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Spherically symmetric solutions of modified field equations in $f(R)$ theories of gravity

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Spherically symmetric static empty space solutions are studied in $f(R)$ theories of gravity. We reduce the set of modified Einstein's equations to a single equation and show how one can construct exact solutions in different $f(R)$ models. In particular, we show that for a large class models, including e.g. the $f(R) = R - \mu^4/R$ model, the Schwarzschild-de Sitter metric is an exact solution of the field equations. The significance of these solutions is discussed in light of solar system constraints on $f(R)$ theories of gravity.

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I. INTRODUCTION

The accelerating expansion of the universe has transformed our view of the universe from a matter filled cosmos to one dominated by dark energy. Modern day cosmological observations are in contradiction with a matter dominated critical density universe whose expansion is decelerating and instead we find evidence for expansion that is accelerating. Direct evidence supporting the cosmic acceleration comes from the supernovae observations [1] and other observations, such as the cosmic microwave background [2] and large scale structure [3], provide more indirect evidence. Combining all of the observations, a cosmological concordance model has emerged: a critical density universe dominated by cold dark matter and cosmological constant-like dark energy.

The most commonly considered candidate for dark energy is the cosmological constant (for a review, see e.g. [4]), but numerous alternative mechanisms for generating the cosmic acceleration have been considered. Very roughly, one can divide the different alternative explanations of cosmic acceleration into two classes: those that include cosmic fluids with exotic equations of state and those that modify gravity. In terms of the Friedmann equation one can, again very roughly, consider the former to modify the right-hand side (rhs) of the equation, the stress-energy tensor, and the latter the left-hand side, the Einstein tensor.

Modifications of general relativity (GR) as a source of cosmic acceleration have been recently considered in numerous works. One particular class of models that has drawn a significant amount attention is the $f(R)$ gravity models (see e.g. [5–11] and references therein). These models are a particular class of higher derivative gravity theories that include higher order curvature invariants as functions of the Ricci scalar. Such theories avoid the Ostrogradski's instability [12] that can otherwise prove to be problematic for general higher derivative theories [13].

A number of challenges have been identified in building phenomenologically viable models of $f(R)$ gravity theories. Such possible obstacles include instabilities within matter [14], outside matter [15], stability of the vacuum [16], and constraints arising from known properties of gravity in our solar system (see e.g. [17–19] and references therein). In addition, identifying the specific functional form of $f(R)$ from cosmological observations is problematic since the background expansion does not determine $f(R)$ uniquely [20].

In a number of works, the solar system constraints on $f(R)$ theories of gravity are derived by first conformally transforming the theory to a scalar-tensor theory and then considering the parametrized post-Newtonian limit [21,22]. This procedure does not seem to be without controversy, however [23,24]. In this light, it is interesting to consider solutions of the modified Einstein's equations of $f(R)$ theory. Armed with the metric, one can hope to study orbital motion directly without resorting to conformal transformations. As a first step in this direction, we consider vacuum solutions of the modified Einstein's equations in this paper. We show how one can reduce the set of equations into a single equation that one can then utilize to construct explicit solutions. As an example we show that a large class of $f(R)$ models has the Schwarzschild-de Sitter (SdS) metric as an exact solution. In addition, we construct other solutions corresponding to different metrics.

II. $f(R)$ GRAVITY FORMALISM

The action for $f(R)$ gravity is (see e.g. [25])

$$S = \int d^4x\sqrt{-g}(f(R) + \mathcal{L}_m), \qquad (1)$$

where we have set $8\pi G = 1$. The field equations resulting from this action in the metric approach, i.e. assuming that the connection is the Levi-Cività connection and variating with respect to the metric $g_{\mu\nu}$, are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = T^c_{\mu\nu} + T^m_{\mu\nu}, \qquad (2)$$

where the stress-energy tensor of the gravitational fluid is

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$$T^c_{\mu\nu} = \frac{1}{F(R)}\left\{\frac{1}{2}g_{\mu\nu}(f(R) - RF(R)) + F(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\mu\nu}g_{\alpha\beta})\right\} \tag{3}$$

with $F(R) \equiv df(R)/dR$.

The standard minimally coupled stress-energy tensor $\tilde{T}^m_{\mu\nu}$, derived from the matter Lagrangian $\mathcal{L}_m$ in the action (1), is related to $T^m_{\mu\nu}$ by

$$T^m_{\mu\nu} = \tilde{T}^m_{\mu\nu}/F(R). \tag{4}$$

In empty space (vacuum), the equations of motion reduce to

$$F(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = 0. \tag{5}$$

Contracting these vacuum equations we obtain simply

$$F(R)R - 2f(R) + 3\Box F(R) = 0. \tag{6}$$

This equation is useful, because it allows us to express $f(R)$ in terms of its derivatives. If $T^m_{\mu\nu} \neq 0$ there is an additional trace of stress-energy tensor $T^{m\mu}_{\mu}$ in the rhs of Eq. (6).

III. SPHERICALLY SYMMETRIC VACUUM SOLUTIONS

We are interested in spherically symmetric, time independent solutions of the empty space field equations. From properties of maximally symmetric subspaces, we know that the metric reads as (in spherically symmetric coordinates)

$$g_{\mu\nu} = \begin{pmatrix} s(t,r) & 0 & 0 & 0 \\ 0 & -p(t,r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{7}$$

The (01)-component of the Einstein equations is satisfied if both $\dot{p} = 0$ and $s(t,r)$ is separable with respect to its variables. This means that $\dot{R} = 0$ and, as in the well-known case of the Schwarzschild metric, the time dependence can be totally removed from the metric by redefinition of time. Here we make the same simplifying assumption and thus we consider henceforth only time independent solutions, i.e. $s = s(r)$ and $p = p(r)$.

The corresponding scalar curvature is

$$R = \frac{2}{r^2 p}\left(1 - p + \left(\frac{s'}{s} - \frac{p'}{p}\right)\left(r - \frac{r^2}{4}\frac{s'}{s}\right) + \frac{r^2}{2}\frac{s''}{s}\right), \tag{8}$$

where we have defined $' \equiv d/dr$.

Using the contracted equation, Eq. (6), the modified Einstein's equations become

$$FR_{\mu\nu} - \nabla_\mu\nabla_\nu F = \tfrac{1}{4}g_{\mu\nu}(FR - \Box F). \tag{9}$$

Since the metric only depends on r, one can view Eq. (9) as a set of differential equations for $F(r)$, $s(r)$, and $p(r)$. In this case both sides are diagonal and hence we have four equations. In addition, we have a consistency relation for $F(r)$,

$$RF' - R'F + 3(\Box F)' = 0, \tag{10}$$

which arises by differentiating the contracted equation, Eq. (6) with respect to r. Any solution of Eq. (9) must satisfy this relation in order to be also a solution of the original modified Einstein's equations, Eq. (5).

From Eq. (9) it is obvious that the combination $A_\mu \equiv (FR_{\mu\mu} - \nabla_\mu\nabla_\mu F)/g_{\mu\mu}$ (with fixed indices) is independent of the index μ and therefore $A_\mu - A_\nu = 0$ for all μ, ν. This allows us to write two equations:

$$2\frac{X'}{X} + rF'\frac{X'}{X} - 2rF'' = 0 \tag{11}$$

$$-4s + 4X - 4rs\frac{F'}{F} + 2r^2s'\frac{F'}{F} + 2rs\frac{X'}{X} - r^2s'\frac{X'}{X} + 2r^2s'' = 0, \tag{12}$$

where we have defined $X(r) \equiv p(r)s(r)$. From Eq. (11) one can solve for X'/X algebraically and substitute into Eq. (12) to obtain X:

$$X(r) = s\left(1 + r\frac{F'}{F} - r^2\frac{F''}{2F + rF'}\right) + \frac{1}{2}r^2s'\left(r\frac{F''}{2F + rF'} - \frac{F'}{F}\right) - \frac{1}{2}r^2s''. \tag{13}$$

Consistency then requires that this form of $X(r)$ satisfies Eqs. (11) and (12), giving an equation relating F and s. In addition, the modified Einstein's equations give four equations relating F and s. However, all of the equations have a common factor of the form:

$$s^{(3)} + s''\frac{4F^2 + 4rFF' + r^2F'^2 - 3r^2FF''}{rF(2F + rF')} - s'\frac{8F^4 + r^4F'^4 + r^3FF'^2(4F' + rF'') + r^2F^2(6F'^2 - 3r^2F''^2 + rF'(rF^{(3)} - 2F'')) + 2rF^3(4F' + r(rF^3 - F''))}{r^2F^2(2F + rF')^2} + 2s\frac{r^3F'^4 + r^2FF'^2(3F' + rF'') + r^2F^2(-3rF''^2 + F'(F'' + rF^{(3)})) + F^3(2r(2F'' + rF^3) - 4F')}{r^2F^2(2F + rF')^2} = 0. \tag{14}$$

Therefore, any pair $s(r)$, $F(r)$ satisfying this equation will be a solution of the modified Einstein's equations. In addition, if Eq. (14) is satisfied, also the consistency relation, Eq. (10) is automatically satisfied. From s and F, one can then calculate $R(r)$ and in principle construct the corresponding $f(R)$ by using Eq. (6). Note that the resulting $f(R)$ is not unique due to the presence of an integration constant. In addition, a larger degeneracy can also exist, e.g. for the SdS solution discussed below, $s(r)$ and $F(r)$ do not determine the $f(R)$ theory uniquely, even when discounting the integration constant.

A. Solutions with constant curvature

Looking for constant curvature solutions, $R = R_0$, the field equations reduce to

$$sp' + ps' = 0, \tag{15}$$

$$1 - p + \frac{r}{2}\left(\frac{p'}{p} + \frac{s'}{s}\right)\left(\frac{r}{2}\frac{s'}{s} - 1\right) - \frac{r^2}{2}\frac{s''}{s} = 0, \tag{16}$$

which are straightforwardly solvable:

$$p(r) = \frac{c_0}{s(r)} \qquad s(r) = c_0 + \frac{c_1}{r} + c_2 r^2, \tag{17}$$

where c_i are integration constants. For conventional definitions of space and time we require $c_0 > 0$. The scalar curvature Eq. (8) for this solution is $R = 12c_2/c_0$. Redefining the time coordinate, $t \to t/\sqrt{c}0$ with $c_1 \to c_1/\sqrt{c}0$ and $c_2 \to c_2/\sqrt{c}0$, we can always choose $c_0 = 1$.

The Schwarzschild solution in the presence of a cosmological constant, Schwarzschild-de Sitter -spacetime (SdS) arising using $f(R) = R + \Lambda$, has the form

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -1/A(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}, \tag{18}$$

where $A(r) \equiv 1 - 2M/r - \Lambda r^2/3$ and Λ is the cosmological constant. The scalar curvature in this case is a constant, $R = -4\Lambda$. For a finite mass distribution, the parameter M can in this case be identified as total material mass

$$M_c = \int^{r_m} d^3x T_0^{m0}. \tag{19}$$

Note that this integral also contains a part of gravitational energy inside the radius r_m of the mass distribution [26], while the total energy within the same radius also includes vacuum energy, $E_{\rm tot} = \int d^3x(T_0^{m0} + \Lambda)$.

Comparing the solution, Eq. (17), to the SdS metric, Eq. (18), one sees that the two metrics are identical with $c_1 = -2M$, $c_2 = -\Lambda/3$. From Eq. (6) it is clear that this metric is a solution for any form of $f(R)$ for which there exists a constant (real) R_0 such that $R_0 f'(R_0) - 2f(R_0) = 0$. In other words, the SdS metric is an exact solution for a set of functions $f(R)$ that satisfy $R_0 f'(R_0) - 2f(R_0) = 0$ such that R_0 is real. For example, for the $f(R) = R - \mu^4/R$ model, it is easy to see that the SdS metric is a solution when $R_0^2 = 3\mu^4 = 144c_2^2$, i.e. $c_2 = \sqrt{3\mu^4/144}$. The same exact result also holds for the other commonly considered model $f(R) = R - \mu^4/R + \epsilon R^2$.

The physical interpretation of the parameters M and Λ are not as straightforward as in the case of general relativity. The naive identification $M = M_c$ is problematic and a more careful analysis is required. In the presence on spherically finite symmetric mass distribution the empty space solution we are studying needs to be matched to the solution valid inside the mass distribution at $r = r_m$. Since the field equations are in general higher order differential equations than in GR, more integration constants need to be determined. In particular, this means that values of the metric components inside the mass distribution depend explicitly on the details of the mass distribution, making matching with the outside solution nonunique. This can be explicitly seen by studying e.g. solutions of spherical shells of different thicknesses: the boundary values at r_m depend explicitly on the thickness of the shell. This also demonstrates that the Birkhoff theorem is no longer valid since the external solution depends on the internal mass distribution. This is a general property of all (empty space) solutions of $f(R)$ theories whenever F differs from a constant. If we define the central mass by the gravitational effect it gives rise to the external space, the parameter M becomes defined as the central mass, but it does not coincide with (19).

B. General solutions with $p(r)s(r) = \text{const}$

From Eq. (11), it is clear that when $X = \text{const} \equiv X_0$, $F''(r) = 0$, and hence $F(r) = Ar + B$. Equation (14) can now be solved, giving

$$s(r) = X_0 + \frac{Ac_1}{2B^2} - \frac{c_1}{3Br} - r\frac{A(B^2X_0 + Ac_1)}{B^3} + r^2\frac{3A^2B^2X_0 + 2B^4c_2 + 2A^2(B^2X_0 + Ac_1)\ln|B/r + A|}{2B^4}, \tag{20}$$

where c_i are constants. Requiring a SdS-type solution, we must choose $X_0 = 1$, which then sets $A = 0$ and we may write also $c_1 = 6BM$, reducing Eq. (20) to $s(r) = 1 - 2M/r + c_2r^2$ with constant curvature. If we were to choose $c_1 = 0$ instead, the mass term would be absent. It is, however, unclear whether these solutions correspond to maximally symmetric (spatial) spaces or to some other type of spherically symmetric (but nonsingular) cases.

Requiring that the SdS-type metric is a solution is hence equivalent to requiring constant scalar curvature and the conclusions of the previous section apply.

As in the constant curvature case, time can always be rescaled so that $X_0 = 1$. [Alternatively we can choose the time scaling of $s(r)$ so that $X_0 + Ac_1/2B^2 = 1$, which generally leads to $X_0 \neq 1$. For examples, see Sec. III D.] Taking $X_0 = 1$ we find that for small values of the radial coordinate r, the leading terms of the general solution (20) read $s(r) \sim 1 + Ac_1/2B^2 - c_1/3\,\mathrm{Br}$ where we again identify $c_1 = 6BM$, leading to the correct form of Newtonian potential. However, additional corrections to the geodetic motion appear because in general $Ac_1/2B^2 \neq 0$, giving rise to additional parameter constraints. In the large r limit, the leading contribution comes from the r^2-term. It should be noted, however, that $s(r)$ may have large finite zeros like in the SdS solution, making the limit $r \to \infty$ physically uninteresting.

C. Asymptotic solutions

In order to have a better handle on the question of uniqueness of the solutions, we consider asymptotic solutions for which $s(r) \to 1$ at large r, mimicking the standard Schwarzschild solution. Our *Ansätze* are

$$s(r) = 1 - 2M/r \qquad F(r) = F_0 r^n.$$

Inserting these into the modified Einstein's equations, by requiring that the highest order term vanishes, one finds that $n = 0$ or $F(r) = \mathrm{const}$ is the leading term in F. Hence, there are no new solutions that tend to constant scalar curvature in the large r limit along with $s(r)$, suggesting that any new solutions will be radically different from the Schwarzschild (de Sitter) solution.

D. Exact solutions

We have also found a number of exact solutions to Eq. (14) and, by considering different *Ansätze* for $s(r)$ or $F(r)$, one can easily find and construct more solutions. A number of interesting solutions along with the corresponding forms of $f(R)$ are

	$s(r)$	$F(r)$	$X(r)$	$f(R)$	R
I	$1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2$	$1 - \frac{1}{3M} r$	2	$R \pm \frac{2}{3M}\sqrt{-R - 2\Lambda} + \Lambda$	$-\frac{1}{r^2} - 2\Lambda$
II	$1 - \frac{1}{3}\Lambda r^2$	$F_0 r$	2	$\pm 2\sqrt{-R - 2\Lambda}$	$-\frac{1}{r^2} - 2\Lambda$
III	$s_0 r^m$	$F_0 r^n,\ n = 2\frac{m(m-1)}{m-2}$	$-2\frac{n^4 - n^3 - 4n - 2}{(n+2)^2} s_0 r^m$	$\frac{2F_0}{2-n}\left(\frac{3n(2-n)}{n^2-2n-2}\right)^{n/2} R^{1-n}$	$\frac{3(2-n)n}{n^2-2n-2} r^{-2}$

These solutions, in particular *I* and *II*, could be considered as suitable asymptotic limit, either $R \to 0$ or $R \to -\infty$, of a more general $f(R)$ having linear $f(R) \propto R$ term.

IV. DISCUSSION AND CONCLUSIONS

We have seen that the set of Einstein's equations reduces to a single nonlinear differential equation relating $s(r)$ and $F(r)$. In spite of the complicated form, a number of solutions is straightforwardly found. The applicability of the general solutions could probably be tested e.g. by exploiting parameter constraints appearing from the solar system and comparing these with those arising from cosmology.

Considering the SdS solution that is present in a large class of models, e.g. the $R - \mu^4/R$ model with $R_0^2 = 3\mu^4$ or $\Lambda^2 = 3\mu^4/16$, the parameter Λ can be constrained by a number of different observations in the solar system. Such are the gravitational redshift measurements, gravitational time delay measurements by the Cassini spacecraft, and the perihelion shift of Mercury (see e.g. [27]). The tightest constraint arises from the perihelion shift of Mercury, for which it is found [27] that $|\Lambda| < 10^{-41}\ \mathrm{m}^{-2}$ (the cosmologically observed value is roughly $10^{-52}\ \mathrm{m}^{-2}$). The solar system observations are hence not able to effectively constrain such a metric compared to the cosmologically relevant values. For the other solutions we have found, solar system observations are likely to be more efficient (e.g. for $X \neq 1$).

In terms of the equivalent scalar-tensor theory, the SdS solution corresponds to a constant field solution. It is straightforward to see that the effective scalar mass, $m^2 = V''(\phi)$, is positive when $f'(R_0)/f''(R_0) - 2f(R_0)/f'(R_0) > 0$. This is equivalent to requiring that the vacuum state is stable with respect to small perturbations [16]. However, it is not at all clear what, if any, role the effective scalar plays since the metric solution is independent of the scalar mass. This question will be addressed in further work.

Important questions in addressing the validity of the metric solutions presented here are the stability and uniqueness of the solution. For example, in order for the SdS metric to be physically relevant, it must be stable with respect to small perturbations; i.e. instead of a test mass, one needs to consider the effect of a massive body on the metric. Uniqueness of $f(R)$ is also an interesting question assessing the physical relevance of a solution. Asymptotic considerations indicate that Schwarzschild-type metrics lead to constant curvature suggesting that new solutions will deviate strongly from the standard Schwarzschild metric.

By considering the boundary conditions of the general SdS solution, we have noted that the Birkhoff theorem is not valid for nontrivial $f(R)$ theories as the solutions around spherically symmetric mass distribution depends on the shape of the distribution. Hence, the straightforward Schwarzschild-ian relation between the gravitational effect and total energy of a finite mass distribution has been broken. By requiring that it holds, we are restricted to some class of special mass distributions. Also we are led to ask which distributions are physically relevant and what kind of distributions are likely to form from collapsing matter. These most interesting but also technically extremely tricky questions certainly require further examination because they may offer additional constraints on allowed $f(R)$ models.

Our results show that, in addition to the SdS metric, $f(R)$ theories typically also have new different solutions. Although further work is needed to determine their physical relevance, they offer an interesting new avenue of research that can guide us in assessing the significance of $f(R)$ theories of gravity as a possible solution to the dark energy problem.

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