Counting black hole microscopic states in loop quantum gravity

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Counting of microscopic states of black holes is performed within the framework of loop quantum gravity. This is the first calculation of the pure *horizon* states using statistical methods, which reveals the possibility of additional states missed in the earlier calculations, leading to an increase of entropy. Also for the first time a microcanonical temperature is introduced within the framework.

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I. INTRODUCTION

A nonperturbative framework of quantum gravity using holonomy as fundamental variables, popularly known as loop quantum gravity, has been in vogue for some years now, see [1] for a recent survey. In this framework, a start was made in [2] in the direction of quantizing a black hole and thereby counting its microstates. In this approach, a black hole is characterized *effectively* by an isolated horizon, see [3] and references therein. The quantum states arise from quantizing the phase space of an isolated horizon whose cross sections, which are two-spheres, are punctured by suitable spin networks. The spin quantum numbers j, m, which characterize the punctures, also label the quantum states. The entropy is obtained by counting the manifold possibilities of such quantum states, or essentially the labels, that are consistent with a fixed area of the cross section [2].

A calculation of microscopic states was carried out in [4] using a recursion relation technique. Soon after, in [5], an explicit combinatorial method was introduced, which in addition to counting states also gives the dominant configuration of spins, namely, the configuration yielding the maximum number of states. However, the two counting calculations gave slightly different results. See also [6] for a recent survey, which supports the result of [5]. The root of this difference has been briefly discussed in [5]: while [4] takes into account only the spin projection (m) labels of the microstates, thus counting what we refer to as the pure *horizon* states in the present work, [5,6] take into account the spin j, which in some sense characterizes the bulk states, as well as the m-labels. Any counting involves two constraints to be met. While one of them, the spin projection constraint (see below for details), which arises from an interplay of the bulk and the horizon Hilbert spaces, can be expressed solely in terms of the m-labels, the other constraint involving the area of the horizon, cannot bypass the j-labels. The calculation of [5] was essentially based upon the intuition that a quantum isolated horizon can never be completely characterized by states of the horizon (or surface) Hilbert space, the bulk states play an essential rôle.

The first part of the present work uses the *combinatorial method* of [5] to count the number of the pure *horizon* states which were sought to be counted in [4]. This leads to an increased number. Unlike the result of [4], this number is consistent with the thesis presented in [7], in the sense that $j = 1$ allows three values of m. In the second part of the work, we introduce a microcanonical temperature for each null normal vector field defined on the horizon. It involves the Immirzi parameter and the surface gravity corresponding to a null normal vector field. We comment on the possible connection between processes involving vanishing of punctures and Hawking radiation.

II. COUNTING OF STATES

We set our units such that $4\pi\gamma\ell_P^2 = 1$, where γ is the so-called Barbero-Immirzi parameter involved in the quantization and ℓ_P the Planck length. Equating the classical area A of the horizon to the eigenvalue (for a specific spin configuration of punctures on the horizon) of the area operator we find

$$A = 2\sum_p \sqrt{j_p(j_p + 1)}, \tag{1}$$

where the p-th puncture carries a spin $j_p > 0$, more accurately an irreducible spin representation labeled by j_p, contributing a *quantum of area* $2\sqrt{j_p(j_p + 1)}$ to the total area eigenvalue.

Let the configuration be such that s_j is the number of punctures carrying spin j. So in (1) the sum over punctures can now be replaced by the sum over spins

$$A = 2\sum_j s_j\sqrt{j(j + 1)}. \tag{2}$$

Such a spin configuration will be called *permissible* if it obeys (2) together with the *spin projection constraint* which will be introduced shortly.

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A. The combinatorial method

First, we briefly review the calculation of [5] before going on to apply the method to the counting of horizon states. Given a configuration labeled by s_j, different projections m of j give $\prod_j (2j+1)^{s_j}$ quantum states. But the s_js themselves can be chosen in $(\sum s_j)!/\prod s_j!$ ways since the punctures are considered distinguishable. Therefore, the total number of quantum states given by such a configuration s_j is

$$d_{s_j} = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j (2j+1)^{s_j}. \tag{3}$$

To obtain the total number of states for all configurations (3) is to be summed over all configurations. We estimate the sum by maximizing $\ln d_{s_j}$ by varying s_j subject to (2). In the variation we assume that $s_j \gg 1$ for each j and only such configurations dominate the counting. Such an assumption breaks down if $A \sim o(1)$. The variational equation $\delta \ln d_{s_j} = \lambda \delta A$, where λ is the Lagrange multiplier, gives

$$\frac{s_j}{\sum s_j} = (2j+1) e^{-2\lambda\sqrt{j(j+1)}}. \tag{4}$$

Clearly, for consistency (i.e. summing both sides over all j), λ must obey (cf. [8])

$$1 = \sum_j (2j+1) e^{-2\lambda\sqrt{j(j+1)}}. \tag{5}$$

The counting however should also incorporate the spin *projection* constraint. In order to implement this constraint the configuration must be given finer labels. Let $s_{j,m}$ denote the number of punctures carrying spin j with projection m. With these new variables the area and the spin projection constraints take the respective simple forms

$$A = 2\sum_{j,m} s_{j,m}\sqrt{j(j+1)}, \qquad 0 = \sum_{j,m} m s_{j,m}. \tag{6}$$

A configuration $s_{j,m}$ will be called permissible if it satisfies both of these Eqs. (6). The total number of quantum states for these configurations is

$$d_{s_{j,m}} = \frac{(\sum_{j,m} s_{j,m})!}{\prod_{j,m} s_{j,m}!}. \tag{7}$$

To obtain the dominant permissible configuration that contributes the largest number of quantum states, we maximize $\ln d_{s_{j,m}}$ by varying $s_{j,m}$ subject to (6). The result can be expressed in terms of two Lagrange multipliers λ, α:

$$\frac{s_{j,m}}{\sum s_{j,m}} = e^{-2\lambda\sqrt{j(j+1)} - \alpha m}. \tag{8}$$

Consistency requires that λ and α be related to each other by $\sum_j e^{-2\lambda\sqrt{j(j+1)}} \sum_m e^{-\alpha m} = 1$. In order that (8) satisfies the spin projection constraint we require $\sum_j e^{-2\lambda\sqrt{j(j+1)}} \sum_m m e^{-\alpha m} = 0$. This is possible if and only if $\sum_m m e^{-\alpha m} = 0$ for each j, which essentially implies $\alpha = 0$ (the value $2i\pi$ is excluded by positivity of $s_{j,m}$). Therefore, the condition (5) on λ remains unchanged. Note that each $s_{j,m}$ is proportional to the area A because of the area constraint.

The total number of quantum states for all permissible configurations is clearly $d(A) = \sum_{s_{j,m}} d_{s_{j,m}}$. To estimate $d(A)$ we expand $\ln d$ around the dominant configuration (8), which we shall denote by $\bar{s}_{j,m}$. Thus $\ln d = \ln d_{\bar{s}_{j,m}} - \frac{1}{2}\sum \delta s_{j,m} K_{j,m;j'm'} \delta s_{j'm'} + o(\delta s_{j,m}^2)$ where K is the symmetric matrix $K_{j,m;j'm'} = \delta_{jj'}\delta_{mm'}/\bar{s}_{j,m} - 1/\sum_{k,l}\bar{s}_{k,l}$. All variations $\bar{s}_{j,m} + \delta s_{j,m}$ must satisfy the two conditions (6) which yield two conditions $\sum \delta s_{j,m}\sqrt{j(j+1)} = 0$ and $\sum \delta s_{j,m} m = 0$. Taking into account these equations we can express the total number of states as

$$\begin{aligned} d &= d_{\bar{s}_{j,m}} \sum_{-\infty}^{\infty} e^{-1/2\sum \delta s_{j,m} K_{j,m;j'm'} \delta s_{j'm'}} \\ &\quad \cdot \delta\Big(\sum \delta s_{j,m}\sqrt{j(j+1)}\Big)\delta\Big(\sum \delta s_{j,m} m\Big) \\ &= C d_{\bar{s}_{j,m}} \Bigg[\prod_{j,m} \sqrt{A}\Bigg] \Bigg/ A, \end{aligned} \tag{9}$$

where C is a constant independent of A. The denominator takes the particular form because the two constraints remove two factors of $\sqrt{A}$, which would be present otherwise in the Gaussian sum. It may be noted that K has a zero eigenvalue, but this is taken care of by the area constraint and all other eigenvalues of K are proportional to $1/A$. Inserting (8) into (7) and dropping factors of $o(1)$ we obtain

$$d_{\bar{s}_{j,m}} = \frac{(\sum \bar{s}_{j,m})^{1/2}}{\prod_{j,m}(2\pi \bar{s}_{j,m})^{1/2}} e^{\lambda A}. \tag{10}$$

Plugging these expressions into d we finally obtain

$$d = \frac{\alpha}{\sqrt{A}} e^{\lambda A}, \quad \text{where } \alpha \sim o(1), \tag{11}$$

leading to the formula [5]

$$S = \lambda \frac{A}{4\pi\gamma\ell_P^2} - \frac{1}{2}\ln\frac{A}{4\pi\gamma\ell_P^2} \tag{12}$$

for entropy. The origin of the $\sqrt{A}$ in d or $\frac{1}{2}\ln A$ in $\ln d$ can be easily traced in this approach: it is the condition $\sum m s_{j,m} = 0$. This shows that the coefficient of the log-correction is robust and does not depend on the details of the configurations at all. It is directly linked with the boundary conditions the horizon must satisfy.

B. Application of the method to horizon states

The above calculation was based on the understanding that j is a relevant quantum number. An alternative plan, adopted in [2,4], is instead to count the states of the horizon Hilbert space alone. For this purpose, one has to consider the number s_m of punctures carrying spin projection m, ignoring what spins j they come from. One can distinguish between s_j and s_m from the context. It is clear that

$$s_m = \sum_j s_{j,m}, \qquad j = |m|, |m| + 1, |m| + 2, \ldots. \tag{13}$$

For the s_m configuration the number of states is $d_{s_m} = (\sum_m s_m)!/\prod_m s_m!$ and the total number of states is obtained by summing over all configurations. As in the earlier cases, the sum can be approximated by maximizing $\ln d_{s_m}$ subject to the conditions (6). The calculation resembles the previous one in spirit, but there are important differences as discussed below. The constrained extremization conditions for variation of $s_{j,m}$ are

$$-\left[\ln\frac{s_m}{\sum_m s_m} + 2\lambda\sqrt{j(j+1)} + \alpha m\right] = 0. \tag{14}$$

Clearly, the above equations cannot hold for arbitrary j even for a fixed m, because inconsistencies will arise for nonzero λ. In fact, for any fixed m the above equation admits at most one j—let us denote it by $j(m)$. For $j \neq j(m)$, the first derivative becomes nonzero. Such a situation can arise if and only if $\ln d_{s_m}$ is maximized at the boundary (in the space of all permissible configurations) for all $j \neq j(m)$ and at an interior point for $j = j(m)$. This means that for the dominant configuration $s_{j,m} = 0$ for all $j \neq j(m)$: the corresponding first derivative is then only required to be zero or negative because in any variation $s_{j,m}$ can only increase from its zero value. Thus, $s_m = s_{j(m),m}$ for the dominant configuration.

Then (14) gives

$$\frac{s_m}{\sum_m s_m} = e^{-2\lambda\sqrt{j(m)(j(m)+1)}-\alpha m}. \tag{15}$$

As before, $\alpha = 0$ in order that the dominant configuration satisfies the spin projection constraint. The parameter λ is determined by a consistency condition involving $j(m)$. Since the entropy increases with λ, and lower $j(m)$ gives higher λ, the maximum entropy is obtained when $j(m)$ is minimum, i.e., $j(m) = j_{\min}(m)$, the minimum value for j for a given m. For all $m \neq 0$, we have $j_{\min}(m) = |m|$. But for $m = 0$, we must have $j_{\min}(m) = 1$, since $j = 0$ is excluded.

The configuration (15) with $j(m) = j_{\min}(m)$ implies that the entropy is given by (12) in terms of λ, which is now determined by the altered consistency relation

$$1 = \sum_{j\neq 1} 2e^{-2\lambda\sqrt{j(j+1)}} + 3e^{-2\lambda\sqrt{2}}, \tag{16}$$

where each $j \neq 1$ is associated with $m = \pm j$ only, but $j = 1$ also has $m = 0$. Note that for λ zero or negative, such relations would be impossible to satisfy, hence no such solutions exist.

This equation for λ differs from that of [4] in allowing $m = 0$ for $j = 1$ and thus yields a slightly greater value 0.790 instead of 0.746. The difference arises because we have used the area constraint directly, using the definition of the area involving j. In contrast, [4] used an inequality involving m,

$$A \geq 2\sum_m s_m\sqrt{|m|(|m|+1)}, \tag{17}$$

which can be derived on the basis of the inequality $j \geq |m|$, but is not saturated for punctures with $j = 1$, $m = 0$, which the maximization conditions allow. The value 0.790 of λ is naturally less than the value 0.861 obtained by taking both j and m to be relevant quantum numbers [5].

It is to be noted that our counting of horizon states allows *three* spin states for $j = 1$ and is thus consistent with the general ideas in [7] which reported an intriguing connection between the spin degeneracy and an observed factor of ln3 occurring in the classical quasinormal modes of black holes. Reference [7] recommends *only* $j = 1$, which could be accommodated by setting $s_{j,m} = 0$ for all m except 0 and 1. Our earlier calculation of bulk states [5] was also consistent with [7] and coincides with the present calculation for $j = 1$. In contrast, the counting of [4] allows only *two* projection states for $j = 1$ and is therefore, inconsistent with [7].

III. TOWARDS THE DEFINITION OF A TEMPERATURE

First we make some comments on the interpretation of the laws of the mechanics of a weakly isolated horizon (WIH) as thermodynamic laws.

(1) The zeroeth law of WIH states that the surface gravity $\kappa_{(\ell)}$ associated with each "fixed" null normal vector ℓ^a (which generates the WIH) is constant on the horizon. However for a given isolated horizon, ℓ^a is fixed only up to a constant rescaling. Under such a rescaling $\ell^a \mapsto c\ell^a$, where c is a positive number, both $\kappa_{(\ell)}$ and the 'horizon-mass' $M_{(\ell)}$ (which also depends on the choice of ℓ^a, see [3] for details) are rescaled by the same constant c, whereas the horizon-area A does not alter. In fact the first law of a nonrotating WIH, which states that the change of the horizon-mass

$$\Delta M_{(\ell)} = \frac{\kappa_{(\ell)}}{8\pi G}\Delta A, \tag{18}$$

where ΔA is the associated change of the horizon-area, depends explicitly on ℓ^a (although the above scaling argument show that the form (18) is independent of ℓ^a). Thus, both zeroeth and first laws of

WIH make an explicit reference to a "fixed" null normal vector field ℓ^a. This fact is to be kept in mind whenever we draw analogies between a WIH and a thermodynamic system. Unless some ℓ^a is fixed, confusions will arise in the thermodynamic interpretation of a WIH.

(2) This is regarding the quantum statistical mechanics of a WIH. Now that there is a quantum mechanical entropy of a WIH, we have definite quantum states of a WIH. However, a realistic statistical interpretation of a WIH, even as a microcanonical ensemble, requires states of the "environment", *viz.*, the states of the bulk of the spacetime of which the WIH is a subsystem, the bulk and the WIH together forming an isolated system. The microcanonical ensemble assumes a weak interaction between the bulk and the horizon such that the horizon-area A is constant (more precisely, it lies in a small interval $[A - \epsilon, A + \epsilon]$ where $\epsilon \ll A$). The trace over the bulk spacetime states provides a density matrix for the WIH (which for a microcanonical ensemble is trivial, proportional to the identity matrix). This is related to the comment made in the introduction that a quantum WIH can never be fully described by surface states alone, the bulk states act like a heat-bath as indicated above.

It is not at all difficult to arrive at an expression of a microcanonical temperature based on the formal analogy with thermodynamics. We already found that for a fixed ℓ^a the surface gravity $\kappa_{(\ell)}$ should be related to the temperature and the first law (18) is to be interpreted as the first law of thermodynamics. Since the entropy is given by (12), its variation is (ignoring the log-correction for now) $\Delta S = \lambda \Delta A / 4\pi\gamma\ell_P^2$ and equating $T_{(\ell)}\Delta S$ with the RHS of (18) we get an expression

$$T_{(\ell)} = \frac{\hbar\gamma\kappa_{(\ell)}}{2\lambda}. \tag{19}$$

This is the microcanonical temperature of a WIH having a fixed null normal vector field.

To interpret the microcanonical ensemble as a canonical or a grand-canonical ensemble we need to allow interactions between the WIH and the bulk. For each permissible configuration $s_j \equiv \sum_m s_{j,m}$ the area spectrum is $A = 8\pi\gamma\ell_P^2 \sum s_j\sqrt{j(j+1)}$. Now imagine a quantum mechanical process that changes the configuration s_j to another permissible configuration $s_j + \Delta s_j$, that causes the area to change by $\Delta A = 8\pi\gamma\ell_P^2 \sum \Delta s_j\sqrt{j(j+1)}$. (This change Δs_j should not be confused with δs_j we used earlier. Here permissibility of the new configuration $s_j + \Delta s_j$ does not imply $\sum \Delta s_j\sqrt{j(j+1)} = 0$: while the permissibility of s_j is associated with the area A, the one of $s_j + \Delta s_j$ is associated with the area $A + \Delta A$, where ΔA is a physical change of area.) Thus, from (18) we obtain

$$\Delta M_{(\ell)} = \hbar\kappa_{(\ell)}\gamma \sum \Delta s_j\sqrt{j(j+1)}. \tag{20}$$

This is a key result showing how the mass/energy of the WIH can leak to the bulk of the spacetime. This involves the creation and annihilation of punctures. In a microcanonical ensemble these processes take place only under the strict permissibility conditions (which basically ensure that the area and the energy cannot change). But in a canonical or grand-canonical ensemble these restrictions are to be removed. A detailed study is required in this direction to interpret the temperature (19) in a canonical ensemble.

Since a WIH involves an infinite family of null normal vectors, it also admits an infinite family of corresponding temperatures, fixed for each fixed ℓ^a. Moreover, (19) shows that the relation $\gamma\pi = \lambda$ which yields the semiclassical expression of entropy, also gives the semiclassical expression of temperature $T_{(\ell)} = \hbar\kappa_{(\ell)}/2\pi$. It is interesting to ask what alteration the log correction to the entropy (12) induces in the temperature. A simple calculation shows that $T_{(\ell)} = (\hbar\kappa_{(\ell)}\gamma/2\lambda)(1 + 2\pi\gamma\ell_P^2/\lambda A)$. So while the entropy receives a universal log-correction, the temperature is corrected only by a power-law. Unlike the case of the entropy, the coefficient of the power-law correction is not universal—it depends on the underlying quantum theory. However, the value of γ that gives the semiclassical sector of quantum gravity also makes the coefficient independent of λ.

IV. DISCUSSION

We have followed the combinatorial approach of [5] to count horizon states and have found that there are more of these than indicated by the approximate analysis of [4]. The increased number is of course still not as large as the total number of microscopic states found in [5] where not only m but also j was regarded as a relevant label for a microscopic state. However, the correction brings the number of states distinguished by m closer to the number of states labeled by j, m and also makes it consistent with [7].

Thereafter we have sought to introduce a temperature corresponding to each choice of the null normal vector field ℓ^a. The discussion in the previous section suggests that the area ensemble may be regarded as an energy ensemble for each fixed ℓ^a. Standard statistical mechanical arguments then may permit us to view the microcanonical ensemble as a canonical or grand-canonical ensemble. At thermal equilibrium the quantum mechanical process changing the horizon-area suggests the following picture: quantum states associated with the punctures get annihilated from the surface Hilbert space by transforming into bulk states. If the bulk is taken to be asymptotically flat then such bulk states appear to be the usual Fock states. Reversibly, the Fock states from the bulk of the spacetime must transform into the puncture-states and these two processes must take place at the same rate. These processes are quite analogous, though not identical, to the particle creation and annihilation processes we encounter in quantum field theories. In quantum field theories in flat space the creation and destruction of one-particle states are performed by certain linear operators in the Fock space. Furthermore, such one-particle states are labeled by their energy and momenta, so a fixed stationary background metric is required. However here we are considering creation and annihilation of punctures in changing s_j to $s_j + \Delta s_j$. Moreover, no background metric is present. Punctures are also labeled by the spin quantum numbers. The linear operators that can create or destroy punctures should be related to the spin-raising and spin-lowering operators in the bulk Hilbert space of loop quantum gravity. Such operators have indeed been constructed while obtaining the area-spectrum [9]. For the time being, it is an open problem to show that such processes exist in the Hilbert space within the framework of loop quantum gravity. Of course, the bigger question is whether, when the reverse process (bulk states to surface states) is ignored, the forward process (surface states to bulk states) appears as black-body radiation.

One can also arrive at a generalized statement of the second law that the combined entropy of the horizon and the bulk does not decrease. While the microscopic degrees of freedom associated with the horizon are punctures, those of the bulk remain the standard matter and field particles. In order that a thermal equilibrium is reached, these 2 degrees of freedom must transform into each other continuously. It remains to be seen how such a picture emerges in quantum geometry.

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