansion (2), except in the case, for instance, of sudden or weak singularities [9], which have been seen to be traversable [8] for geodesic observers.

In order to avoid dealing with signs, we consider singularities in the past, $t > t_0$. Of course the analysis is valid also for singularities in the future, since the equations are time symmetric, and occasionally we will comment what would happen if the singularity lies in the past.

We consider causal geodesics parametrized by their proper time τ, $(t(\tau), r(\tau), \theta(\tau), \phi(\tau))$. This means that the velocity u of the parametrization $(\dot t, \dot r, \dot\theta, \dot\phi)$ satisfies

$$\delta = -g_{ij}\dot x^i \dot x^j, \qquad x^i, x^j = t, r, \theta, \phi, \tag{3}$$

where δ takes the zero value for lightlike geodesics and the value one for timelike geodesics. It takes the value minus one for spacelike geodesics, but since we need just causal curves for our analysis, we will discard the $\delta = -1$ case. The dot stands for derivation with respect to proper time.

Condition (3) defines proper time up to a change of scale and a translation, $\tilde\tau = A\tau + B$, and therefore the parametrization is also called affine parametrization.

Geodesic equations are quasilinear in the acceleration ($\ddot t$, $\ddot r$, $\ddot\theta$, $\ddot\phi$) and depend on the metric components g_{ij} through the Christoffel symbols

$$\ddot x^i + \Gamma^i_{jk}\dot x^j \dot x^k = 0, \tag{4}$$

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il}\{g_{lj,k} + g_{lk,j} - g_{jk,l}\}. \tag{5}$$

For a FLRW cosmology they may be written as

$$\ddot t = -\frac{aa'}{1-kr^2}\dot r^2 - aa' r^2(\dot\theta^2 + \sin^2\theta\dot\phi^2), \tag{6a}$$
$$\ddot r = -2\frac{a'}{a}\dot t\,\dot r - \frac{kr}{1-kr^2}\dot r^2 + (1-kr^2)r(\dot\theta^2 + \sin^2\theta\dot\phi^2), \tag{6b}$$
$$\ddot\theta = -2\frac{a'}{a}\dot t\,\dot\theta - \frac{2}{r}\dot r\,\dot\theta + \sin\theta\cos\theta\dot\phi^2, \tag{6c}$$
$$\ddot\phi = -2\frac{a'}{a}\dot t\,\dot\phi - \frac{2}{r}\dot r\,\dot\phi - 2\cot\theta\dot\theta\,\dot\phi, \tag{6d}$$

where the comma stands for derivation with respect to coordinate time t.

Taking into account that orbits of geodesics in spherically symmetric spacetimes remain in equatorial hypersurfaces, they can be fit in a hypersurface $\theta = \pi/2$ by choosing coordinates accordingly. This allows a simplication of the system of equations,

$$\ddot t = -\frac{aa'}{1-kr^2}\dot r^2 - aa' r^2\dot\phi^2, \tag{7a}$$
$$\ddot r = -2\frac{a'}{a}\dot t\,\dot r - \frac{kr}{1-kr^2}\dot r^2 + (1-kr^2)r\dot\phi^2, \tag{7b}$$
$$\ddot\phi = -2\frac{a'}{a}\dot t\,\dot\phi - \frac{2}{r}\dot r\,\dot\phi\,. \tag{7c}$$

Finally, due to the existence of isometries, the following conserved quantities of geodesic motion exist:

$$P_1 = a(t)\left\{f(r)\cos\phi\,\dot r - \frac{r}{f(r)}\sin\phi\,\dot\phi\right\}, \tag{8a}$$
$$P_2 = a(t)\left\{f(r)\sin\phi\,\dot r + \frac{r}{f(r)}\cos\phi\,\dot\phi\right\}, \tag{8b}$$
$$L = a(t)r^2\dot\phi. \tag{8c}$$

They allow us to reduce (7) to a simple set of first order differential equations:

$$\dot{t}^2 = \delta + \frac{P^2 + kL^2}{a^2(t)}, \tag{9a}$$

$$\dot{r} = \frac{P_1 \cos\phi + P_2 \sin\phi}{a^2(t) f(r)}, \tag{9b}$$

$$\dot{\phi} = \frac{L}{a^2(t) r^2}. \tag{9c}$$

These constants of geodesic motion are related to angular momentum and linear momentum,

$$P^2 = P_1^2 + P_2^2, \tag{10}$$

and even allow us to obtain the equation of the orbits, namely,

$$f(r) r = \frac{L}{P_2 \cos\phi - P_1 \sin\phi}, \tag{11}$$

which is just the equation of a straight line in polar coordinates in the flat case $k = 0$:

$$P_2 x - P_1 y = L,$$

as it was to be expected.

It may be seen that the distinction between different types of universes (flat, open, closed) appears only through the constant k in (9a) and in the function $f(r)$ in (9b). This function may be factored away in that equation just by taking

$$\dot{R} = \frac{P_1 \cos\phi + P_2 \sin\phi}{a^2(t)}, \qquad R = \begin{cases} \operatorname{arcsinh} r & k = -1 \\ r & k = 0 \\ \arcsin r & k = 1 \end{cases}.$$

Therefore, the relevant information for geodesics is encoded in the scale factor $a(t)$. We make use of this fact on analyzing the behavior of causal geodesics.

From (9) we learn that just the equation for t needs to be solved, since the equations for the other coordinates are reduced to quadratures once the solution of (9a) is known.

We can forget the equation for ϕ along a geodesic since, due to homogeneity and isotropy of the FLRW universe, the origin may be located at any point on the geodesic and hence this appears as a straight line with zero angular velocity:

$$\dot{t} = \sqrt{\delta + \frac{P^2}{a^2(t)}}, \tag{12a}$$

$$\dot{r} = \pm \frac{P}{a^2(t) f(r)}. \tag{12b}$$

Possibly a quicker way to reach this result is considering beforehand that geodesics are straight lines due to the homogeneity and isotropy of the spacetime and that $\partial_R = \partial_r / f(r)$ is a generator of an isometry along one of these lines. Hence,

$$\pm P = u \cdot \frac{\partial_r}{f(r)} = a^2(t) f(r) \dot{r},$$

is a conserved quantity of geodesic motion. The equation for $\dot{t}$ is derived from the normalization condition (3),

$$\delta = \dot{t}^2 - a^2(t) f^2(r) \dot{r}^2.$$

We consider future-pointing geodesics and therefore we take $\dot{t} > 0$.

Now we may begin to draw information about causal geodesics from their equations. Following [5] we take causal geodesic completeness as a minimum condition for a spacetime to be considered singularity free. Therefore, we analyze the cases where causal geodesics are incomplete, that is, where they cannot be extended to arbitrarily large values of their proper time τ.

However, we must bear in mind that, since we have no guarantee that the coordinate chart that allows us to write the metric in the form (1) covers the whole universe, some conclusions about incompleteness may not be correct if the spacetime is extendible to a larger one. That is, a geodesic may leave the portion of spacetime depicted by our coordinates in finite proper time, but not the universe itself. Therefore, some of the singularities we may encounter may not be real, since the universe can be extended.

This is the case of Milne universe, which is the case of (1) for $k = -1$, $a(t) = t$. This universe can be reduced to a portion of empty Minkowski spacetime by the coordinate transformation

$$T = t\sqrt{1 + r^2}, \qquad R = rt,$$

which covers just the region inside the null cone $T^2 = R^2$. It is, therefore, a geodesically complete and singularity-free spacetime, but it appears singular in the Milne form, since it can be extended to the whole Minkowski spacetime.

III. LIGHTLIKE GEODESICS

The lightlike case is fairly simple and can be explicitly integrated for the time coordinate,

$$a(t)\dot{t} = P \Rightarrow \int_{t_0}^{t} a(t') dt' = P(\tau - \tau_0).$$

Close to t_0, the leading term in the power expansion of Eq. (12a) is the one with the lowest exponent η_0. In many cases, in order to analyze the singular behavior of geodesics near t_0, we just require the first term of the power expansion

$$a(t) \simeq c_0 |t - t_0|^{\eta_0},$$

which provides the time coordinate at first order in terms of proper time τ, after integrating (12a),

$$\int_{t_0}^{t} c_0|t'-t_0|^{\eta_0}dt' \simeq P(\tau-\tau_0),$$
$$t \simeq t_0 + \left\{\frac{(1+\eta_0)P}{c_0}\right\}^{1/(1+\eta_0)}(\tau-\tau_0)^{1/(1+\eta_0)}, \tag{13}$$

for $\eta_0 \neq -1$. If $\eta_0 = -1$, the leading term is exponential

$$t \simeq t_0 + Ce^{P\tau/c_0}.$$

Other cases which require a different treatment, involving more terms of the power expansion, are those with $\eta_0 = (1-n)/n$ with n a positive natural number. We elaborate further on those cases below.

From either the expression of t in (13) or $\dot{t}$ and its derivatives one gets

$$\dot{t} \simeq \frac{P}{c_0}|t-t_0|^{-\eta_0}, \tag{14a}$$

$$\ddot{t} \simeq -\frac{\eta_0 P}{c_0}|t-t_0|^{-\eta_0-1}\dot{t} \simeq -\frac{\eta_0 P^2}{c_0^2}|t-t_0|^{-2\eta_0-1}, \tag{14b}$$

$$t^{n)} \simeq \lambda_n|t-t_0|^{1-n-n\eta_0}, \tag{14c}$$

with

$$\lambda_n = (-1)^{n+1}\eta_0\cdots((n-1)\eta_0+n-2)\left(\frac{P}{c_0}\right)^n. \tag{15}$$

Several possibilities arise, since (14c) implies that if

$$\eta_0 \leq \frac{1-n}{n}, \qquad \frac{5n-3}{3(1-n)} \leq w \leq -1, \tag{16}$$

there is no blow up in any of the derivatives of order lower than or equal to n. The latter condition is not too stringent, that is, such cases appear often, so it is of utmost relevance to get further insight into the details of the different subcases one can distinguish:

(i) $\eta_0 > 0$: This case includes all classical matter contents (for flat universes, scalar field $\eta_0 = 1/3$, radiation $\eta_0 = 1/2$, dust $\eta_0 = 2/3\ldots$, with $w > -1$). Since the exponent $1/(1+\eta_0)$ is lower than one, t is not differentiable at t_0 and the derivative $\dot{t}$ blows up.

(ii) $\eta_0 \in (-1/2, 0)$: It corresponds to $w < -7/3$ for flat power-law models. In this case $\dot{t}$ does not blow up at t_0, but $\ddot{t}$ does.

(iii) $\eta_0 \in (-2/3, -1/2)$: It corresponds to $w \in (-7/3, -2)$ for flat power-law models. In this case $\ddot{t}$ does not blow up at t_0, but $\dddot{t}$ does.

(iv) $\eta_0 \in (\frac{1-n}{n}, \frac{2-n}{n-1})$: It corresponds to $w \in (\frac{5n-8}{3(2-n)}, \frac{5n-3}{3(1-n)})$ for flat power-law models. The derivative $t^{n-1)}$ does not blow up at t_0, but $t^{n)}$ does.

(v) $\eta_0 < -1$: It corresponds to $w \in (-5/3, -1)$ for power-law models. According to (13), the time coordinate along the lightlike geodesic is dominated by a negative power of proper time τ. This means that these geodesics never reach t_0, since $t - t_0$ only vanishes when τ tends to infinity. Typically, best fit phantom models have a value of w within this range, so it seems very likely that if the universe is phantom its geodesics are going to have this peculiar behavior.

The limit cases where η_0 is of the form $(1-n)/n$ $(0, -1/2, -2/3, -3/4, \ldots, -1)$ fall out of this classification since the derivative $t^{n+1)}$ vanishes, as it follows from (15). If every exponent η_i in the generalized power expansion of $a(t)$ were of this form, none of the derivatives of the time coordinate along lightlike geodesics would blow up. These are extremely fine-tuned cases, so we will not consider them any further, and we will then turn back to the cases $\eta_i = (1-n)/n$ for $i = 0$ only.

For this analysis we have to resort to the next term of the power expansion, $c_1(t-t_0)^{\eta_1}$, but if the term does not provide sufficient information, one would have to keep adding terms until a satisfactory expression is obtained.

Up to second order

$$\dot{t} = \frac{P}{a(t)} \simeq \frac{P}{c_0}|t-t_0|^{-\eta_0} - \frac{Pc_1}{c_0^2}|t-t_0|^{\eta_1-2\eta_0} + \cdots$$

we see that, after the contribution of the first term to the derivative $t^{n+1)}$ vanishes,

$$t^{n+1)} \sim |t-t_0|^{\eta_1-(n+2)\eta_0-n},$$

the leading term for lightlike geodesic behavior in these cases is the one with η_1.

Let us take a look to the new cases arising:

(i) $\eta_0 = 0$. In this case the scale factor is neither zero nor does it tend to infinity at t_0 and, in principle, Eqs. (12a) are regular at t_0 as it was pointed out in [8]. This is the case of the models proposed by Barrow [9]. How regular/singular geodesics are in these spacetimes depends on the next exponent, $\eta_1 > 0$, in the expansion of the scale factor. For any such value $\dot{t}$ does not blow up at t_0, but some remarks are in order.

If $\eta_1 \in (n-2, n-1)$, $n > 1$, the derivative $t^{n-1)}$ does not blow up at t_0 but $t^{n)}$ does. The phenomenon called sudden singularity [9], which has aroused much interest [9,11–13],[1] corresponds to the models with $\eta_1 > 1$, which have therefore nonsingular $\ddot{t}$ at t_0, and were shown to have well-defined geodesics around t_0 in [8].

If η_1 is a natural number, the same reasoning is

[1]In [9] and some of the references inspired by that work [11,14], the treatment is purely phenomenological and the focus is on a quest for *ad hoc* parametrizations of $a(t)$ leading to such singular behavior to arise (see [13] for an approximate reconstruction of the equation of state or [14] for statefinder parameters). Interestingly, sudden singularities can appear in more solid contexts, for instance in some braneworld models [15].

applied to the first exponent η_N which is not natural. If every exponent η_i is natural, then no derivative of lightlike geodesics diverges at t_0. This is the case of course of a nonvanishing analytical $a(t)$ in the vicinity of t_0, such as, for instance, in de Sitter, $a(t) = \cosh t$, and anti-de Sitter universes, $a(t) = \cos t$.

(ii) $\eta_0 = -1/2$. For all of these cases $\ddot{t}$ does not blow up at t_0. Since $\dddot{t} \sim |t - t_0|^{\eta_1}$, the third derivative blows up if η_1 is negative.
Since $t^{n)} \sim |t - t_0|^{\eta_1+(3-n)/2}$ in this case, if $\eta_1 \in ((n-4)/2, (n-3)/2)$, $n > 2$, the derivative $t^{n-1)}$ does not blow up at t_0 but $t^{n)}$ does.

(iii) $\eta_0 = -2/3$. In these models $\dddot{t}$ is finite at t_0. Since $\ddddot{t} \sim |t - t_0|^{\eta_1+1/3}$, the fourth derivative blows up if $\eta_1 < -1/3$.
Since $t^{n)} \sim |t - t_0|^{\eta_1+(5-n)/3}$, if $\eta_1 \in ((n-6)/3, (n-5)/3)$, $n > 3$, the derivative $t^{n-1)}$ does not blow up at t_0 but $t^{n)}$ does.

Therefore we see different levels of singular behavior at $t = t_0$ along lightlike geodesics. As η_0 decreases to get closer to the value -1, the regularity of the geodesic improves in the sense that one has to go to derivatives of higher order to find singular behavior. This limit model has been named superphantom [16] in the case of flat power-law cosmologies with $w = -5/3$. For η_0 lower than or equal to -1 these geodesics do not even reach the event at t_0.

As we have already pointed out, the geodesic equation for r does not add any further information, as we see in the equation of the orbit

$$\left|\frac{dR}{dt}\right| = \frac{|\dot{R}|}{\dot{t}} = \frac{1}{a(t)} \simeq \frac{|t-t_0|^{-\eta_0}}{c_0},$$

which is integrable close to t_0 if $\eta_0 < 1$. That is, it adds no new information, since cases with $\eta_0 \geq 1$ have already been shown to be singular. These results are summarized in Table I.

TABLE I. Derivatives of lightlike geodesics at t_0. A slash indicates the cases where t_0 is never reached.

η_0	η_1	$\dot{t}$	$\ddot{t}$	$\dddot{t}$	$t^{n)}$
$(0, \infty)$	(η_0, ∞)	∞	∞	∞	∞
0	$(0, 1)$	finite	∞	∞	∞
	$(1, 2)$	finite	finite	∞	∞
	$(2, 3)$	finite	finite	finite	∞
$(-1/2, 0)$	(η_0, ∞)	finite	∞	∞	∞
$-1/2$	$(-1/2, 0)$	finite	finite	∞	∞
	$(0, 1/2)$	finite	finite	finite	∞
$(-2/3, -1/2)$	(η_0, ∞)	finite	finite	∞	∞
$-2/3$	$(-2/3, -1/3)$	finite	finite	finite	∞
$(\frac{1-n}{n}, \frac{2-n}{n-1})$	(η_0, ∞)	finite	finite	finite	∞
$(-\infty, -1]$	(η_0, ∞)	/	/	/	/

IV. STRENGTH OF SINGULARITIES ALONG LIGHTLIKE GEODESICS

As we have shown in the previous section, the "strength" of singularities decreases qualitatively as the exponent η_0 decreases. It would be interesting to check these differences in behavior with the usual definitions of strong singularities.

The idea of a strong singularity was first introduced by Ellis and Schmidt [17]. A singularity is meant to be strong if tidal forces exert a severe disruption on finite objects falling into it. There have been several attempts to provide a rigorous mathematical definition for this idea.

The finite volume is considered to be spanned by three orthogonal Jacobi fields which form an orthonormal basis with the velocity of the incomplete geodesic. According to Tipler [18], the singularity is strong if the volume tends to zero as the geodesic approaches the value of proper time where it meets its end. Królak's definition [19] is less restrictive, since it just requires the derivative of the volume with respect to proper time to be negative. This definition has been further refined in [20].

However, these definitions are meant for focusing gravitational forces, that is, $R_{ij}u^i u^j$ must be non-negative for timelike and lightlike observers with velocity u. Hence, these definitions leave out the possibility of big rip singularities, but they can be extended to these cases just reversing signs. For instance, for negative $R_{ij}u^i u^j$ Tipler's definition requires a volume tending to infinity as the geodesic meets its end and Królak's definition requires a positive derivative of the volume close to the end of the geodesic.

Fortunately, one is not to construct the basis of Jacobi fields in order to check these definitions. There are necessary and sufficient conditions due to Clarke and Królak [21] related to integrals of Riemann tensor components along the incomplete geodesic. They are not affected by the inclusion of big rip singularities since they just require components of the Riemann tensor to blow up along causal geodesics. In order to apply them we need the components of the Riemann tensor

$$\frac{R^t{}_{rtr}}{f^2} = \frac{R^t{}_{\theta t\theta}}{r^2} = \frac{R^t{}_{\phi t\phi}}{r^2\sin^2\theta} = aa'',$$

$$R^r{}_{ttr} = R^\theta{}_{tt\theta} = R^\phi{}_{tt\phi} = \frac{a''}{a},$$

$$\frac{R^r{}_{\theta r\theta}}{r^2} = \frac{R^r{}_{\phi r\phi}}{r^2\sin^2\theta} = -\frac{R^\theta{}_{rr\theta}}{f^2} = \frac{R^\theta{}_{\phi\theta\phi}}{r^2\sin^2\theta} = -\frac{R^\phi{}_{rr\phi}}{f^2}$$
$$= -\frac{R^\phi{}_{\theta\theta\phi}}{r^2} = a'^2 + k,$$

omitting the ones that can be obtained by the symmetries of the tensor, and Ricci tensor components

$$R_{tt} = -\frac{3a''}{a},$$
$$\frac{R_{rr}}{f^2} = \frac{R_{\theta\theta}}{r^2} = \frac{R_{\phi\phi}}{r^2\sin^2\theta} = aa'' + 2(a'^2 + k).$$

For FLRW models conditions are simpler, since the Weyl tensor vanishes and therefore only conditions related to the Ricci tensor are relevant. According to [21], a lightlike geodesic meets a strong singularity, according to Tipler's definition, at proper time τ_0 if and only if the integral of the Ricci tensor

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' R_{ij}u^i u^j \tag{17}$$

diverges as τ tends to τ_0.

For Królak's definition the condition is less restrictive: a lightlike geodesic meets a strong singularity at proper time τ_0 if and only if the integral

$$\int_0^\tau d\tau' R_{ij}u^i u^j \tag{18}$$

diverges as τ tends to τ_0.

In our case, the velocity of the geodesic is

$$u^t = \dot{t} = \frac{P}{a}, \qquad u^r = \dot{r} = \pm\frac{P}{fa^2},$$

and therefore the component of the Ricci tensor measured along the geodesic is

$$R_{ij}u^i u^j = 2P^2\left(\frac{a'^2 + k}{a^4} - \frac{a''}{a^3}\right).$$

At first order in our generalized power expansion, we have two cases, depending on whether the curvature term is leading or not

$$R_{ij}u^i u^j \simeq \frac{2P^2\eta_0}{c_0^2|t - t_0|^{2(\eta_0+1)}} + \frac{2kP^2}{c_0^4|t - t_0|^{4\eta_0}} + \cdots,$$

(i) First, the case $\eta_0 \le -1$, which is complete, since lightlike geodesics never reach $t = t_0$, as we have seen.

(ii) For $\eta_0 \in (-1, 1)$, $k \neq 0$ or $\eta_0 > -1$, $k = 0$, using (13),

$$R_{ij}u^i u^j \simeq \frac{2P^2\eta_0}{c_0^2}|t - t_0|^{-2(\eta_0+1)} \simeq \frac{2\eta_0}{(\eta_0 + 1)^2}\frac{1}{|\tau - \tau_0|^2},$$

produces a logarithmic divergence with Tipler's definition and an inverse power divergence with Królak's for $0 \neq \eta_0 > -1$ and therefore we have a strong singularity in these cases.

(iii) There is a subcase left, $\eta_0 = 0$, for which the approximation at first order leaves

$$R_{ij}u^i u^j \simeq 2P^2\left(\frac{k}{c_0^4} - \frac{c_1\eta_1(\eta_1 - 1)}{c_0^3|t - t_0|^{2-\eta_1}}\right) \simeq \frac{2kP^2}{c_0^4} - \frac{2P^{\eta_1}c_1\eta_1(\eta_1 - 1)}{c_0^{\eta_1+1}|\tau - \tau_0|^{2-\eta_1}},$$

which provides no divergent integral with Tipler's definition, since $\eta_1 > 0$, but provides one with Królak's one if $\eta_1 \in (0, 1)$. This generalizes the result of [8], since there it was shown that sudden singularities, a special case with $\eta_0 = 0$, $\eta_1 > 1$, were not in fact singularities.

For $\eta_1 = 1$ we have to use still another term of the expansion,

$$R_{ij}u^i u^j \simeq 2P^2\left(\frac{k + c_1^2}{c_0^4} - \frac{c_2\eta_2(\eta_2 - 1)}{c_0^3|t - t_0|^{2-\eta_2}}\right) \simeq \frac{2(k + c_1^2)P^2}{c_0^4} - \frac{2P^{\eta_2}c_2\eta_2(\eta_2 - 1)}{c_0^{\eta_2+1}|\tau - \tau_0|^{2-\eta_2}},$$

which shows that this subcase does not produce a divergent integral since $\eta_2 > 1$.

(iv) For $\eta_0 = 1$,

$$R_{ij}u^i u^j \simeq 2P^2\frac{c_0^2 + k}{c_0^4}|t - t_0|^{-4},$$

we see that both integrals are divergent, since the exponent is smaller than -2, unless $k = -1$, $c_0 = 1$, values for which the Ricci tensor vanishes at first order, because at this order it is a Milne model. We are to resort then to the next term, $\eta_1 > 1$,

$$R_{ij}u^i u^j \simeq -\frac{2P^2\eta_1(\eta_1 - 3)c_1}{|t - t_0|^{5-\eta_1}} \simeq -\frac{2^{(\eta_1-3)/2}P^{(\eta_1-1)/2}\eta_1(\eta_1 - 3)c_1}{|\tau - \tau_0|^{(5-\eta_1)/2}},$$

which produces no divergent integral for Tipler's definition but for Królak's one the singularity is strong if $\eta_1 < 3$.

We need another term, $\eta_2 > 3$, to check the regu-

TABLE II. Degree of singularity of null geodesics around t_0.

η_0	η_1	k	c_0	*Tipler*	*Królak*
$(-\infty, -1]$				Regular	Regular
$(-1, 0)$	(η_0, ∞)	$0, \pm1$	$(0, \infty)$	Strong	Strong
0	$(0, 1)$			Weak	Strong
	$[1, \infty)$			Weak	Weak
$(0, 1)$	(η_0, ∞)			Strong	Strong
1	$(1, \infty)$	$0, 1$		Strong	Strong
	$(1, \infty)$		$(0, 1) \cup (1, \infty)$	Strong	Strong
	$(1, 3)$	-1	1	Weak	Strong
	$[3, \infty)$			Weak	Weak
$(1, \infty)$	(η_0, ∞)	$0, \pm1$	$(0, \infty)$	Strong	Strong

larity of the $\eta_1 = 3$ subcase with Królak's definition,

$$R_{ij}u^i u^j \simeq -\frac{2P^2\eta_2(\eta_2-3)c_2}{|t-t_0|^{5-\eta_2}},$$

and we find that it is similar to the η_1 contribution. Hence it does not diverge for $\eta_2 > 3$.

(v) Finally, in the cases with $k \neq 0$ and $\eta_0 \geq 1$ the leading term in the Ricci tensor component is the curvature one

$$R_{ij}u^i u^j \simeq \frac{2kP^2}{a^4} \simeq \frac{2kP^2}{c_0^4|t-t_0|^{4\eta_0}} \simeq \frac{2k(1+\eta_0)^{-4/(1+\eta_0)}P^{2-4\eta_0/(1+\eta_0)}}{c_0^{4-4\eta_0/(1+\eta_0)}|\tau-\tau_0|^{4\eta_0/(1+\eta_0)}},$$

which also provides divergent integrals since the exponent of the denominator is larger than 2. Hence, the singularities are strong in these cases too.

We conclude that for $\eta_0 > -1$ lightlike geodesics in all models meet a strong singularity except for the cases $\eta_0 = 0$ (and $\eta_1 \geq 1$ with Królak's definition) and $\eta_0 = 1$, $k = -1$, $c_0 = 1$ ($\eta_1 \geq 3$ with Królak's definition), which are regular. These are the only cases, together with $\eta_0 \leq -1$, that are null geodesically complete, even though the curvature is singular also for these models, as it was shown in [3].

We notice that the different behavior of geodesics for positive and negative η_0 does not quite influence the strength of the curvature singularity at t_0. Generically models with a big rip have null geodesics with derivatives that do not blow up at t_0, whereas all derivatives of null geodesics in models with a big bang or crunch are infinite. These results are summarized in Table II.

V. TIMELIKE GEODESICS

For timelike geodesics the relevant equation is, at first order of the power expansion,

$$\dot{t} = \sqrt{1+\frac{P^2}{a^2}} \simeq \sqrt{1+\frac{P^2}{c_0^2}(t-t_0)^{-2\eta_0}}, \tag{19}$$

which can be solved explicitly in terms of hypergeometric functions,

$$(t-t_0)F\left(\frac{1}{2}, -\frac{1}{2p}, 1-\frac{1}{2p}; -\frac{P^2}{c_0^2}(t-t_0)^{-2\eta_0}\right) \simeq \tau - \tau_0,$$

where F is the hypergeometric function, but we shall not use this expression.

It is clear, as it happened for lightlike geodesics, that for $\eta_0 > 0$ the geodesic is singular at $t = t_0$, since $\dot{t}$ blows up there, unless P is zero, which is a trivial regular case,

$$t - t_0 = \tau - \tau_0, \qquad r = r_0,$$

the comoving congruence of fluid wordlines.

On the contrary, this derivative is well defined and takes the value one if η_0 is negative. In this case we may approximate $\dot{t}$ in the vicinity of t_0,

$$\dot{t} \simeq 1 + \frac{P^2}{2c_0^2}(t-t_0)^{-2\eta_0}. \tag{20}$$

In order to carry out the analysis of the behavior of these geodesics, we need expressions for higher derivatives of coordinate time t,

$$\ddot{t} = -\frac{P^2 a'}{a^3\sqrt{1+P^2a^{-2}}}\dot{t} = -\frac{P^2a'}{a^3} \simeq -\frac{P^2\eta_0(t-t_0)^{-2\eta_0-1}}{c_0^2}, \tag{21a}$$

$$\dddot{t} = P^2\left(\frac{3a'^2}{a^4} - \frac{a''}{a^3}\right)\dot{t} \tag{21b}$$

$$\simeq \frac{P^2\eta_0(2\eta_0+1)(t-t_0)^{-2\eta_0-2}}{c_0^2},$$

$$t^{n)} \simeq \tilde{\lambda}_n(t-t_0)^{-2\eta_0-n+1}. \tag{21c}$$

with

$$\tilde{\lambda}_n = \frac{(-1)^{n+1}P^2 2\eta_0\cdots(2\eta_0+n-2)}{2c_0^2}. \tag{21d}$$

From these expressions we may draw valuable information about timelike geodesics around t_0:

(i) $\eta_0 > 0$: As it happened in the lightlike case, timelike geodesics are singular at t_0, since the derivative $\dot{t}$ blows up. For flat power-law models it corresponds to $w > -1$.

(ii) $\eta_0 \in (-1/2, 0)$: Again as it happened for lightlike geodesics, the derivative $\ddot{t}$ blows up at t_0, whereas $\dot{t}$ does not. It corresponds to $w < -7/3$ for flat power-law models.

(iii) $\eta_0 \in (-1, -1/2)$: For these cases we find the first difference with lightlike geodesics. The derivative $\dddot{t}$ blows up at t_0, but $\ddot{t}$ does not. They correspond to flat power-law models with $w \in (-7/3, -5/3)$.

(iv) $\eta_0 \in (\frac{1-n}{2}, \frac{2-n}{2})$, $n > 1$: Whereas lightlike geodesics did not reach t_0 in finite proper time in these models, timelike geodesics do, with regular $t^{n-1)}$ but with infinite $t^{n)}$ at t_0. The corresponding flat power-law coefficients would be $w \in (\frac{3n-2}{3(2-n)}, \frac{3n+1}{3(1-n)})$.

Again for the limit cases $\eta_0 = \frac{1-n}{2}(0, -1/2, -1\ldots)$ the contribution of the term with exponent η_0 to the derivative $t^{n+1)}$ vanishes and we have to resort to the next term in the expansion with a nonvanishing contribution to higher derivatives,

TABLE III. Derivatives of timelike geodesics at t_0.

η_0	η_1	$\dot t$	$\ddot t$	$\dddot t$	$t^{n)}$
$(0, \infty)$	(η_0, ∞)	∞	∞	∞	∞
0	$(0, 1)$	finite	∞	∞	∞
	$(1, 2)$	finite	finite	∞	∞
	$(2, 3)$	finite	finite	finite	∞
$(-1/2, 0)$	(η_0, ∞)	finite	∞	∞	∞
$-1/2$	$(-1/2, 1/2)$	finite	finite	∞	∞
	$(1/2, 3/2)$	finite	finite	finite	∞
$(-1, -1/2)$	(η_0, ∞)	finite	finite	∞	∞
-1	$(-1, 0)$	finite	finite	finite	∞
$(\frac{1-n}{2}, \frac{2-n}{2})$	(η_0, ∞)	finite	finite	finite	∞

$$\dot t \simeq 1 + \frac{P^2}{2c_0^2}(t-t_0)^{-2\eta_0} - \frac{P^2 c_1}{c_0^3}(t-t_0)^{\eta_1-3\eta_0} - \frac{P^4}{8c_0^4}(t-t_0)^{-4\eta_0} + \cdots, \tag{22}$$

which is the term with exponent $\eta_1 - 3\eta_0$ and therefore the relevant contribution to the derivative $t^{n+1)}$ is of the form $(t-t_0)^{\eta_1-3\eta_0-n}$.

Let us analyze some of these cases:

(i) $\eta_0 = 0 < \eta_1$: The discussion is entirely similar to the one for lightlike geodesics. The scale factor does not vanish at t_0 and therefore these are sudden or weak singularities. Since $\dot t \sim |t-t_0|^{\eta_1}$, they have finite $\dot t$ at t_0 and the derivative $\ddot t$ is finite for $\eta_1 \geq 1$. If $\eta_1 \in (n-2, n-1)$, $n > 2$, the derivative $t^{n-1)}$ is also finite at t_0 but $t^{n)}$ is not.

(ii) Again, if η_1 is natural, we would have to resort to the first exponent η_N which is not natural and if all of them are natural, then no derivative of timelike geodesics diverges at t_0.

(iii) $\eta_0 = -1/2$. In these cases $\ddot t$ is finite at t_0. Since $\dddot t \sim |t-t_0|^{\eta_1-1/2}$, the third derivative is finite if $\eta_1 \geq 1/2$. When $\eta_1 \in (n-7/2, n-5/2)$, $n > 3$, the derivative $t^{n-1)}$ is finite at t_0 and $t^{n)}$ is not.

(iv) $\eta_0 = -1$. Now $\dddot t$ is finite at t_0 and $\ddddot t \sim |t-t_0|^{\eta_1}$. Hence the fourth derivative is finite if η_1 is positive. For $\eta_1 \in (n-5, n-4)$, $n > 4$, the derivative $t^{n-1)}$ is finite at t_0 whereas $t^{n)}$ is not.

Summarizing, geodesic behavior is similar for timelike and lightlike geodesics in models with $\eta_0 > -1/2$, but there is a different pattern for the rest of the models. Differentiability of timelike geodesics improves as η_0 decreases, but there are only isolated cases for which they are completely regular and this makes a difference with the lightlike case. There are no timelike geodesics which take an infinite proper time to reach t_0, as it happens with null geodesics with $\eta_0 \leq -1$.

Equation (12b) for r does not add further information on the behavior of timelike geodesics either. We may tackle the equation of the orbit of the geodesics

$$\left|\frac{dR}{dt}\right| = \frac{|\dot R|}{\dot t} = \frac{P}{a(t)\sqrt{P^2 + a^2(t)}},$$

in a similar fashion.

If $\eta_0 > 0$, we get, close to t_0,

$$\left|\frac{dR}{dt}\right| \leq \frac{1}{a(t)} \simeq \frac{1}{c_0}|t-t_0|^{-\eta_0},$$

which is not integrable if $\eta_0 \geq 1$, but these are all already singular cases.

If $\eta_0 \leq 0$, we get, close to t_0,

$$\left|\frac{dR}{dt}\right| \leq \frac{P}{a^2(t)} \simeq \frac{P}{c_0^2}|t-t_0|^{-2\eta_0},$$

which is integrable for $\eta_0 < 1/2$.

Therefore, no new singular behavior appears on considering the geodesic equation for r. The radial coordinate is singular where t is already singular. These results are summarized in Table III.

VI. STRENGTH OF SINGULARITIES ALONG TIMELIKE GEODESICS

Again, it would be quite interesting to know whether the singularities encountered by timelike geodesics are strong or not according to the usual definitions.

Conditions like (17) and (18) are not so simple for timelike geodesics, since there are not both necessary and sufficient conditions in this case. Those conditions become just sufficient if the Weyl tensor vanishes.

A timelike geodesic meets a strong singularity, according to Tipler's definition, at proper time τ_0 if the integral of the Ricci tensor

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' R_{ij}u^i u^j \tag{23}$$

diverges as τ tends to τ_0.

In contrast, for Królak's definition, a timelike geodesic meets a strong singularity at proper time τ_0 if the integral

$$\int_0^\tau d\tau' R_{ij}u^i u^j \tag{24}$$

diverges as τ tends to τ_0.

First we use the comoving fluid worldline congruence, with velocity $u = \partial_t$. In this case, $\dot t = 1$, proper time and coordinate time are the same $t - t_0 = \tau - \tau_0$. The component of the Ricci tensor measured by observers along this congruence,

$$R_{ij}u^i u^j = -\frac{3a''}{a} \simeq -\frac{3\eta_0(\eta_0-1)}{|t-t_0|^2} = -\frac{3\eta_0(\eta_0-1)}{|\tau-\tau_0|^2},$$

produces a logarithmic divergence with Tipler's definition and an inverse power divergence with Królak's one, so we may conclude that the singularities are strong for all models with $1 \neq \eta_0 \neq 0$. Therefore we have:

(i) For $1 \neq \eta_0 \neq 0$ the geodesics in the fluid congruence meet a strong singularity at t_0.

(ii) For $\eta_0 = 0$, we need another term in the expansion

$$\frac{3a''}{a} \simeq \frac{3c_1\eta_1(\eta_1-1)}{c_0|t-t_0|^{2-\eta_1}} = \frac{3c_1\eta_1(\eta_1-1)}{c_0|\tau-\tau_0|^{2-\eta_1}},$$

and we see that in these cases the integrals do not diverge with Tipler's definition, but they do with Królak's for $\eta_1 \in (0,1)$.

(iii) For $\eta_0 = 0$, $\eta_1 = 1$, we still need another term

$$\frac{3a''}{a} \simeq \frac{3c_2\eta_2(\eta_2-1)}{c_0|t-t_0|^{2-\eta_2}} = \frac{3c_2\eta_2(\eta_2-1)}{c_0|\tau-\tau_0|^{2-\eta_2}},$$

in order to check that integrals do not diverge for these models with both definitions, since $\eta_2 > 1$.

(iv) For $\eta_0 = 1$, we resort to the second term in the expansion,

$$\frac{3a''}{a} \simeq \frac{3c_1\eta_1(\eta_1-1)}{c_0|t-t_0|^{3-\eta_1}} = \frac{3c_1\eta_1(\eta_1-1)}{c_0|\tau-\tau_0|^{3-\eta_1}},$$

and we find that for these models the integrals do not diverge with Tipler's definition, but they do with Królak's for $\eta_1 \in (1,2]$.

These results are summarized in Table IV.

This result may be further refined using timelike radial geodesics, for which

$$u^t = \dot{t} = \sqrt{1+\frac{P^2}{a^2}}, \qquad u^r = \dot{r} = \pm\frac{P}{fa^2},$$

$$R_{ij}u^iu^j = -\frac{3a''}{a} + 2P^2\left(\frac{a'^2+k}{a^4} - \frac{a''}{a^3}\right).$$

Taking a look at the geodesic Eq. (19) for t, we notice three different possibilities:

(i) $\eta_0 < 0$: Since $\dot{t} \simeq 1$, $t - t_0 \simeq \tau - \tau_0$ close to t_0.

(ii) $\eta_0 = 0$: Now $\dot{t} \simeq \sqrt{1+P^2/c_0^2} = \alpha$ close to t_0 and so $t - t_0 \simeq \alpha(\tau - \tau_0)$.

(iii) $\eta_0 > 0$: For these cases $\dot{t} \simeq P/a$ close to t_0 as for lightlike geodesics.

Accordingly, there are several cases:

(i) $\eta_0 < 0$: At lowest order, the P-dependent terms

$$\frac{a'^2+k}{a^4} - \frac{a''}{a^3} \simeq \frac{\eta_0}{c_0^2|\tau-\tau_0|^{2(\eta_0+1)}},$$

produce no divergent integral with Tipler's definition but they do with Królak's one for $\eta_0 \in [-1/2, 0)$, but it does not matter, since the first term was already seen to be divergent, as it is the same as for the fluid congruence in all these cases.

(ii) $\eta_0 = 0$: The P-dependent terms are essentially the same as for lightlike geodesics and we reach therefore the same conclusion: these models produce no divergent integral with Tipler's definition, but with Królak's one they do if $\eta_1 \in (0,1)$. The same happens with the first term, which is the same as for the fluid congruence.

(iii) $\eta_0 = 1$: The P-dependent terms are the same as for lightlike geodesics. Hence, these cases are all singular but except maybe for $k = -1$, $c_0 = 1$. Models with $k = -1$, $c_0 = 1$, and $\eta_1 < 3$ are singular with Królak's definition. On the other hand, the first term

$$\frac{3a''}{a} \simeq \frac{3c_1\eta_1(\eta_1-1)}{c_0|t-t_0|^{3-\eta_1}} \simeq \left(\frac{2P}{c_0}\right)^{\eta_1-3/2}\frac{3c_1\eta_1(\eta_1-1)}{c_0|\tau-\tau_0|^{3-\eta_1/2}}$$

does not diverge.

(iv) $1 \neq \eta_0 > 0$: The P-dependent term for these geodesics is the same as for lightlike geodesics and therefore it is divergent in all cases, though the first term

$$\frac{3a''}{a} \simeq \frac{3\eta_0(\eta_0-1)}{|t-t_0|^2} \simeq \left(\frac{c_0}{P(1+\eta_0)}\right)^{2/1+\eta_0}\frac{3\eta_0(\eta_0-1)}{|\tau-\tau_0|^{2/1+\eta_0}}$$

does not diverge with Tipler's definition and only with Królak's for $\eta_0 \in (0,1)$.

Therefore we have so far exactly the same models with strong singularities as we found for lightlike geodesics plus the $\eta_0 \leq -1$ models, which are null, but not timelike, geodesically complete. That is, we know that all models with $0 \neq \eta_0 \neq 1$ have strong singularities at t_0. We do not know what happens with models with $\eta_0 = 0$, though those with $\eta_1 \in (0,1)$ have strong singularities according to Królak. And the same happens with models with $\eta_0 = 1$, $k = -1$, $c_0 = 1$, though those with $\eta_1 \in (1,3)$ have also strong singularities according to Królak.

Since the condition on integrals of the Ricci tensor is not also a necessary condition for the appearance of strong singularities, we have to check other ways to get information about the $\eta_0 = 0$ and $\eta_0 = 1$ models.

TABLE IV. Degree of singularity of the fluid congruence of timelike geodesics around t_0.

η_0	η_1	*Tipler*	*Królak*
$(-\infty, 0)$	(η_0, ∞)	Strong	Strong
0	$(0, 1)$	Weak	Strong
	$[1, \infty)$	Weak	Complete
(0,1)	(η_0, ∞)	Strong	Strong
1	$(1, 2]$	Weak	Strong
	$(2, \infty)$	Weak	Weak
$(1, \infty)$	(η_0, ∞)	Strong	Strong

TABLE V. Degree of singularity of radial timelike geodesics around t_0.

η_0	η_1	k	c_0	*Tipler*	*Królak*
$(-\infty, 0)$	(η_0, ∞)			Strong	Strong
0	$(0, 1)$			Weak	Strong
	$[1, \infty)$	$0, \pm 1$	$(0, \infty)$	Weak	Weak
(0,1)	(η_0, ∞)			Strong	Strong
1	$(1, \infty)$	0,1		Strong	Strong
	$(1, \infty)$		$(0, 1) \cup (1, \infty)$	Strong	Strong
	(1,3)	-1	1	Weak	Strong
	$[3, \infty)$			Weak	Weak
$(1, \infty)$	(η_0, ∞)	$0, \pm 1$	$(0, \infty)$	Strong	Strong

For Tipler's definition [21], if a causal geodesic with velocity u meets a strong singularity, then the integral

$$I^i_j(\tau) = \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' |R^i_{kjl} u^k u^l|, \tag{25}$$

diverges as τ tends to τ_0 for some i, j. The components are referred to a parallelly transported orthonormal frame.

Again Królak's definition is less restrictive and just requires that the integral

$$I^i_j(\tau) = \int_0^\tau d\tau' |R^i_{kjl} u^k u^l| \tag{26}$$

diverges as τ tends to τ_0 for some i, j.

We begin again with the fluid worldline congruence, $u^t = 1$, for which the only nonvanishing components of the Riemann tensor,

$$R^i_{\mathrm{tit}} = -\frac{a''}{a}, \qquad i = r, \theta, \phi,$$

produce a necessary condition which is the same as the already studied sufficient condition.

Therefore, geodesics in the fluid worldline congruence meet a strong singularity if and only if $1 \neq \eta_0 \neq 0$ (or $\eta_0 = 0$, $\eta_1 \in (0, 1)$ and $\eta_0 = 1$, $\eta_1 \in (1, 2)$ with Królak's definition).

Radial timelike geodesics show more strong singularities, as we may see. We complete the orthonormal basis formed by u,

$$u = \dot{t}\partial_t + \dot{r}\partial_r = \sqrt{1 + \frac{P^2}{a^2}}\partial_t \pm \frac{P}{a^2 f}\partial_r,$$

and a vector v,

$$v = af\dot{r}\partial_t + \frac{\dot{t}}{af}\partial_r = \pm\frac{P}{a}\partial_t + \frac{1}{af}\sqrt{1 + \frac{P^2}{a^2}}\partial_r,$$

adding the corresponding unitary vectors parallel to ∂_θ and ∂_ϕ. The parallel transport requirement is trivially satisfied.

The θ and ϕ components of the Riemann tensor

$$\begin{aligned} R^\theta_{k\theta l} u^k u^l &= R^\theta_{t\theta t}\dot{t}^2 + R^\theta_{r\theta r}\dot{r}^2 + (R^\theta_{t\theta r} + R^\theta_{r\theta t})\dot{t}\,\dot{r} \\ &= -\frac{a''}{a} - P^2\left\{\frac{a''}{a^3} - \frac{a'^2 + k}{a^4}\right\} = R^\phi_{k\phi l} u^k u^l \end{aligned}$$

have similar terms as $R_{ij}u^i u^j$ and therefore produce the same results as the corresponding sufficient condition.

Finally, the v components

$$\begin{aligned} R_{ikjl} v^i u^k v^j u^l = R_{trtr}\bigg\{&\frac{P^4}{a^6 f^2} + \frac{1}{a^2 f^2}\left(1 + \frac{P^2}{a^2}\right)^2 \\ &- 2\frac{P^2}{a^4 f^2}\left(1 + \frac{P^2}{a^2}\right)\bigg\} = -\frac{a''}{a} \end{aligned}$$

provide a term that has already been discussed and therefore we may conclude also that sufficient conditions for the appearance of strong singularities along timelike geodesics are also necessary, as it happened for lightlike ones. These results are summarized in Table V.

VII. DISCUSSION

We have obtained a thorough classification of singular events in FLRW cosmological models in terms of the exponents of a generalized power expansion of the scale factor in coordinate time around a cosmological milestone at t_0. The behavior of causal geodesics has been obtained in the vicinity of the event. The first difference that has been found is that whereas the velocity of causal geodesics blows up at big bang and big crunch singularities, it is finite at big rip singularities, as well as acceleration and other derivatives, depending on the first exponent in the expansion, η_0. For sudden singularities the velocity is finite and the acceleration may be finite or not, depending on the next exponent η_1.

However this difference of regularity between big bang/crunch and big rip singularities does not prevent the strong character of both types of cosmological milestones with both Tipler and Królak's definitions of strong singularities. There is only a curious feature in big rip singularities in models with $\eta_0 \leq -1$ (which are precisely those favored by observations): lightlike geodesics do not reach the curvature singularity at t_0 in finite proper time and therefore these spacetimes are null geodesically complete close to the singular event. Hence photons never experience big rip singularities and the Universe would last eternally for them. This feature, however, is lost on dealing with timelike geodesics, which reach t_0 in finite proper time and meet a strong singularity.

The only models which allow regular behavior close to t_0 are those with $\eta_0 = 0$ and with $\eta_0 = 1$, $k = -1$, $c_0 = 1$. The latter ones are Milne universes at first order, which are essentially Minkowski spacetime after extending the model beyond t_0. The former ones include models with a nonvanishing analytical scale factor, such as de Sitter universes, and models with sudden singularities, which have finite velocity, but nonfinite acceleration or higher derivatives of the parametrization of the geodesics depending on the exponent η_1. The larger this exponent is, the better the properties of the model are. These cosmologies prevent the formation of strong singularities according to Tipler's definition, which requires the crushing to zero or disrupting to infinity of finite volume objects evolving along causal geodesics. With Królak's definition, which requires just a positive derivative of volume for big bang and big rip singularities and a negative derivative for big rip singularities, strong singularities are avoided in models with $\eta_1 \geq 1$. This definition seems more appropriate, since causal geodesics in models with $\eta_1 \in (0, 1)$ do not have finite acceleration and therefore geodesic equations would be singular at t_0, though the curves may be extended beyond that event.

We may compare these results with those studied by Cattoën and Visser in [3], where just singularities in curvature were considered, without taking into account their strength nor the behavior of causal geodesics. Those authors found that the only models without polynomial curvature singularities are those with $\eta_0 = 0$, $\eta_1 \geq 2$ or $\eta_1 = 1$, $\eta_2 \geq 2$, and those with $\eta_0 = 1$, $k = -1$, $c_0 = 1$, $\eta_1 \geq 3$. Dealing with derivative curvature singularities, the list reduces to models with $\eta_0 = 0$ and natural exponents η_i, $i \geq 1$ and those with $\eta_0 = 1$, $k = -1$, $c_0 = 1$, and natural exponents $\eta_i \geq 3$, $i \geq 1$. Derivative curvature singularities are not reflected in our classification since derivatives of the Riemann tensor appear neither in geodesic equations nor in Jacobi equations. The apparent discrepancy between our results and the presence of polynomial curvature singularities lies on the fact that either geodesics do not reach that singularity or that they reach it, but the curvature growth is not enough to form a strong singularity.

Finally, another consequence is that singularities appear just in models with vanishing, divergent, or nonsmooth scale factors. From the mathematical point of view at least it is worth mentioning that regular models are an open set within the family of smooth homogeneous and isotropic spacetimes, as it happened for instance with inhomogeneous scalar field Abelian diagonal G_2 models [22]. On the contrary, singular models are not an open set, since the vanishing requirement is not generic.

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