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High energy behavior of gravity at large N

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A first step in the analysis of the renormalizability of gravity at large **N** is carried out. Suitable resummations of planar diagrams give rise to a theory in which there is only a finite number of primitive, superficially divergent, Feynman diagrams. The mechanism is similar to the one which makes the 3D Gross-Neveu model renormalizable at large **N**. The connections with gravitational confinement and Kawai-Lewellen-Tye relations are briefly analyzed. Some potential problems in fulfilling the Zinn-Justin equations are pointed out.

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I. INTRODUCTION

The quantum theory of renormalizable interactions plays the main role in our present understanding of fundamental physical laws. It gave rise to the formulation of the standard model of elementary particles which is still widely used today. Of course, there are many problems which have not been solved yet (such as confinement in QCD); some of these problems are likely to be "technical" problems in the sense that a better understanding of the actual standard theory should be enough to give the correct solutions. As far as other problems (such as the hierarchy problem, the cosmological constant, the quantum version of the gravitational interaction, and so on) are concerned, the standard theory is likely to be inadequate. In particular, the lack of a precise understanding of nonperturbative phenomena occurring in the strongly coupled phase of gravity (which should clarify important and still poorly understood features of early cosmology) is unpleasant: since gravity is perturbatively nonrenormalizable, it is not possible to make detailed predictions in such a phase. The two main candidates for the final theory of quantum gravity, *superstring theory* and *loop quantum gravity* (which are not necessarily to be thought of as mutually exclusive), are still too complicated to be fully understood. Thus, it is worth exploring new ways in which "physical effects beyond the standard model" could already be manifest in low energy (low with respect to the Planck scale) physics. The dominant point of view of string theorists is to see the Einstein-Hilbert action as an effective action in which the "heavy degrees of freedom" have been integrated out. Unfortunately, in order to make this idea predictive and to clarify how it is possible to improve in practice the UV behavior of gravity, it is important to understand in more detail the dynamics of such degrees of freedom and, at the present stage, this is a rather difficult task.

It is not enough to say that "gravity is an effective field theory": even if this is the case, to perform meaningful computations in the strongly coupled regime with the technical tools at our disposal (perturbative expansions of various kind, renormalization groups, and so on), physical effects "beyond the standard model" (related, for instance, to string theory) which enable such meaningful (and, hopefully, predictive) computations still have to be clarified. To be more specific, string theory predicts various types of geometrical corrections to the "bare" Einstein-Hilbert action:

$$S_{\rm corr} = S_{\rm EH} + \sum_n (\alpha')^n \int_M f_n(R_{\mu\nu\rho\sigma}) d\mu$$

where $S_{\rm EH}$ is the Einstein-Hilbert action, α' is the string length, and f_n are higher order curvature invariants. Even if a large but finite number of terms are added, the corrected action remains nonrenormalizable and, consequently, it is not yet clear how to perform meaningful quantum computations (unless one could sum the whole series: a hopeless task indeed). In a sense, the analysis of the α' corrections as a tool to shed light on nonperturbative phenomena in gravity is like the analysis of the standard perturbation expansion in QCD to understand confinement. Clearly, in QCD there is little hope to understand nonperturbative phenomena with ordinary perturbation theory. In the same way, in gravity one should expect physics beyond the standard model to manifest itself in a different, perhaps more subtle, way. Renormalizability is not a mere aesthetic requirement; it is, in fact, the need to have a theory which is predictive in the strongly coupled phase: one should expect (it is better to say "hope") that physics beyond the standard model will improve the Einstein-Hilbert gravity precisely in this direction.

A sound theoretical framework is the *holographic principle* introduced in [1,2] (for two detailed reviews, see [3,4]) which is at the basis of the string-theoretical correspondence between supergravity in anti-de Sitter space and conformal field theory on its boundary (henceforth AdS/CFT) [5]. In a recent paper [6] a large **N** expansion for the gravitational interaction was formulated which shed new light on the relations between higher spins, the holographic

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principle, and nonperturbative phenomena such as (a sort of) gravitational confinement. In **SU**(**N**) *gauge theory*, the large **N** expansion introduced by 't Hooft in [7,8] and refined by Veneziano [9] is indeed one of the most powerful nonperturbative techniques available to investigate the strongly coupled phase. (It provides the issues of *confinement*, *chiral symmetry breaking*, and the relation with *string theory* with a rather detailed understanding; a clear analysis of the role of baryons at large **N** has been given in [10]: for two detailed pedagogical reviews, see [11].)

One of the main properties of the large **N** expansion is that many nontrivial models [such as the $O(N)$ ϕ^4 model in five space-time dimensions and the Gross-Neveu model in three space-time dimensions which are not renormalizable in the standard perturbative expansion] become, in fact, renormalizable (see, for example, [12]): the Green functions are not analytic anymore in the small coupling constant region (see, for example, [13–15]). Thus, the large **N** expansion can be seen as a strong coupling expansion which, besides clarifying nonperturbative phenomena such as *confinement* and *chiral symmetry breaking*, greatly improves the UV behavior of theories that, at first glance, would appear meaningless at high energies. Here, it will be argued that, under very reasonable assumptions, this could also happen in gravity. The large **N** expansion suggests suitable resummations of planar diagrams which lead to a UV softening of gravity: the relation between the Newton constant and the mass of higher spin field(s) seems to be quite similar to the relation between the Fermi coupling constant and the mass of the $W_{\pm}$ bosons of the electroweak interactions.

The paper is organized as follows: in Sec. II the diagrammatic formulation of general relativity as a constrained topological theory (the topological theory suitable to describe gravity is the BF theory whose name comes from the fact that the principal fields are a differential 2-form called B and a connection 1-form A whose curvature is called F) is shortly described and the well-known result about the perturbative nonrenormalizability of gravity is described in this formalism. In Sec. III, the large **N** resummations which improve the UV behavior of gravity are introduced: for the sake of clarity, in Sec. III the complications connected to ghosts will be neglected keeping manifest, however, the main physical features leading to the UV softening. In Sec. IV, the ghost effects which could prevent the UV softening are analyzed. In Sec. V, a possible physical interpretation of the large **N** "UV softening" is proposed and the connections with the Kawai-Lewellen-Tye (KLT) relations equation are pointed out. Eventually, the conclusions are drawn.

II. BF GRAVITY AND FEYNMAN RULES

In this section the BF formulation of gravity and the corresponding Feynman rules will be briefly described.

The topological BF theory [16] in four dimensions is defined by the following action:

$$S[A,B] = \int_M B^{IJ} \wedge^* (F_{IJ}(A)) = \frac{1}{4}\int_M \varepsilon^{\alpha\beta\gamma\delta} B^{IJ}_{\alpha\beta} F_{\gamma\delta IJ} d^4x, \qquad (1)$$

$$B^{IJ} = \tfrac{1}{2} B^{IJ}_{\alpha\beta} dx^\alpha \wedge dx^\beta, \qquad F_{IJ} = \tfrac{1}{2} F_{\alpha\beta IJ} dx^\alpha \wedge dx^\beta,$$
$$F_{\alpha\beta IJ} = (\partial_\alpha A_\beta - \partial_\beta A_\alpha)_{IJ} + A^L_{\alpha I} A_{\beta LJ} - A^L_{\beta I} A_{\alpha LJ}, \qquad (2)$$

where M is the four-dimensional space-time, $*$ is the Hodge dual, the Greek letters denote space-times indices, $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally skew-symmetric Levi-Civita symbol in four-dimensional space-times, and I, J, and K are the internal Lorentz indices which are raised and lowered with the Minkowski metric η_{IJ}: $I, J = 1, \ldots, \mathbf{N}$. Thus, the basic fields are a $so(\mathbf{N}-1,1)$-valued differential 2-form B_{IJ} and a $so(\mathbf{N}-1,1)$ connection 1-form $A_{\alpha LJ}$, the internal gauge group being $SO(\mathbf{N}-1,1)$. Also, the Riemannian theory can be considered in which the internal gauge group is $SO(\mathbf{N})$ and the internal indices are raised and lowered with the Euclidean metric δ_{IJ}; in any case, both B_{IJ} and $A_{\alpha LJ}$ are in the adjoint representation of the (algebra of the) internal gauge group. The equations of motion are

$$F = 0, \qquad \nabla_A B = 0, \qquad \nabla_A = \nabla = d + [A,.] \qquad (3)$$

where ∇_A is the covariant derivative with respect to the connection $A_{\alpha LJ}$. The above equations tell us that $A_{\alpha LJ}$ is, locally, a pure gauge and B^{IJ} is covariantly constant. When $\mathbf{N} = 4$ and B^{IJ} has the form

$$B^{IJ} = \tfrac{1}{2}\varepsilon^{IJ}_{KL} e^K \wedge e^L, \qquad (4)$$

the action (1) is nothing but the Palatini form of the Einstein-Hilbert action. Equation (4) can be enforced by adding to the action (1) a suitable constraint: the basic action in the BF formalism is

$$GS_{\rm GR} = S[A,B] - \int_M (\phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H(\phi)), \qquad (5)$$

where G is the gravitational coupling constant, μ is a fixed differential 4-form, and $H(\phi)$ is a scalar which may have one of the following expressions:

$$H_1 = \phi^{IJ}_{IJ}, \qquad H_2 = \phi_{IJKL}\varepsilon^{IJKL}, \qquad H_3 = a_1 H_1 + a_2 H_2, \qquad (6)$$

with a_i being real constants (see [17–21]). It is worth noting here that the Lagrange multiplier ϕ has four internal indices. The form (5) of the Einstein-Hilbert action is a natural starting point to formulate the "gravitational" large **N** expansion [6] since the connection formulation allows us to adopt the double-line notation. (However, the fundamental representation of $so(\mathbf{N}-1,1)$ being real, the lines of internal indices carry no arrows.)

The classical action in Eq. (5) is left invariant by $so(\mathbf{N}-1,1)$-gauge transformations and by diffeomorphisms. The analysis of the Becchi-Rouet-Stora-Tyutin (henceforth BRST) invariance (which, as it is well known, is the quantum counterpart of the classical gauge invariance) of the gravitational action in the BF formalism can be found in [22]. As far as the scope of the present paper is concerned, a technical complication is that the natural kinetic term [see Eq. (14) below] is invariant under a further transformation

$$\delta_3 A_\mu = 0, \qquad \delta_3 B_{\mu\nu} = \nabla_{[\mu}\omega_{\nu]}$$

[where ω_ν is a $so(\mathbf{N}-1,1)$-valued 1-form] which has to be gauge fixed too in order to derive the propagators (a detailed discussion of this issue can be found in [23,24]).

The most suitable way to proceed is to follow [24] in which the authors (in the case of the BF formulation of Yang-Mills theory) introduced an auxiliary nonphysical field η_μ (a Lie-algebra-valued 1-form), in the combination

$$B' = B - \nabla\eta,$$

whose role is to keep, at the same time, both the local degrees of freedom of Yang-Mills theory and the symmetries of the BF theory (physically, η represents the longitudinal components of B). It is worth noting here that, because of the Bianchi identities, one has

$$S[A, B'] = S[A, B].$$

In the Yang-Mills case, this procedure is lawful as is because the (classical) action of the BF Yang-Mills theory is

$$S^{\mathrm{cl}}_{\mathrm{YM}} = S[A,B] - e^2 \int_M \mathrm{tr}(B')_{\mu\nu}(B')^{\mu\nu} \tag{7}$$

(where e is the Yang-Mills coupling constant). In gravity, the second term on the right-hand side of the above equation is replaced by the constraint in Eq. (5),

$$\int_M (\phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H(\phi)).$$

The consequence is that, in order to produce the Yang-Mills kinetic term for η, it is convenient to add the second term on the right-hand side of Eq. (7) to the gravitational action. This is possible since it has been shown in [23,24] that the small e limit does not present any problem (the theory is "perturbative" in e). In other words, the second term on the right-hand side of Eq. (7) can be regarded as a true vertex: therefore, one can consider the gravitational case as the small e limit of the action [25] (8).

Indeed, η has only a technical role since it just represents the longitudinal components of B, and one of η's transformation laws is given by a shift [see below, Eq. (10)]. However, at this stage of the analysis, it seems to be unavoidable to add the Yang-Mills term [the second term on the right-hand side of Eq. (7)] in order to obtain the Feynman rules, vertices, and propagators. On the other hand, due to the comparison with Yang-Mills theory, this scheme is mandatory if one wants to clearly identify the "guilty of the perturbative nonrenormalizability of gravity" in the BF scheme: but, for this enlargement of the gravitational action, it would not be possible to "large **N**" improve the UV behavior of gravity.

The classical symmetries of the "enlarged" classical BF gravitational action

$$GS_{\mathrm{cl}} = S[A,B] - \int_M (\phi_{ab}(B')^a \wedge (B')^b + \mu H(\phi)) - e^2 \int_M \mathrm{tr}(B')^a (B')_a \tag{8}$$

[where a, b, c, and so on are indices in the adjoint representation of $so(\mathbf{N}-1,1)$] are

$$\delta_1 A_\mu = (\nabla\theta_{(1)})_\mu, \qquad \delta_1 B_{\mu\nu} = [B_{\mu\nu}, \theta_{(1)}], \qquad \delta_1 \eta_\mu = [\eta_\mu, \theta_{(1)}], \tag{9}$$

$$\delta_2 A_\mu = 0, \qquad \delta_2 B_{\mu\nu} = \nabla_{[\mu}\omega_{\nu]}, \qquad \delta_2 \eta_\mu = \omega_\mu, \tag{10}$$

$$\delta_3 A_\mu = 0, \qquad \delta_3 B_{\mu\nu} = [F_{\mu\nu}, \theta_{(3)}], \qquad \delta_3 \eta_\mu = \nabla_\mu \theta_{(3)}, \tag{11}$$

where $\theta_{(i)}$ are $so(\mathbf{N}-1,1)$-valued gauge scalars. δ_1 is a simple gauge transformation so that the action is invariant. As far as δ_2 and δ_3 are concerned, the transformations of B cancel out the transformations of η in the second and in the third terms on the right-hand side of Eq. (8). The BF term is left invariant by δ_1 because of the Bianchi identities and by δ_2 because it reduces to a trivial gauge transformation on the usual F^2 term of (the standard formulation of) Yang-Mills theory. It is worth noting that the symmetries δ_2 and δ_3 are reducible, as it is clear if one considers in Eqs. (10) and (11)

$$(\nabla\theta_{(3)})_\mu = \omega_\mu.$$

To obtain the Feynman rules, the gauge-fixing and ghost terms related to the above symmetries have to be included. A convenient gauge-fixing term is

$$\partial_\mu A^\mu = 0, \qquad \partial_\mu \eta^\mu = 0, \qquad \partial_\mu B^{\mu\nu} = 0 \tag{12}$$

so that the corresponding gauge-fixing term of the action is

$$\begin{aligned} S_{gf} = \int_M & \{\bar{c}(-\partial_\mu \nabla^\mu)c + h_A(\partial_\mu A^\mu) \\ & + \bar{\psi}^\nu \partial^\mu \{-[B_{\mu\nu}, c] + \nabla_{[\mu}\psi_{\nu]} + [F_{\mu\nu}, \rho]\} \\ & + h_B(\partial_\mu B^{\mu\nu}) + \bar{\rho}\partial^\mu\{-[\eta_\mu, c] + \psi_\mu + \nabla_\mu \rho\} \\ & + h_\eta(\partial_\mu \eta^\mu) + u(\partial^\mu h_B) + h_{\bar{\psi}}(\partial_\nu \bar{\psi}^\nu) \\ & + \bar{\xi}^\mu \partial\{[\psi_\mu, c] + \nabla_\mu \xi\} + h_\psi(\partial_\nu \psi^\nu)\}, \end{aligned} \tag{13}$$

where $(c, \bar{c}, h_A)$, $(\psi, \bar{\psi}, h_B)$, and $(\rho, \bar{\rho}, h_\eta)$ are, respectively, the ghost, the antighost, and the Lagrange multiplier for δ_1 and δ_2; $(\xi, \bar{\xi}, h_\psi)$ the ghost, the antighost and the Lagrange multiplier for the zero mode of the topological symmetry δ_3 and $(u, h_{\bar{\psi}})$ a pair of fields which takes into account a further degeneracy associated with $\bar{\psi}$. It is worth stressing here that all the fields appearing in the gauge-fixing term (13) are in the adjoint representation of the gauge group: this will be important when the role of the ghost loop effects (which, as it will be shown in the next section, do not prevent the UV softening) will be discussed. The following tables summarize the ghost numbers and dimensions of the fields:

Fields	A	B	η	c	$\bar{c}$	ψ	$\bar{\psi}$	h_A	h_B
Dimension	1	2	1	0	2	1	1	2	1
Ghost number	0	0	0	1	-1	1	-1	0	0

Fields	ξ	$\bar{\xi}$	h_ψ	ρ	$\bar{\rho}$	h_η	u	$h_{\bar{\psi}}$
Dimension	0	2	2	0	2	2	2	2
Ghost number	2	-2	-1	1	-1	0	0	1

The natural choice is to consider, as the Gaussian part of the fields A and B, the off-diagonal kinetic term

$$S_0 = \frac{1}{\kappa}\int_M (\varepsilon^{\alpha\beta\gamma\delta} B^a_{\alpha\beta}\partial_\gamma A_{\delta a}), \tag{14}$$

plus the quadratic terms for ghosts in the gauge-fixing term (13). The $A \rightarrow B$ propagator (which propagates A_μ into $B_{\nu\gamma}$) has the following structure (a simple method to find them can be found in [26,27]):

$$\Delta^{(a,b)}_{(A,B)\mu\nu\gamma} = -\delta^{ab}\frac{1}{2}\varepsilon_{\mu\nu\gamma\alpha}\frac{p^\alpha}{p^2}. \tag{15}$$

The internal index structures of the propagators tell us that, as one should expect, the internal index is conserved along the gravitational internal color lines. The $A \rightarrow A$, $B \rightarrow B$ and the $\eta \rightarrow \eta$ propagators are

$$\begin{aligned} \Delta^{(a,b)}_{(A,A)\mu\nu} &= \delta^{ab}\frac{1}{p^2}\left(\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right), \\ \Delta^{(a,b)}_{(B,B)\mu\nu\gamma\rho} &= -\delta^{ab}\varepsilon_{\mu\nu\alpha\lambda}\varepsilon_{\gamma\rho\beta\lambda}\frac{p^\alpha p^\beta}{p^2}, \end{aligned} \tag{16}$$

$$\Delta^{(a,b)}_{(\eta,\eta)} = \left(\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\delta^{ab}. \tag{17}$$

The ghost propagators can be deduced from the gauge-fixing term in Eq. (13):

$$\begin{aligned} \Delta^{(a,b)}_{(\psi,\bar{\psi})\mu\nu} &= -i\delta^{ab}\frac{1}{p^2}\left(\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right), \\ \Delta^{(a,b)}_{(c,\bar{c})} &= -i\delta^{ab}\frac{1}{p^2}, \\ \Delta^{(a,b)}_{(\xi,\bar{\xi})} &= -i\delta^{ab}\frac{1}{p^2}, \qquad \Delta^{(a,b)}_{(\rho,\bar{\rho})} = -i\delta^{ab}\frac{1}{p^2}. \end{aligned}$$

Graphically, all the propagators in the double-line notation are represented by two parallel internal "gravitational" color lines [along which the internal $so(\mathbf{N}-1, 1)$ index is conserved] with no arrows [7,9].

The coupling with matter fields (as discussed in [6]) shows that the number of internal "gravitational color" lines associated with each matter field is connected with its spin: the higher the spin, the more internal lines are needed [that is, the higher the representation of the internal gauge group $so(\mathbf{N}-1, 1)$ is] to describe the matter field in the double-line notation (so that the Lagrange multiplier field ϕ should be considered as a nonpropagating higher spin field). For the sake of simplicity, in this paper the purely gravitational case will be considered; however, the inclusion of matter fields should not destroy the main conclusions since, usually, matter fields do not worsen the UV behavior.

The theory has the following matter vertices:

$$V_1(A^a_\mu, A^b_\nu, B^c_{\alpha\beta}) = Gf^{abc}\varepsilon_{\mu\nu\alpha\beta}, \tag{18}$$

$$V_2(B^a_{\mu\nu}, B^b_{\alpha\beta}, \phi^{cd}) = G\delta_{ac}\delta_{bd}\varepsilon^{\mu\nu\alpha\beta}, \tag{19}$$

where f^{abc} are the structure constants (the Newton constant G, as usual, has been absorbed in the fields in such a way that it appears only in the vertices which are depicted in Fig. 1). The first vertex is also present in the BF formulation of Yang-Mills theory while the second one only pertains to general relativity and is likely to be the main vertex responsible for the quantum realization of the holographic principle [6]. The ghost vertices in the gauge-

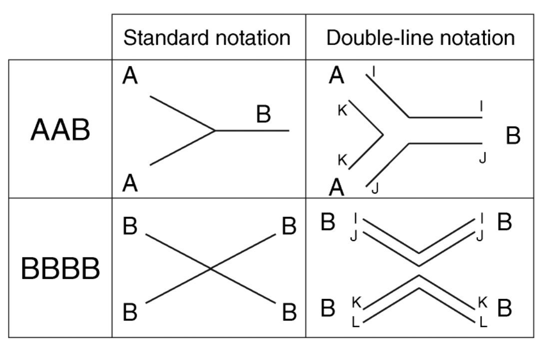

FIG. 1. Here, the double-line structures of the two matter vertices are displayed: the AAB vertex is in common with the BF-Yang-Mills theory; the 4-uple B vertex is peculiar of gravity.

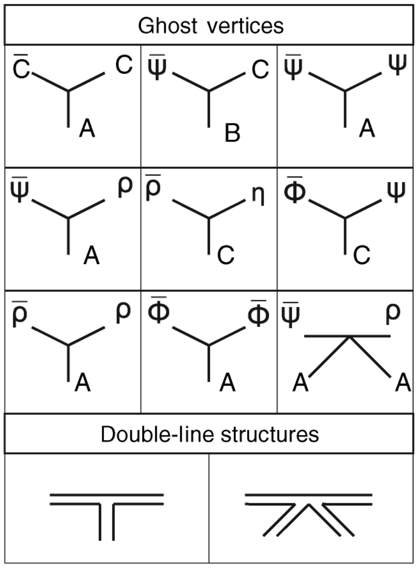

FIG. 2. In this picture the ghost vertices have been drawn: they have only two types of connected internal index structures.

fixing term are

$$V_3(A^a_\mu, c^b, \bar{c}^d) = -Gf^{abd}p^\mu,$$
$$V_4(\bar{\psi}^{a\sigma}, B^b_{\gamma\nu}, c^d) = -Gf^{abd}p^\nu\delta^\gamma_\sigma, \tag{20}$$

$$\frac{V_5(A^a_\mu, \psi^b_\nu, \bar{\psi}^d_\sigma)}{G} = -f^{abd}p^{[\mu}\delta^{\nu]\sigma},$$
$$\frac{V_6(A^a_\mu, \rho^b, \bar{\psi}^d_\sigma)}{G} = f^{abd}p^{[\mu}\delta^{\nu]\sigma}p_\nu, \tag{21}$$

$$V_7(A^a_\mu, \rho^b, \bar{\rho}^d) = -Gf^{abd}p^\mu,$$
$$V_8(c^a, \eta^b_\nu, \bar{\rho}^d) = Gf^{abd}\varepsilon_{\mu\nu\alpha\beta}, \tag{22}$$

$$V_9(A^a_\mu, \xi^b, \bar{\xi}^d) = Gf^{abd}p^\mu,$$
$$V_{10}(c^a, \psi^b_\mu, \bar{\xi}^d) = Gf^{abd}p^\mu \tag{23}$$

$$V_{11}(A^m_\omega, A^n_\tau, \rho^b, \bar{\psi}^d_\sigma) = G^2 f^{abd}p^{[\mu}\delta^{\nu]\sigma}\delta^\tau_{[\nu}\delta^\omega_{\mu]}f^{mna}. \tag{24}$$

It is worth noting that the internal index structures of the ghost vertices are standard, that is, they have a connected structure (see Fig. 2): this fact will play an important role in the following.

The aim of this paper is to show that, in the large **N** expansion, there is only a finite number of superficially divergent diagrams. However, large **N** resummations (even if they improve the UV behavior of the theory) could give rise to some problems in fulfilling the Zinn-Justin equation which is needed to ensure that the infinities can be absorbed in counterterms not violating the symmetries of the original action. This point will be discussed in slightly more detail later on.

The "nonrenormalizable" vertex

The perturbative nonrenormalizability of (super)gravity [28] was an important result: even if, at first glance, this could be rather obvious by power counting, there are many examples (such as gravity in three dimensions [29]) of theories which are trivially renormalizable (being exact in the cohomology of the operator associated with the BRST symmetry) and, in fact, would not appear in this way by naive power-counting arguments. Moreover, the results in [30] about the one-loop finiteness of the Einstein-Hilbert action lead to the expectations that the powerful symmetries of gravity could give rise to some "miracle," at least in supergravity: indeed, in the standard perturbative formulation, such a miracle does not occur. It is important to understand this result in the present formalism; the question is, which is the "wrong" vertex responsible for the perturbative nonrenormalizability of the theory? The answer is as follows: if one would drop the vertex in Eq. (19), one would obtain the topological BF Yang-Mills theory which, obviously, is renormalizable. Thus, the vertex responsible for the perturbative nonrenormalizability of the theory is the one in Eq. (19). In gravity, the problem is mainly connected to the 4-uple B vertex. Indeed, B has bad asymptotic behavior [see Eq. (16)]: such a propagator is also present in the BF Yang-Mills theory [27]. However, in the Yang-Mills case, the B field is always attached to the better-behaved A field through the good AAB vertex. Consequently, loops with only B fields cannot arise. Such loops in which only the B fields appear are at the origin of the perturbative nonrenormalizability of gravity. The 4-uple vertex for B (see the left column of Fig. 5 in which there is a typical nonrenormalizable diagram which cannot arise in the BF formulation of Yang-Mills theory) gives rise to badly behaved in the UV loops in which only B fields appear: at any order in perturbation theory, new diagrams diverging in the UV appear in the expansion.

III. LARGE N RESUMMATIONS AND EFFECTIVE PROPAGATORS

Here, it will be shown that the large **N** expansion suggests a useful resummation of a certain class of planar diagrams which greatly improves the UV behavior of gravity. For the sake of clarity, we will first present a simplified version of the argument [31] (in which, however, the main physical ideas are manifest) leading to the UV improvement. In the next section, the effects of ghosts in the loops will be discussed.

It is worth briefly recalling what happens in the large **N** expansion of the 5D $O(\mathbf{N})$ ϕ^4 model [12]. The Lagrangian is

$$L_\phi = \frac{1}{2}(\partial_\mu\phi_j\partial^\mu\phi^j + m^2\phi_j\phi^j) + \frac{\lambda}{8}(\phi_j\phi^j)^2,$$
$$j = 1, \ldots, \mathbf{N}, \tag{25}$$

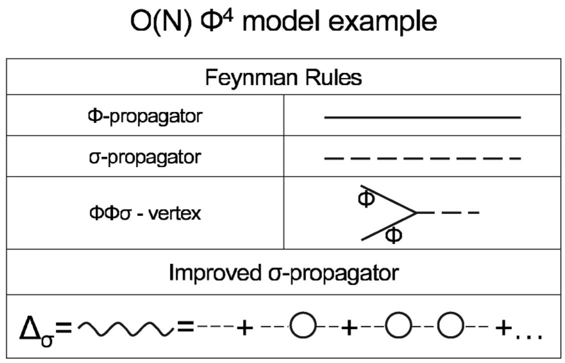

FIG. 3. In this picture the large **N** improvement of the scalar $\phi^4 - O(N)$ model is depicted: the constant propagator of the Lagrange multiplier is UV softened since the leading large **N** contribution is given by the geometric sum of bubble diagrams.

ϕ_j being an N-component scalar field, λ the coupling constant, and m the mass. It is convenient to introduce a Lagrange multiplier field σ as follows:

$$L_{\phi,\sigma} = \tfrac{1}{2}(\partial_\mu \phi_j \partial^\mu \phi^j + (m^2 + i\sigma\sqrt{\lambda})\phi_j\phi^j) + \tfrac{1}{2}\sigma^2. \quad (26)$$

Of course, due to the equation of motion of σ, this Lagrangian is equivalent to the previous one. In the momentum space, the bare propagator $D_\sigma(k)$ of σ is constant:

$$D_\sigma(k) = 1.$$

The Lagrangian in Eq. (26) in five space-time dimensions gives rise to a nonrenormalizable theory. On the other hand, by the taking into account that at large **N** the dominating diagrams are the well-known *bubble diagrams*, it is possible to obtain an improved propagator $\bar{D}_\sigma(k)$ at the leading order in $1/\mathbf{N}$ (see Fig. 3):

$$\bar{D}_\sigma(k) = \frac{1}{1 + g\Pi(k)}, \qquad \Pi(k) \underset{k\to\infty}{\longrightarrow} k, \qquad g = \lambda\mathbf{N}.$$

Such an effective propagator improves the UV behavior of the theory in such a way that there are only three superficially divergent diagrams and the theory becomes renormalizable in the $1/\mathbf{N}$ expansion. This miracle happens in the following way: many diverging diagrams in the standard perturbative expansion (which are of different order in the standard coupling constant λ) are, in fact, of the same order in $1/\mathbf{N}$; this fact tells us that such diagrams should be summed together (such a resummation is easy, being a geometric series). The difference with respect to the standard perturbative renormalizability is that the propagator is not analytic anymore in the region of the small effective coupling constant g (see, for instance, [13–15]).

As far as gravity is concerned, one has to cure mainly the "nonrenormalizable" vertex in Eq. (19). In the scalar $O(\mathbf{N})$ case, the large **N** expansion tells us that a suitable class of diverging diagrams (the "bubble diagrams") should be summed together; of course, from the technical point of view, the gravitational case is more difficult. However,

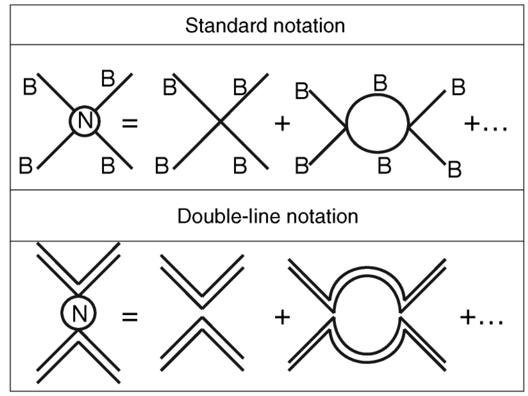

FIG. 4. In this picture the *tree-level large* **N** 4-uple B vertex has been drawn. Because of the presence of the Lagrange multiplier with four internal color indices, it is clear that the tree-level large **N** 4-uple B vertex is the (geometric) sum of an infinite number of terms which have no closed color loop but have an always-increasing number of loops of the standard Feynman expansion. Indeed, this phenomenon is peculiar of gravity.

similarities in the two cases are indeed present. It is easy to see that the dominant contributions to the 4-uple B vertex come from bubblelike diagrams which, from the internal index point of view, are "tree diagrams" (see Fig. 4): that is, they are planar diagrams with no closed color loops so that they contribute to the "large **N**-bare" vertex which is the relevant quantity as far as power-counting arguments are concerned [32]. Thus, one is simply computing the treelike term of the topological expansion at genus zero. These *large*-**N** *tree diagrams* can be formally summed as a geometric series, as will be shown in a moment.

This is very much in the spirit of the large **N** expansion [7,9]: at any fixed genus, the contributions are weighted by the effective coupling constant

$$g_{\text{eff}} = G\mathbf{N} \quad (27)$$

(kept fixed at large **N**) to the power of the number of closed color loops L and, consequently, the tree diagrams should be summed together as it happens in standard Feynman diagram calculations. To be more precise, at any given genus in the topological expansion, the large **N** diagrams are weighted by the effective coupling constant in Eq. (27). To any closed color loop in the topological expansion, it corresponds by a factor g_{eff} in the same way as the powers of the Planck constant weight the usual loops in the standard Feynman diagrams. Any interesting physical observable $\langle O\rangle$ can be expanded at large **N** as follows:

$$\langle O\rangle = \sum_{g,b} \mathbf{N}^{2-2g-b} \sum_L (g_{\text{eff}})^L O_{g,b,L} \quad (28)$$

where g is the genus, b is the number of closed matter loops, L is the number of closed color loops, and $O_{g,b,L}$ is the sum of fat diagrams contributing at genus g, with b matter loops and L closed color loops. In the same way, in the standard Feynman expansion, for physical quantities such as amplitudes, one has

$$\langle O\rangle_{(F)} = \sum_{L_F} (\hbar)^{L_F} O^{(F)}_{L_F}$$

where $\hbar$ is the Planck constant, L_F is the number of loops, and $O^{(F)}_{L_F}$ is the sum of the Feynman diagrams contributing at L_F loops: the bare propagators and vertices have, by definition, $L_F = 0$ (the similarity between the topological expansion with g and b fixed and the standard Feynman expansion is apparent). Thus, in very much the same way, in the large **N** expansion the bare propagators and vertices must not contain any closed color loops [33]. The key point is that, to count the number of primitive superficially divergent diagrams, one only needs treelike propagators and vertices (see, for instance, [34]).

UV-softening of the 4-uple B vertex

The "guilty of perturbative nonrenormalizability" 4-uple B vertex is UV softened by large **N** effects: the factor which dresses V_2 is related to the geometric "bubblelike" series in Fig. 4. It is worth stressing here that the above bubblelike series is made of "treelike large **N**" diagrams: that is, it is the sum of diagrams with no closed internal color lines ($L = 0$). A peculiar feature of gravity is the possibility of constructing a treelike large **N** quantity (which, therefore, enters at the leading order in the large **N** expansion) which, in fact, contains an infinite number of standard Feynman loops. In the large **N** expansion of Yang-Mills theory, the treelike large **N** vertices and propagators coincide with the standard treelike Feynman vertices and propagators since the vertices have connected structures.

In the standard Feynman expansion, the basic building blocks are treelike vertices and propagators (that is, vertices and propagators without Feynman loops): starting from these vertices and propagators, well-known arguments (and, in particular, the Bogoljubov-Parasiuk-Hepp-Zimmermann theorem) tell us whether or not the perturbation expansion has a finite number of primitive, superficially divergent diagrams. In very much the same way, the analogous question in the large **N** expansion has to be answered by looking at the treelike large **N** vertices and propagators [35] [as it is clear from well-studied $O(\mathbf{N})$-vectorial examples: see, for instance, [12]]. Thus, to understand whether or not gravity in the large **N** expansion has a finite number of primitive, superficially divergent diagrams, one has to use treelike large **N** vertices and propagators: thus the 4-uple B vertex (which is a treelike quantity in the Feynman expansion) has to be replaced by the bubblelike series in Fig. 4 which correctly accounts for all the treelike large **N** contributions.

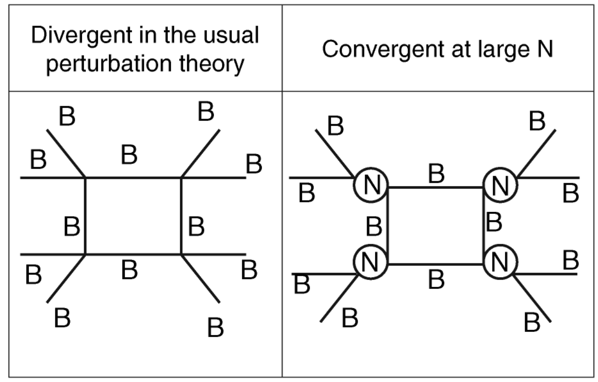

FIG. 5. In this picture a typical diagram is shown which is divergent in the usual Feynman expansion and is, in fact, convergent at large **N**. It is manifest that three "large **N** improved" 4-uple B vertices are enough to make a loop integral UV finite. This should be compared with standard loop integrals with only B propagators which trigger the perturbative nonrenormalizability of gravity; such loops are no longer a problem in the large **N** expansion.

Any potentially dangerous 4-uple B vertex is dressed by a factor vanishing in the UV as $1/q^2$. In Fig. 5 we depict a diagram which diverges in the standard perturbative expansion and is finite in the large **N** improved expansion. This UV improvement appears to be quite consistent with the Weinberg *asymptotic safety scenario* [36]. The large **N** dressed vertex is

$$\mathbf{V}_{4B}(q) = V_2 \frac{1}{1 - \Pi(q^2)}, \tag{29}$$

where $\Pi(q^2)$ is the B self-energy giving rise to the geometric series:

$$\Pi(q^2) = G^2 \int_\Lambda d^4p \Delta_{(B,B)}(p)\Delta_{(B,B)}(p-q) = G^2(\varepsilon^4)^{\xi\eta}_{\alpha\beta} \int_\Lambda d^4p \frac{p^\alpha p^\beta (p-q)_\xi (p-q)_\eta}{p^2(p-q)^2} \underset{\Lambda\to\infty}{\longrightarrow}, \tag{30}$$

$$\underset{\Lambda\to\infty}{\longrightarrow} (I_1 G^2 \Lambda^2) q^2 + (G^2 I_2 \Lambda^4) + \ldots$$

$$(\varepsilon^4)^{\xi\eta}_{\alpha\beta} = \varepsilon_{\mu\nu\alpha\lambda}\varepsilon_{\gamma\rho\beta\lambda}\varepsilon^{\mu\nu\xi\chi}\varepsilon^{\gamma\rho\eta\chi}, \tag{31}$$

where $\Delta_{(B,B)}$ is the B propagator. In the momentum integrals, a cutoff Λ (to be removed in a suitable way) has been introduced, terms which are subleading for $\Lambda \to \infty$ and $\mathbf{N} \to \infty$ have been neglected, and I_1 and I_2 are two nonvanishing real constants (whose precise values are not important as far as the present discussion is concerned so that, from now on, they will be set equal to 1). It is worth noting that, in the geometric series giving rise to $\Pi(q^2)$, no factor **N** appears since there are no closed color loops [$\Pi(q^2)$ is a sum of the large **N** tree diagrams]. It is useful to rewrite Eq. (29),

$$\mathbf{V}_{4B}(q) = V_2 \frac{M^2}{M_0^2 - q^2 + \cdots},$$
$$M_0^2 = M^2 + \delta M_0^2 = M^2 - b\Lambda^2, \qquad M^2 = \frac{1}{(G\Lambda)^2}, \tag{32}$$

where in the denominator of Eq. (32) subleading terms when $\Lambda \to \infty$ have been neglected [37]. The most convenient way to remove the cutoff is

$$\frac{\Lambda}{\mathbf{N}} \underset{\Lambda,\mathbf{N}\to\infty}{\to} \text{finite value} \Rightarrow M^2 = \frac{1}{(G\Lambda)^2} \to \text{fixed}, \tag{33}$$

where M could be interpreted as a sort of renormalized Planck mass. The divergent term $\delta M_0^2 \sim -b\Lambda^2$ has the typical form which can be removed by a tadpole contribution.

The same phenomenon also occurs in simpler models in which the large **N** technique works (see, for instance, [38]). In such models the quadratically divergent contribution to the mass is already included in the so-called *gap equation*. The gap equation accounts for the tadpole embodying it in the effective coupling constant(s). However, to do this, it is necessary to write down and solve the saddle point equations at large **N** for the full Lagrangian. In the gravitational case this seems to be rather difficult so that one is forced to remove the divergent contribution to the mass "by hand" with counterterms. The typical tadpole diagram [39] contributing to the quadratically divergent contribution to the mass is in Fig. 6: one can easily see that it is of order Λ^2 due to the UV behavior of the B propagator.

Once the quadratically divergent term in the denominator is removed [40], one gets

$$\mathbf{V}_{4B}(q) = V_2 \frac{M^2}{M^2 - q^2 + \cdots} \tag{34}$$

where, in the denominator, terms which are subleading in the large **N** limit have been neglected.

It is now possible to compute the superficial degree of divergence of the large **N** diagrams in which one simply has to use the 4-uple B vertex in Eq. (34) instead of V_2. Since any 4-uple B vertex now decreases by 2 the superficial degree of divergence Ω of a loop integral, the three vertices of the type of Eq. (34) are enough to make a loop integral convergent; thus, $\Omega \leq 4 - E$ (where E is the number of external legs in the diagrams) as in the BF–Yang-Mills case [27] (in Fig. 5 there is a meaningful example). The large **N** prescription gives rise to a gravitational theory with only a finite number of primitive, superficially divergent diagrams (in good agreement with the Weinberg asymptotic safety scenario [36]): interestingly enough, this opens the unexpected possibility to control the UV behavior gravity exploring, for instance, the inflationary phase of cosmology.

However, in theories with local symmetries, one also has to show that the infinities of the theory have the suitable symmetries which allow us to cancel such infinities by adding counterterms not violating the original symmetries of the action (see, for instance, [34]). It is not clear at this stage of the analysis if such "large **N** reorganization" prevents the fulfillment of the Zinn-Justin equation; for instance, the large **N** expansion could not commute with the BRST and Slavnov operators. This is a rather involved technical problem; I hope to return to this issue in a future publication.

It would be very interesting to compare more closely the above results with the ones obtained in a series of papers [41–43] in which the authors began the analysis of the Einstein-Hilbert in the standard metric variables using the method of the *nonperturbative renormalization group*. The authors found sound evidences supporting the existence of a nontrivial UV fixed point (which, obviously, would confirm the Weinberg asymptotic safety scenario [36]). The present results, although derived in a completely different way, seem to be consistent with such a scheme.

The above UV improvement is conceptually very similar to what happens in the change from the Fermi effective model of weak interaction to the Glashow-Weinberg-Salam model of electroweak interactions: in this case the nonrenormalizable Fermi coupling constant G_F is replaced by the $W^{\pm}$ propagators (for a detailed pedagogical exposition, see [34])

$$G_F \to \frac{g_W^2}{M_W^2 - q^2},$$

where g_W is the (dimensionless) electroweak coupling constant and M_W is the mass of the W bosons. In the gravitational case, the large **N** expansion suggests that, in the strongly coupled phase of gravity, a similar phenomenon should occur:

$$G \to G \frac{M^2}{M^2 - \Pi(q^2)}.$$

Thus, the sentence "gravity is an effective field theory" would acquire a rather precise meaning. Such a nonrenormalizability should be removed by nonperturbative effects, displayed by the large **N** expansion, which correctly take into account the heavy degrees of freedom in a way very similar to what happens in the electroweak model.

IV. POSSIBLE COMPLICATIONS

The main effects which have been neglected in the previous discussion are related to the ghosts. In particular, ghost loops could cancel out, asymptotically, the B loops neutralizing the previous UV softening. In fact, the large **N** expansion itself provides one with a natural recipe to deal with this problem.

In the gravitational case, due to the "higher spin" Lagrange multiplier ϕ_{ab}, it is possible to construct planar diagrams without closed color loops (thus contributing to

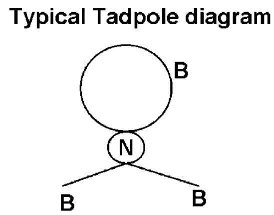

FIG. 6. In this picture the typical tadpole diagram has been drawn. In simpler models [such as the scalar $\phi^4 - O(N)$ model] one can take care of the tadpoles by using the so-called *gap equation*. In gravity, it seems difficult to write and solve the gap equation so one has to remove by hand the typical tadpole quadratic divergences.

the large **N** bare propagators) containing an infinite number of loops of the standard Feynman expansion [44]. The question is, do the ghost contributions to the 4-uple B vertex survive at tree level in the large **N** expansion in such a way as to prevent the above analyzed UV softening? In other words, is it possible to construct a contribution to the 4-uple B vertex by using the ghost vertices in Eqs. (20)–(24) *without closed internal lines*? The key point is that the only way in which ghost effects could neutralize the previously considered UV softening is by changing the treelike large **N** vertices and, in particular, the 4-uple B vertex (since the large **N** power counting only depends on treelike large **N** vertices and propagators): ghost effects have to come into play at tree level, otherwise the power counting does not change. It is easy to see that all the possible ghost-mediated corrections to the 4-uple B vertex contain at least one closed color loop [since the vertices in Eqs. (20)–(24) have connected structures] and, therefore, do not contribute to the bare large **N** propagators and vertices. Let us consider the ghost contribution to the B vertex in Fig. 7: indeed, at genus zero, it contains at least one closed color loop. More generally, in the presence of

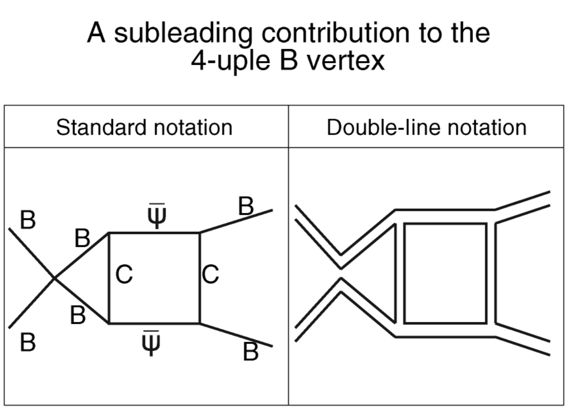

FIG. 7. In this picture a typical ghost contribution to the 4-uple B vertex has been drawn. It is clear that, in the large **N** expansion, the ghost contributions are always subleading since, due to their connected structures, they contain at least one closed color loop (consequently, they do not contribute to the *tree-level large* **N** 4-uple B vertex).

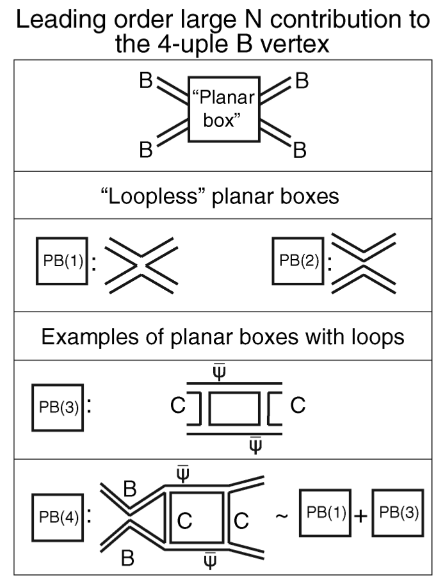

FIG. 8. In this picture the double-line structures of the possible contributions to the 4-uple B vertex have been drawn. Only two structures could contribute at the tree-level large **N** 4-uple B vertex: PB(1) (which, however, is not present in the theory) and PB(2). All other planar boxes with four external B lines have at least one closed color loop inside; therefore, they are not relevant as far as UV power counting is concerned.

fields living in the adjoint representation of the gauge group and vertices with the ghost vertices of the BF formulation of gravity, it is not possible to construct treelike large **N** diagrams without closed color loops contributing to the 4-uple B vertex (see Fig. 8). Therefore, ghost effects do not affect the UV softening of the previously considered bubblelike series.

The role of η

Up to now, the role of η has not been discussed. Its introduction (as in the Yang-Mills case) is dictated by technical reasons. To deduce the large **N** effects on the interactions of η and B, one can consider the large **N** improvements for the 4-uple vertex without separating B' in B and $\nabla\eta$. Obviously, since B and $\nabla\eta$ can be "reassembled" in a unique field, the vertices between $\nabla\eta$ and B experience, at large **N**, similar UV improvements as in Eq. (34). In other words, η represents the longitudinal components of B' so that the separation of B' in B and $\nabla\eta$ is an artifact; the physical field is $B + \nabla\eta$. This implies that one can trivially deduce the large **N** effects on the interactions between $\nabla\eta$ and B by analyzing the large **N** improvements of the effective 4-uple vertex for the physical field $B + \nabla\eta$ [45] and then separating it into B and $\nabla\eta$. On the other hand, since η can be gauged to zero (one of its transformation laws being given by a shift) its properties do not affect the UV behavior.

As was already mentioned, at this stage of the analysis, it is not clear if and how one could avoid the introduction of the auxiliary η field from the very beginning. Indeed, the introduction of η and of the "Yang-Mills" term [46] is not a mere avoidable technical complication: it is, in fact, the more transparent way to identify the "guilty" vertex and to cure it by means of the large **N** expansion.

V. A POSSIBLE PHYSICAL INTERPRETATION

I will now discuss a possible interpretation of the previous results in terms of a sort of gravitational confinement (in which the gravitational color, that is, the spin, is confined) which should be dual, in the sense of the gauge/gravity correspondence (for two detailed reviews, see [47,48]) to the standard gauge theoretical confinement.

In many other cases, such as the $O(\mathbf{N})$ scalar (in five space-time dimensions) and fermionic (in three space-time dimensions) models with quartic interactions, the seemingly "magic" properties of the large **N** resummations are, in fact, related in a very simple way to physical properties of the models. In the above-mentioned cases, the large **N** expansion is able to explore the strongly coupled phase of the theory in which phenomena such as the appearance of "colorless" bound states in the spectrum, spontaneous breaking of symmetries, and so on occur. Thus, a natural question is, which phenomenon is behind the improvement of the UV behavior of gravity at large **N**? In the present formulation of the gravitational action, matter fields should be described as scalar fields living in a suitable representation of the internal gauge group according to their spin [6]. At large **N**, in the strongly coupled phase of gravity (as it is also suggested by the gauge/gravity correspondence), the physical spectrum should be dominated by colorless bound states. In this context, colorless particles are simply scalar particles which, therefore, should dominate the spectrum in the UV region [49]. Gravitational confinement would dramatically improve the renormalizability of the theory. The reason is that, if the gravitational interaction confines at the Planck scale, one is left with only scalar fields at high energy: this would soften the UV behavior of the theory. This can be seen by forgetting for a moment the present formalism (in which the spin of particles is represented by an internal index in a suitable representation of the internal gauge group) and returning to the standard "space-time" interpretation of spinful particles. The contribution to the scattering amplitude $A_J(s, t)$ in the t channel (where t and s are Mandelstam variables) at high energies of particles with spin equal to J is roughly

$$A_J(s, t) \approx -e^2 \frac{s^J}{t - m_J^2},$$

m_J being the mass of the particle and e being a suitable coupling constant. When $J < 1$, loop integrals in which such a spin J particle appears are convergent; when $J = 1$ there are logarithmic divergences (which, in this case, can be renormalized), and when $J > 1$ there are nonrenormalizable divergences. This fact is behind the perturbative nonrenormalizability of the Einstein-Hilbert action. In fact, if there is a critical energy scale, beyond which only scalars are left in the spectrum due to the gravitational confinement, the above problems are naturally solved by the dynamics of the gravitational field; thus, the gravitational interaction could be nonperturbatively renormalizable.

It is worth noting that the main vertex responsible for the good UV properties of gravity at large **N** is, presumably, a sort of gravitational confinement. On the other hand, in gauge theories, suitable order parameters for confinement are the Wilson loops. Therefore, one should expect that the gravitational analogue of the Wilson loops should play a fundamental role in understanding the strongly coupled phase of gravity. Interestingly enough, such operators have been introduced in [50] in the context of the Ashtekar formalism for gravity [51] (for an updated review, see [52]). Thus, an intriguing relation between loop quantum gravity and the present large **N** expansion for gravity is apparent.

This is a good point to think about the following question: what would be the physical meaning of renormalizability at large **N**? As far as the standard perturbative expansion is concerned, a pragmatic answer is that a given theory (without local symmetries) is renormalizable (in the standard Feynman expansion) when the number of primitive divergent Feynman diagrams is finite. This issue can be dealt with by looking at the UV behavior of bare propagators and vertices (on which the standard perturbative expansion is based). The deep physical meaning of such a property is manifest in the Wilson point of view (to discuss the Wilson point of view in a gravitational context is far beyond the scope of this paper). As far as the present scheme is concerned, it is enough to say that a UV renormalizable theory is a theory which allows meaningful and predictive computations (in the UV) because the fields appearing in the Lagrangian are suitable to describe the UV phase [53]. When this does not happen, there are two possibilities: either the theory is wrong or the UV degrees of freedom are nontrivial, nonlocal combinations of the ones appearing in the Lagrangian [54].

Thus, a pragmatic answer to the question about the physical meaning of renormalizability at large **N** [55] could be as follows: a theory which is UV renormalizable at large **N** (and is not UV renormalizable in the standard perturbative expansion such as the Gross-Neveu model in three dimensions or the ϕ^4 model in five dimensions [12]) is a theory which is formulated in terms of fields not suitable to describe the UV phase. However, it is not wrong: the large **N** recipe tells us how to sum a class of diagrams of the standard expansion in order to obtain a meaningful expansion (with a finite number of primitive, superficially divergent diagrams). In the already-mentioned examples, this has a very precise meaning: in the UV phase, new degrees of freedom appear which are bound states of the original fields (see, for instance, [38]).

This is exactly the picture which seems to emerge from the present scheme: scalar bound states which soften the UV behavior of amplitudes of particles carrying spin 2 or greater. However, as I will discuss in a moment, in gravity there is the further complication of local symmetries: one should also prove that local symmetries are preserved by the large **N** resummations.

Connections with the KLT relations

It is worth mentioning here an interesting relation (which is worth being further investigated) with the so-called *KLT* relations [56]. The KLT relations were first obtained in a string-theoretical framework: they allow us to express closed string amplitudes in terms of open string amplitudes. Schematically, the factorization of the vertex operators

$$V^{\text{closed}} = V^{\text{open}}_{\text{left}} \times \bar{V}^{\text{open}}_{\text{right}}$$

(where the V's are vertex operators of the closed string, and open string left modes and right modes) is related to the results in [57], stating that correlations of vertex operators factorize at the level of integrands (that is, before the world sheet integrations are performed). Kawai, Lewellen, and Tye were able to prove a stronger result: the complete closed string amplitudes factorize into products of open string amplitudes even after the world sheet integrations. Indeed, the KLT relations go well beyond string theory itself: they imply, for instance, highly nontrivial relations in the low energy limit among tree amplitudes of gravity in four dimensions which factorize (in suitable nonlinear gauges) into gauge theoretical tree amplitudes in four dimensions. These results have been generalized to include loops by using unitarity relations (for a pedagogical review, see [58]); however, a complete proof of these very useful factorization results is not available yet. Usually, one deals with the KLT relations in the standard metric formalism in which, having in mind small deviations from a flat metric

$$g_{\mu\nu} \sim \eta_{\mu\nu} + h_{\mu\nu},$$

the basic variable is the metric fluctuation $h_{\mu\nu}$. In the standard metric formulation of gravity, the structure of the many vertices present in the theory is very complicated. To fully exploit the KLT relations, one needs to choose a gauge in which, roughly speaking, the right index and the left index of $h_{\mu\nu}$ are never contracted with each other (otherwise no "left-right" factorization

$$h_{\mu\nu} \sim \epsilon^{-}_{\mu} \otimes \epsilon^{+}_{\nu}$$

would be apparent [59]). Such (nonlinear) gauge choices are highly nontrivial, and the many vertices of the theory in the metric formalism sometimes obscure the physical origin behind the KLT relations. In the present framework

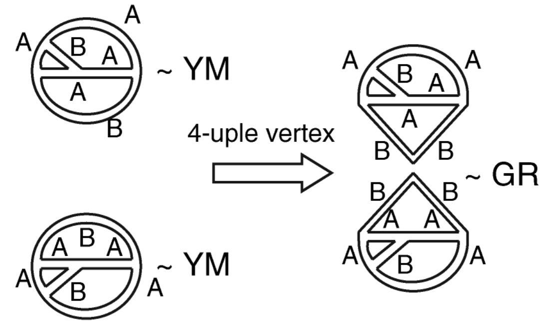

FIG. 9. In this picture a typical gravitational large **N** diagram which could be responsible for the KLT relations has been drawn. The important role which the 4-uple B vertex (which is disconnected from the "internal lines" point of view) could have is manifest in explaining such important relations between gravity and gauge theory (which, in the BF formalism, are distinguished precisely by the above-mentioned 4-uple B vertex).

such features are rather manifest; in particular, the presence of a Lagrange multiplier field ϕ_{ab} which, in the double-line notation, is represented by four internal lines leads directly to amplitudes fulfilling the generalized KLT relations. The reason is that in the BF formulation of gravity the propagators and vertices can be chosen to be equal to the propagators and vertices appearing in the BF formulation of Yang-Mills theory: the only, crucial, difference is the higher spin Lagrange multiplier field. Such a field gives rise, quite generically, to gravitational amplitudes which are manifestly factorized into pieces in which only "YM fields" (that is, fields which are also present in the BF Yang-Mills Lagrangian) appear (see, for instance, Fig. 9). In other words, ϕ_{ab} "allows" us to attach amplitudes of the BF Yang-Mills theory to obtain amplitudes of the BF formulation of gravity; for this reason, the present scheme seems to be suitable to fully exploit and, hopefully, establish, in general, the KLT relations.

VI. CONCLUSIONS AND PERSPECTIVES

In this paper an analysis of the renormalizability of gravity at the large **N** expansion for general relativity has been carried out. It is based on the BF formulation of general relativity in which the Einstein-Hilbert action is split into a topological term plus a constraint. It has been shown that the large **N** expansion dictates resummations of a suitable class of planar diagrams which lead to a great improvement of the UV behavior of gravity: only a finite number of superficially divergent diagrams are present at large **N**. This is an important step in proving renormalizability. The next steps are the analysis of the infinities and their fulfillment of the symmetry constraints such as the Slavnov-Taylor identities and the Zinn-Justin equation. The analysis of the KLT relations in this scheme is also worth being further investigated.

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[1] L. Susskind, J. Math. Phys. (N.Y.) **36**, 6377 (1995).
[2] G. 't Hooft, gr-qc/9310026.
[3] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000), and references therein.
[4] R. Bousso, Rev. Mod. Phys. **74**, 825 (2002), and references therein.
[5] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[6] F. Canfora, Nucl. Phys. **B731**, 389 (2005).
[7] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[8] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[9] G. Veneziano, Nucl. Phys. **B117**, 519 (1976).
[10] E. Witten, Nucl. Phys. **B160**, 57 (1979).
[11] Y. Makeenko, hep-th/0001047; A. V. Manohar, hep-ph/9802419.
[12] G. Parisi, Nucl. Phys. **B100**, 368 (1975).
[13] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963).
[14] T. D. Lee, Phys. Rev. **128**, 899 (1962).
[15] P. J. Redmond and J. L. Uretski, Phys. Rev. Lett. **1**, 147 (1958).
[16] Topological, in this setting, means that the theory has no local degrees of freedom and the expectation values which can be computed are related to topological invariants of the manifold where the theory lives.
[17] R. Capovilla, J. Dell, and T. Jacobson, Classical Quantum Gravity **8**, 59 (1991).
[18] R. Capovilla, J. Dell, T. Jacobson, and L. Mason, Classical Quantum Gravity **8**, 41 (1991).
[19] R. Capovilla, M. Montesinos, V. A. Prieto, and E. Rojas, Classical Quantum Gravity **18**, L49 (2001); **18**, 1157 (2001).
[20] R. De Pietri and L. Freidel, Classical Quantum Gravity **16**, 2187 (1999).
[21] M. P. Reisenberger, Classical Quantum Gravity **16**, 1357 (1999).
[22] H. Y. Lee, A. Nakamichi, and T. Ueno, Phys. Rev. D **47**, 1563 (1993).
[23] F. Fucito, M. Martellini, S. P. Sorella, A. Tanzini, L. C. Q. Vilar, and M. Zeni, Phys. Lett. B **404**, 94 (1997).
[24] A. Cattaneo, P. Cotta-Ramusino, F. Fucito, M. Martellini, M. Rinaldi, A. Tanzini, and M. Zeni, Commun. Math. Phys. **197**, 571 (1998).
[25] It is worth stressing here that the results of the present paper, and, in particular, the large **N** improvement of the UV behavior of gravity, do not depend on e. The second term on the right-hand side of Eq. (7) has been added only to obtain a theory with the same propagators as the BF Yang-Mills theory: this scheme allows us to clearly single out the "bad UV-behaved" vertex of the theory. At this stage of the analysis it is not yet clear if it is possible to get the same UV improvement without adding the above term to the gravitational action.
[26] J. H. Horne, Nucl. Phys. **B318**, 22 (1989).
[27] M. Martellini and M. Zeni, Phys. Lett. B **401**, 62 (1997).
[28] M. Goroff and A. Sagnotti, Nucl. Phys. **B266**, 709 (1986).
[29] E. Witten, Commun. Math. Phys. **117**, 353 (1988); Nucl. Phys. **B311**, 46 (1988).
[30] G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincare **20**, 69 (1974).
[31] Here, simplified means that the possible complications due to ghost effects in the loops (which, in principle, could prevent the UV softening) are not taken into account.
[32] From a large **N** perspective, it is mandatory to sum together these diagrams because, from the internal index point of view, they can be considered on perfectly equal footing.
[33] In the Yang-Mills case, the bare propagators and vertices in the large **N** expansion coincide with the bare propagators in the standard Feynman expansion. Needless to say, the two expansions are, in any case, very different.
[34] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1996), Vols. I–II.
[35] In the Yang-Mills case, both the expansions (Feynman and large **N**) are power-counting renormalizable.
[36] S. Weinberg, in *General Relativity: An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979); hep-th/9702027.
[37] One should also sum over cyclic permutations of the internal indices as well as over the Mandelstam variables in order to preserve crossing symmetry.
[38] M. Moshe and J. Zinn-Justin, Phys. Rep. **385**, 69 (2003).
[39] Of course, the tadpole diagrams also form a geometric series.
[40] If one does not remove such a term (which appears in the denominator of the improved vertex), the improved 4-uple B vertex would actually vanish when removing the cutoff. Therefore, one would obtain that the gravitational action would be equal to the large **N** Yang-Mills action in the limit of the small coupling constant. Even if this produced a renormalizable theory in the large **N** expansion, it seems more natural to remove the quadratically divergent term in the denominator as it is usually done in similar situations [38].
[41] M. Reuter, Phys. Rev. D **57**, 971 (1998).
[42] O. Lauscher and M. Reuter, Phys. Rev. D **65**, 025013 (2001); Classical Quantum Gravity **19**, 483 (2002).
[43] M. Reuter and F. Saueressig, Phys. Rev. D **65**, 065016 (2002).
[44] As was already remarked, this feature tells apart gravity from gauge theories: in gravity, the bare large **N** propa-

gators and vertices do not coincide with the standard bare propagators and vertices in the Feynman expansion.

[45] This is given by the already discussed bubblelike geometric series ''mediated'' by the ''higher spin'' Lagrange multiplier ϕ_{ab} which allows a ''bubblelike'' series with the same internal-line structure in Fig. 4. Thus, the 4-uple vertex of the physical field $B + \nabla\eta$ is dressed by the same factor which was discussed in the previous section.

[46] The introduction of the Yang-Mills term has not played any explicit role in the whole discussion. Its only role is, by the way, to arrive at a theory which has the same propagators of the BF Yang-Mills theory.

[47] M. Bertolini, Int. J. Mod. Phys. A **18**, 5647 (2003).
[48] P. Di Vecchia, A. Liccardo, R. Marotta, and F. Pezzella, Int. J. Mod. Phys. A **20**, 4699 (2005).

[49] Gravitational confinement would be of great cosmological importance since it would provide the standard inflationary scenario with a sound basis: such a mechanism could lead to the expected decrease in the number of degrees of freedom expected in a holographic theory.

[50] C. Rovelli and L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nucl. Phys. **B331**, 80 (1990).
[51] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).
[52] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).

[53] That is, the fields in the Lagrangian represent small fluctuations around the UV vacuum so that the perturbation expansion works.

[54] For instance, in QCD quarks and gluons are good UV degrees of freedom but they are not good IR degrees of freedom. This, of course, does not imply that QCD is wrong: it is an indication that the IR degrees of freedom are nontrivial combinations of the UV degrees of freedom.

[55] Large **N** renormalizability (in the cases in which there are not local symmetries) can be ''detected'' as standard renormalizability using power-counting arguments by looking at the UV behavior of large **N** propagators and vertices (that is, vertices and propagators which have been corrected in order to encode the leading corrections in the expansion).

[56] H. Kawai, D. C. Lewellen, and S. H. Tye, Nucl. Phys. **B269**, 1 (1986); **B288**, 1 (1987).
[57] Z. Koba and H. B. Nielsen, Nucl. Phys. **B12**, 517 (1969).
[58] Z. Bern, Living Rev. Relativity **5**, 5 (2002).

[59] $\epsilon_\mu^\pm$ have to be thought of as the ''gluons'' of the gauge theory responsible for the factorization of the gravitational amplitudes (the plus and minus signs refer to the elicities).