

**Non-Abelian solutions in AdS<sub>4</sub> and  $d = 11$  supergravity**Robert B. Mann,<sup>1,2</sup> Eugen Radu,<sup>3</sup> and D. H. Tchrakian<sup>3,4</sup><sup>1</sup>*Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*<sup>2</sup>*Perimeter Institute for Theoretical Physics, Ontario N2J 2W9, Canada*<sup>3</sup>*Department of Mathematical Physics, National University of Ireland, Maynooth, Ireland*<sup>4</sup>*School of Theoretical Physics—DIAS, 10 Burlington Road, Dublin 4, Ireland*

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We consider solutions of the four-dimensional Einstein-Yang-Mills (EYM) system with a negative cosmological constant  $\Lambda = -3g^2$ , where  $g$  is the non-Abelian gauge coupling constant. This theory corresponds to a consistent truncation of  $\mathcal{N} = 4$  gauged supergravity and may be uplifted to  $d = 11$  supergravity. A systematic study of all known solutions is presented as well as new configurations corresponding to rotating regular dyons and rotating non-Abelian black holes. The thermodynamics of the static black hole solutions is also discussed. The generic EYM solutions present a nonvanishing magnetic flux at infinity and should give us information about the structure of a conformal field theory (CFT) in a background SU(2) field. We argue that the existence of these configurations violating the no-hair conjecture is puzzling from the anti-de Sitter (AdS)/CFT point of view.

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**I. INTRODUCTION**

Recently a tremendous amount of interest has been focused on anti-de Sitter (AdS) spacetime. This interest is mainly motivated by the proposed correspondence between physical effects associated with gravitating fields propagating in AdS spacetime and those of a conformal field theory (CFT) on the boundary of AdS spacetime [1,2].

A precise formulation of the AdS/CFT correspondence is made in equating the generating function of the correlation functions in the CFT with the string/gravity partition function on the AdS space [1,3]

$$\begin{aligned} Z_{\text{AdS}}[h, \Psi_0] &= \int_{[h, \Psi_0]} D[g] D[\Psi] e^{-I[g, \Psi]} \\ &= \left\langle \exp \left( \int_{\partial \mathcal{M}_d} d^d x \sqrt{\gamma} \mathcal{O}_{[h, \Psi_0]} \right) \right\rangle \\ &= Z_{\text{CFT}}[h, \Psi_0]. \end{aligned} \quad (1)$$

The induced boundary metric and matter fields are, respectively, denoted by  $h$  and  $\Psi_0$  symbolically, with  $\mathcal{O}$  a conformal operator defined on the boundary of AdS <sub>$d+1$</sub> . The integration is over configurations  $[g, \Psi]$  of metric and matter fields that approach  $[h, \Psi_0]$  when one goes from the bulk of AdS <sub>$d+1$</sub>  to its boundary. This conjecture has been verified for several important examples, encouraging the expectation that an understanding of quantum gravity in an asymptotically AdS spacetime (AAAdS) can be obtained by studying its holographic CFT dual, defined on the boundary of spacetime at infinity.

In this context, various AAAdS solutions of the Einstein equations coupled with matter fields have been studied in the literature. However, although the gauged supergravity AdS theories generically contain Yang-Mills (YM) fields, most of the studies in the literature have been restricted to the case of Abelian matter content in the bulk. At the same

time, a number of results obtained for an asymptotically flat (AF) spacetime clearly indicate that a variety of well-known, and rather intuitive, features of self-gravitating Maxwell fields are not shared by non-Abelian gauge fields. In particular, and in contrast to the Abelian situation, self-gravitating YM fields can form particlelike configurations [4]. Moreover, the Einstein-Yang-Mills (EYM) equations also admit black hole solutions that are not uniquely characterized by their mass, angular momentum, and YM charges [5]. Therefore the uniqueness theorem for electrovacuum black hole spacetimes ceases to exist for EYM systems.

As proven by some authors [6,7], even the simple spherically symmetric EYM-SU(2) system with a negative cosmological constant  $\Lambda$  in four spacetime dimensions presents a number of surprising results. A variety of well-known features of AF self-gravitating non-Abelian solutions are not shared by their AAAdS regular and black hole counterparts.

Although the picture one finds is very much  $\Lambda$ -dependent, it is still possible to identify some general features. For example, there is always a continuum of regular and black hole solutions in terms of the adjustable shooting parameters that specifies the initial conditions at the origin or at the event horizon, rather than discrete points. Depending on the value of  $\Lambda$ , the spectrum has a finite number of continuous branches. Furthermore, there exist nontrivial solutions that are stable against spherically symmetric linear perturbations, corresponding to stable configurations with nonvanishing non-Abelian charges. The solutions are classified by non-Abelian electric and magnetic charges and the Arnowitt-Deser-Misner (ADM) mass. When the parameter  $\Lambda$  approaches zero, an already-existing branch of monopoles and dyon solutions collapses to a single point in the moduli space [8]. At the same time new branches of solutions emerge. A general study of these

configurations together with a stability analysis is presented in [9,10].

These spherically symmetric solutions have been generalized in various directions. Axially symmetric solutions are discussed in [11–13], Newman-Unti-Tamburino (NUT)-charged configurations and topological black holes with non-Abelian fields were considered in [14,15]. Spherically symmetric, five-dimensional solutions of the EYM system with negative cosmological constant were examined in [16], this result being extended recently to  $d > 5$  [17].

Although further research is clearly necessary, at least some of the EYM-SU(2) solutions, emerging as consistent reduction of  $d = 11$  supergravity on a seventh-dimensional sphere [18,19], have relevance in AdS/CFT context. In this case, the ratio between the four-dimensional cosmological constant and the gauge coupling constant  $g$  is fixed by  $\Lambda/g^2 = -3$ . Apparently the bulk/boundary correspondence for AAdS EYM configurations has not received much attention in the literature. Although all such solutions containing non-Abelian fields are classical they may have a role to play in the full quantum theory. If the AdS/CFT correspondence conjecture is indeed correct, it should either be able to account for solutions to the EYM system from the CFT viewpoint or be able to demonstrate why the conjecture does not apply to them.

The lack of attention given to AAdS EYM solutions is presumably due to the notorious absence of closed form solutions in the presence of non-Abelian matter fields in the bulk. (Very few such exact solutions exist e.g. [20], which features an effective negative cosmological constant but is not AAdS, and, [21] for  $\Lambda = 0$ .) However, one can analyze their properties by using a combination of analytical and numerical methods, which is enough for most purposes. Euclidean solutions of the  $\Lambda = -3g^2$  EYM model have been discussed in [22], in a different context, however. The authors of Ref. [22] considered Euclidean wormhole solutions, with an  $S^3$  conformal infinity.

The Lorentzian solutions we consider here are very different. Their conformal infinity is the product of time and a two-dimensional sphere, plane, or hyperboloid. In the black hole case, these are the non-Abelian counterparts of the well-known AdS<sub>4</sub> Einstein-Maxwell solutions. For a  $R \times S^2$  boundary structure, there are also globally regular, particlelike solutions. The existence of configurations violating the no-hair conjecture is puzzling from the AdS/CFT point of view. Since several distinct solutions with the same set of boundary at infinity data may exist, it is not clear how the dual CFT distinguishes between them.

In the first part of this paper we discuss the features of the AAdS non-Abelian solutions with  $\Lambda = -3g^2$ . Although some of these solutions are already known in literature, their properties have not been discussed for this particular value of the cosmological constant. New types of EYM configurations with a nonvanishing angular momen-

tum are presented as well. All these configurations have a higher dimensional interpretation, solving the equations of motion of  $d = 11$  supergravity.

The second part of this paper attempts a discussion of these solutions in an AdS/CFT context. The boundary stress tensor and the associated conserved charges are computed in Sec. IV, where we also present a discussion of black hole thermodynamics. We further argue that studying the  $\Lambda = -3g^2$  EYM-AdS system should give us information about the structure of the CFT in a background SU(2) field. We conclude with Sec. V. The appendices contain a discussion of technical aspects of solutions' construction.

## II. GENERAL FRAMEWORK

We start by discussing the embedding of the EYM solutions in M theory. It is known that the standard  $\mathcal{N} = 4$  SO(4) gauged supergravity in  $d = 4$  can be viewed as a reduction of  $d = 11$  supergravity on  $S^7$  [18]. The bosonic sector of this theory contains two SU(2) fields  $F_{\mu\nu}$ ,  $\tilde{F}_{\mu\nu}$ , a dilaton  $\phi$ , and an axion  $\chi$  with the Lagrangian density

$$\begin{aligned} \mathcal{L} = & \mathcal{R} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{2\phi} \partial_\mu \chi \partial^\mu \chi - V(\phi, \chi) \\ & - \frac{1}{2} e^{-\phi} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} \frac{e^\phi}{1 + \chi^2 e^{2\phi}} \tilde{F}_{\mu\nu}^a \tilde{F}^{a\mu\nu} \\ & - \frac{1}{2\sqrt{-g}} \chi \epsilon_{\mu\nu\rho\sigma} F^{a\mu\nu} F^{a\rho\sigma} \\ & + \frac{1}{2\sqrt{-g}} \frac{\chi e^{2\phi}}{1 + \chi^2 e^{2\phi}} \epsilon_{\mu\nu\rho\sigma} \tilde{F}^{a\mu\nu} \tilde{F}^{a\rho\sigma}, \end{aligned} \quad (2)$$

where the potential  $V(\phi, \chi)$  is

$$V(\phi, \chi) = -2g^2(4 + 2 \cosh \phi + \chi^2 e^\phi). \quad (3)$$

It can easily be seen that  $\chi = \phi = 0$  is a consistent truncation of this theory for  $F_{\mu\nu} = \tilde{F}_{\mu\nu}$  and, as a result one finds the action

$$I = \frac{1}{4} \int d^4x \sqrt{-g} [\mathcal{R} - 2\Lambda - 2 \text{Tr}(F_{\mu\nu} F^{\mu\nu})], \quad (4)$$

where the field strength tensor  $F_{\mu\nu} = \frac{1}{2} \tau^a F_{\mu\nu}^a$  is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \quad (5)$$

and the gauge field  $A_\mu = \frac{1}{2} \tau^a A_\mu^a$ , with  $\tau^a$  an SU(2) basis written in terms of Pauli matrices (the value of the gauge coupling  $g$  has been set to one without loss of generality). The gauge field transforms as  $A'_\mu = U A_\mu U^\dagger + i(\partial_\mu U) U^\dagger$  under a SU(2) gauge transformations  $U$ .

The value of the cosmological constant as read from  $V(\phi = 0, \chi = 0)$  is  $\Lambda = -3$ .

As usual, to ensure well-defined Euler-Lagrange field equations, one adds to the action principle (4) the

Hawking-Gibbons surface term [23],  $I_{\text{surf}} = -\frac{1}{2} \times \int_{\partial \mathcal{M}} d^3x \sqrt{-h} K$  where  $K$  is the trace of the extrinsic curvature for the boundary  $\partial \mathcal{M}$  and  $h$  is the induced metric of the boundary. This term does not affect the equations of motion but it is relevant in the discussion of solutions' mass and boundary stress tensor.

Variation of the action (4) with respect to the metric  $g^{\mu\nu}$  leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 2T_{\mu\nu}, \quad (6)$$

where the YM stress-energy tensor is

$$T_{\mu\nu} = 2 \text{Tr}(F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}). \quad (7)$$

Variation with respect to the gauge field  $A_\mu$  leads to the YM equations

$$D_\mu F^{\mu\nu} = 0, \quad (8)$$

where  $D_\mu = \partial_\mu + i[A_\mu, \cdot]$ .

In this paper we will consider solutions of the EYM equations possessing at least two Killing vectors  $\xi = \partial_t$ ,  $\eta = \partial_\varphi$ , corresponding to a stationary, axially symmetric spacetime. For the time translation symmetry, we choose a natural gauge such that the matter fields have no time dependence,  $\partial A / \partial t = 0$ . However, a rotation around the  $z$ -axis can be compensated by a gauge rotation  $\mathcal{L}_\varphi A = D\Psi$  [24], with  $\Psi$  being a Lie-algebra valued gauge function. This introduces an integer  $n$  in the matter ansatz (which is a constant of motion) and implies the existence of a potential  $W$  with

$$F_{\mu\varphi} = D_\mu W, \quad (9)$$

where  $W = A_\varphi - \Psi$ . The integer  $n$  represents the winding number with respect to the azimuthal angle  $\varphi$ . While  $\varphi$  covers the trigonometric circle once, the fields wind  $n$  times around. The qualitative features of the solutions obtained in the Abelian sector of the theory are insensitive to  $n$ ; however, the winding number determines to some extent the properties of the non-Abelian solutions.

Using the formulas in [18] with the two equal gauge fields,  $A^a = \tilde{A}^a$ , we can uplift any configuration that extremizes the action principle (4) to  $d = 11$  supergravity. The 11-dimensional metric ansatz reads

$$ds_{11}^2 = ds_4^2 + 4d\xi^2 + \cos^2\xi \sum_a (\Theta^a - A_\mu^a dx^\mu)^2 + \sin^2\xi \sum_a (\tilde{\Theta}^a - A_\mu^a dx^\mu)^2. \quad (10)$$

The antisymmetric tensor field  $\hat{F}_{(4)}$ , which appears in the action principle of the  $d = 11$  supergravity, can be read from [18,19]

$$\begin{aligned} \hat{F}_{(4)} = & -3\epsilon_{(4)} + \sqrt{2} \sin\xi \cos\xi d\xi \wedge (\Theta - \tilde{\Theta})^a \wedge *F_2^a \\ & + \frac{\sqrt{2}}{4} \epsilon_{abc} [\cos^2\xi (\Theta - A)^a \wedge (\Theta - A)^b \\ & + \sin^2\xi (\tilde{\Theta} - A)^a \wedge (\tilde{\Theta} - A)^b] \wedge *F_2^c, \end{aligned} \quad (11)$$

where  $\Theta^a, \tilde{\Theta}^a$  are SU(2) right invariant one forms on two 3-spheres  $S^3, \tilde{S}^3$ .

### III. $\Lambda = -3$ EYM SOLUTIONS

The field equations (6) and (8) contain a large variety of solutions. First, any solution  $(g_{\mu\nu}, \mathcal{A}_\mu)$  of the Einstein-Maxwell (EM) theory with  $\Lambda = -3$  can be viewed as a solution of the EYM Eqs. (6) and (8) by taking  $A_\mu = \mathcal{A}_\mu T$  where  $T$  belongs to the Lie algebra of the non-Abelian gauge group [25]. The properties of these AAdS Abelian solutions have been discussed by various authors (for generic  $\Lambda$ ), as well as their relevance in AdS/CFT context.

Here we restrict ourselves to the pure non-Abelian case. As expected, a number of EYM configurations with  $\Lambda = 0$  are found to possess AAdS counterparts, with very different properties, however. For  $\Lambda < 0$ , there are also rotating regular EYM solutions, which do not survive in the AF limit.

In this section we present a discussion of the EYM solutions which arise as a truncation of  $\mathcal{N} = 4$  SO(4) gauged supergravity, both regular and black hole configurations being considered. Some of these solutions have been already presented in the literature, however, without discussing the case of interest  $\Lambda = -3$ . Also, to simplify the general picture we will not consider dyon solutions, except for the rotating case where Einstein equations require the presence of a YM electric field.

The mass-energy of these solutions is computed by using the formalism presented in Sec. IV. Also, we will present here general features of these solutions, without entering into technical details, which are the subject of Appendices A, B, C, and D.

#### A. Regular configurations

##### 1. The $\Lambda = -3$ Bjouraker-Hosotani monopole solutions

We start with the simplest case, corresponding to spherically symmetric solutions. These are the AdS generalizations of the Bartnik-McKinnon configurations found in [7] by Bjouraker and Hosotani, within a metric ansatz

$$ds^2 = \frac{dr^2}{H(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - \frac{H(r)}{p^2(r)} dt^2, \quad (12)$$

where

$$H(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda}{3}r^2,$$

$m(r)$  corresponding to the local mass-energy density.

The static, spherically symmetric SU(2) YM ansatz is obtained for a winding number  $n = 1$  and can be parametrized by one real function  $\omega(r)$

$$A = \frac{1}{2}\{w(r)\tau_1 d\theta + (\cot\theta\tau_3 + w(r)\tau_2)\sin\theta d\varphi\}, \quad (13)$$

which implies the field strength tensor expression

$$F = \frac{1}{2}\{w'dr \wedge (\tau_1 d\theta + \tau_2 \sin\theta d\varphi) - (1 - w^2)\tau_3 d\theta \wedge \sin\theta d\varphi\}, \quad (14)$$

where a prime denotes a derivative with respect to  $r$ . For this purely magnetic ansatz, the EYM equations take the form

$$m' = \omega'^2 H + \frac{(w^2 - 1)^2}{2r^2}, \quad (15)$$

$$w'' = \frac{w'}{H} \left( \frac{2\Lambda r}{3} - \frac{2m}{r^2} + \frac{(w^2 - 1)^2}{r^3} \right) + \frac{w(w^2 - 1)}{r^2 H},$$

the equation for  $p$  decoupling from the rest,  $p'/p = -2w'^2/r$ .

No exact solutions of this system are known yet, so the equations must be solved numerically. The solutions with a regular origin have the following behavior at  $r = 0$

$$w(r) = 1 - br^2 + O(r^4), \quad m(r) = 2b^2 r^3 + O(r^4),$$

$$p(r) = p_0(1 - 4b^2 r^2) + O(r^4), \quad (16)$$

where  $b, p_0$  are real constants. We are here interested in solutions with AdS asymptotics, which implies the following expansion at large  $r$

$$m(r) = M + \left( \frac{\Lambda C_1^2}{3} - \frac{1}{2}(\omega_\infty^2 - 1)^2 \right) \frac{1}{r} + O\left(\frac{1}{r^2}\right),$$

$$\omega(r) = \omega_\infty + \frac{C_1}{r} + O\left(\frac{1}{r^2}\right), \quad (17)$$

$$p(r) = 1 + \frac{C_1^2}{r^4} + O\left(\frac{1}{r^5}\right),$$

where  $\omega_0, M$ , and  $C_1$  are constants determined by numerical calculations.  $M$  corresponds to the ADM mass of the solutions, while  $\omega_0$  determines the value of the magnetic charge,  $Q_m = |1 - \omega_0^2|$  (see Appendix C). For  $\Lambda = 0$ ,  $\omega_0 = \pm 1$  are the only allowed values,  $(M, b)$  being restricted to a discrete family indexed by the node number of the gauge function  $\omega(r)$  [4].

As found in [7], the AAdS solutions are much less restricted. By varying the parameter  $b$  which enters the expansion at the origin (16), a continuum of monopole solutions are found. These configurations are regular in the entire space. The overall picture depends on the value of the cosmological constant; for  $\Lambda = -3$ , solutions with AAdS asymptotics are found for only one branch with  $-0.557 < b < 1.31$ . In Fig. 1, we plot the asymptotic quantities  $M$  and  $\omega_0$  as well as the value of the metric function  $p$  at the origin as a function of the parameter  $b$ . One can see that  $p(0)$  diverges as  $b \rightarrow b_{\max}$  (i.e.  $g_{tt}(0) \rightarrow$

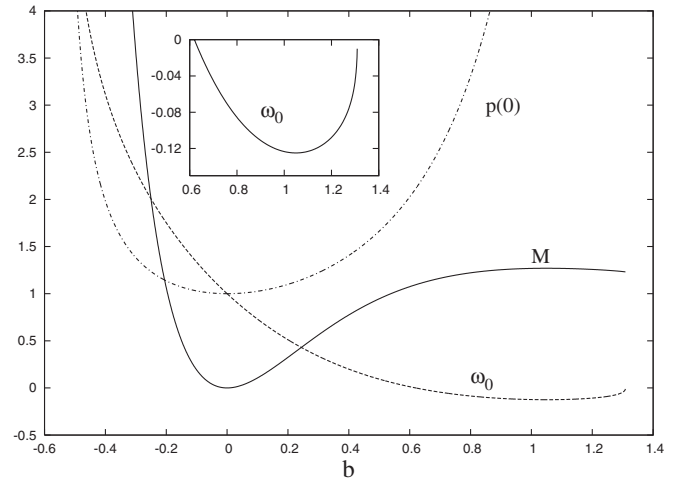


FIG. 1. The mass-parameter  $M$ , the asymptotic value of the gauge function  $\omega_0$ , and the value of the metric function  $p(r)$  at the origin are represented as a function of the parameter  $b$  for spherically symmetric, globally regular monopole solutions.

0), while  $M(b_{\max}), \omega(b_{\max})$  remain finite, while all these parameters appear to diverge as  $b \rightarrow b_{\min}$ . The critical solutions have been studied in [10] and are not of interest here.

A configuration is uniquely characterized by the asymptotic parameters  $M, \omega_0$ . As a characteristic feature of  $\Lambda = -3$  globally regular configurations, we notice the existence of zero- and one-node monopole solutions only. The nodeless solutions are of particular interest because, as discussed in [7], they are stable against linear perturbations. The solution with  $b \simeq 0.619$  has  $\omega_0 = 0$  and, similar to the 't Hooft-Polyakov monopole, has magnetic charge  $Q_m = 1$ . In this case the AdS boundary conditions play a role similar to the Higgs field in the AF case. As seen in Fig. 1, for values of  $b$  in the interval  $0.62 < b < 1.31$  the gauge

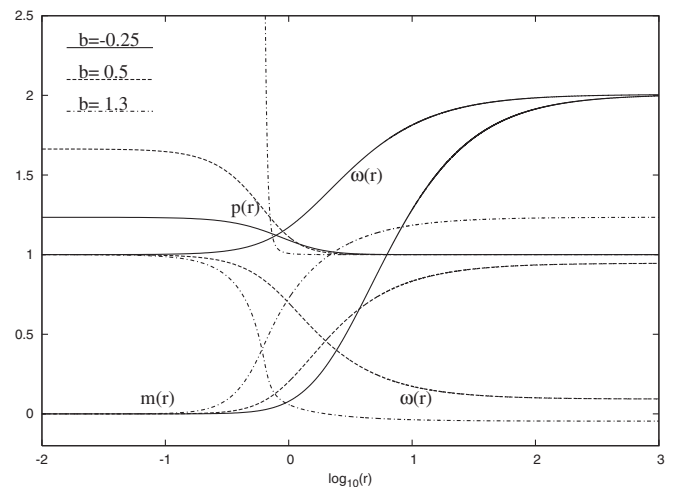


FIG. 2. The profiles of typical spherically symmetric globally regular monopole solutions are plotted for several values of the parameter  $b$  which enters the expansion at the origin of the gauge potential.

function  $\omega$  crosses the  $r$ -axis, approaching a negative value at infinity. Although for  $0.635 \leq b \leq 1.309$  there are two configurations with the same asymptotic value of the gauge function, these are distinguished by the value of  $M$ , with  $M(b_2) < M(b_1)$  if  $b_2 < b_1$ .

Typical configurations are displayed in Fig. 2 for three different values of the parameter  $b$ . The solution with  $b = 1.3$  has  $\omega_0 = -0.045$ , corresponding to a near critical configuration with  $g_{tt}(0) = -0.0032$ .

## 2. Axially symmetric monopoles

It is well known that, for  $\Lambda = 0$ , an SU(2) YM theory coupled to gravity also possesses AF static axially symmetric globally regular solutions [26], labeled by a winding number  $n > 1$ . The AAdS generalizations of these configurations were reported in [11], featuring very different properties, as expected. Here we present a discussion of these solutions for the special case  $\Lambda = -3$ .

As discussed in Appendix A, the minimal axially symmetric YM ansatz is parametrized by four functions  $H_i(r, \theta)$ . These magnetic potentials satisfy a suitable set of boundary conditions at the origin, at infinity, and on the symmetry axis imposed by finite energy, regularity, and symmetry requirements. The spherically symmetric YM ansatz is recovered for a unit winding number  $n = 1$ , two vanishing gauge potentials  $H_1 = H_3 = 0$  while the other two are equal  $H_2 = H_4 = \omega(r)$ .

For large  $r$ , the configuration becomes spherically symmetric, with  $H_2 = H_4 = \omega_0$ , the asymptotic value of the other two functions being zero. The magnetic charge of these solutions is  $Q_M = n|1 - \omega_0^2|$ . The expression of the gauge ansatz and the boundary conditions are presented in Appendices A and B; see also Appendix D for a discussion of the numerical procedure we used to find these solutions.

The static axially symmetric EYM solutions are obtained within a metric ansatz

$$ds^2 = \frac{m}{f} \left( \frac{dr^2}{1 - \frac{\Lambda}{3} r^2} + r^2 d\theta^2 \right) + \frac{l}{f} r^2 \sin^2 \theta d\varphi^2 - f \left( 1 - \frac{\Lambda}{3} r^2 \right) dt^2, \quad (18)$$

where the metric functions  $f$ ,  $m$ , and  $l$  are only functions of  $r$  and  $\theta$ . Here  $r$  is the radial coordinate,  $t$  is a global time coordinate,  $(\theta, \varphi)$  being the usual coordinates on the sphere, with  $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi < 2\pi$ . The expansion of the metric functions as  $r \rightarrow \infty$  is

$$\begin{aligned} f &= 1 + \frac{f_1 + f_2 \sin^2 \theta}{r^3} + O\left(\frac{1}{r^5}\right), \\ m &= 1 + \frac{m_1 + m_2 \sin^2 \theta}{r^3} + O\left(\frac{1}{r^5}\right), \\ l &= 1 + \frac{l_1 + l_2 \sin^2 \theta}{r^3} + O\left(\frac{1}{r^5}\right), \end{aligned} \quad (19)$$

which leads to AAdS solutions. Here  $f_1, f_2$  are undetermined constants, while  $l_1 = m_1 = 2f_1/3$ ,  $l_2 = 6f_2/17$ ,  $m_2 = 14f_2/17$ . As discussed in Sec. IV, the mass  $M$  of these configurations is encoded in the parameters  $f_1, f_2$

$$M = \frac{\Lambda}{3} \left( \frac{2f_1}{3} + \frac{8f_2}{17} \right). \quad (20)$$

A spherically symmetric spacetime is recovered for  $l = m$  and  $f_2 = m_2 = l_2 = 0$  (i.e. no angular dependence). The coordinate transformation between the resulting line element and the more familiar Schwarzschild-like form (12) is discussed in Ref. [11].

The results of the numerical integration indicate that every  $\Lambda = -3$  spherically symmetric regular solution appears to present axially symmetric generalizations. For any given winding number, we find only one branch of solutions classified by the mass and the value of the parameter  $\omega_0$  (we have studied solutions up to  $n = 4$ ). Similar to the  $n = 1$  case, the functions  $H_2$  and  $H_4$  are nodeless or present one node only. The gauge potentials  $H_1, H_3$  always present a complicated  $\theta$ -dependence, while one finds usually a small angular dependence for  $H_2, H_4$ . The metric functions  $f, m$ , and  $l$  do not exhibit a strong angular dependence, while  $m$  and  $l$  have a rather similar shape. As expected, the angular dependence of the metric and matter functions increases with  $n$ . Also, the values at the origin of the metric functions decreases with  $n$ . The typical profiles of the metric and gauge functions are similar to those exhibited in [11] and we will not present them here.

In Fig. 3 we plot the mass  $M$  as a function of the  $\omega_0$  for various winding numbers. We see that the  $n > 1$  branches generally follow the picture found for  $n = 1$  (with higher values of mass, however).

The energy density of the matter fields  $\epsilon$ , shows a pronounced peak along the  $\rho$ -axis and decreases mono-

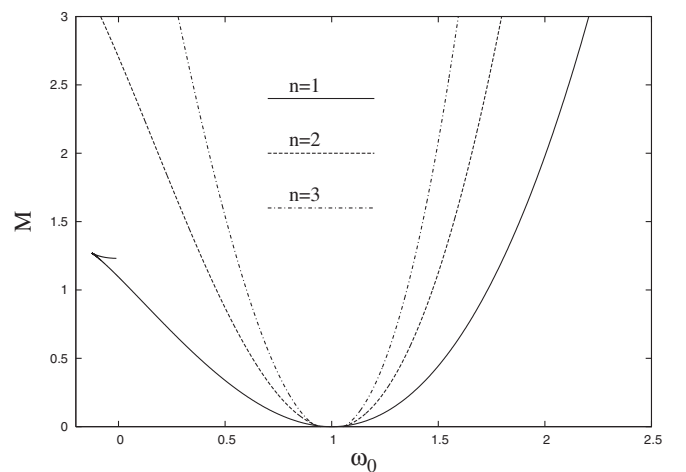


FIG. 3. Mass  $M$  is plotted as a function of  $\omega_0$  for globally regular gravitating monopole static solutions. The winding number  $n$  is also marked. The configurations with  $n = 2, 3$  are axially symmetric.

tonically along the  $z$ -axis (with  $z = r \cos\theta$ ,  $\rho = r \sin\theta$ ). The contours of equal energy density  $\epsilon = -T_t^t$  are two-torii and squashed two-spheres. The peak of the energy density along the  $\rho$ -axis slightly shifts outward with increasing  $n$  and increases in height. For a fixed value of  $\omega_0$ , the mass of the solutions,  $M(n)$ , increases with  $n$ . For example, for  $\omega_0 = 0.05$ , one has  $M(1) = 1.014$ ,  $M(2) = 2.467$ ,  $M(3) = 4.747$ , and so on. As a general feature, the particlelike non-Abelian solitons are less massive than the extremal Reissner-Nordström-AdS (RNAdS) solutions with the same magnetic charge.

We have found it difficult to obtain axially symmetric generalizations of the spherically symmetric solutions near limits of the  $b$ -interval, with large errors for the functions. A different metric parametrization appears to be necessary.

### 3. Rotating regular solutions

An interesting physical question is whether these static non-Abelian regular solutions can be generalized to include an angular momentum. For the AF case, contrary to results from perturbation theory [27], no rotating generalizations of the BK solutions seem to exist. In this case, the  $A_t$  components of the gauge field act like an isotriplet Higgs field with negative metric, and by themselves would cause the other components of the gauge field to oscillate as  $r \rightarrow \infty$  [28], which would imply an infinite mass. Rotating solutions are found by including in the theory a triplet Higgs field with a vacuum expectation value greater than the asymptotic value of  $A_t$  [29].

For  $\Lambda < 0$ , there are no boundary conditions to exclude a non-Abelian solution with nonzero electric potential; dyon EYM solutions already exist in the spherically symmetric case [7]. The existence of dyon solutions without a Higgs field is a feature for AdS spacetime; if  $\Lambda \geq 0$  the electric part of the gauge fields is forbidden [7,30]. This makes possible the existence of rotating regular configurations, too. A discussion of these solutions in a fixed AdS background is presented in [13], where it is argued that they survive in the presence of gravity. However, the general picture appears to be very complicated, crucially depending on the value of the cosmological constant. Here we analyze the solutions' properties for  $\Lambda = -3$ .

These rotating EYM solutions are found for a metric form generalizing (18) for an extradiagonal metric component  $g_{\varphi t}$ , which satisfies also the circularity condition [31]

$$ds^2 = \frac{m}{f} \left( \frac{dr^2}{1 - \frac{\Lambda}{3} r^2} + r^2 d\theta^2 \right) + \frac{l}{f} r^2 \sin^2 \theta \left( d\phi + \frac{\Omega}{r} dt \right)^2 - f \left( 1 - \frac{\Lambda}{3} r^2 \right) dt^2, \quad (21)$$

where the four metric functions  $f$ ,  $m$ ,  $l$ , and  $\Omega$  depend only on the coordinates  $r$  and  $\theta$ . At infinity, the asymptotic form of the metric functions  $f$ ,  $l$ ,  $m$  is still given by (19) while the metric function associated with rotation decays as

$$\Omega = \frac{j_1 + j_2 \sin^2 \theta}{r^3} + O\left(\frac{1}{r^4}\right), \quad (22)$$

$j_1, j_2$  being two real constants. As discussed in Sec. IV, the mass-energy  $M$  of the rotating solutions is still given by (20), while their angular momentum is

$$J = \frac{\Lambda}{3} \left( \frac{j_1}{2} + \frac{2j_2}{5} \right). \quad (23)$$

The YM ansatz contains in this case six functions: four magnetic potentials  $H_i$  ( $i = 1, \dots, 4$ ) and two electric potentials  $H_5, H_6$ . The boundary conditions satisfied by the magnetic potentials are similar to static case. For the electric potentials we impose at infinity  $H_5 = V \cos\theta$ ,  $H_6 = V \sin\theta$ . The parameter  $V$  corresponds to the asymptotic magnitude of the electric potential  $A_t(\infty)$  and determines the properties of these solutions. In the Abelian case, by using a suitable gauge transformation one can set  $V = 0$  (or any other value) without any loss of generality. In the non-Abelian theory, however, such a gauge transformation would render the whole configuration time dependent. In this case,  $V$  enters the asymptotic expansion for the non-Abelian field strength and has a physical relevance (although it does not contribute to the electric YM charge). As discussed in Appendix C, the value  $V = 0$  implies a purely magnetic, static configuration  $A_t = \Omega = 0$ . Technical details on these solutions, including the boundary conditions are presented in Appendices B and D.

Globally regular rotating solutions are found by starting with a purely magnetic EYM configuration with a given  $\omega_0$  and increasing the value of  $V$ . Here we consider only rotating solutions with the lowest winding number  $n = 1$ , although we obtained a number of configurations with  $n = 2$  also.

The branch structure of the spherically symmetric solutions is preserved in the presence of rotation. The rotating configurations depend on two continuous parameters: the value  $\omega_0$  of the magnetic potentials  $H_2, H_4$  at infinity (which fixed the magnetic charge  $Q_m = n|1 - \omega_0^2|$ ) and the magnitude of the electric potential at infinity  $V$ .

For a given value of the magnetic charge, a branch of rotating dyon solutions emerges smoothly from every spherically monopole solution and extends up to some maximal value of  $V$  beyond which gravity becomes too strong for regular dyons to persist. We notice that the value at the origin of the metric function  $f$  decreases with increasing  $V$  and tends to zero as  $V$  approaches the critical value  $V_{\max}$  (which is  $\omega_0$  dependent), corresponding to the formation of a horizon. With increasing  $V$ , the dyon becomes more and more deformed. The mass, angular momentum, and electric charge increase with  $V$  and we find again a maximal value for the magnitude of the electric potential at infinity  $V_{\max}$ . Alternatively, we may keep fixed the magnitude of the electric potential at infinity and vary the parameter  $\omega_0$ . Again, it has proven difficult to obtain rotating generalizations of the spherically symmetric solu-

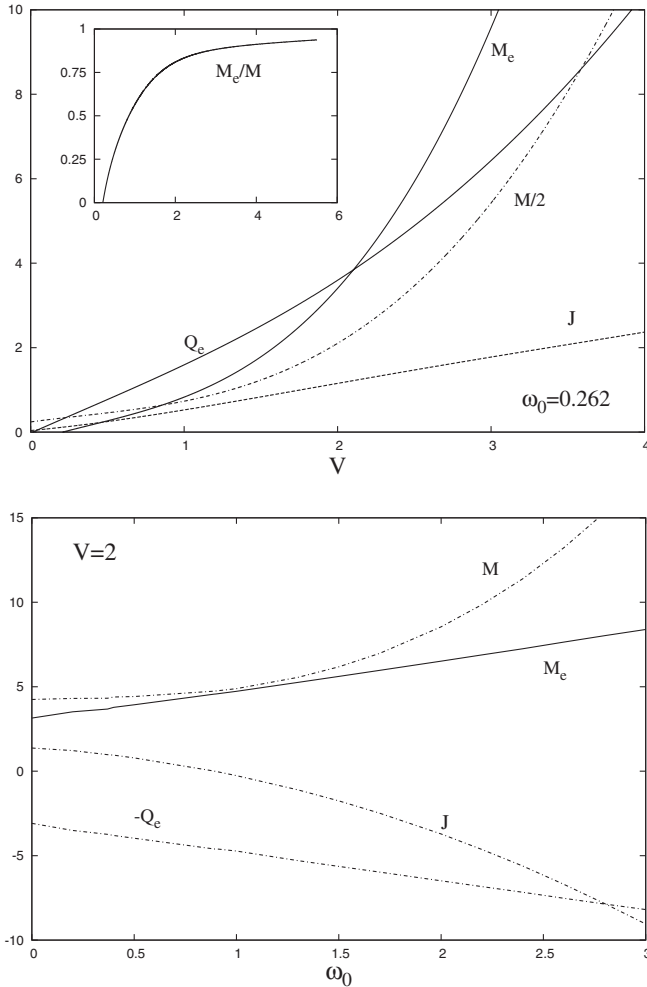


FIG. 4. The mass  $M$  and the angular momentum  $J$  of non-Abelian globally regular rotating solutions are shown as a function on the parameter  $V$  [Fig. 4(a), for  $\omega_0 = 0.262$ ] and the parameter  $\omega_0$  [Fig. 4(b),  $V = 2$ ]. Also shown are the electric charge  $Q_e$  and the contribution  $M_e$  of the non-Abelian electric field to the total energy of the system.

tions near the limits of the  $b$ -interval. In Fig. 4 we present the properties of typical branches of solutions for a fixed value of  $\omega_0$  [Fig. 4(a)] and for a fixed  $V$  [Fig. 4(b)].

All solutions we have found present nonvanishing non-Abelian electric and magnetic charges, representing rotating dyons. A vanishing  $Q_e$  implies a nonrotating, purely magnetic configuration. However, we find dyon solutions with vanishing total angular momentum (e.g.  $J = 0$  for  $\omega_0 = 0.895$ ,  $V = 2$ ) that are not static (locally  $T'_\varphi \neq 0$ ).

Similar to the static case, the functions  $H_2$  and  $H_4$  are nodeless or present one node only, although they have a small  $\theta$  dependence, while  $H_1$ ,  $H_3$  and the electric potentials depend on  $\theta$  in a complicated way. The metric functions  $f$ ,  $l$ ,  $m$  present a rather small angular dependence.

For all configurations, the energy density of the solutions has a strong peak along the  $\rho$  axis, and it decreases

monotonically along the symmetry axis, without being possible to clearly distinguish any individual component.

## B. Black hole configurations

The spherically symmetric black holes found by Winstanley in [6] were the first non-Abelian solutions with AdS asymptotics presented in the literature. More details on these configurations have been presented in [7], including dyonic black holes. These solutions obviously violate the no-hair conjecture and present static axially symmetric generalizations that are absent in the Abelian sector.

Also, it is well known that for  $\Lambda < 0$ , the EM theory has black hole solutions for which the topology of the horizon is an arbitrary genus Riemann surface. The EYM-SU(2) counterparts of these solutions with a nonspherical event horizon topology have been discussed in [15].

The properties of the AAdS non-Abelian solutions are strikingly different from those valid in the AF case. For example, black holes with  $\Lambda < 0$  exist for continuous intervals of the parameter space (the value of the gauge field on the event horizon), rather than discrete points. Also, there are configurations for which the gauge field has no zeros. Moreover, some of these configurations are stable within a perturbation theory approach.

Since the features of these configurations depend on the value of the cosmological constant, we present in this section an analysis of their properties for  $\Lambda = -3$ . Apart from some classes of configurations already known in the literature, we present here numerical arguments for the existence of rotating black holes with non-Abelian hair, generalizing for an SU(2) field the Kerr-Newman-AdS solution.

### 1. $n = 1$ static solutions

We start by discussing the better known spherically symmetric solutions and their topological black hole counterparts.

These solutions are obtained within a metric ansatz generalizing (12) for a nonspherically symmetric topology of the event horizon

$$ds^2 = \frac{dr^2}{H(r)} + r^2 d\Omega_k^2 - \frac{H(r)}{p^2(r)} dt^2, \quad (24)$$

where

$$H(r) = k - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \quad (25)$$

Here  $d\Omega_k^2 = d\theta^2 + f_k^2(\theta)d\varphi^2$  is the metric on a two-dimensional surface  $\Sigma$  of constant curvature  $2k$ .  $r$  is the radial coordinate for which  $r \rightarrow \infty$  defines the asymptotic region. The discrete parameter  $k$  takes the values 1, 0, and  $-1$  and implies the form of the function  $f_k(\theta)$

$$f_k(\theta) = \begin{cases} \sin\theta, & \text{for } k = 1 \\ \theta, & \text{for } k = 0 \\ \sinh\theta, & \text{for } k = -1. \end{cases} \quad (26)$$

The topology of a constant  $(t, r)$  slice is  $H_g^2$ . When  $k = +1$ , the universe takes on the familiar spherically symmetric form, and the  $(\theta, \varphi)$  sector has constant positive curvature. When  $k = 0$ , the  $\Sigma$  is a flat surface, while for  $k = -1$ , the  $(\theta, \varphi)$  sector is a space with constant negative curvature, also known as a hyperbolic plane. When  $\Sigma$  is closed, we denote its area by  $V_k$ . Also, we set  $V_k/4\pi = 1$  in all numerical data we present in this paper.

The construction of the SU(2) connection is presented in [15]. Taking into account the symmetries of the line element (24) we find

$$A = \frac{1}{2} \left\{ \omega(r) \tau_1 d\theta + \left( \frac{d \ln f_k}{d\theta} \tau_3 + \omega(r) \tau_2 \right) f_k d\varphi \right\}, \quad (27)$$

which reduces to (13) for  $k = 1$ .

As a result, we obtain a simplified YM curvature

$$F = \frac{1}{2} \{ \omega' \tau_1 dr \wedge d\theta + f_k \omega' \tau_2 dr \wedge d\varphi + (w^2 - k) f_k \tau_3 d\theta \wedge d\varphi \}. \quad (28)$$

The EYM equations reduce to

$$m' = \omega'^2 H + \frac{(\omega^2 - k)^2}{2r^2}, \quad \left( \frac{H\omega'}{p} \right)' = \frac{\omega(\omega^2 - k)}{pr^2}, \quad \frac{p'}{p} = -\frac{2}{r} \omega'^2. \quad (29)$$

We find the following expansion near the event horizon which is located at  $r = r_h > 0$ ,

$$\begin{aligned} m(r) &= \frac{r_h}{2} \left( k - \frac{\Lambda r_h^2}{3} \right) + m'(r_h)(r - r_h), \\ \omega(r) &= \omega_h + \omega'(r_h)(r - r_h), \end{aligned} \quad (30)$$

where

$$\begin{aligned} m'(r_h) &= \frac{(\omega_h^2 - k)^2}{2r_h^2}, \\ \omega'(r_h) &= \frac{r_h \omega_h (\omega_h^2 - k)}{(k - \Lambda r_h^2) r_h^2 - (\omega_h^2 - k)^2}, \end{aligned} \quad (31)$$

(since Eqs. (29) are invariant under the transformation  $\omega \rightarrow -\omega$ , it is enough to consider only values of  $\omega_h > 0$ ). For  $k = 1$ ,  $w(r) = \pm 1$  corresponds to vacuum Schwarzschild-AdS solution, while  $w(r) = 0$  is the Abelian RNAdS solution, with unit magnetic charge. Note that, apart from embedded Abelian configurations, no extremal solutions with reasonable asymptotics exist in this case.

The condition for a regular event horizon is  $H'(r_h) > 0$  and places a bound on  $\omega_h$

$$2m'(r_h) = \frac{(\omega_h^2 - k)^2}{r_h^2} < k - \Lambda r_h^2, \quad (32)$$

and implies positiveness of the quantity  $\omega'(r_h)$ . For  $\Lambda = -3$ , this relation implies  $\omega_h < \omega_{h(\max)}$ , with

$$\omega_{h(\max)}^2 = k + r_h \sqrt{3r_h^2 + k}, \quad (33)$$

(with a minimal event horizon radius  $r_h = 1/\sqrt{3}$  for  $k = -1$ ). The asymptotics as  $r \rightarrow \infty$  are still given by (17) for any value of  $k$ . As argued in Sec. IV, the configurations' mass is given by  $MV_k/4\pi$ , with  $M$  the asymptotic value of the metric function  $m(r)$ .

The Hawking temperature of the solutions is evaluated from the surface gravity  $\kappa$  as given by  $T_H = \kappa/(2\pi)$ , with

$$\kappa^2 = -(1/4) g^{tt} g^{ij} (\partial_i g_{jt}) (\partial_j g_{it}) |_{r=r_h}. \quad (34)$$

This implies

$$T_H = \frac{k - 2m'(r_h) - \Lambda r_h^2}{4\pi r_h p(r_h)}. \quad (35)$$

For every considered value of  $r_h$ , we find regular black hole solutions with  $w_h$  taking values in only one interval  $0 < \omega_h < \omega_h^c$ , where  $\omega_h^c$  is always smaller than  $\omega_{h(\max)}$ . There are also solutions for which  $\omega_0 > 1$  although  $\omega_h < 1$ . The behavior of the metric functions  $m$  and  $p$  are qualitatively similar for any value of  $k$ . However the gauge field behavior depends on the topology of the event horizon. For  $k = 0, -1$ , in contrast to the spherically symmetric case, we find only nodeless solutions. This can be analytically proven by integrating the equation for  $\omega$ ,  $(H\omega'/p)' = \omega(\omega^2 - k)/pr^2$  between  $r_h$  and  $r$ ; thus we obtain  $\omega' > 0$  for every  $r > r_h$ . For  $k = 1$ , solutions where  $\omega$  crosses the axis can exist for small enough values of  $r_h$ . For large values of  $r_h$  (e.g.  $r_h = 1$ ), only nodeless solutions are

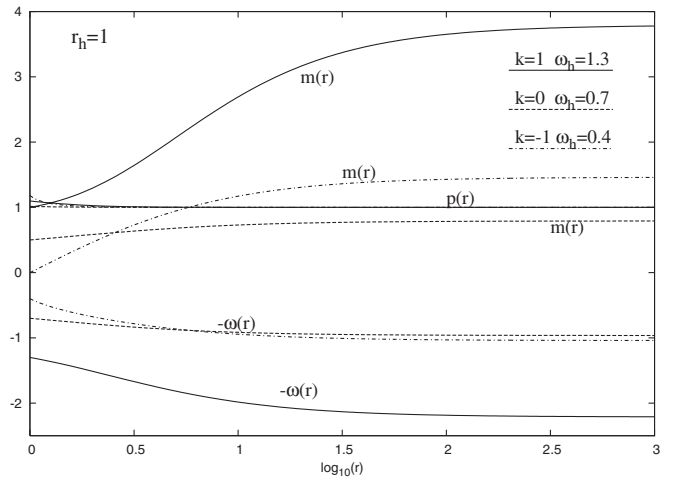


FIG. 5. Typical profiles of  $n = 1$  black hole monopole solutions. The solutions with  $k = 0, -1$  correspond to topological black holes with non-Abelian hair.



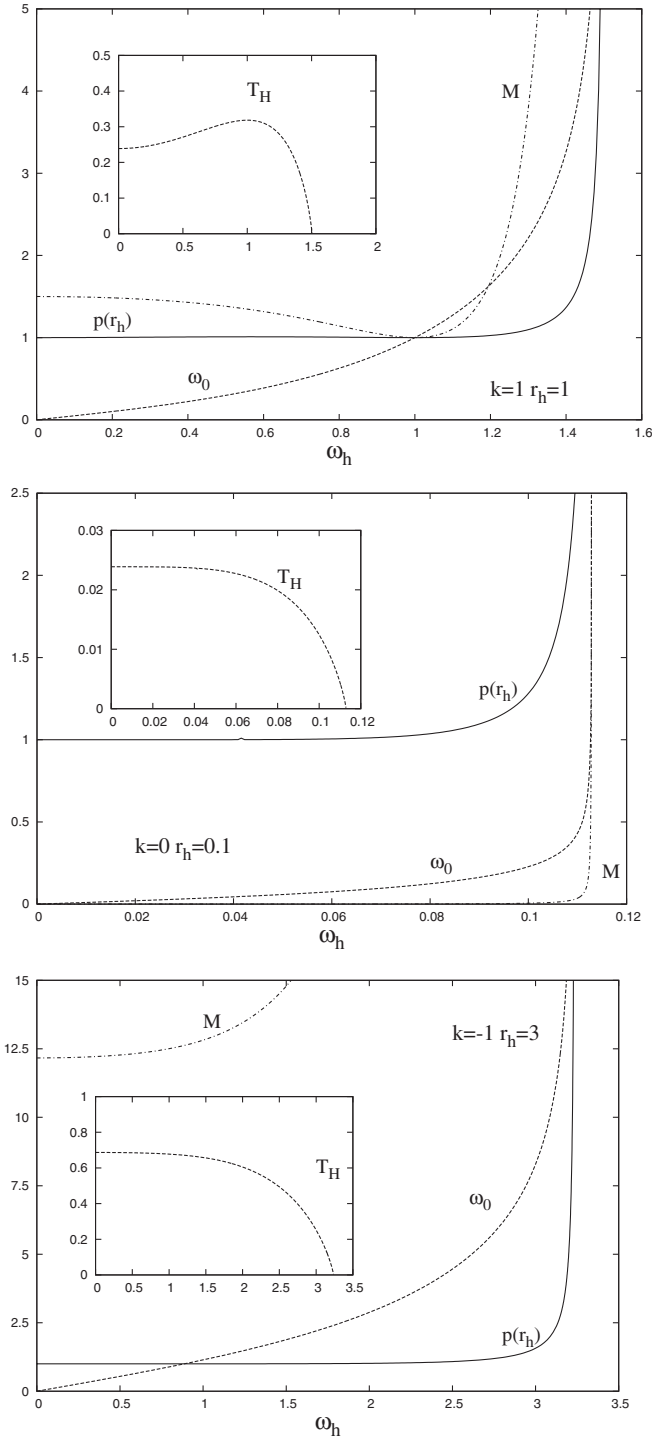


FIG. 6. The mass-parameter  $M$ , the asymptotic value of the gauge function  $\omega_0$ , the value of the metric function  $p(r)$  at the event horizon, and the Hawking temperature are represented as a function of the value of the gauge function  $\omega$  at the event horizon for typical monopole black hole solutions.

found in this case too. In this case, for sufficiently small  $\omega_h$ , all field variables remain close to their values for the Abelian configuration with the same  $r_h$ . Significant differences occur for large enough values of  $\omega_h$  and the effect of

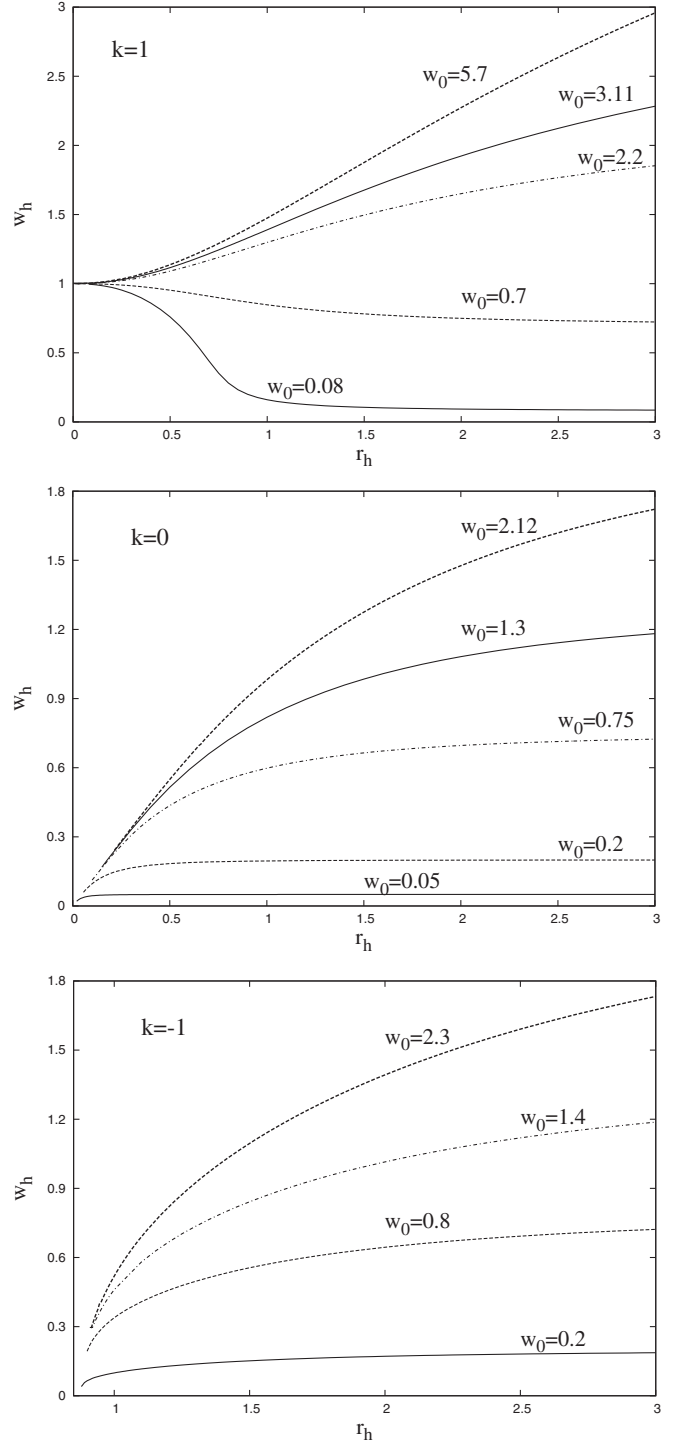


FIG. 7. The value of the non-Abelian gauge potential at the event horizon  $w_h$  is plotted as a function of the event horizon radius  $r_h$  for spherically symmetric and topological black hole monopole solutions and several values of the magnetic potential at infinity.

the non-Abelian field on the geometry becomes more and more pronounced.

In contrast to the picture found in [15], the  $k = -1, \Lambda = -3$  configurations always have  $m > 0$ . The black holes

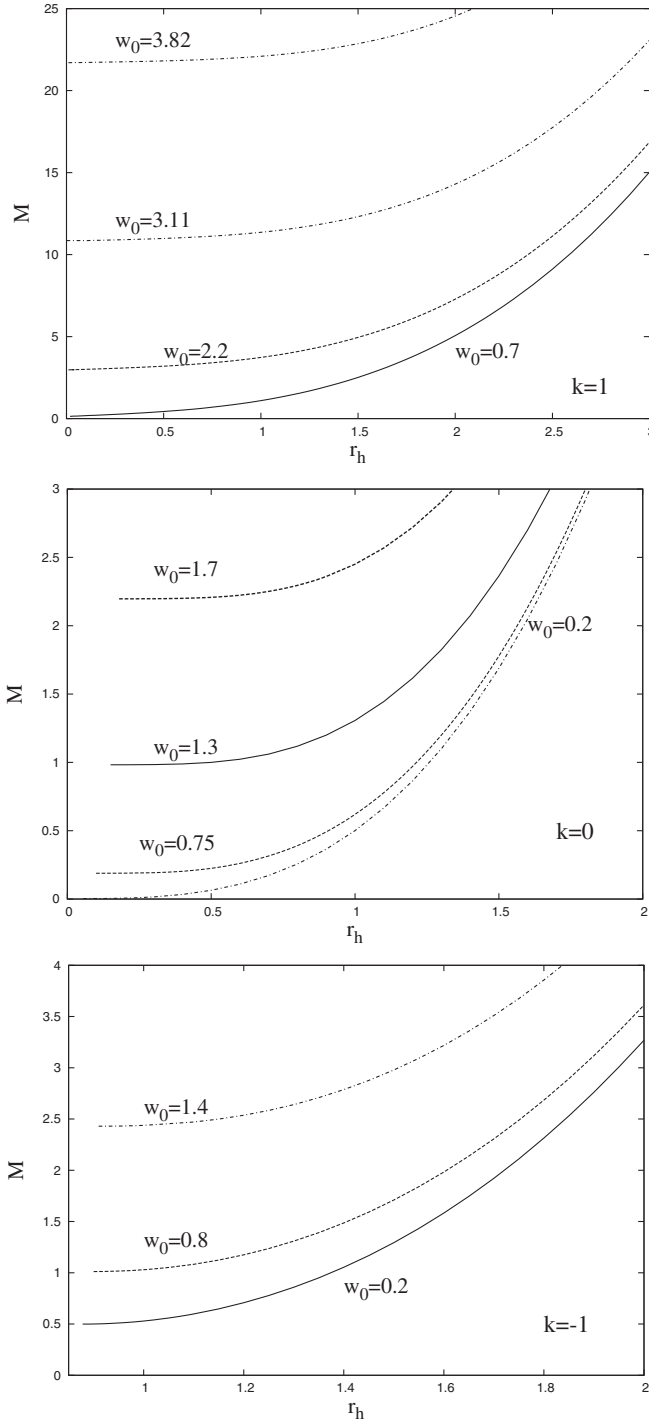


FIG. 8. The black hole mass  $M$  is plotted as a function of event horizon radius  $r_h$  for  $k = 1, 0, -1$  black hole monopole solutions with  $n = 1$  and several values of the magnetic potential at infinity.

therefore only occur with positive values of the mass. Typical solutions in this case are presented in Fig. 5. In Fig. 6,  $M$ ,  $\omega_0$ , the value of the metric function  $p(r)$  on the event horizon and the Hawking temperature are plotted as a function of  $\omega_h$  for  $k = 0, \pm 1$  black holes and several values of  $r_h$ . Note that as the temperature approaches

zero (i.e. as the solutions approach extremality), these physical quantities all diverge.

In Figs. 7 and 8 we plot the mass  $M$  and the value of the gauge potential at the event horizon  $w_h$  for several values of the asymptotic value of the magnetic gauge potential as a function of the event horizon radius. For  $k = 1$ , the corresponding solution with a regular origin is approached as  $r_h \rightarrow 0$ . For topological black holes, we noticed the existence of a minimal event horizon radius  $r_c$ , for any given  $w_0$ . The Hawking temperature vanishes as  $r_h \rightarrow r_c$  and a naked singularity develops, while the mass stays finite.

The discussion in [6,15] on the stability of these black hole solutions within the perturbation theory can easily be applied to  $\Lambda = -3$ . It follows that all  $k = 0$  solutions are stable; the  $k = -1$  solutions with  $\omega_0 > 1$  are also stable as well as the nodeless spherically symmetric solutions.

## 2. Static, axially symmetric black holes

Similar to the regular case, the  $k = 1, n = 1$  solutions discussed above admit static axially symmetric generalizations. (Static  $k = 0, -1$  topological black holes with a winding number  $n > 1$  are also likely to exist but the corresponding EYM ansatz has not yet been considered in the literature.)

The situation for a non-Abelian field is very different from the EM theory, where the static black hole solution is spherically symmetric (or, for  $\Lambda < 0$  belongs to one of the three cases (26), with the same amount of symmetry).

The properties of the AAdS axially symmetric EYM black holes were addressed in [12], however without consideration of the case  $\Lambda = -3$ . The metric ansatz in this case is given again by (18), with a gauge potential  $A_i$  written in terms of four functions  $H_i$ . We require the horizon of the black hole to reside at a surface of constant radial coordinate  $r = r_h$ , where  $g_{tt}(r_h) = 0$ . Similar to the procedure in the regular case, axially symmetric solutions are obtained by extending the  $n = 1$  configurations to higher values of the winding number. These solutions are AAdS and have a regular event horizon but for  $n > 1$  they are not spherically symmetric and the event horizon gets deformed away from spherical symmetry.

The boundary conditions at infinity and on the symmetry axis are similar to those used in the regular case; in particular, the asymptotic expansion (19) is still valid (see also Appendix B).

The surface gravity  $\kappa$  turns out to be constant at the horizon, as required by the zeroth law of black hole thermodynamics. To derive its expression we use the asymptotic expansion near the event horizon in  $\delta = (r - r_h)/r_h$

$$f(r, \theta) = f_2(\theta)\delta^2 + O(\delta)^3,$$

$$m(r, \theta) = m_2(\theta)\delta^2 + O(\delta)^3,$$

$$l(r, \theta) = l_2(\theta)\delta^2 + O(\delta)^3.$$

From  $T_H = \kappa/(2\pi)$ , we find for the Hawking temperature

$$T_H = \frac{f_2(\theta)(1 - \Lambda r_h^2/3)}{2\pi r_h \sqrt{m_2(\theta)}}, \quad (36)$$

which is constant as a consequence of the  $(r, \theta)$  Einstein equation implying  $f_2 m_{2,\theta} = 2m_2 f_{2,\theta}$ .

For the line element (24), the area  $A$  of the event horizon is given by

$$A = 2\pi \int_0^\pi d\theta \sin\theta \frac{\sqrt{l_2(\theta)m_2(\theta)}}{f_2(\theta)} r_h^2, \quad (37)$$

which allows a computation of the black hole entropy  $S = A/4$ .

The axially symmetric black hole solutions depend on two continuous parameters  $(r_h, \omega_0)$  as well as the winding number  $n > 1$ .

The behavior of the solutions is in many ways similar to that of the axially symmetric solitons. Again, starting from a spherically symmetric black hole configuration with given  $\omega_0$  we obtain higher winding number generalizations. Axially symmetric generalizations seem to exist for every spherically symmetric black hole solution. For a fixed winding number, the solutions form a branch, which can be indexed by the mass and the non-Abelian magnetic charge  $Q_m = n|1 - \omega_0^2|$ . This branch follows the picture found for  $n = 1$  (with higher values of mass, however). This is in sharp contrast to the  $\Lambda = 0$  case, where only a discrete set of solutions is found [32]. Also, the Kretschman scalar  $K = R_{ijkl}R^{ijkl}$  remains finite for every  $(r \geq r_h, \theta)$ . One finds that the deviation from spherical symmetry increases with growing  $n$ .

Once we have a solution, the horizon variables such as  $T_H$ ,  $A$  are calculated in a straightforward way from (36) and (37). The mass of the solution is computed by using the relation (20), extracting the values of the coefficients  $f_1, f_2$  from the asymptotics of the metric functions. In Fig. 9 we plot the mass  $M$ , the Hawking temperature, and the entropy as a function of  $\omega_0$  for black hole monopole solutions with  $r_h = 1$  and  $n = 1, 2, 3$ .

The gauge functions  $H_2, H_3, H_4$  start always at (angle dependent) nonzero values on the event horizon. For  $r_h = 1$ , we find only solutions where  $H_2, H_4$  do not cross the  $r$  axis. The gauge function  $H_2$  is always almost spherically symmetric, while the gauge functions  $H_1$  and  $H_3$  are much smaller than the functions  $H_2$  and  $H_4$ .

For the considered solutions, the metric functions  $m, f, l$  do not exhibit a strong angular dependence. These functions start with a zero value on the event horizon and approach rapidly the asymptotic values. The functions  $m$  and  $l$  have a rather similar shape, while the ratio  $m/l$  indicating the deviation from spherical symmetry is typically close to 1, except in a region near the horizon. The typical profiles of the metric and gauge functions we find

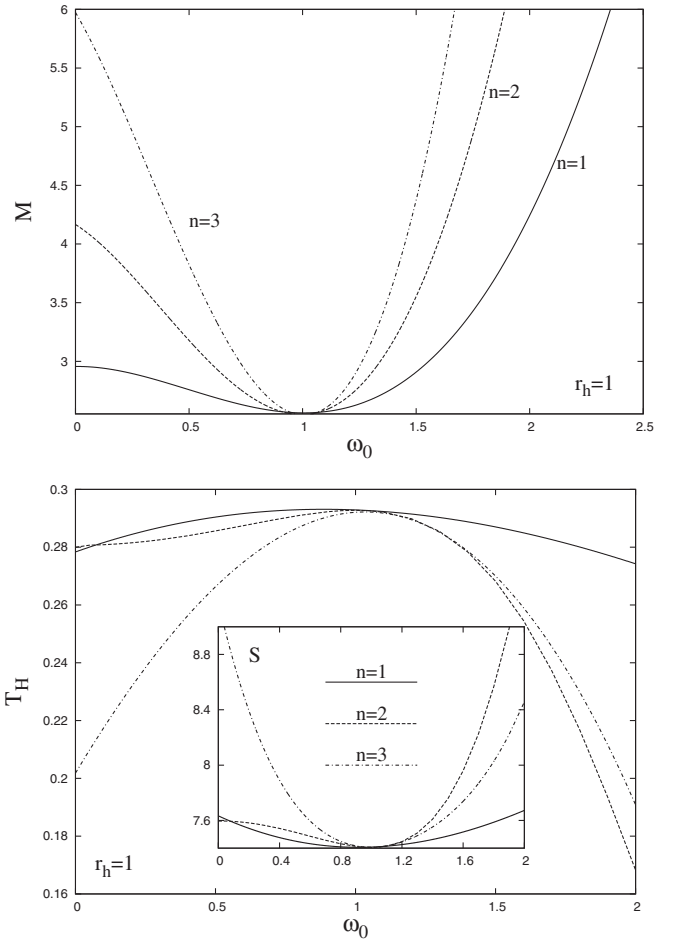


FIG. 9. The mass  $M$  [Fig. 9(a)] and the Hawking temperature and the entropy [Fig. 9(b)] are plotted as a function of  $\omega_0$  for static black hole monopole solutions with  $r_h = 1$ . The winding number  $n$  is also marked. The configurations with  $n = 2, 3$  are axially symmetric.

for  $\Lambda = -3$  are similar to those presented in [12] for other values of the cosmological constant.

The horizon has  $S^2$  topology, but geometrically is not a sphere, since its circumference along the equator  $L_e$  turns out to be different from that along a meridian  $L_p$

$$L_e = \int_0^{2\pi} d\varphi \sqrt{\frac{l}{f}} r \sin\theta \Big|_{r=r_h, \theta=\pi/2} = 2\pi r_h \sqrt{\frac{l_2(\theta)}{f_2(\theta)}} \Big|_{\theta=\pi/2}, \quad (38)$$

$$L_p = 2 \int_0^\pi d\theta \sqrt{\frac{m}{f}} r \Big|_{r=r_h, \varphi=\text{const}} = 2r_h \int_0^\pi d\theta \sqrt{\frac{m_2(\theta)}{f_2(\theta)}}. \quad (39)$$

However, for these static solutions, one finds a small deviation from spherical symmetry (as measured by the ratio  $L_e/L_p$ ), at the level of few percent.

### 3. Rotating black holes

AF rotating hairy black holes in EYM theory were obtained in [33], within the standard Lewis-Papapetrou parametrization of the metric and a gauge field ansatz consistent with the circularity and Froebenius conditions. These solutions possess three global charges: mass, angular momentum, and non-Abelian electric charge. Although they possess nontrivial magnetic gauge fields outside the event horizon, they do not carry a non-Abelian magnetic charge.

Obviously, the static AAdS solutions should also possess rotating counterparts, representing non-Abelian generalizations of the Kerr-Newman-AdS solution. However, the construction of such hairy rotating solutions represents a very difficult task since it involves the solution of a large number of coupled nonlinear partial differential equations for the metric and gauge field functions and a much richer set of possible boundary conditions as compared to the AF case.

Here we present the first set of rotating black hole configurations with  $\Lambda = -3$ . The ansatz we used in this case is similar to that employed for the regular rotating solitons, with the same asymptotic expansion as  $r \rightarrow \infty$ . In particular, the expressions (19) and (22) are valid in this case too, as well as the expressions (20) and (23) of the mass-energy and angular momentum. The boundary conditions and details on the numerical integration are presented in Appendices B and D.

For a given winding number, the rotating non-Abelian black hole solutions depend on four continuous parameters: two geometric parameters— $(r_h, \Omega_h)$  representing the event horizon radius and the value of the metric function  $\Omega$  at the horizon, respectively, and two parameters associated with the gauge field:  $\omega_0$  which gives the magnetic charge and  $V$  which is the asymptotic value of the electric non-Abelian potential  $A_t(\infty)$ . Not surprisingly, different from the AF case, rotating black holes are found also for  $V \neq 0$ .

The complete classification of the solutions in the space of these four physical parameters is a considerable task, whose scope is beyond the aim of this paper. We have studied mainly rotating configurations with  $r_h = 1$  and several values of  $\omega_0$ , although a number of solutions have been found for other values of  $r_h$ . Although rotating black hole solutions should exist for any value of  $n$ , we restrict here to a unit value of the winding number.

The properties of the horizon can be computed similar to the static case. The surface gravity is obtained from

$$\kappa^2 = -1/4(D_\mu \chi_\nu)(D^\mu \chi^\nu), \quad (40)$$

where the Killing vector  $\chi = \xi - (\Omega_h/r_h)\eta$  ( $\xi = \partial_t$ ,  $\eta = \partial_\varphi$ ) is orthogonal to and null on the horizon. It can be proven that the expansion near the event horizon (36) remains valid in the rotating case, which implies the expression (36) for the Hawking temperature. We further consider the area  $A$  of the black hole horizon computed

according to (37), and the deformation of the horizon, quantified by the ratio  $L_e/L_p$  of the circumferences along the equator and the poles.

To construct a rotating solution, we start from the corresponding static black hole configuration with  $\Omega_h = 0$ ,  $V = 0$ . As we increase  $(\Omega_h, V)$  from zero via the boundary conditions, while keeping  $r_h$  fixed, a first branch of solutions forms. For a given  $V$ , this branch ends at a critical value  $\Omega_h$ , which depends on the value of  $(\omega_0, r_h)$ , and the numerical errors increase dramatically for  $\Omega_h > \Omega_{h(\text{cr})}$  rendering the solutions increasingly less reliable. As  $\Omega_h \rightarrow \Omega_{h(\text{cr})}$ , the geometry remains regular with no event horizon appearing for  $r > r_h$ , and, the mass and angular momentum approach finite values. As found in [33] for  $\Lambda = 0$ , a second branch of solutions bends backward toward  $\Omega_h = 0$ ; there the mass and angular momentum diverge with  $\Omega_h^{-1}$  in the limit  $\Omega_h \rightarrow 0$ . Therefore we expect a similar picture in the AAdS case. However the numerical construction of such configurations presents a considerable numerical challenge beyond the scope of the present work. Also, the existence of other branches of AAdS rotating solutions, not necessarily connected to the static configurations, might be possible.

The picture gets simpler if we study the dependence of the solutions as a function of  $V$  (the magnitude of the electric YM potential at infinity) for fixed  $(\omega_0, r_h, \Omega_h)$ . In this case the solutions share a number of common properties with the rotating regular counterparts. In Fig. 10 we plot the mass, angular momentum, electric charge, and the contribution of the electric field to the total mass as a function of  $V$  for a fixed value of  $\omega_0$ . As seen in this picture, the mass, angular momentum, and electric

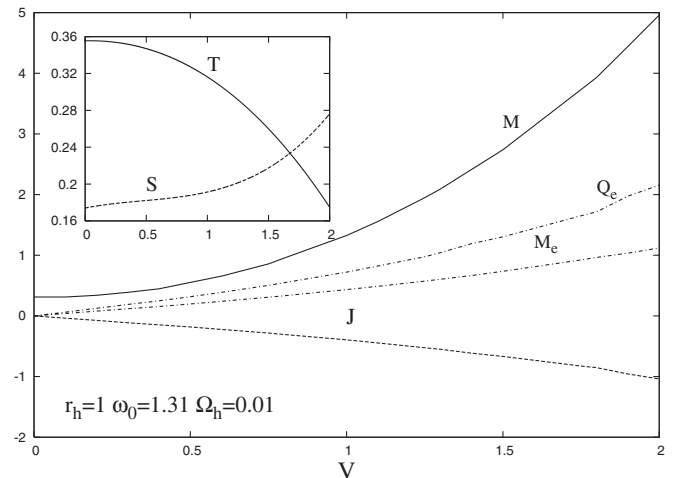


FIG. 10. The mass  $M$ , the angular momentum  $J$ , the electric charge  $Q_e$ , and the contribution  $M_e$  of the electric field to the total energy of the system of non-Abelian black hole rotating solutions are shown as a function on the parameter  $V$  for fixed values of  $r_h$ ,  $\omega_0$ , and  $\Omega_h$ . Also shown are the entropy and the Hawking temperature.

charges increase with  $V$  and we find again a maximal value for the magnitude of the electric potential at infinity.

The functions  $H_2$  and  $H_4$  are always nodeless, although they have a small  $\theta$  dependence, while  $H_1$ ,  $H_3$ , and the electric potentials depend on  $\theta$ -angle in a complicated way. The metric functions  $f$ ,  $l$ ,  $m$  present a rather small angular dependence, the metric function  $\Omega$  presenting a strong dependence of  $\theta$  for small values of  $V$ .

For all configurations, the energy density of the solutions has a strong peak along the  $\rho$  axis, and it decreases monotonically along the symmetry axis. In contrast to the rotating regular case, we found no locally rotating solutions with vanishing total angular momentum, although such configurations are likely to exist.

Further details on these rotating black hole solutions, as well as a discussion of the dependence of the solutions on the value of the cosmological constant, will be presented elsewhere.

## IV. A COMPUTATION OF PHYSICAL QUANTITIES

### A. The counterterm formalism

The mass, angular momentum, and action of the solutions discussed in Sec. III is found by using the counterterm formalism proposed by Balasubramanian and Kraus [34] to compute conserved quantities for a spacetime with a negative cosmological constant. This technique was inspired by AdS/CFT correspondence and consists in adding suitable counterterms  $I_{\text{ct}}$  to the action. These counterterms are built up with curvature invariants of a boundary  $\partial\mathcal{M}$  (which is sent to infinity after the integration) and thus obviously they do not alter the bulk equations of motion.

The following counterterms are sufficient to cancel divergences in four dimensions [35], for vacuum solutions with a negative cosmological constant (to agree with the standard conventions in literature, we set the usual factors  $1/16\pi G$  in the action principle (4) and  $1/8\pi G$  for the Gibbons-Hawking boundary term)

$$I_{\text{ct}} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{-h} \left[ \frac{2}{\ell} + \frac{\ell}{2} R \right]. \quad (41)$$

Here  $R$  is the Ricci scalar for the boundary metric  $h$ , while  $\ell^2 = -3/\Lambda = 1$ .

Using these counterterms one can construct a divergence-free stress tensor from the total action  $I = I_{\text{bulk}} + I_{\text{surf}} + I_{\text{ct}}$  by defining

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{\mu\nu}} \\ &= \frac{1}{8\pi G} \left( K_{\mu\nu} - K h_{\mu\nu} - \frac{2}{\ell} h_{\mu\nu} + \ell E_{\mu\nu} \right), \end{aligned} \quad (42)$$

where  $E_{ab}$  is the Einstein tensor of the intrinsic metric  $h_{ab}$ . The efficiency of this approach has been demonstrated in a broad range of examples, the counterterm subtraction method being developed for its own interest and applica-

tions. If there are matter fields on  $\mathcal{M}$  additional counterterms may be needed to regulate the action (see e.g. [36] for such an example in EYM-dilaton theory). However, we find that for a pure SU(2) non-Abelian matter content in four dimensions, the prescription (41) removes all divergences (a different situation is found for the five-dimensional AAdS non-Abelian solutions where the counterterm method fails and logarithmic divergences are presented in the total action and the expression of mass [16]).

Having obtained the boundary energy-momentum tensor, one can determine the conserved charges corresponding to the Killing vectors as explained in [34,35]. The usual prescription is to first pick a spacelike surface  $\Sigma$  on the boundary with metric  $\sigma_{ab}$ . The boundary metric is written in the following form [34]

$$h_{\mu\nu} dx^\mu dx^\nu = -N_\Sigma^2 dt^2 + \sigma_{ab} (dx^a + N_a^c dt) (dx^b + N_c^b dt). \quad (43)$$

The conserved charge associated to a symmetry generated by the Killing vector  $\chi^\mu$  is

$$Q_\chi = \int_\Sigma d^2x \sqrt{\sigma} T_{\mu\nu} n^\mu \chi^\nu, \quad (44)$$

where  $n^\mu$  is a timelike unit normal to  $\Sigma$ .

The conserved charge associated with time translation is the mass, the angular momentum being the charge associated with the Killing vector  $\partial_\varphi$ .

### B. $n = 1$ static configurations

We consider first the case of  $n = 1$  spherically symmetric and topological black hole static configurations. The results we find by using the asymptotic expressions (17) for the boundary stress tensor at large  $r$  are

$$\begin{aligned} T_{\theta\theta} &= \frac{1}{8\pi G} \frac{\ell M}{r} + \frac{1}{32\pi G} \frac{\ell(\ell^2 - 4(w_0^2 - k)^2)}{r^2} + O\left(\frac{1}{r^3}\right), \\ T_{\varphi\varphi} &= \frac{1}{8\pi G} \frac{\ell M}{r} f_k^2(\theta) + \frac{1}{32\pi G} \frac{\ell(\ell^2 - 4(w_0^2 - k)^2)}{r^2} f_k^2(\theta) \\ &\quad + O\left(\frac{1}{r^3}\right), \\ T_{tt} &= \frac{1}{8\pi G} \frac{2M}{\ell r} + \frac{1}{32\pi G} \frac{\ell^4 - 4\ell^2(w_0^2 - k)^2 - 8C_1^2}{r^2} \\ &\quad + O\left(\frac{1}{r^3}\right). \end{aligned} \quad (45)$$

Thus the leading order terms in this expression boundary stress tensor are similar to the (topological-) Schwarzschild-AdS<sub>4</sub> black holes. The presence of non-Abelian matter is manifest in the second order of this expansion only.

It can easily be verified that the mass of these solutions computed from (44) is given by  $MV_k/4\pi G$ , where  $V_k$  is the area of the surface  $\Sigma$ . Obviously, these solutions have a vanishing angular momentum.

### C. Axially symmetric configurations

We consider now the general case of an axially symmetric, rotating spacetime described by the line element (21). All relevant expressions can easily be derived by using the asymptotic form of the metric functions (19) and (22). The boundary metric in this case is the Einstein universe,  $ds^2 = \ell^2(d\theta^2 + \sin^2\theta d\varphi^2) - dt^2$ .

One finds in this way the large- $r$  expansion

$$\begin{aligned} T_{\theta\theta} &= -\frac{1}{8\pi G\ell r} \left( \frac{2f_1}{3} + \frac{6f_2}{17} \sin^2\theta \right) + O\left(\frac{1}{r^2}\right), \\ T_{\varphi\varphi} &= -\frac{\sin^2\theta}{8\pi G\ell r} \left( \frac{2f_1}{3} + \frac{18f_2}{17} \sin^2\theta \right) + O\left(\frac{1}{r^2}\right), \\ T_{tt} &= -\frac{1}{8\pi G\ell^3 r} \left( \frac{4f_1}{3} + \frac{24f_2}{17} \sin^2\theta \right) + O\left(\frac{1}{r^2}\right), \\ T_{t\varphi} &= -\frac{3}{16\pi G\ell r} (j_1 + j_2 \sin^2\theta) \sin^2\theta + O\left(\frac{1}{r^2}\right). \end{aligned} \quad (46)$$

Direct computation shows that this stress tensor is traceless. This result is expected from the AdS/CFT correspondence, since even dimensional bulk theories with  $\Lambda < 0$  are dual to odd dimensional CFTs that have a vanishing trace anomaly. The corresponding expressions for the static case are found by taking  $j_1 = j_2 = 0$  in the above relations. For spherically symmetric configuration found within the metric ansatz (19),  $f_2 = 0$  and the angular dependence vanishes, as expected.

By using this relation, we find that the mass and angular momenta of the solutions are given by the relations (20) and (23), respectively. For static solutions, the parameter  $f_2$  describes the deviation of the solutions from the spherical symmetry.

### D. Euclidean action and entropy

One can use the counterterm expression (41) to compute the regularized gravitational action and to prove that the entropy of the AAdS hairy black holes is one quarter of the event horizon area.

Here we start by constructing the path integral [23]

$$Z = \int D[g] D[\Psi] e^{-iI[g, \Psi]}, \quad (47)$$

integrating over all metrics and matter fields between some given initial and final hypersurfaces,  $\Psi$  corresponding to the SU(2) potentials. By analytically continuing the time coordinate  $t \rightarrow i\tau$ , the path integral formally converges, and in the leading order one obtains

$$Z \simeq e^{-I_{\text{cl}}}, \quad (48)$$

where  $I_{\text{cl}}$  is the classical action evaluated on the equations of motion of the gravity/matter system. Since the Euclidean approach becomes problematic for non-Abelian solutions with an electric potential [37], we restrict here to compute the action of static, purely magnetic solutions. We should also remark that the variation of the

action (4) gives the correct equations of motion only if the gauge potential  $A_\mu$  is held fixed on the boundary  $\partial M$ . This imposes the boundary condition  $\delta A_\mu = 0$  on  $\partial M$ , which for purely magnetic solutions fixes the value of magnetic charge.

The globally regular solutions have an arbitrary periodicity  $\beta$  of the Euclidean time coordinate. In the black hole case, the value of  $\beta$  is found by demanding regularity of the Euclideanized manifold as  $r \rightarrow r_h$ . It can easily be verified that the Hawking temperature expression  $T_H = 1/\beta$  found in this way coincides with that given by the surface gravity computation.

The physical interpretation of this formalism is that the class of regular stationary metrics forms an ensemble of thermodynamic systems at equilibrium temperature  $T_H$  [38].  $Z$  has the interpretation of partition function and we can define the free energy of the system  $F = -\beta^{-1} \log Z$ .

Therefore

$$\log Z = -\beta F = S - \beta M, \quad (49)$$

or

$$S = \beta M - I_{\text{cl}}, \quad (50)$$

straightforwardly follows, with  $S$  the entropy of the system.

To compute  $I_{\text{cl}}$ , we make use of the Einstein equations, replacing the  $R - 2\Lambda$  volume term with  $2R'_t - 16\pi GT'_t$ . For a purely magnetic ansatz ( $A_t = 0$ ), the term  $T'_t$  exactly cancels the matter field Lagrangian in the bulk action  $L_m = -1/2 \text{Tr}(F_{\mu\nu} F^{\mu\nu})$ . The Ricci component  $R'_t$  is computed by integrating the Killing identity  $\nabla^a \nabla_b K_a = R_{bc} K^c$ , for the Killing vector  $K^a = \delta_t^a$ . The divergent contribution given by the surface integral term at infinity in  $R'_t$  is also canceled by  $I_{\text{surface}} + I_{\text{ct}}$ , yielding a finite expression of the action. For the metric ansatz (24) describing spherically symmetric and topological black holes static configurations one finds

$$I_{\text{cl}} = \beta \frac{MV_k}{4\pi} - \frac{r_h^2 V_k}{16\pi G} \quad (51)$$

while the corresponding expression for static axially symmetric configurations described by the metric ansatz (18) is

$$\begin{aligned} I_{\text{cl}} &= \beta \left( \frac{\Lambda}{3G} \left( \frac{2f_1}{3} + \frac{8f_2}{17} \right) - \frac{r_h}{4G} \left( 1 - \frac{\Lambda r_h^2}{3} \right) \right. \\ &\quad \left. \times \int_0^\pi d\theta \sin\theta \sqrt{l_2(\theta)} \right). \end{aligned} \quad (52)$$

The corresponding expressions in the globally regular case are found by taking  $r_h \rightarrow 0$  in the above relations.

Replacing now in (50) (where  $M$  is the mass-energy computed in Section III B 2 and III B 3, we find  $S = 0$  in the absence of an event horizon, while the entropy of the black hole solutions is one quarter of the event horizon area, as expected.

**E. The thermodynamics of  $n = 1$  static black holes**

Based on the numerical results presented in Sec. III, we attempt here a discussion of the thermodynamic properties the  $\Lambda = -3$  EYM black solutions. To our knowledge these have not been previously considered.

For simplicity, we will restrict our considerations to  $n = 1$  static configurations (i.e. spherically symmetric or topological black holes). Thus we shall analyze black hole thermodynamics in a canonical ensemble, holding the temperature  $T$  and the magnetic potential at the boundary at infinity (i.e. the magnetic charge) fixed. The associated potential is the Helmholtz free energy  $F$ .

The response function whose sign determines the thermodynamic stability is the heat capacity

$$C = T \left( \frac{\partial S}{\partial T} \right)_{Q_m}. \tag{53}$$

Stability follows from  $C \geq 0$  given the fact that black holes radiate at higher temperatures when they are smaller.

The behavior of the specific heat  $C$  can be easily understood from the state equation  $S = S(T)$  at fixed magnetic charge  $Q_m$ . In Fig. 11 we plot these curves for several values of  $Q_m$  for  $k = 1, 0$ , and  $-1$  black holes with unit winding number.

For  $k = 1$ , the usual Schwarzschild-AdS behavior is reproduced: the curves first decrease toward a minimum, corresponding to the branch of small unstable black holes, then increase along the branch of large stable black holes. This is in strong contrast with the behavior of the Abelian RNAdS solutions, which present only one branch and approach the extremal limit as  $T \rightarrow 0$  [see Fig. 11(a)]. Note that the  $k = 1$  solutions with small values around zero of the gauge potential at infinity appear to present a complicated thermodynamic structure. However these solutions are unstable to small perturbations and we will not consider them here.

As seen in Fig. 11(b) and 11(c), the heat capacity (53) is always positive for hairy black holes with zero or negative curvature horizon, (this appears to be valid for any  $w_0$ ). As a result, the  $k = 0, 1$  topological black hole solutions are always thermodynamically locally stable.

It is instructive to plot also the free energy  $F = I/\beta$  as a function of temperature and various values of the magnetic charge (Fig. 12). One can see again that for  $k = 1$  there are always two black hole radii associated with each temperature, for any value of  $Q_m$ . Correspondingly, the smaller branch is unstable having negative specific heat. However, the action of the  $k = 1$  solutions becomes positive for some critical value of the event horizon radius, for any value of the non-Abelian magnetic charge.

In the vacuum case, this indicates the existence of a phase transition. When the free energy is negative, the Schwarzschild-AdS black hole phase is dominant over the thermal AdS background phase. When the free energy changes its sign, the Hawking-Page phase transition be-

tween the AdS black hole and the thermal AdS background takes place [39].

As seen in Fig. 8, for  $k = 1$  monopole configurations the globally regular solution has minimal energy in its asymptotic class of solutions (with a given  $Q_m$ ) and so is the

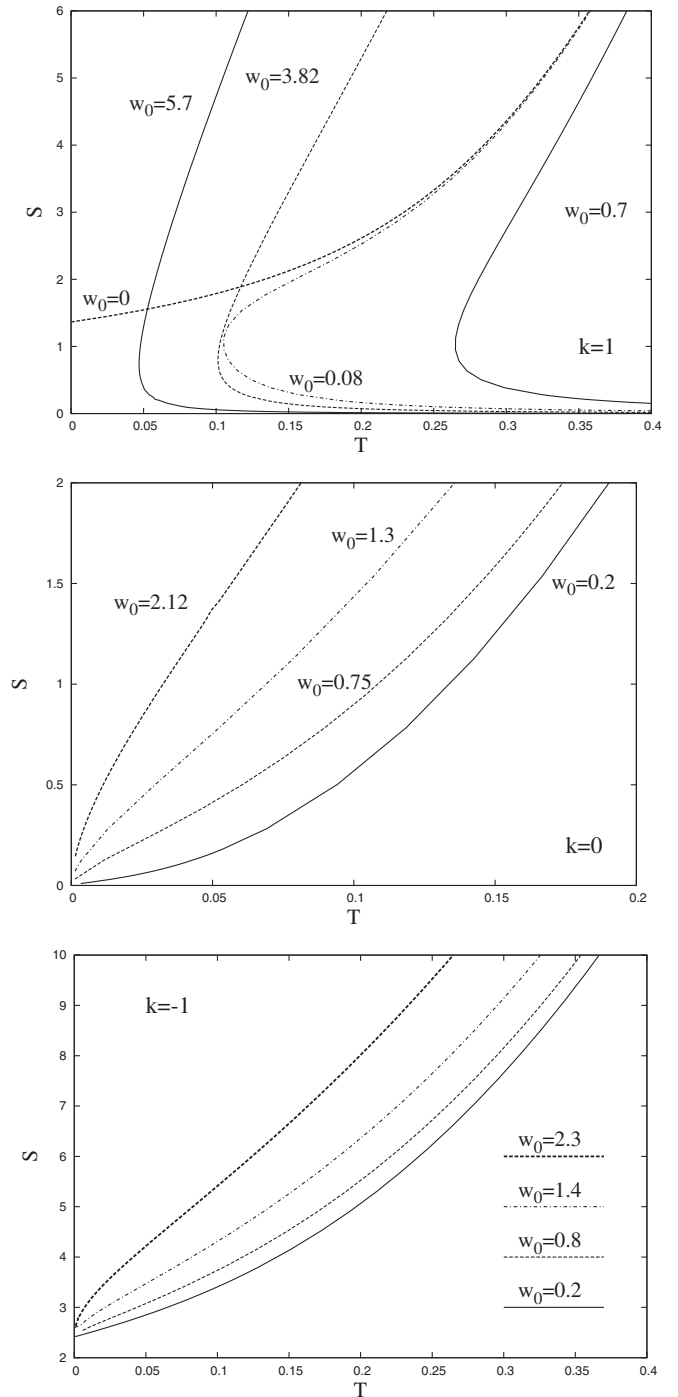


FIG. 11. The entropy is plotted as a function of temperature for  $k = 1, 0, -1$  black hole monopole solutions and several values of the magnetic potential at infinity. Here and in Fig. 12(a), the  $k = 1$  curve with  $w_0 = 0$  corresponds to RNAdS Abelian black holes with unit magnetic charge.

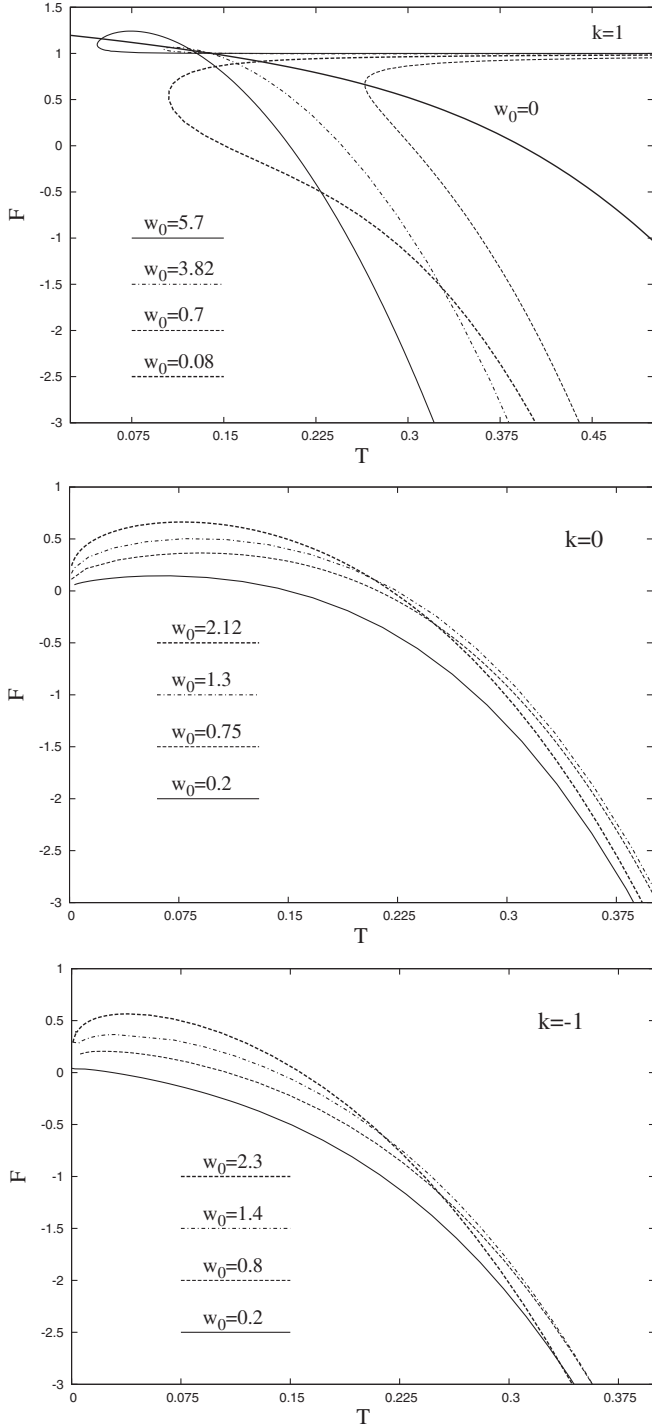


FIG. 12. The free energy  $F = M - TS$  is plotted as a function of temperature for  $k = 1, 0, -1$  black hole monopole solutions and several values of the magnetic potential at infinity.

thermal background. Thus a phase transition should exist between the large black hole solutions and the corresponding globally regular configurations.

In the absence of matter fields, the action of  $k = 0, -1$  black holes is always negative. Therefore these configurations can also be globally stable and there is no phase

transition. The inclusion of YM fields changes this behavior and the low temperature solutions have a positive action.

### F. On the boundary CFT

Restricting to static solutions, we find from  $\gamma_{\mu\nu} = \lim_{r \rightarrow \infty} \frac{\ell^2}{r^2} h_{\mu\nu}$  the following background metric upon which the dual field theory resides

$$\gamma_{\mu\nu} dx^\mu dx^\nu = \ell^2 (d\theta^2 + f_k^2(\theta) d\varphi^2) - dt^2. \quad (54)$$

For  $k = 1$  this describes a  $2 + 1$ -dimensional Einstein universe; for  $k = 0$  it is  $2 + 1$  flat space, while  $k = -1$  describes the three-dimensional open static universe.

It would be desirable to compute some quantities in this background and to compare the results with the bulk predictions. The main problem is that, even in the vacuum case, the  $\text{AdS}_4/\text{CFT}_3$  correspondence is much less understood than the  $\text{AdS}_5/\text{CFT}_4$  case. For example, in [40], Klebanov and Polyakov proposed a duality between a theory of massless higher spin gauge fields in  $\text{AdS}_4$  spacetime on the one hand and the  $O(N)$  vector model at large  $N$  in three dimensions on the other. However, although the details of the boundary CFT will depend on the details of the bulk supergravity theory, the generic properties are expected to be independent of the precise features of the theory.

Here we should remark that as found in Sec. III, the non-Abelian matter field in the bulk does not approach asymptotically a pure gauge configuration. For static solutions, the boundary form of the unit winding number non-Abelian potential is

$$A^{(0)} = \frac{1}{2} \left\{ \omega_0 \tau_1 d\theta + \left( \frac{d \ln f_k(\theta)}{d\theta} \tau_3 + \omega_0 \tau_2 \right) f_k(\theta) d\varphi \right\}, \quad (55)$$

with  $w_0$  a real constant. The corresponding boundary gauge field expression for  $k = 1$  and a winding number  $n > 1$  is

$$A^{(0)} = \frac{1}{2} (1 - w_0) \{ \tau_\varphi^n d\theta - n \sin\theta \tau_\theta^n d\varphi \}, \quad (56)$$

where  $\tau_\varphi^n(\theta, \varphi)$ ,  $\tau_\theta^n(\theta, \varphi)$  are suitable combinations of the Pauli matrices, whose form is given in Appendix A. One can see that the winding number enters the boundary gauge field expression.

From the  $\text{AdS}/\text{CFT}$  correspondence, we expect the non-Abelian hairy black holes to be described by some thermal states in a dual theory formulated in a background given by (54). The spherically and axially symmetric solitons will correspond to zero-temperature states in the same theory. This CFT will interact with a background  $\text{SU}(2)$  field given by (55) and (56).

As conjectured in Ref. [22], the dual field theory is the field theory of a stack of  $N$  coincident  $M2$  branes with a background external field coupled to the  $R$ -symmetry cur-



rent. The bosonic sector of the Lagrangian for a single  $M2$  brane is [22]

$$I = \int d^3x \sqrt{-\gamma} (D_a \Phi D^a \Phi + \frac{1}{8} R \Phi^2), \quad (57)$$

where  $D = \partial - iA^{(0)}$  is the gauge covariant derivative, and  $R$  is the Ricci scalar of the boundary CFT metric  $\gamma_{ab}$ .

Computing quantum effects for a generic  $SU(2)$  background field is a difficult task. Simpler results are found only for the case  $w_0 = 0$ . For spherical symmetry, the expressions (55) and (56) describe the field of a Dirac monopole with  $n$  units of magnetic charge

$$A^{(0)} = \frac{\tau_3}{2} n \cos\theta d\varphi. \quad (58)$$

(For  $n > 1$  this is proven by transforming the gauge connection (A5) to a special gauge such that  $A_\varphi$  has only a  $\tau_3$  component.)

A computation of the effective action  $I_{\text{eff}}$  for a charged singlet scalar field propagating in a zero-temperature Euclideanized  $d = 3$  Einstein-universe background and interacting with the  $U(1)$  field given by (58) is presented in Appendix E, based on a zeta-function approach. From the basic relation (1), we expect  $I_{\text{eff}}$  to present a qualitative agreement with the corresponding bulk computation. First we note that a straightforward evaluation of (E3) and (E14) gives positive values for  $I_{\text{eff}}(n)$ , as expected from the bulk results (with the bulk action of globally regular solutions  $I_B = \beta M$ ). It is also interesting to compute the ratio  $I_{\text{eff}}(n)/I_{\text{eff}}(1)$  and to compare with the bulk results. One finds e.g.  $I_{\text{eff}}(2)/I_{\text{eff}}(1) \simeq 3.89$ ,  $I_{\text{eff}}(3)/I_{\text{eff}}(1) \simeq 8.52$  while  $I_B(2)/I_B(1) \simeq 2.46$ ,  $I_B(3)/I_B(1) \simeq 4.72$ . The discrepancy between the bulk and boundary results increases with  $n$ .

However, on the AdS side, there are two distinct bulk configurations with zero temperature and magnetic charge  $n$ . First, there is the non-Abelian soliton solution with charge  $n$  discussed in the previous section. The second solution is an extremal RNAdS black hole with zero temperature and the same value of magnetic charge. It is not clear how the CFT knows to distinguish between these different bulk solutions. In any case, this is not possible within the model (57).

There is also another problem with the action principle (57). Since the  $d = 4$  Einstein-Maxwell system has electric-magnetic duality [41], one expects this duality to also be manifest on the CFT side. However, its realization in  $d = 3$  is a rather subtle question (see e.g. the discussion in [42]). For example, the  $U(1)$  dual of (58) is a  $A^{(0)} = \tau_3 c dt/2$  with  $c^2 = n\sqrt{(-1 + \sqrt{1 + 12n^2})/6}$  for an extremal RNAdS solution. We can also perform a computation similar to that in Appendix E, for an electric  $A^{(0)}$ . One can see that, in the zero-temperature limit, the parameter  $c$  will not enter the final results, which are similar to the vacuum case. Thus we conclude that the model (57) is too

simple to mimic the expected features of the boundary CFT.

However, we can use the AdS/CFT ‘‘dictionary’’ to predict qualitative features of a quantum field theory in the background (54). For example, the expectation value of the dual CFT stress tensor can be calculated using the relation [43]

$$\sqrt{-\gamma} \gamma^{ab} \langle \tau_{bc} \rangle = \lim_{r \rightarrow \infty} \sqrt{-h} h^{ab} T_{bc}. \quad (59)$$

Applying this prescription to the  $n = 1$  static solutions, we find the standard form for the stress tensor of a  $(2 + 1)$ -dimensional CFT

$$\langle \tau^{ab} \rangle = \frac{M}{8\pi\ell^2} [3u^a u^b + \gamma^{ab}], \quad (60)$$

where  $u^a = \delta_t^a$ .

A similar computation can be done for an axially symmetric configuration in the bulk. Considering the more general rotating case, we find the field theory stress tensor

$$\langle \tau_b^a \rangle = \langle \tau_b^a \rangle^{(\text{st})} + \langle \tau_b^a \rangle^{(\text{rot})}, \quad (61)$$

where  $\langle \tau_b^a \rangle^{(\text{st})}$  is a contribution which survives in the static limit

$$\langle \tau_b^a \rangle^{(\text{st})} = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad (62)$$

where

$$A = \frac{1}{8\pi\ell^2} \left( M + \frac{8f_2}{17} \frac{1}{G\ell^2} \right), \quad (63)$$

$$B = -\frac{1}{8\pi G\ell^4} \frac{6f_2}{17} \sin^2\theta,$$

and  $x^1 = \theta$ ,  $x^2 = \varphi$ ,  $x^3 = t$ . Here  $M$ ,  $f_2$  are continuous variables which encode the bulk parameters.  $\langle \tau_b^a \rangle^{(\text{rot})}$  is the part of the CFT induced by the rotation in the bulk and has the expression

$$\langle \tau_{ab} \rangle^{(\text{rot})} = C \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{with} \quad (64)$$

$$C = \frac{1}{8\pi G} \frac{3\ell^2}{2} (j_1 + j_2 \sin^2\theta) \sin^2\theta.$$

The CFT stress tensor is covariantly conserved and manifestly traceless. Even for static configurations in the bulk, a winding number  $n > 1$  of the bulk configurations implies  $f_2 \neq 0$  and thus a  $\theta$ -dependence of the dual theory stress tensor (although the boundary metric is spherically symmetric). This is a unique property of AAdS gravitating non-Abelian configurations, since the boundary stress tensor of an Abelian solution with the same global charges has  $f_2 = 0$ . This also suggests the dual theory should also be sensi-

tive to the integer  $n$ , and is much more complex than the simple model (57).

The form (61) of the dual CFT stress tensor is puzzling from yet another point of view, since there is no global charge associated with the parameter  $f_2$ . However, the expression (61) means that the dual CFT is able to discern between  $n > 1$  non-Abelian and embedded Abelian bulk solutions with the same set of boundary data.

## V. CONCLUSIONS

In this paper we have discussed the basic features of the non-Abelian solutions of an EYM-SU(2) theory with a negative cosmological constant  $\Lambda = -3g^2$  (where  $g$  is the gauge coupling constant of the YM theory). This theory corresponds to a consistent truncation of  $d = 4\mathcal{N} = 4$  SO(4) gauged supergravity. Except for the NUT-charged solutions, we have considered the  $\Lambda = -3g^2$  version of all monopole configurations previously discussed in the literature for a generic value of the cosmological constant. This includes both particlelike globally regular and black hole solutions. These configurations are asymptotically AdS, possessing a regular origin or a regular event horizon. Apart from spherically symmetric solutions (or topological black holes) we presented arguments for the existence of  $\Lambda = -3g^2$  static, axially symmetric solutions. These configurations have no counterparts in the Abelian theory. They generalize to higher winding number the known spherically symmetric solutions, presenting a nontrivial angle-dependence of matter fields and metric functions.

The main feature of the EYM AAdS configurations is the existence of a nonvanishing non-Abelian magnetic flux on the sphere at infinity. The thermodynamics of the static black hole solutions has been also discussed to some extent. Apart from static solutions, we discussed AAdS configurations with a nonvanishing angular momentum which have not been presented before in the literature.

All known EYM asymptotically flat configurations are likely to present  $\Lambda = -3g^2$  generalizations. Thus we expect the existence AAdS counterparts of the EYM configurations discussed in [44], satisfying a complicated angle-dependent set of boundary conditions at infinity. It would be interesting to construct AAdS non-Abelian solutions which possess only discrete symmetries [45] and to find the corresponding boundary stress tensor.

One should remark that all solutions discussed here may be uplifted to  $d = 11$  supergravity. However, the solutions we discussed here are generically not supersymmetric. Supersymmetric solutions are likely to exist, but we expect them to present naked singularities (this is the case of the Abelian counterparts with the same amount of symmetry). Note that the planar Bogomol'nyi-Prasad-Sommerfield (BPS) solution of the  $\Lambda = -3g^2$  EYM model found in closed form in [12] has a naked singularity.

Apart from a discussion of the physical properties of various bulk EYM configurations, we attempted a prelimi-

nary discussion of these solutions in an AdS/CFT context. The  $\Lambda = -3g^2$  EYM configurations should give us information about the structure of a dual CFT in a background SU(2) field. However, a naive computation of the effective action of a charged scalar field in a U(1) magnetic background field gave some inconclusive results. Further progress in this direction would require a better knowledge of the structure of the dual theory defined on the  $2 + 1$ -dimensional boundary metric. The results we discussed in Sec. IV appear to indicate that the dual CFT is able to distinguish between various bulk solutions with the same set of boundary data at infinity.

We think that these issues deserve further study, which should lead to a deeper understanding of the AdS/CFT correspondence.

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## APPENDIX A: THE AXIALLY SYMMETRIC GAUGE FIELD ANSATZ

The construction of an axially symmetric YM ansatz has been discussed by many authors starting with Manton [46] and Rebbi and Rossi [47]. The most general axially symmetric YM-SU(2) ansatz contains nine magnetic and three electric potentials and can be easily obtained in cylindrical coordinates  $x^\mu = (\rho, \varphi, z)$

$$A_\mu = \frac{1}{2}A_\mu^{(\rho)}(\rho, z)\tau_\rho^n + \frac{1}{2}A_\mu^{(\varphi)}(\rho, z)\tau_\varphi^n + \frac{1}{2}A_\mu^{(z)}(\rho, z)\tau_z^n, \quad (\text{A1})$$

where the only  $\varphi$ -dependent terms are the SU(2) matrices (composed of the standard  $(\tau_1, \tau_2, \tau_3)$  Pauli matrices)

$$\begin{aligned} \tau_\rho^n &= \cos n\varphi \tau_1 + \sin n\varphi \tau_2, \\ \tau_\varphi^n &= -\sin n\varphi \tau_1 + \cos n\varphi \tau_2, \quad \tau_z^n = \tau_3. \end{aligned} \quad (\text{A2})$$

This ansatz contains an integer  $n$ , representing the winding number with respect to the azimuthal angle  $\varphi$ .

Transforming to spherical coordinates  $r, \theta, \varphi$ , with  $(\rho = r \sin\theta, z = r \cos\theta)$ , it proves convenient to introduce, without any loss of generality, a new SU(2) basis  $(\tau_r^n, \tau_\theta^n, \tau_\varphi^n)$ , with

$$\tau_r^n = \sin\theta \tau_\rho^n + \cos\theta \tau_z^n, \quad \tau_\theta^n = \cos\theta \tau_\rho^n - \sin\theta \tau_z^n. \quad (\text{A3})$$

The general expression (A1) takes the following form in spherical coordinates

$$A_\mu = \frac{1}{2}A_\mu^{(r)}(r, \theta)\tau_r^n + \frac{1}{2}A_\mu^{(\theta)}(r, \theta)\tau_\theta^n + \frac{1}{2}A_\mu^{(\varphi)}(r, \theta)\tau_\varphi^n, \quad (\text{A4})$$

where  $A_\mu^a dx^\mu = A_r^a dr + A_\theta^a d\theta + A_\varphi^a d\varphi + A_t^a dt$ . This ansatz is axially symmetric in the sense that a rotation around the  $z$ -axis can be compensated by a gauge rotation  $\mathcal{L}_\varphi A = D\Psi$  [24], with  $\Psi$  being a Lie-algebra valued gauge function. For the ansatz (A4),  $\Psi = n \cos\theta \tau_r^n / 2 - n \sin\theta \tau_\theta^n / 2$ . Therefore we find  $F_{\mu\varphi} = D_\mu W$ , where  $W = A_\varphi - \Psi$ .

We use in this paper a reduced YM ansatz, employed also in all previous studies on EYM solutions, with five of the gauge potentials taken identically zero

$$A_r^{(r)} = A_r^{(\theta)} = A_\theta^{(r)} = A_\theta^{(\theta)} = A_\varphi^{(\varphi)} = A_t^{(\varphi)} = 0.$$

The consistency of this reduction can easily be proven at the level of the YM field equations. A suitable parametrization of the six nonzero components of  $A_\mu^a$  which factorizes the trivial  $\theta$ -dependence is [33]

$$\begin{aligned} A_\mu dx^\mu = & \left[ \frac{H_1}{r} dr + (1 - H_2)d\theta \right] \frac{\tau_\varphi^n}{2} \\ & - n \sin\theta \left[ H_3 \frac{\tau_r^n}{2} + (1 - H_4) \frac{\tau_\theta^n}{2} \right] \left( d\varphi - \frac{\Omega}{r} dt \right) \\ & + \left[ H_5 \frac{\tau_r^n}{2} + H_6 \frac{\tau_\theta^n}{2} \right] dt. \end{aligned} \quad (\text{A5})$$

One may consistently take  $\Omega = 0$  in this ansatz; however the inclusion of this metric function simplifies the set of boundary conditions for the rotating configurations. For  $\Omega = H_5 = H_6 = 0$ , the static axially symmetric ansatz used in previous studies on AAdS static EYM solutions is recovered; further, by taking  $n = 1$ ,  $H_1 = H_3 = 0$ ,  $H_2 = H_4 = w(r)$  one finds a spherically symmetric static ansatz which reduces to (13) after a suitable gauge transformation.

The ansatz (A5) satisfies also some additional discrete symmetries [47,48] (in particular the parity reflection symmetry) and it is also invariant under Abelian gauge transformations  $U$

$$U = \exp\left(\frac{i}{2} \tau_\varphi^n \Gamma(r, \theta)\right). \quad (\text{A6})$$

To fix this residual gauge degree of freedom we choose the usual gauge condition [32,33]

$$r\partial_r H_1 - \partial_\theta H_2 = 0.$$

The nonvanishing components of the field strength tensor are given by

$$\begin{aligned} F_{r\theta}^\varphi &= -\frac{1}{r} [H_{1,\theta} + rH_{2,r}], \\ F_{r\varphi}^r &= -\frac{\sin\theta}{r} [rH_{3,r} - H_1 H_4], \\ F_{r\varphi}^\theta &= \frac{\sin\theta}{r} [rH_{4,r} + H_1(H_3 + \cot\theta)], \\ F_{\theta\varphi}^r &= -\sin\theta [H_{3,\theta} + H_3 \cot\theta + H_2 H_4 - 1], \\ F_{\theta\varphi}^\theta &= \sin\theta [H_{4,\theta} + \cot\theta(H_4 - H_2) - H_2 H_3], \\ F_{tr}^r &= -\frac{1}{r} \left[ rH_{5,r} + H_1 H_6 - \frac{\Omega}{r} \sin\theta (H_1(1 - H_4) - H_3 \right. \\ & \quad \left. + rH_{3,r}) - \Omega_{,r} \sin\theta H_3 \right], \\ F_{rt}^\theta &= \frac{1}{r} \left[ rH_{6,r} - H_1 H_5 + \frac{\Omega}{r} \sin\theta (H_1 H_3 + (1 - H_4) \right. \\ & \quad \left. + rH_{4,r}) - \Omega_{,r} \sin\theta (1 - H_4) \right], \\ F_{\theta t}^r &= \left[ H_{5,\theta} - H_2 H_6 + \frac{\Omega}{r} \sin\theta (H_2(1 - H_4) - \cot\theta H_3 \right. \\ & \quad \left. - H_{3,\theta}) - \frac{\Omega_{,\theta}}{r} \sin\theta H_3 \right], \\ F_{\theta t}^\theta &= \left[ H_{6,\theta} + H_2 H_5 - \frac{\Omega}{r} \sin\theta (H_2 H_3 + \cot\theta (1 - H_4) \right. \\ & \quad \left. - H_{4,\theta}) - \frac{\Omega_{,\theta}}{r} \sin\theta (1 - H_4) \right], \\ F_{\varphi t}^\varphi &= \sin\theta \left[ H_5 H_4 + H_6 (H_3 + \cot\theta) \right. \\ & \quad \left. - \frac{\Omega}{r} \sin\theta (\cot\theta (1 - H_4) + H_3) \right]. \end{aligned} \quad (\text{A7})$$

One can easily verify the matter ansatz is compatible with the metric form (21), since the energy-momentum tensor (7) satisfies  $T_{tr} = T_{t\theta} = T_{\varphi r} = T_{\varphi\theta} = 0$ .

## APPENDIX B: BOUNDARY CONDITIONS

### 1. Static axially symmetric solutions

These solutions are obtained for a truncation  $A_t = 0$  of the ansatz (A5) and a metric given by (18).

To obtain AAdS axially symmetric configurations with a regular origin or event horizon and with the proper symmetries, we must impose the appropriate boundary conditions. The boundaries of the system are the origin/event horizon and spacelike infinity, the  $z$ -axis and, because of parity reflection symmetry satisfied by the matter fields, the  $\rho$ -axis. The boundary conditions at infinity and along the  $z$ - and the  $\rho$ -axis (i.e. for  $\theta = 0, \pi/2$ ) are similar for both globally regular and black hole solutions.

We start by setting the boundary conditions at infinity compatible with the AAdS assumption. For the metric functions one imposes

$$f|_{r=\infty} = m|_{r=\infty} = l|_{r=\infty} = 1, \quad (\text{B1})$$

while the boundary conditions for the matter part are

$$H_2|_{r=\infty} = H_4|_{r=\infty} = \omega_0, \quad H_1|_{r=\infty} = H_3|_{r=\infty} = 0, \quad (\text{B2})$$

where there are no obvious conditions on the value of  $\omega_0$ . For a solution with parity reflection symmetry (the only type we consider in this paper), the boundary conditions along the axes are

$$\begin{aligned} H_1|_{\theta=0,\pi/2} &= H_3|_{\theta=0,\pi/2} = 0, \\ \partial_\theta H_2|_{\theta=0,\pi/2} &= \partial_\theta H_4|_{\theta=0,\pi/2} = 0, \\ \partial_\theta f|_{\theta=0,\pi/2} &= \partial_\theta m|_{\theta=0,\pi/2} = \partial_\theta l|_{\theta=0,\pi/2} = 0. \end{aligned} \quad (\text{B3})$$

Therefore we need to consider the solutions only in the region  $0 \leq \theta \leq \pi/2$ . Regularity on the  $z$ -axis requires also

$$H_2|_{\theta=0} = H_4|_{\theta=0}, \quad m|_{\theta=0} = l|_{\theta=0}. \quad (\text{B4})$$

For globally regular solutions, the boundary conditions imposed at the origin are

$$\begin{aligned} \partial_r f|_{r=0} &= \partial_r m|_{r=0} = \partial_r l|_{r=0} = 0, \\ H_2|_{r=0} &= H_4|_{r=0} = 1, \quad H_1|_{r=0} = H_3|_{r=0} = 0. \end{aligned} \quad (\text{B5})$$

The boundary conditions satisfied by the black hole solutions at the event horizon are

$$\begin{aligned} f|_{r=r_h} &= m|_{r=r_h} = l|_{r=r_h} = 0, \quad H_1|_{r=r_h} = 0, \\ \partial_r H_2|_{r=r_h} &= \partial_r H_3|_{r=r_h} = \partial_r H_4|_{r=r_h} = 0. \end{aligned} \quad (\text{B6})$$

## 2. Rotating solutions

These solutions are found for metric form (21) and the YM ansatz (A5). Similar to the static case, we consider solutions with parity reflection symmetry i.e.  $0 \leq \theta \leq \pi/2$ .

A systematic analysis reveals that the boundary conditions for the magnetic potentials  $H_1, \dots, H_4$  and the metric functions  $f, l, m$  presented in the static case remain valid in the presence of rotation. For the supplementary functions  $H_5, H_6, \Omega$  we imposed

$$\begin{aligned} \Omega|_{r=\infty} &= 0, \quad H_5|_{r=\infty} = V \cos\theta, \\ H_6|_{r=\infty} &= V \sin\theta \end{aligned} \quad (\text{B7})$$

at infinity, and

$$\begin{aligned} \partial_\theta \Omega|_{\theta=0,\pi/2} &= 0, \quad \partial_\theta H_5|_{\theta=0} = H_5|_{\theta=\pi/2} = 0, \\ H_6|_{\theta=0} &= \partial_\theta H_6|_{\theta=\pi/2} = 0 \end{aligned} \quad (\text{B8})$$

on the axis. For globally regular solutions, the following set of boundary conditions is imposed at the origin

$$H_5|_{r=0} \sin\theta + H_6|_{r=0} \cos\theta = 0,$$

$$\partial_r H_5|_{r=0} \cos\theta - \partial_r H_6|_{r=0} \sin\theta = 0, \quad \Omega|_{r=0} = 0. \quad (\text{B9})$$

The black hole solutions are found imposing at the event horizon

$$\begin{aligned} r_h H_5|_{r=r_h} + \cos\theta \Omega_h &= 0, \quad r_h H_6|_{r=r_h} - \sin\theta \Omega_h = 0, \\ \Omega|_{r=r_h} &= \Omega_h. \end{aligned} \quad (\text{B10})$$

## APPENDIX C: GENERAL RELATIONS

Solutions of the field equations are also classified by the non-Abelian electric and magnetic charges  $Q_e$  and  $Q_m$ . The definition of conserved currents and charges in a non-Abelian Yang-Mills theory is a problem approached by different authors in the last decades (see e.g. [49–52]), with various solutions. A gauge invariant definition for the non-Abelian charges was proposed in [51] (see also [33])

$$Q_e = \frac{1}{4\pi} \oint d\theta d\varphi |\tilde{F}_{\theta\varphi}|, \quad Q_m = \frac{1}{4\pi} \oint d\theta d\varphi |F_{\theta\varphi}|, \quad (\text{C1})$$

where the vertical bars denote the Lie-algebra norm and the integrals are evaluated as  $r \rightarrow \infty$ . The expression for the magnetic charge implied by this definition is  $Q_m = |k - \omega_0^2|V_k/4\pi$  for spherically symmetric and topological black hole solutions and  $Q_m = n|1 - \omega_0^2|$  for axially symmetric configurations.

The magnetic charge defined in this way is equal (up to a sign) to the expression found by using the usual (gauge dependent) definition

$$Q_m = \frac{1}{4\pi} \int dS_k \sqrt{-g} \text{Tr}\{\tilde{F}^{kt} T\}, \quad (\text{C2})$$

(with  $T = \tau_3$  for the gauge ansatz (27) and  $T = \tau_r$  for the axially symmetric generalization (A5)).

In evaluating the electric charge expression one uses the asymptotic expansion of the electric potential

$$\begin{aligned} H_5 &\sim \cos\theta(V + (c_1 \sin^2\theta + c_2)/r), \\ H_6 &\sim \sin\theta(V + (c_3 \sin^2\theta + c_4)/r), \end{aligned} \quad (\text{C3})$$

(with  $c_i$  real constants).

The energy density of the solutions is given by the  $tt$ -component of the energy-momentum tensor  $T_\mu^\nu$ . Of interest here is the electric part of this component,  $\text{Tr}\{F_{\mu t} F^{\mu t}\}$  and its integral

$$-M_e = \int \text{Tr}\{F_{\mu t} F^{\mu t}\} \sqrt{-g} d^3x, \quad (\text{C4})$$

which measures the contribution of the non-Abelian electric field to the mass/energy of the system. Similar to the purely Abelian part, by using the YM Eqs. (8) this integral can be expressed as a total divergence

$$\begin{aligned}
-M_e &= \int \text{Tr}\{F_{\mu t} F^{\mu t}\} \sqrt{-g} d^3x \\
&= \oint_{\infty} \text{Tr}\{A_t F^{\mu t}\} dS_{\mu} - \oint_{eh} \text{Tr}\{A_t F^{\mu t}\} dS_{\mu}. \quad (C5)
\end{aligned}$$

Thus, for globally regular configurations, a vanishing magnitude of the electric potentials at infinity implies a purely magnetic solution. In contrast, one finds rotating black hole solutions with  $V = 0$  which are supported by the event horizon contribution.

Here we remark that the angular momentum of any solution admits also an expression in terms of surface integrals of the matter fields [28]. This can easily be proven by using the existence of a potential  $W$  (9) and the YM equation

$$\begin{aligned}
J &= \int T_{\varphi}^t \sqrt{-g} d^3x = \int 2 \text{Tr}\{F_{r\varphi} F^{rt} + F_{\theta\varphi} F^{\theta t}\} \sqrt{-g} d^3x \\
&= \oint_{\infty} 2 \text{Tr}\{W F^{\mu t}\} dS_{\mu} - \oint_{eh} 2 \text{Tr}\{W F^{\mu t}\} dS_{\mu}. \quad (C6)
\end{aligned}$$

The above relation takes a particularly simple form for globally regular configurations. By using the gauge field asymptotics, the total angular momentum in terms of matter field coefficients is

$$J = \frac{4\pi n}{3} \left( -\frac{2}{5} c_1 - c_2 + \omega_0 \left( \frac{8}{5} c_3 + 2c_4 \right) \right). \quad (C7)$$

In the black hole case, (C7) should be supplemented with the event horizon contribution. This relation can also be used to test the accuracy of the numerical results.

#### APPENDIX D: NUMERICAL ALGORITHM

The spherically symmetric and topological black hole solutions can easily be found by using a standard differential equations solver. Starting with a suitable set of boundary conditions at the origin/event horizon the equations are integrated towards  $r \rightarrow \infty$  using an automatic step procedure and accuracy  $10^{-12}$ . The integration stops when the AdS spacetime asymptotics are reached with a prescribed accuracy.

The situation is much more complicated for axially symmetric configurations, which requires the solving of seven (in the static case) or ten nonlinear partial differential equations. All axially symmetric solutions presented in this paper have been found by using a similar approach to that employed by Kleihaus and Kunz in their studies of AF non-Abelian configurations. We use in the numerical procedure a suitable combination of the EYM equations such that the differential equations for the metric and gauge functions are diagonal in the second derivatives with respect to  $r$ . These equations are then discretized on a  $(r, \theta)$  grid with  $N_r \times N_{\theta}$  points. The angular coordinate  $\theta$  runs from 0 to  $\pi/2$  and the radial coordinate goes from  $r_h$  to some large enough value  $r_{\max}$  (typically  $r_{\max} \simeq 2 \times 10^3 \div 5 \times 10^3$ ). For any type of solution, we tested that the relevant quan-

ties are insensitive to the cutoff value  $r_{\max}$ . The grid spacing in the  $r$ -direction is nonuniform, while the values of the grid points in the angular direction are given by  $\theta_k = (k-1)\pi/(2(N_{\theta}-1))$ . Typical grids have sizes  $150 \times 30$  points. We monitored also the remaining Einstein equations which are not directly solved, assuring that they are satisfied with a reasonable accuracy.

In this scheme, a new radial variable is introduced which maps the semi infinite region  $[r_c, \infty)$  to the closed region  $[0, 1]$  (with  $r_c = 0$  or  $r_h$ ). For the globally regular solutions, our choice for this transformation was  $x = r/(r+1)$ . For the derivatives this leads to the substitutions

$$\begin{aligned}
rF_{,r} &\rightarrow x(1-x)F_{,x}, \\
r^2F_{,r,r} &\rightarrow x^2((1-x)^2F_{,x,x} - 2(1-x)F_{,x}) \quad (D1)
\end{aligned}$$

for any function  $F$  in the differential equations. For the black hole solutions, we employed a new coordinate  $x$  defined as  $x = 1 - r_h/r$ , which leads to the following substitutions in the differential equations

$$\begin{aligned}
rF_{,r} &\rightarrow (1-x)F_{,x}, \\
r^2F_{,r,r} &\rightarrow (1-x)^2F_{,x,x} - 2(1-x)F_{,x} \quad (D2)
\end{aligned}$$

for any function  $F$ .

The resulting system is solved iteratively until convergence is achieved. All numerical calculations for axially symmetric configurations are performed by using the program FIDISOL (written in Fortran), based on the iterative Newton-Raphson method. A detailed presentation of the FIDISOL code is presented in [53]. This code requests the system of nonlinear partial differential equations to be written in the form  $P(r, \theta, u, u_r, u_{\theta}, u_{r\theta}, u_{rr}, u_{\theta\theta}) = 0$ , (where  $u$  denotes the unknown functions) subject to a set of boundary conditions on a rectangular domain. The user must deliver to FIDISOL the equations, the boundary conditions, the Jacobian matrices for the equations and the boundary conditions, and some initial guess functions. The numerical procedure works as follows: for an approximate solution  $u^{(1)}$ ,  $P(u^{(1)})$  does not vanish. Next step is to consider an improved solution  $u^{(2)} = u^{(1)} + \Delta u$ , supposing that  $P(u^{(1)} + \Delta u) = 0$ . The expansion in the small parameter  $\Delta u$  gives in the first order  $0 = P(u^{(1)} + \Delta u) \simeq P(u^{(1)}) + \frac{\partial P}{\partial u}(u^{(1)})\Delta u$ . This equation can be used to determine the correction  $\Delta u^{(1)} = \Delta u$ . Repeating the calculations iteratively ( $u^{(3)} = u^{(2)} + \Delta u^{(2)}$  etc.), the approximate solutions will converge, provided the initial guess solution is close enough to the exact solution. The iteration stops after  $i$  steps if the Newton residual  $P(u^{(i)})$  is smaller than a prescribed tolerance. Therefore it is essential to have a good first guess, to start the iteration procedure. Our strategy therefore is to use a known solution as guess and then vary some parameter to produce the next solution.

To obtain axially symmetric solutions, we start always with the  $n = 1$  solution as initial guess and increase the

value of the relevant parameters slowly. For static solutions, the parameter we vary is the winding number  $n$ . The physical values of  $n$  are integers. Rotating configurations are found e.g. by increasing the magnitude of the electric potential at infinity (for black holes, we vary also  $\Omega_\mu$ ). The iteration is done in small steps and eventually converges with a good enough accuracy. Repeating the procedure one obtains in this way solutions for requested values of the relevant parameters. For some of the configurations, we interpolate the resulting configurations and use them as a starting guess on a finer grid.

FIDISOL automatically provides also an error estimate for each function, which is the maximum of the discretization error divided by the maximum of the function [53]. For the solutions discussed in this paper, the typical numerical error for the functions is estimated to be on the order of  $10^{-3}$ . The output of the code was analyzed and visualized mainly with MATHEMATICA.

### APPENDIX E: CHARGED SCALAR FIELD IN AN EINSTEIN UNIVERSE WITH A U(1) DIRAC MONOPOLE BACKGROUND

We consider the following action principle for a non-minimally coupled scalar field interacting with a background U(1)-field  $A^{(0)}$

$$I[\phi] = - \int (D_\mu \phi D^\mu \phi^* + M^2 \phi \phi^* + \xi R \phi \phi^*) \times \sqrt{g(x)} d^3x, \quad (\text{E1})$$

where  $D = \partial - iA^{(0)}$ ,  $M$  is the scalar field mass, and  $\xi$  determines the coupling with the scalar curvature  $R = 2k/\ell^2$ .

The zeta-function approach implies the computation of the eigenfunctions  $\phi_N$  and the eigenvalues  $\lambda_N$  of the differential second-order self-adjoint operator  $A = -D^\mu D_\mu + M^2 + \xi R$ . Thus we consider the series with  $s \in \mathbb{C}$  (the prime on the sum means that any possible null eigenvalues are omitted)

$$\zeta(s|A) = \sum'_N \lambda_N^{-s}. \quad (\text{E2})$$

As is well known, this series converges provided  $\text{Re } s > D/2$ . It is possible to continue the above sum into a meromorphic function of  $s$  that is regular at  $s = 0$  [54]. In a path-integral approach, the effective action for a scalar field can be formally expressed as the functional determinant of the operator  $A$  as

$$I_{\text{eff}} = -\frac{1}{2} \ln \det(A/\mu^2), \quad (\text{E3})$$

where  $\mu$  is an arbitrary renormalization mass scale coming from the path-integral measure. This determinant however is a formally divergent quantity and needs to be regularized.

In a zeta-function renormalization framework, the regularized determinant reads

$$\ln \det(A/\mu^2) = -\zeta'(0|A) - \zeta(0|A) \ln \mu^2. \quad (\text{E4})$$

We note that since  $\zeta(0|A) = 0$  in odd dimensions (which is our case), the dependence on the renormalization scale drops out.

The eigenvalue equation  $A\phi_N = \lambda_N\phi_N$  can be solved by using the ansatz

$$\phi_N = e^{i(m\varphi - \omega t)} F(\theta), \quad (\text{E5})$$

where  $F(\theta)$  is a solution of the equation

$$\frac{1}{\ell^2 f_k(\theta)} \frac{d}{d\theta} \left( f_k(\theta) \frac{dF}{d\theta} \right) - \frac{m^2}{\ell^2 f_k^2(\theta)} (m + Q_m f'_k(\theta))^2 F + (\omega^2 + M^2 + \xi R) F = \lambda_N F, \quad (\text{E6})$$

satisfying certain boundary conditions at the limits of the  $\theta$  interval. Here we will restrict to the  $k = 1$  case of a zero-temperature  $2 + 1$ -dimensional Einstein universe background and a U(1) field  $A^{(0)} = Q_m \cos\theta d\varphi$ . Note that  $m$  should be an integer for  $k = 1$ .

The substitution  $F = g/\sqrt{\sin\theta}$ ,  $\theta = 2x$  transforms (E6) into the quantum mechanical problem of the Schrödinger equation with a Pöschl-Teller potential

$$-\frac{d^2 g}{dx^2} + \left( (m - Q_m)^2 - \frac{1}{4} \right) \frac{g}{\sin^2 x} + \left( (m + Q_m)^2 - \frac{1}{4} \right) \frac{g}{\cos^2 x} = (4Q_m^2 + 1 - 4\ell^2(\omega^2 + M^2 + \xi R - \lambda_N))g = 0, \quad (\text{E7})$$

whose solutions are well known, see e.g. [55]. This leads to an eigenvalue expression

$$\lambda_N = \omega^2 + \frac{1}{4}(1 + |m + Q_m| + |m - Q_m| + 2n)^2 + M^2 + \xi R - Q_m^2 - 1/4, \quad (\text{E8})$$

where  $(N = m, n, \omega)$ . The corresponding eigenfunctions are

$$\phi_N(x) = \left( \cos \frac{\theta}{2} \right)^{|m+Q_m|+1/2} \times \left( \sin \frac{\theta}{2} \right)^{|m-Q_m|} P^{(|m-Q_m|, |m+Q_m|)}(\cos\theta) e^{i(m\varphi - \omega t)}, \quad (\text{E9})$$

with  $P^{(a,b)}(x)$  the Jacobi polynomials.

After integrating over  $\omega$ , we arrive at a sum on the form

$$\zeta(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{1}{2} (1 + |m + Q_m| + |m - Q_m| + n)^2 + M^2 + \xi R - Q_m^2 - \frac{1}{4} \right)^{-s+1/2}. \quad (\text{E10})$$

Restricting to the case of a massless, conformally coupled field, the resulting zeta function can be written as

$$\zeta(s) = \sqrt{\pi} \frac{\ell^{2s-1}}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\Gamma(s+k-\frac{1}{2})}{k!} Q_m^{2k} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left( n + \frac{1}{2} (1 + |m + Q_m| + |m - Q_m|) \right)^{-2s-2k+1}. \quad (\text{E11})$$

Neglecting an additional contribution arising as a result of interchanging the order of summation, one performs first the sum over  $(m, n)$ , finding the simple approximate expression

$$\begin{aligned} \zeta(s) \sim \sqrt{\pi} \frac{\ell^{2s-1}}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\Gamma(s+k-\frac{1}{2})}{k!} Q_m^{2k} & \left( 2\zeta_H\left(2s+2k-2, Q_m + \frac{3}{2}\right) \right. \\ & \left. + (2Q_m + 1) \left( \zeta_H\left(2s+2k-1, Q_m + \frac{1}{2}\right) - \zeta_H\left(2s+2k-1, Q_m + \frac{3}{2}\right) \right) \right) \end{aligned} \quad (\text{E12})$$

with  $\zeta_H(s, a)$  the Hurwitz zeta functions, which are meromorphic functions with a unique simple pole at  $s = 1$ . In deriving this relation we used also [56]

$$\sum_{m,n=0}^{\infty} [m+n+a]^{-s} = \zeta_H(s-1, a) - (a-1)\zeta_H(s, a). \quad (\text{E13})$$

The zeta function (E12) is analytic throughout the complex  $s$ -plane except for  $s = 3/2 - n$  (with  $n = 0, 1, \dots$ ) where simple poles appear, while  $\zeta(0) = 0$ . The derivative of this function evaluated at  $s = 0$  is

$$\begin{aligned} \zeta'(0) \sim \frac{\sqrt{\pi}}{\ell} & \left\{ -2\sqrt{\pi} \left[ \zeta_H\left(-2, Q_m + \frac{3}{2}\right) + (2Q_m + 1) \left( \zeta_H\left(-1, Q_m + \frac{1}{2}\right) - \zeta_H\left(-1, Q_m + \frac{3}{2}\right) \right) \right] \right. \\ & + \sqrt{\pi} Q_m^2 \left[ 2\zeta_H\left(0, Q_m + \frac{3}{2}\right) + (2Q_m + 1) \left( \Psi\left(Q_m + \frac{3}{2}\right) - \Psi\left(Q_m + \frac{1}{2}\right) \right) \right] \\ & \left. + \sum_{k=0}^{\infty} \frac{\Gamma(k-\frac{3}{2})}{(k+2)!} Q_m^{2k+4} \left[ 2\zeta_H\left(2k+2, Q_m + \frac{3}{2}\right) + (2Q_m + 1) \left( \zeta_H\left(2k+3, Q_m + \frac{1}{2}\right) - \zeta_H\left(2k+3, Q_m + \frac{3}{2}\right) \right) \right] \right\}, \end{aligned} \quad (\text{E14})$$

where  $\Psi(x) = d \log \Gamma(x) / dx$ .

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