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Gauge field theory for the Poincaré-Weyl group

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On the basis of the general principles of a gauge field theory, the gauge theory for the Poincaŕe-Weyl group is constructed. It is shown that tetrads are not true gauge fields, but represent functions of true gauge fields: Lorentzian, translational, and dilatational ones. The equations for gauge fields are obtained. Geometrical interpretation of the theory is developed demonstrating that as a result of localization of the Poincaré-Weyl group the space-time becomes a Weyl-Cartan space. The geometrical interpretation of a dilaton field as a component of the metric tensor of a tangent space in Weyl-Cartan geometry is also proposed.

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I. INTRODUCTION

It is well known that the gauge treatment of physical interactions underlies modern fundamental physics [1,2]. Application of the gauge approach to gravitational interactions was developed in [3,4] for the Lorentz group and in [5–9] for the Poincaré group (see also reviews [10–14], the book [15], and the literature cited therein). Nevertheless, up to the present time, the interest to the gauge treatment of gravitational interaction [13–18] does not decrease. Recall that the first gauge theory was proposed yet in 1918 by Weyl (see [19]), who introduced a gauge field appropriate to the group of variations of scales (calibres), which were arbitrarily at each point of the space-time. Variation of length scales, in the mathematical sense, is equivalent to expansion or compression (dilatations) of the space. Connection of the dilatations group with the Poincaré group results in expansion of the Poincaré group to the Poincaré-Weyl group.

The importance of consideration of the Poincaré-Weyl group is related to the role the Weyl scale symmetry plays in the quantum field theory. Violation of this symmetry at the quantum level results in the appearance of the Weyl's anomaly connected with the following problems: the definition of the counterterms structure and asymptotic freedom in the quantum field theory, supersymmetry, calculation of critical dimensions $n = 26$ and $n = 10$ in the strings theory, gravitational instantons, Hawking's phenomenon of the black holes evaporation, the problems of inflation, the cosmological constant, creation of particles and black holes in the early universe [20], etc. In investigation of some of the above problems, the known techniques of the Becchi-Rouet-Stora-Tyutin (BRST)-symmetry [21,22] with application to Weyl gauge scale transformations are employed [23].

The gauge theory for the Poincaré-Weyl group was constructed in [24,25]. The main assumption of the authors of these works is the idea (going back to Kibble's work [5]) that, for the group of translations, tetrads $h^a{}_\mu$ play the role of gauge fields. In our opinion, this point of view obviously contradicts the fact that gauge fields should not transform as tensors under gauge transformations, while tetrads are transformed as tensor components with both tetrad and coordinate indices. We note that it was pointed out in [11,13] that treatment of tetrads as gauge fields is inadequate.

In the present work, the gauge theory for the Poincaré-Weyl group [26] constructed without this inadequacy is presented. Our consideration is based on the method of introduction of gauge fields for the groups connected to transformations of space-time coordinates, developed in [6,7,15]. The first and the second Noether theorems are used that allow to introduce the gauge fields, dynamically realizing the appropriate conservation laws. In our approach, the quantities $h^a{}_\mu$ are not the gauge fields, they rather represent some functions of the true gauge fields. The proper choice of potentials of the gravitational field as true gauge fields is obviously important in view of the realization of the quantization procedure for the gravitational field understood as a gauge field for the Poincaré-Weyl group. Moreover, within the framework of the general gauge procedure, the Dirac's scalar field [27] and the Utiyama's "measure" scalar field [28] are naturally introduced. These scalar fields play an essential role in constructing the gravitational field Lagrangian.

The paper is organized as follows. In Sec. II, the Poincaré-Weyl group and its action on physical fields are discussed. In Sec. III, the Noether theorem for the

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Poincaré-Weyl group is formulated with appropriate laws of conservation of an energy-momentum, spin and orbital angular momenta, and also a dilatational current. In Sec. IV, following [15], four initial postulates of the theory are formulated: the principle of local invariance, the principle of stationary action, the principle of minimality of gauge interactions, and the postulate of the existence of a free gauge field. In Sec. V, the gauge invariant Lagrangian for the interaction of external and gauge fields is derived from the main principles. In Sec. VI, the gauge invariant Lagrangian of free gauge fields is derived. In Sec. VII, interactions of gauge fields are analyzed with consideration for the full group of symmetries that include the Poincaré-Weyl group as a component of the direct product. In Sec. VIII, geometrical interpretation of the theory is developed. It is found that, with localization of the Poincaré-Weyl group, a space with the Weyl-Cartan geometry arises. Finally, in Sec. IX, the equations for the gauge field are derived in the geometrical form. In the conclusion, the basic results of the work are discussed, the resulting Lagrangian of the gravitational field is proposed, which enables all the gauge fields of the gravitational field gauge theory considered to be dynamically realized.

II. THE POINCARÉ-WEYL GROUP

Let space-time $\mathcal{M}$ with a metric tensor $\breve{g}$ have the structure of a flat space with the geometry defined according to F. Klein's Erlangen program by the global definition of action of the Poincaré-Weyl group $\mathcal{PW}(\omega, \varepsilon, a)$ as a fundamental group (with E. Cartan's terminology [29]) of this geometry. The fundamental group determines geometry of the space as a system of invariants, i.e. relations and geometrical images, which remain constant in space under the action of the given group (see [30]). A similar geometry arises in a space-time filled with radiation and ultra relativistic particles. We assume that curvature, torsion, and nonmetricity of the space $\mathcal{M}$ are equal to zero all over the space.

In $\mathcal{M}$, a special coordinate system x^i $(i = 1, 2, 3, 4)$ (analogues to the Cartesian coordinates in Minkowski space) can be globally introduced with the metric tensor components equal to

$$g_{ij} = \beta^2 g^M_{ij}, \qquad \beta = \mathrm{const} > 0, \tag{2.1}$$

where $g^M_{ij} = \mathrm{diag}(1, 1, 1, -1)$ are components of the metric tensor of the Minkowski space.

We represent infinitesimal transformations of the group $\mathcal{PW}$ as follows:

$$\begin{gathered}\delta x^i = \omega^m I_m{}^i{}_j x^j - \varepsilon x^i + a^i = -(\omega^m \mathring{M}_m + \varepsilon \mathring{D} + a^k P_k) x^i,\\ \mathring{M}_m = -I_m{}^l{}_j x^j \frac{\partial}{\partial x^l}, \qquad I_m{}^{ij} = I_m{}^{[ij]},\\ \mathring{D} = x^l \frac{\partial}{\partial x^l}, \qquad P_k = -\delta^l_k \frac{\partial}{\partial x^l}.\end{gathered} \tag{2.2}$$

Operators $\mathring{M}_m$ and P_k are generators of 4-rotations (Lorentz subgroup L_4) and 4-shifts (a subgroup of translations T_4), and the operator $\mathring{D}$ is a generator of dilatations (a subgroup of dilatations D_4) of the space $\mathcal{M}$. These operators satisfy the following commutation relations:

$$\begin{gathered}[\mathring{M}_m, \mathring{M}_n] = c_m{}^q{}_n \mathring{M}_q, \qquad [\mathring{M}_m, \mathring{D}] = 0, \qquad [\mathring{D}, \mathring{D}] = 0,\\ [P_k, P_l] = 0, \qquad [\mathring{M}_m, P_k] = I_m{}^l{}_k P_l, \qquad [P_k, \mathring{D},] = P_k,\end{gathered} \tag{2.3}$$

where $c_m{}^q{}_n$ are structure constants of the subgroup of 4-rotations L_4.

Here three types of indices are introduced. The indices of type $m, n, p, q, \ldots$ are numbered parameters of L_4. The indices of type $i, j, k, l, a, b, c, \ldots$ are numbered parameters of the subgroup of 4-translations T_4. The subgroup of dilatations D_4 has only one parameter ε, and its index in the corresponding generator $I_\phi{}^i{}_j = -\delta^i_j$ we shall often mark as ϕ (a symbol of an empty set). In order to simplify formulas we shall introduce indices of type $z, R, P, Q, \ldots$, which unify all types of indices. Introducing the generalized notations for parameters, $\{\omega^z\} = \{\omega^m, \varepsilon, a^k\}$, we represent the transformations (2.2) as follows,

$$\begin{gathered}\delta x^i = \omega^z X^i_z, \qquad X^i_m = I_m{}^i{}_j x^j,\\ X^i = -x^i, \qquad X^i_k = \delta^i_k.\end{gathered} \tag{2.4}$$

Let an arbitrary field ψ^A be given in $\mathcal{M}$, and its infinitesimal transformation under the action of the group $\mathcal{PW}(\omega, \varepsilon, a)$ looks like $(I_z{}^A{}_B = \{I_m{}^A{}_B, w\delta^A_B, 0\})$

$$\delta\psi^A = \omega^m I_m{}^A{}_B \psi^B + \varepsilon w \psi^A = \omega^z I_z{}^A{}_B \psi^B, \tag{2.5}$$

where w is a weight of ψ^A under the action of the subgroup of dilatations D_4. Operators $I_m{}^A{}_B$ satisfy the commutation relations: $I_m{}^A{}_C I_n{}^C{}_B - I_n{}^A{}_C I_m{}^C{}_B = c_m{}^q{}_n I_q{}^A{}_B$.

The action of group $\mathcal{PW}$ on the metric tensor is described as follows:

$$\delta g_{ij} = -\omega^z I_z{}^l{}_i g_{lj} - \omega^z I_z{}^l{}_j g_{il} = 2\varepsilon g_{ij}. \tag{2.6}$$

Components g_{ij} are not invariant under the action of $\mathcal{PW}$. As a result of (2.1), under the action of $\mathcal{PW}$ the following transformation holds, $\beta^2 \to \beta^2 + 2\varepsilon$, $\delta\beta = \varepsilon\beta = \omega^z \delta^\phi_z \beta$.

We introduce an arbitrary curvilinear system of coordinates $\{x^\mu\} = \{x^\mu(x^i)\}$ in $\mathcal{M}$ $(\partial_\mu = \partial/\partial x^\mu)$:

$$dx^i = \mathring{h}^i{}_\mu dx^\mu, \qquad \mathring{h}^i{}_\mu = \partial_\mu x^i, \qquad \mathring{h}^\mu{}_k \mathring{h}^k{}_\nu = \delta^\mu_\nu \tag{2.7}$$

Then for the metric tensor we have:

$$\begin{gathered}ds^2 = \mathring{g}_{\mu\nu} dx^\mu dx^\nu, \qquad \mathring{g}_{\mu\nu} = \breve{g}(\vec{e}_\mu, \vec{e}_\nu) = g_{ij} \mathring{h}^i{}_\mu \mathring{h}^j{}_\nu,\\ \mathring{g} = \det(\mathring{g}_{\mu\nu}) = \det(g_{ij})(\mathring{h})^2, \qquad \mathring{h} = \det(\mathring{h}^a{}_\mu).\end{gathered} \tag{2.8}$$

In a flat manifold fibers attached to each point of the manifold are identical to a base of the manifold. Curvilinear coordinates x^μ of the points belonged to the base $\mathcal{M}$ are transformed under the group of general transformations of curvilinear coordinates. But the quantities x^k belong to fibers of $\mathcal{M}$, which are transformed by the Poincaré-Weyl group.

The curvilinear system of coordinates in the flat space is introduced with the purpose to separate the problems connected with the invariance of the theory under gauge transformations, and the problems following from the requirement of covariance of the theory with respect to the group of the general transformations of coordinates. In papers [5–7,24,25], the curvilinear system of coordinates was not introduced before starting the localization procedure. As a result, transformations of coordinates under the action of gauge groups after localization became general transformations of coordinates, and this broke the mathematical structure of the gauge groups.

As a consequence of Eqs. (2.5) and (2.6), both $\psi^A(x)$ and g_{ij} are not transformed as representations of the subgroup of translations T_4, but change under the action of T_4 only as a result of the transformation of the argument x^k. Therefore, the action of the operator of shift in $\mathcal{M}$, for example, on field $\psi^A(x)$ (when $\delta x^k = a^k$) is realized as follows:

$$\begin{aligned}\psi^A(x+\delta x) &= \psi^A(x) + \delta x^\mu \frac{\partial}{\partial x^\mu}\psi(x)\\ &= \psi^A(x) - a^k P_k \psi^A(x),\\ P_k &= -\overset{\circ}{h}{}^\mu{}_k \partial_\mu.\end{aligned} \tag{2.9}$$

Under the action of $\mathcal{PW}(\omega, \varepsilon, a)$, variation $\bar{\delta}$ of the form of a field $\psi^A(\bar{\delta} = \psi'^A(x) - \psi^A(x))$:

$$\begin{aligned}\bar{\delta}\psi^A &= \delta\psi^A - \delta x^\mu \partial_\mu \psi^A = \delta\psi^A + \delta x^k P_k \psi^A\\ &= \delta\psi^A + \omega^z X_z^k P_k \psi^A = \delta\psi^A - \omega^z X_z^k \overset{\circ}{h}{}^\mu{}_k \partial_\mu \psi^A,\end{aligned}$$

commutes with the operator of differentiation. The following commutation relations hold:

$$[\bar{\delta}, P_k] = 0, \qquad [\delta, P_k] = (P_k \delta x^l) P_l, \tag{2.10}$$

and this is valid owing to an identity:

$$\bar{\delta}\overset{\circ}{h}{}^l{}_\nu = \partial_\nu \delta x^l - \overset{\circ}{h}{}^l{}_\mu \partial_\nu \delta x^\mu - \delta x^\mu \partial_\mu \overset{\circ}{h}{}^l{}_\nu = 2\partial_{[\nu} \overset{\circ}{h}{}^l{}_{\mu]} \delta x^\mu = 0.$$

III. NOETHER THEOREM FOR THE POINCARÉ-WEYL GROUP

The field ψ^A in $\mathcal{M}$ is described in curvilinear coordinates by the action

$$J = \int_\Omega (dx)\mathcal{L}, \qquad \mathcal{L} = \sqrt{|\overset{\circ}{g}|} L(\psi^A, P_k\psi^A, \beta^2 g_{ij}^M). \tag{3.1}$$

The variation of the action integral under the action of the Poincaré-Weyl group (2.2) (the variation of the integration volume Ω being taken into account) is equal to

$$\begin{aligned}\delta J &= \delta \int_\Omega (dx)\mathcal{L}\\ &= \int_\Omega (dx)\Big(\sqrt{|\overset{\circ}{g}|}(\partial_\mu \delta x^\mu) L + \delta\sqrt{|\overset{\circ}{g}|} L + \sqrt{|\overset{\circ}{g}|}\delta L\Big) = 0.\end{aligned} \tag{3.2}$$

In the curvilinear system of coordinates, the equality holds $\delta\sqrt{|\overset{\circ}{g}|} = -\sqrt{|\overset{\circ}{g}|}(\partial_\mu \delta x^\mu)$ as the consequence of Eq. (2.8). Therefore, by virtue of arbitrariness of the volume Ω, Eq. (3.2) yields $\delta L = 0$:

$$\delta L = \frac{\partial L}{\partial \psi^A}\delta\psi^A + \frac{\partial L}{\partial P_k \psi^A}\delta P_k \psi^A + \frac{\partial L}{\partial \beta}\delta\beta = 0.$$

Calculating variations according to Eqs. (2.5), (2.6), (2.7), (2.8), (2.9), and (2.10) and using equality $P_k X_z^i = -I_z{}^i{}_k$, we receive the following identity as the consequence of randomness of ω^z:

$$\begin{aligned}&\frac{\partial L}{\partial \psi^A} I_z{}^A{}_B \psi^B + \frac{\partial L}{\partial P_k \psi^A}(I_z{}^A{}_B P_k \psi^B - I_z{}^l{}_k P_l \psi^A)\\ &\qquad + \frac{\partial L}{\partial \beta}\beta \delta_z^\phi = 0.\end{aligned} \tag{3.3}$$

The last term here arises only for the subgroup of dilatations (when $z = \phi$). Independence of the Lagrangian density (3.1) of x^k yields the identity:

$$P_k L = \frac{\partial L}{\partial \psi^A} P_k \psi^A + \frac{\partial L}{\partial P_l \psi^A} P_k P_l \psi^A. \tag{3.4}$$

Identities (3.3) and (3.4) are the "strong" identities, which hold independently of whether the field equations for ψ^A are satisfied. When these equations are satisfied, the above identities are equivalent to the existence of conservation laws. Indeed, it is possible to represent the variation of the action (3.2) as

$$\begin{aligned}\delta J = \int_\Omega (dx)\Big[&\frac{\delta \mathcal{L}}{\delta \psi^A}\bar{\delta}\psi^A + \frac{\partial \mathcal{L}}{\partial \beta}\bar{\delta}\beta\\ &+ \partial_\mu\Big(\mathcal{L}\overset{\circ}{h}{}^\mu{}_k \delta x^k - \overset{\circ}{h}{}^\mu{}_k \frac{\partial \mathcal{L}}{\partial P_k \psi^A}\bar{\delta}\psi^A\Big)\Big] = 0,\end{aligned} \tag{3.5}$$

where the variational derivative has the standard structure. If the field equations, $\delta\mathcal{L}/\delta\psi^A = 0$, are satisfied, the variation (3.5), with account of (2.1) and $\partial_\mu(\sqrt{|\overset{\circ}{g}|}\overset{\circ}{h}{}^\mu{}_a) = 0$, is equal to

$$\begin{aligned}\delta J = \int_\Omega (dx)\Big(&\frac{\partial \mathcal{L}}{\partial \beta}\delta\beta\\ &+ \sqrt{|\overset{\circ}{g}|}\overset{\circ}{h}{}^\mu{}_k \partial_\mu (a^l t^k{}_l + \omega^m M^k{}_m + \varepsilon \Delta^k)\Big) = 0,\end{aligned} \tag{3.6}$$

where notations for the energy-momentum tensor $t^k{}_l$, total $M^k{}_m$, and spin $S^k{}_m$ momenta, and also for the total dilatation current Δ^k and dilatation current J^k of the field ψ^A are introduced:

$$t^k{}_l = L\delta^k_l - \frac{\partial L}{\partial P_k \psi^A} P_l \psi^A, \tag{3.7}$$

$$M^k{}_m = S^k{}_m + I_m{}^l{}_i x^i t^k{}_l, \qquad S^k{}_m = -\frac{\partial L}{\partial P_k \psi^A} I_m{}^A{}_B \psi^B, \tag{3.8}$$

$$\Delta^k = J^k - x^l t^k{}_l, \qquad J^k = -\frac{\partial L}{\partial P_k \psi^A} w\psi^A. \tag{3.9}$$

Parameters a^l, ω^m, and ε are constant, though arbitrary, and the integration volume Ω is also arbitrary. Therefore, since variation (3.6) is identically equal to zero, and with the account to Eq. (2.6), the following equalities follow in a curvilinear system of coordinates:

$$P_k t^k{}_l = 0, \qquad P_k M^k{}_m = 0, \qquad \sqrt{|\overset{\circ}{g}|} P_k \Delta^k = \beta \frac{\partial \mathcal{L}}{\partial \beta}. \tag{3.10}$$

Equalities (3.10) are the result of the first Noether theorem. The first two equalities yield the conservation laws of the energy-momentum $t^k{}_l$ and the total momentum $M^k{}_m$ of the field ψ^A. For the conservation of the dilatation current Δ^k, it is necessary that an additional condition $\partial\mathcal{L}/\partial\beta = 0$ is fulfilled as a consequence of the equation of the field ψ^A (about the dilatational invariant Lagrangians with explicit dependence on the parameter β, see [15,31]).

Using the field equations, it is possible to show that the first equality (3.10) is equivalent to the identity (3.4), and the second and the third equalities (3.10) are together equivalent to the identity (3.3).

IV. THE PRINCIPLE OF LOCAL INVARIANCE

We suppose now that the group $\mathcal{PW}(\omega, \varepsilon, a)$ is a localized group $\mathcal{PW}(x)$, that is we consider its parameters $\{\omega^z\} = \{\omega^m, \varepsilon, a^k\}$ as arbitrary smooth enough (belonging to class C^2) functions of coordinates $\omega^z(x)$.

Consider the invariance of action integral (3.1) under $\mathcal{PW}(x)$. Assuming the quantities $\omega^z(x)$ and $\partial_\mu \omega^z(x)$ are arbitrary and independent functions of coordinates, from Eq. (3.6) we obtain conditions $t^k{}_l = 0$, $M^k{}_m = 0$, $\Delta^k = 0$. Thus, the action integral (3.1) is locally invariant if and only if the conservations laws are valid by virtue of the fact that the appropriate currents (3.7), (3.8), and (3.9) are identically equal to zero.

It is possible to avoid this physically unsatisfactory result, if some additional fields named *gauged* (or *compensating*) fields enter the Lagrangian density (3.1). They should have the property that the additional terms, arising in the action integral (3.1) owing to the transformation of field ψ^A under the action of the localized group $\mathcal{PW}(x)$, disappear being compensated by accordingly transformed gauge fields. Therefore, the gauge fields should be transformed under the action of $\mathcal{PW}(x)$ as nontensorial quantities extracting under this transformation the terms proportional to the derivative over parameters of the group $\mathcal{PW}(x)$. In the case of the group $\mathcal{PW}(x)$, variation (2.6) becomes equal to

$$\delta g_{ij} = 2\varepsilon(x) g_{ij}, \tag{4.1}$$

where $\varepsilon(x)$ is an arbitrary function. Therefore the metric tensor becomes a function of a space-time point and can be represented as

$$g_{ij} = \beta^2(x) g^M_{ij}, \tag{4.2}$$

hence, derivatives $P_k\beta(x)$ ($\beta(x) > 0$) should be included in the Lagrangian.

The requirement of gauge invariance in application to the Poincaré group has been formulated in [6,7] in the form of a variational principle, which for the case of the localized Poincaré-Weyl group, can be generalized as follows:

Postulate 1 (*The principle of local invariance*)—The action integral

$$J = \int_\Omega (dx) \mathcal{L}(\psi^A, P_k\psi^A, A^R_a, P_k A^R_a, \beta(x), P_k\beta(x)), \tag{4.3}$$

where the Lagrangian density $\mathcal{L}$ describes a field ψ^A, interaction of a field ψ^A with an additional gauge field A^R_a, and a free field A^R_a, is invariant under the action of the localized group $\mathcal{PW}(x)$, the gauge field being transformed as follows

$$\delta A^R_a = U^R_{za}\omega^z + S^{R\mu}_{za}\partial_\mu \omega^z, \tag{4.4}$$

where U and S are some matrix functions.

This variational principle allows to apply the first and the second Noether theorems to gauge theories. Moreover, though it is formulated in rather general terms, this principle is sufficient to determine the structure of the Lagrangian density $\mathcal{L}$ and to find the matrix functions U, S. In the present work, the method of constructing gauge theories, developed in [15–17] for the Poincaré group, is generalized to the gauge theory of the localized Poincaré-Weyl group.

The gauge field equations, as well as the equations of the field ψ^A, are derived basing upon the principle of stationary action, and this should be understood as the second postulate.

Postulate 2 (*The principle of stationary action*)—The equations of the field ψ^A and the gauge fields A^R_a realize an extremum of the action integral (4.3) that describes the field ψ^A, the gauge field A^R_a, and their interaction.

For physical reasons, it should be concluded that the full Lagrangian density $\mathcal{L}$ consists of the Lagrangian density $\mathcal{L}_0$ of free gauge fields and of the Lagrangian density $\mathcal{L}_\psi$ that describes the free matter field ψ^A and the interaction of the field ψ^A with the gauge fields. Action integrals for each of these Lagrangian densities should be separately locally invariant, as it is natural to expect, that the gauge field can exist without the field ψ^A. We formulate the given above physical requirements as the third postulate of the theory of gauge fields.

Postulate 3 (*An independent existence of a free gauge field*)—The locally invariant Lagrangian density $\mathcal{L}_0$ of free gauge fields is included as an additive term in the full Lagrangian density $\mathcal{L}$ of a physical system: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\psi$, where

$$\mathcal{L}_0 = \mathcal{L}_0(A^R_a, P_k A^R_a, \beta(x), P_k\beta(x)), \qquad \frac{\partial \mathcal{L}_0}{\partial \psi^A} = 0, \qquad \frac{\partial \mathcal{L}_0}{\partial P_k \psi^A} = 0.$$

In what follows, we shall always assume that interaction of the field ψ^A with gauge fields does not involve their derivatives. In other words, all derivatives of the gauge fields are included only in the Lagrangian density $\mathcal{L}_0$. The interaction satisfying this condition is called *minimal*. We formulate this condition as the fourth postulate of the gauge fields theory.

Postulate 4 (*The principle of minimality of gauge interaction*)—The Lagrangian density $\mathcal{L}_\psi$ that describes interaction of the material field ψ^A with the gauge fields includes derivatives of the material field ψ^A only. Thus the following conditions are satisfied,

$$\frac{\partial \mathcal{L}_\psi}{\partial P_k A^R_a} = 0, \qquad \frac{\partial \mathcal{L}_\psi}{\partial P_k \beta} = 0.$$

The variation of the action integral (4.3) under the action of the localized Poincaré-Weyl group reads

$$\begin{aligned} 0 = \delta J &= \int_\Omega (dx)((\partial_\mu \delta x^\mu)\mathcal{L} + \delta\mathcal{L}) \\ &= \int_\Omega (dx)\Big(\frac{\delta\mathcal{L}}{\delta\psi^A}\bar\delta\psi^A + \frac{\delta\mathcal{L}}{\delta A^R_a}\bar\delta A^R_a + \frac{\delta\mathcal{L}}{\delta\beta}\bar\delta\beta\Big) \\ &\quad + \int_\Omega (dx)\partial_\mu\Big(\mathcal{L}\mathring{h}^\mu{}_k\delta x^k - \mathring{h}^\mu{}_k\frac{\partial\mathcal{L}}{\partial P_k\psi^A}\bar\delta\psi^A \\ &\quad - \mathring{h}^\mu{}_k\frac{\partial\mathcal{L}}{\partial P_k A^R_a}\bar\delta A^R_a - \mathring{h}^\mu{}_k\frac{\partial\mathcal{L}}{\partial P_k\beta}\bar\delta\beta\Big). \end{aligned} \tag{4.5}$$

According to Eq. (4.1), describing the action of the localized subgroup of dilatations D_4, the metric tensor becomes a function and is itself subject to variation. The principle of stationary action will be satisfied, if the variational equations are valid,

$$\frac{\delta\mathcal{L}}{\delta\psi^A} = 0, \qquad \frac{\delta\mathcal{L}}{\delta A^R_a} = 0, \qquad \frac{\delta\mathcal{L}}{\delta\beta(x)} = 0. \tag{4.6}$$

It can be explicitly shown [15] that the last of these variational field equations is a consequence of the others. We assume that the equation of the field ψ^A is always valid. According to Postulate 3, the full Lagrangian density of gauge fields consists of the free gauge field Lagrangian density and of the Lagrangian density of interaction. The action integral with the interaction Lagrangian density is separately locally invariant, though the variational equation of the gauge field A^R_a is not satisfied. The equation of the field A^R_a is valid only for the full Lagrangian density $\mathcal{L}$. In this last case, basing upon Eq. (4.5) and taking into account Eqs. (4.4) and (4.6), with quantities $\omega^z(x)$, $\partial_\mu\omega^z(x)$, $\partial_\mu\partial_\nu\omega^z(x)$ assumed to be arbitrary and independent functions of the coordinates, we obtain a fundamental set of identities on extremals of the fields ψ^A, A^R_a, and $\beta(x)$:

$$\partial_\mu(\mathring{h}^\mu{}_k\Theta^k{}_z) = 0, \tag{4.7a}$$

$$\mathring{h}^\mu{}_k\Theta^k{}_z - \partial_\nu\mathcal{M}^{\nu\mu}{}_z = 0, \tag{4.7b}$$

$$\mathcal{M}^{(\nu\mu)}{}_z = 0, \tag{4.7c}$$

where the following notations are introduced with regard to Eqs. (2.4) and (4.4):

$$\begin{aligned} \Theta^k{}_z &= \mathcal{L}X^k_z - \frac{\partial\mathcal{L}}{\partial P_k\psi^A}(I_z{}^A{}_B\psi^B + X^l_z P_l\psi^A) \\ &\quad - \frac{\partial\mathcal{L}}{\partial P_k A^R_a}(U^R_{za} + X^l_z P_l A^R_a) \\ &\quad - \frac{\partial\mathcal{L}}{\partial P_k\beta}(\beta\delta^\phi_z + X^l_z P_l\beta), \\ \mathcal{M}^{\nu\mu}{}_z &= \mathring{h}^\nu{}_k\frac{\partial\mathcal{L}}{\partial P_k A^R_a}S^{R\mu}_{za}. \end{aligned} \tag{4.8}$$

The equalities (4.7a)–(4.7c) represent the relations following from the second Noether theorem written down in a curvilinear system of coordinates. It can be easily understood that the first of these relations (representing the conservation law of the appropriate current) is a consequence of two others. Thus it is shown, that introduction of gauge fields leads to a dynamical realization of conservation laws. The quantity (4.8) represents a superpotential for the appropriate conservation current.

V. STRUCTURE OF THE INTERACTION LAGRANGIAN

Following the method developed in [15], we introduce the differential operator M_R ($R = \{m, \phi, k\}$):

$$M_R = \{M_m{}^A{}_B, M_\phi{}^A{}_B, M_k{}^A{}_B\}, \qquad M_k{}^A{}_B = \delta^A_B P_k, \tag{5.1}$$

$$M_m{}^A{}_B = I_m{}^A{}_B + \delta^A_B\mathring{M}_m, \qquad M_\phi{}^A{}_B = w\delta^A_B + \delta^A_B\mathring{D}, \tag{5.2}$$

uniting the operators of total momentum, total dilatation current, and shift.

Let us represent the gauge field A^R_a as a set of three components:

$$A^R_a = \{A^m_a, A_a, A^k_a\},$$

where A^m_a is the gauge field corresponding to the subgroup of 4-rotations (r-field), A_a is the gauge field of the subgroup of dilatations (d-field), and A^k_a is the gauge field of the subgroup of translations (t-field) of the Poincaré-Weyl group.

The following theorem about the structure of the Lagrangian density $\mathcal{L}_\psi$ of interaction between the matter field and the gauge fields represents generalization on the Poincaré-Weyl group $\mathcal{PW}(x)$ of the appropriate theorem proved in [15] for the case of the Poincaré group.

Theorem 1—There exists a gauge field A^R_a with transformation structure (4.4) of Postulate 1 under the action of the localized Poincaré-Weyl group $\mathcal{PW}(x)$ and there are matrix functions Z, U, and S of the gauge field such that the Lagrangian density

$$\mathcal{L}_\psi = \sqrt{|\bar{g}|} L_\psi(\psi^A, D_a\psi^A, \beta(x)), \tag{5.3a}$$

$$\sqrt{|\bar{g}|} = Z\sqrt{|\mathring{g}|}, \tag{5.3b}$$

satisfies the principle of local invariance (Postulate 1) concerning the localized group $\mathcal{PW}(x)$, $\mathcal{L}_\psi$ being formed from the invariant concerning the nonlocalized group $\mathcal{PW}$ Lagrangian density $L(\psi^A, P_k\psi^A)$ by replacement of the differential operator P_k by the gauge derivative operator

$$D_a = -A^R_a M_R, \tag{5.4}$$

where the operator M_R is given by Eq. (5.1). The following representation of the gauge t-field is also valid:

$$A^k_a = D_a x^k. \tag{5.5}$$

Proof—Substituting the expression for the operator M_R in Eq. (5.4), we obtain, according to Eqs. (5.1) and (5.2), the explicit form of the gauge derivative for the group $\mathcal{PW}(x)$ $(D_\mu\psi^A = h^a{}_\mu D_a\psi^A)$:

$$D_a\psi^A = h^\mu{}_a\partial_\mu\psi^A - A^m_a I_m{}^A{}_B\psi^B - wA_a\psi^A, \tag{5.6}$$

$$D_\mu\psi^A = \partial_\mu\psi^A - A^m{}_\mu I_m{}^A{}_B\psi^B - wA_\mu\psi^A. \tag{5.7}$$

Here, new quantities are introduced:

$$Y^k_a = A^R_a X^k_R = A^k_a + A^m_a I_m{}^k{}_l x^l - A_a x^k, \tag{5.8a}$$

$$h^\mu{}_a = \mathring{h}^\mu{}_k Y^k_a, \qquad h^a{}_\mu = (h^{-1})^a{}_\mu = Z^a_k \mathring{h}^k{}_\mu, \tag{5.8b}$$

$$Z^a_k = (Y^{-1})^a_k, \qquad A^m{}_\mu = A^m_a h^a{}_\mu, \qquad A_\mu = A_a h^a{}_\mu. \tag{5.8c}$$

By analogy with Sec. III, we obtain the strong identities expressing conditions of invariance of the action integral for the Lagrangian density (5.3a) against transformations of the localized group $\mathcal{PW}(x)$. The variation of the action integral is as follows

$$\delta\int_\Omega (dx)\mathcal{L}_\psi = \int_\Omega (dx)(\sqrt{|\bar{g}|}(\partial_\mu\delta x^\mu)L_\psi + \delta(\sqrt{|\bar{g}|})L_\psi + \sqrt{|\bar{g}|}\delta L_\psi) = 0. \tag{5.9}$$

We introduce a quantity $\bar{g}_{\mu\nu}$ (tensor g_{ab} has the same components as the tensor g_{ij} (2.1)):

$$\bar{g}_{\mu\nu} = g_{ab}h^a{}_\mu h^b{}_\nu = g_{ab}Z^a_k Z^b_l \mathring{h}^k{}_\mu \mathring{h}^l{}_\nu, \tag{5.10a}$$

$$\bar{g} = \det(\bar{g}_{\mu\nu}) = \mathring{g}Z^2, \qquad Z = \det(Z^a_k), \tag{5.10b}$$

and also demand that matrixes U and S in the gauge field transformation law (4.4) are such that the following transformation rule is satisfied for the quantity $\bar{g}$ in Eq. (5.10b) under the action of the localized group $\mathcal{PW}(x)$:

$$\delta(\sqrt{|\bar{g}|}) = -\sqrt{|\bar{g}|}(\partial_\mu\delta x^\mu). \tag{5.11}$$

The proof of the existence of the quantity $\bar{g}$ with the specified property is given at the end of the section.

Then the equality (5.9) by virtue of arbitrariness of the volume Ω means $\delta L_\psi = 0$:

$$\delta L_\psi = \left(\frac{\partial L_\psi}{\partial\psi^A}\right)_{D\psi=\mathrm{const}}\delta\psi^A + \frac{\partial L_\psi}{\partial D_a\psi^A}\delta D_a\psi^A + \frac{\partial L_\psi}{\partial\beta}\delta\beta = 0. \tag{5.12}$$

Using Eqs. (4.4), (5.6), (5.8a), and (5.8b), let us calculate the variation $\delta D_a\psi^A$ and then substitute it as well as Eqs. (2.5) and (4.1) into Eq. (5.12). In the identity received, we collect factors in front of quantities $\omega^z(x)$ and $\partial_\mu\omega^z(x)$. In view of arbitrariness of these quantities these factors should be separately equal to zero identically. The factor in front of $\partial_\mu\omega^z(x)$ is equal to

$$\frac{\partial L_\psi}{\partial D_a\psi^A}(I_R{}^A{}_B\psi^B + X^l{}_R P_l\psi^A)(S^{R\mu}_{za} - \delta^R_z h^\mu{}_a) = 0.$$

This equality is satisfied identically at

$$S^{R\mu}_{za} = \delta^R_z h^\mu{}_a. \tag{5.13}$$

For various sets of indices, we find values of an unknown matrix S:

$$S^{n\mu}_{ma} = \delta^n_m h^\mu{}_a, \qquad S^{n\mu}_{ka} = 0, \qquad S^{k\mu}_{ma} = 0, \tag{5.14a}$$

$$S^{l\mu}_{ka} = \delta^l_k h^\mu{}_a, \qquad S^\mu_a = h^\mu{}_a, \qquad S_{ma}{}^\mu = 0, \tag{5.14b}$$

$$S^{n\mu}_a = 0, \qquad S^{k\mu}_a = 0, \qquad S_{ka}{}^\mu = 0. \tag{5.14c}$$

Now we take into account that an algebraic structure of the scalar L_ψ should satisfy identity (3.3), which upon replacing P_a by D_a takes the form

$$\left(\frac{\partial L_\psi}{\partial \psi^A}\right)_{D\psi=\text{const}} I_z{}^A{}_B\psi^B + \frac{\partial L_\psi}{\partial D_a\psi^A}(I_z{}^A{}_B D_a\psi^B - I_z{}^b{}_a D_b\psi^A) + \frac{\partial L_\psi}{\partial \beta}\beta\delta_z^\phi = 0.$$

Considering the given identity, let us write out the factor at $\omega^z(x)$ in identity (5.12). As a result, we obtain an expression, which is identically equal to zero for the following set of matrixes U in the transformation law (4.4):

$$U^n_{ma} = c_m{}^n{}_q A^q_a - I_m{}^b{}_a A^n_b, \qquad U^n_a = A^n_a, \tag{5.15a}$$

$$U_{ma} = -I_m{}^b{}_a A_b, \qquad U_a = A_a, \qquad U_{ka} = 0, \tag{5.15b}$$

$$U^k_{ma} = I_m{}^k{}_l A^l_a - I_m{}^b{}_a A^k_b, \qquad U^n_{ka} = 0, \tag{5.15c}$$

$$U^k_{ia} = -I_n{}^k{}_i A^n_a + \delta^k_i A_a, \qquad U^k_a = 0. \tag{5.15d}$$

These formulas can be expressed in a short form as

$$U^R_{za} = c_z{}^R{}_Q A^Q_a - I_z{}^b{}_a A^R_b, \tag{5.16}$$

where each of the indices R, Q, z can take the values each of the indices m, k, ϕ, and the commutation relations (2.3) of the Poincaré-Weyl group $\mathcal{PW}$ should be taken into account. The expressions (5.13) and (5.16) found for an unknown function Z and unknown matrix functions U and S, for which the Lagrangian density satisfies identity (5.9), prove the basic statements of the Theorem 1.

Now we shall prove the formula (5.5). Because of Eq. (2.2), the quantity x^k is transformed according to the vector representation of the group $\mathcal{PW}$. Then, comparing (2.2) and (2.5), we find that $w[x^k] = -1$. In calculation of the gauge derivative $D_a x^k$, we use the Eqs. (2.2), (2.7), (5.6), (5.8a), and (5.8b):

$$\begin{aligned} D_a x^k &= h^\mu{}_a\partial_\mu x^k - A^m_a I_m{}^k{}_l x^l - w[x^k]A_a x^k \\ &= \mathring{h}^\mu{}_i(A^i_a + A^m_a I_m{}^i{}_l x^l - A_a x^i)\partial_\mu x^k - A^m_a I_m{}^k{}_l x^l \\ &\quad + A_a x^k = A^k_a. \end{aligned}$$

Formula (5.5) makes clear the geometrical meaning of the gauge field of the subgroup of translations. This formula is the generalization to the Poincaré-Weyl group of the similar formula, which arises in the gauge approach for the Poincaré group [15–17].

Let us find transformation laws for the components of the gauge field under the action of the localized Poincaré-Weyl group $\mathcal{PW}(\omega, \varepsilon, a)$. The general form of the transformation law is determined on account of the principle of local invariance by the expression (4.4). Let us put in this expression, instead of general indices R and z, three explicit indices of the subgroups of 4-rotations, dilatations, and translations and also substitute the concrete values of the matrix functions U and S from the formulas (5.14a)–(5.14c) and (5.15a)–(5.15d). As a result, we obtain the general rules of transformations for the fields A^m_a, A_a, and A^k_a:

$$\delta A^m_a = \omega^n c_n{}^m{}_q A^q_a - \omega^n I_n{}^b{}_a A^m_b + \varepsilon A^m_a + h^\mu{}_a\partial_\mu\omega^m, \tag{5.17}$$

$$\delta A_a = -\omega^n I_n{}^l{}_a A_l + \varepsilon A_a + h^\mu{}_a\partial_\mu\varepsilon, \tag{5.18}$$

$$\delta A^k_a = \omega^n(I_n{}^k{}_l A^l_a - I_n{}^l{}_a A^k_l) + a^l(-A^n_a I_n{}^k{}_l + A_a\delta^k_l) + h^\mu{}_a\partial_\mu a^k. \tag{5.19}$$

A variation of the quantity $h^\mu{}_a$ is determined based upon formulas (5.8b), the transformation law [32] of the quantity Y^k_a having been determined previously with the help of the variations (5.17), (5.18), and (5.19):

$$\delta Y^k_a = -\omega^n I_n{}^l{}_a Y^k_l + \varepsilon Y^k_a + h^\mu{}_a\partial_\mu\delta x^k. \tag{5.20}$$

As a result one finds

$$\delta h^\mu{}_a = -\omega^n I_n{}^b{}_a h^\mu{}_b + \varepsilon h^\mu{}_a + h^\nu{}_a\partial_\nu\delta x^\mu. \tag{5.21}$$

At last, from Eqs. (5.8b) and (5.8c) we find transformation rules for the quantities $h^a{}_\mu$, $A^m{}_\mu$, and A_μ:

$$\begin{aligned} \delta h^a{}_\mu &= \omega^n I_n{}^a{}_b h^b{}_\mu - \varepsilon h^a{}_\mu - h^a{}_\nu\partial_\mu\delta x^\nu, \\ \delta A^m{}_\mu &= \omega^n c_n{}^m{}_q A^q_\mu + \partial_\mu\omega^m - A^m{}_\nu\partial_\mu\delta x^\nu, \\ \delta A_\mu &= \partial_\mu\varepsilon - A_\nu\partial_\mu\delta x^\nu. \end{aligned} \tag{5.22}$$

After the geometrical interpretation of the theory (Sec. VIII), the quantity $h^\mu{}_a$ will be interpreted as a tetrad potential. We see, that with localization of the group $\mathcal{PW}$ (as well as in the case of the Poincaré group [15]), the quantity $h^\mu{}_a$ is not a gauge field, because it is transformed as a tensor (in agreement with Eqs. (5.21) and (5.22)), in contrast to gauge fields A^m_a, A_a, and A^k_a, which are transformed under the group $\mathcal{PW}(x)$ by nontensorial rules. We note that in [11,13] the fact that tetrads are not true potentials of a gravitational field was also emphasized.

We shall prove now the formula (5.11) for components of the quantity $\bar{g}$ determined by (5.10b). Expressions for matrix functions S and U yield the transformation law (5.22) for the quantity $h^a{}_\mu$. Because of the definition (5.10a), we have

$$\begin{aligned} \delta(\sqrt{|\bar{g}|}) &= \frac{1}{2\sqrt{|\bar{g}|}}\delta|\bar{g}| = \frac{1}{2\sqrt{|\bar{g}|}}|\bar{g}|g^{\mu\nu}\delta g_{\mu\nu} \\ &= \frac{1}{2}\sqrt{|\bar{g}|}g^{\mu\nu}(h^a{}_\mu h^b{}_\nu\delta g_{ab} + 2g_{ab}h^b{}_\nu\delta h^a{}_\mu). \end{aligned}$$

Substituting here the expressions for variations of the quantities g_{ab} and $h^a{}_\mu$ from (4.1) and (5.22), we find

$$\delta(\sqrt{|\bar{g}|}) = \tfrac{1}{2}\sqrt{|\bar{g}|}g^{\mu\nu}(2\varepsilon(x)g_{ab}h^a{}_\mu h^b{}_\nu + 2g_{ab}h^b{}_\nu(\omega^n(x)I_n{}^a{}_c h^c{}_\mu - \varepsilon(x)h^a{}_\mu - h^a{}_\sigma\partial_\mu\delta x^\sigma)) = \tfrac{1}{2}\sqrt{|\bar{g}|}(8\varepsilon(x) + \omega^n(x)I_n{}^a{}_a - 8\varepsilon(x) - 2\partial_\mu\delta x^\mu) = -\sqrt{|\bar{g}|}\partial_\mu\delta x^\mu.$$

Thus we have proved that such matrix functions S and U exist, and they are defined by expressions (5.14a)–(5.14c) and (5.15a)–(5.15d), for which the equality (5.11) holds.

VI. FREE GAUGE FIELD LAGRANGIAN

In order to define the structure of the free gauge field Lagrangian for the group $\mathcal{PW}(x)$, we need to find the functions of the gauge fields such that they are transformed with the group $\mathcal{PW}(x)$ as tensors. Since the gauge derivative $D_a\psi^A$ is a tensor, we calculate the commutator of gauge derivatives of the field ψ^A:

$$2D_{[a}D_{b]}\psi^A = -F^m{}_{ab}I_m{}^A{}_B\psi^B + wF_{ab}\psi^A - F^c{}_{ab}D_c\psi^A.$$

The quantities $F^m{}_{ab}$, F_{ab}, and $F^c{}_{ab}$ are tensors. They are defined by the expressions

$$F^m{}_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|}A^m_{b]} + A^m_c C^c{}_{ab} - c_n{}^m{}_q A^n_a A^q_b, \quad (6.1)$$

$$F_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|}A_{b]} + A_c C^c{}_{ab}, \quad (6.2)$$

$$F^c{}_{ab} = C^c{}_{ab} + 2I_n{}^c{}_{[a}A^n_{b]} + 2A_{[a}\delta^c_{b]}, \quad (6.3)$$

$$C^c{}_{ab} = -2h^c_\tau h^\lambda{}_{[a}\partial_{|\lambda|}h^\tau{}_{b]} = 2h^\lambda{}_a h^\tau{}_b\partial_{[\lambda}h^c{}_{\tau]}.$$

Moreover, the following contraction of the gauge derivative is also a vector

$$Q_a = -g^{bc}D_a g_{bc} = -h^\mu{}_a g^{bc}\partial_\mu g_{bc} - 2g^{bc}A^z_a I_{zbc} = 8(A_a - h^\mu{}_a\partial_\mu \ln\beta(x)). \quad (6.4)$$

With the help of the variations of the gauge fields (5.17), (5.18), and (5.19) and of the variation of the tetrads (5.21) and (5.22), it is possible to show by direct calculations that the expressions (6.1), (6.2), (6.3), and (6.4) are transformed as covariant quantities under the action of the group $\mathcal{PW}(x)$:

$$\delta F^m{}_{ab} = \omega^n(c_n{}^m{}_q F^q{}_{ab} - I_n{}^c{}_a F^m{}_{cb} - I_n{}^c{}_b F^m{}_{ac}) + 2\varepsilon F^m{}_{ab},$$

$$\delta F_{ab} = -\omega^n(I_n{}^c{}_d F_{cb} + I_n{}^c{}_b F_{ac}) + 2\varepsilon F_{ab},$$

$$\delta F^c{}_{ab} = \omega^n(I_n{}^c{}_d F^d{}_{ab} - I_n{}^d{}_a F^c{}_{db} - I_n{}^d{}_b F^c{}_{ad}) + \varepsilon F^c{}_{ab},$$

$$\delta Q_a = -\omega^n I_n{}^b{}_a Q_b + \varepsilon Q_a.$$

These tensor quantities contain derivatives of gauge fields only of the first order, therefore it is natural to call them gauge field strengths. In order to construct the free gauge field Lagrangian density, it is necessary to use the scalars formed from the gauge field strengths. As a result, we come to the conclusion about the structure of the free gauge field Lagrangian density.

Theorem 2—The Lagrangian density

$$\mathcal{L}_0 = \sqrt{|\bar{g}|}L_0(F^m{}_{ab}, F_{ab}, F^c{}_{ab}, Q_a, \beta(x)), \quad (6.5)$$

where L_0 is a scalar function formed from the gauge field strengths (6.1), (6.2), (6.3), and (6.4), satisfies the principle of local invariance.

VII. INTERACTIONS OF GAUGE FIELDS

Let us consider the full group of gauge symmetries $\Gamma(x)$, into which the group $\mathcal{PW}(x)$ enters via direct product. As an example, consider a group

$$\Gamma(x) = \mathcal{PW}(x)\bigotimes SU_3(x)\bigotimes U_1(x),$$

where $SU_3(x)$ is the non-Abelian gauge color group of quantum chromodynamics, and $U_1(x)$ is the gauge group of electrodynamics. Then, applying the general theory of gauge fields [15] to the group $\Gamma(x)$, we obtain, according to Theorem 2, the strength tensor of the unified gauge field

$$F^M{}_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|}A^M_{b]} + A^M_c C^c{}_{ab} - c_N{}^M{}_Q A^N_a A^Q_b, \quad (7.1)$$

where indices M, N, Q run over the values of indices of all infinitesimal operators of all components of the direct product. Further, it is necessary to take into account that various infinitesimal operators of components of the direct product commute. As a result, structural constants and the metric tensor g_{MN} of the group space, as well as the squares of the tensor (7.1) break up into blocks corresponding to components of the direct product.

Then the gauge field strength tensor (7.1) will be represented as a set of components $F^M{}_{ab} = \{F^m{}_{ab}, F^i{}_{ab}, F_{ab}\}$. Here the tensor $F^m{}_{ab}$ represents the group $\mathcal{PW}(x)$ and is given by the expression (6.1). The tensor $F^i{}_{ab}$ describes the color gauge field, and the tensor F_{ab} describes the electromagnetic field. These tensors are given by the expressions

$$F^i{}_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|}A^i_{b]} + A^i_c C^c{}_{ab} - c_k{}^i{}_l A^k_a A^l_b, \quad (7.2)$$

$$F_{ab} = 2h^\lambda{}_{[a}\partial_{|\lambda|}A_{b]} + A_c C^c{}_{ab}, \quad (7.3)$$

where $c_k{}^i{}_l$ are structural constants of the group SU_3. Formulas (7.2) and (7.3) describe interaction of color and electromagnetic fields with the gauge field of the group $\mathcal{PW}(x)$.

With regard to what has been just stated, the often used description of this interaction by replacing usual derivatives by gauge covariant ones, such as, e.g.,

$$F^i{}_{ab} = 2D_{[a}A^i_{b]} + A^i_c C^c{}_{ab} - c_k{}^i{}_l A^k_a A^l_b,$$
$$F_{ab} = 2D_{[a}A_{b]} + A_c C^c{}_{ab},$$

seems to be incorrect.

VIII. GEOMETRICAL INTERPRETATION

It is well known that the theory of gauge fields can be interpreted in terms of differential geometry and of fiber bundles (see [1,11–13,33,34] and the literature therein). Already in the book of Weyl [19], a generalization of the Riemann geometry to the Weyl geometry was formulated as a consequence of the requirement that the theory be invariant against variation of scales. The fiber bundles treatment of the Weyl geometry can be found in [35].

In [5–9,15] it was shown how, as a consequence of localization of the Poincaré group, the Riemann-Cartan geometry arose. Let us show that the results of the previous sections can be interpreted as a realization of the Weyl-Cartan differential geometry in the space-time manifold $\mathcal{M}$. Based upon the above mentioned geometrical interpretation, there appears the identification of the gauge derivative (5.7) and the covariant derivative in the space-time manifold, as well as the interpretation of the frame $\vec{e}_a$ as an orthogonal frame of the space (tangent to the manifold $\mathcal{M}$), in which the localized Poincaré-Weyl group operates: $\vec{e}_a = \vec{e}_\mu h^\mu{}_a$, $\{\vec{e}_\mu\} = \{\partial_\mu\}$

$$[\vec{e}_a, \vec{e}_b] = -C^c{}_{ab}\vec{e}_c, \qquad C^c{}_{ab} = 2h^\lambda{}_a h^\tau{}_b \partial_{[\lambda} h^c{}_{\tau]}. \quad (8.1)$$

Thus the basis $\vec{e}_a$ is a nonholonomic basis, and quantity $C^c{}_{ab}$ is an object of nonholonomicity [30]. The shift operator P_a is redefined: $P_a = h^\mu{}_a\partial_\mu$, $[P_a, P_b] = -C^c{}_{ab}P_c$, and represents the shift operator of the tangent space. The metric tensor g_{ab} appears to be a metric tensor of the tangent space with the element of length defined as $dx^a = h^a{}_\mu dx^\mu$. Then the square of an element of length of this space will be equal

$$ds^2 = g_{ab}dx^a dx^b = g_{ab}h^a{}_\mu h^b{}_\nu dx^\mu dx^\nu = \bar{g}_{\mu\nu}dx^\mu dx^\nu,$$

and this allows one to interpret quantities $\bar{g}_{\mu\nu}$, calculated with the use of the formula (5.10a), as components of the Weyl-Cartan metric tensor of the space-time manifold $\mathcal{M}$ in the coordinate holonomic basis:

$$\bar{g}_{\mu\nu} = \breve{g}(\vec{e}_\mu, \vec{e}_\nu) = \breve{g}(\vec{e}_a, \vec{e}_b)h^a{}_\mu h^b{}_\nu = g_{ab}h^a{}_\mu h^b{}_\nu = \beta^2 g_{\mu\nu}, \quad (8.2)$$
$$g_{\mu\nu} = g^M_{ab}h^a{}_\mu h^b{}_\nu.$$

$$\sqrt{|\bar{g}|} = \beta^4 h, \qquad h = \det(h^a{}_\mu). \quad (8.3)$$

With this interpretation of quantities $\bar{g}_{\mu\nu}$, the formula (5.11) becomes obvious. Quantities $g_{\mu\nu}$ are the coordinate holonomic components of the Riemann-Cartan metric tensor of the space-time manifold $\mathcal{M}$. Two types of indices, arising in the theory, i.e., the tetrad $a, b, \ldots$ and coordinate $\mu, \nu, \ldots$, change each other by means of the quantities $h^a{}_\mu$, which are interpreted as tetrads. It is generally accepted that contractions of quantities with Greek holonomic indices are performed with the use of the Riemann-Cartan metric tensor $g_{\mu\nu}$.

In the formula for the gauge derivative (5.7), expressions for the generators $I_m{}^A{}_B$ of the vector representation $\psi^A = v^a$ of the Poincaré-Weyl group,

$$I_{ij}{}^a{}_b = \delta^a_i g_{jb} - \delta^a_j g_{ib} \qquad (m \to \{i, j\}, i < j),$$

should be used, the weight of the vector field being equal to $w[v^a] = -1$. Then, equating the expression for the gauge derivative of a vector

$$D_\mu v^a = \partial_\mu v^a - A^m{}_\mu I_m{}^a{}_b v^b + A_\mu v^a,$$

to the expression for the covariant derivative of a vector in differential geometry $\nabla_\mu v^a = \partial_\mu v^a + \Gamma^a{}_{b\mu}v^b$, we find the connection coefficients in the nonholonomic basis:

$$\Gamma^a{}_{b\mu} = -A^m{}_\mu I_m{}^a{}_b + \delta^a_b A_\mu. \quad (8.4)$$

In order to define covariant derivative ∇_μ for quantities with coordinate indices, one postulates that

$$\nabla_\lambda h^a{}_\mu = \partial_\lambda h^a{}_\mu + \Gamma^a{}_{b\lambda}h^b{}_\mu - \Gamma^\nu{}_{\mu\lambda}h^a{}_\nu = 0.$$

From this formula, we find the connection coefficients in the holonomic coordinate basis:

$$\Gamma^\lambda{}_{\nu\mu} = h^\lambda{}_a h^b{}_\nu \Gamma^a{}_{b\mu} + h^\lambda{}_a \partial_\mu h^a{}_\nu. \quad (8.5)$$

According to (4.1), under the action of the group $\mathcal{PW}(x)$, the metric tensor of the tangent space is multiplied by an arbitrary function and can be represented as (4.2) [36–38]. Calculating a variation of this expression (4.2) and comparing it with the variation of the metric tensor (4.1), we find that $\delta\beta = \beta\varepsilon(x)$. Thus, the field $\beta(x)$ has the weight $w[\beta(x)] = 1$. This field coincides with the scalar field introduced by Dirac [27], and can be represented as $\beta(x) = \exp\sigma(x)$, where $\sigma(x)$ is the *dilaton* field. The field $\beta(x)$ is also similar to the "measure" scalar field introduced by Utiyama [28]. In fact, the field $\beta(x)$ is a factor that multiplies the components of the tangent space metric tensor in the Weyl-Cartan geometry.

It is known from differential geometry [30] that a nonmetricity tensor is equal to

$$Q_{ab\mu} = -\nabla_\mu g_{ab} = -\partial_\mu g_{ab} + 2\Gamma_{(ab)\mu}.$$

Substituting expression (4.2) in the above formula, as well as in (6.4), we obtain

$$Q_{ab\mu} = \tfrac{1}{4}g_{ab}Q_\mu, \qquad Q_\mu = g^{ab}Q_{ab\mu}, \quad (8.6a)$$
$$Q_\mu = 8(A_\mu - \partial_\mu \ln\beta(x)) = Q_a h^a{}_\mu. \quad (8.6b)$$

If the nonmetricity tensor satisfies the equalities (8.6a), then nonmetricity is the Weyl nonmetricity. In this case, the trace (8.6b) of the nonmetricity tensor is called the Weyl vector, and it is expressed through the vector (6.4).

Let us introduce the quantities

$$F^m{}_{\mu\nu} = F^m{}_{ab} h^a{}_\mu h^b{}_\nu = 2\partial_{[\mu} A^m{}_{\nu]} - c_n{}^m{}_q A^n{}_\mu A^q{}_\nu, \tag{8.7}$$

$$F_{\mu\nu} = F_{ab} h^a{}_\mu h^b{}_\nu = 2\partial_{[\mu} A_{\nu]}, \tag{8.8}$$

$$F^c{}_{\mu\nu} = F^c{}_{ab} h^a{}_\mu h^b{}_\nu = 2\partial_{[\mu} h^c{}_{\nu]} + 2I_n{}^c{}_a h^a{}_{[\mu} A^n{}_{\nu]} + 2A_{[\mu} h^c{}_{\nu]}, \tag{8.9}$$

representing (together with (8.6b)) the gauge field strengths for a new set of dynamic variables $\{A^m{}_\mu, A_\mu, h^a{}_\mu, \beta(x)\}$.

Substituting the expression for the connection coefficients (8.4) in the curvature of the space-time manifold and using the commutation relations of the generators of the Lorentz subgroup, as well as Eqs. (8.7) and (8.8), we obtain a representation for the curvature tensor that corresponds to the decomposition of the Weyl-Cartan curvature tensor into symmetric and antisymmetric parts [30]:

$$\bar{R}^a{}_{b\mu\nu} = 2\partial_{[\mu}\Gamma^a{}_{|b|\nu]} + 2\Gamma^a{}_{c[\mu}\Gamma^c{}_{|b|\nu]} = -I_m{}^a{}_b F^m{}_{\mu\nu} + \delta^a_b F_{\mu\nu}. \tag{8.10}$$

Using (8.5) and taking into account Eqs. (8.4) and (8.9), we obtain the expression for the torsion tensor of the space-time manifold:

$$T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\nu\mu]} = 2h^\lambda{}_a h^b{}_{[\nu}\Gamma^a{}_{|b|\mu]} + 2h^\lambda{}_a \partial_{[\mu} h^a{}_{\nu]} = h^\lambda{}_a F^a{}_{\mu\nu}. \tag{8.11}$$

Expressions (8.1), (8.4), (8.6b), (8.10), and (8.11) determine the relation between the geometrical quantities of the space-time manifold and the formulas of the gauge fields theory for the localized Poincaré-Weyl group.

IX. EQUATIONS OF GAUGE FIELDS

The total Lagrangian density of the set of the field ψ^A and the gauge field is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\psi, \tag{9.1}$$

where $\mathcal{L}_0$ is given by expression (6.5), and $\mathcal{L}_\psi$ by expression (5.3a). For the total Lagrangian density (9.1) the variational gauge field equations (4.6) are satisfied:

$$\frac{\delta\mathcal{L}_0}{\delta A^m_a} = -\frac{\partial\mathcal{L}_\psi}{\partial A^m_a}, \qquad \frac{\delta\mathcal{L}_0}{\delta A_a} = -\frac{\partial\mathcal{L}_\psi}{\partial A_a}, \tag{9.2a}$$

$$\frac{\delta\mathcal{L}_0}{\delta A^k_a} = -\frac{\partial\mathcal{L}_\psi}{\partial A^k_a}, \qquad \frac{\delta\mathcal{L}_0}{\delta\beta(x)} = -\frac{\delta\mathcal{L}_\psi}{\delta\beta(x)}. \tag{9.2b}$$

As it has already been pointed out, the last of these variational field equations is a consequence of the others.

It is more convenient to pass from the variational field equations for the set of independent fields $\{A^m_a, A_a, A^k_a, \beta(x)\}$ to the variational field equations with respect to independent dynamical variables $\{A^m{}_\mu, A_\mu, h^a{}_\mu, \beta(x)\}$:

$$\frac{\delta\mathcal{L}}{\delta A^m{}_\mu} = 0, \qquad \frac{\delta\mathcal{L}}{\delta A_\mu} = 0, \qquad \frac{\delta\mathcal{L}}{\delta h^a{}_\mu} = 0, \qquad \frac{\delta\mathcal{L}}{\delta\beta(x)} = 0. \tag{9.3}$$

By direct calculations it is possible to establish that the two sets of field equations (9.2a), (9.2b), and (9.3) are equivalent, provided that the tetrads $h^a{}_\mu$ are represented by the formulas (5.8a)–(5.8c).

The first of the field equations (9.3) can be represented as

$$\partial_\nu \frac{\partial\mathcal{L}_0}{\partial F^m{}_{\mu\nu}} = \frac{1}{2}\sqrt{|\bar{g}|}(S^\mu_{(0)m} + S^\mu{}_m), \tag{9.4a}$$

$$\sqrt{|\bar{g}|}S^\mu{}_m = -\frac{\partial\mathcal{L}_\psi}{\partial A^m{}_\mu} = \frac{\partial\mathcal{L}_\psi}{\partial D_\mu\psi^A} I_m{}^A{}_B \psi^B, \tag{9.4b}$$

$$\sqrt{|\bar{g}|}S^\mu_{(0)m} = -\frac{\partial\mathcal{L}_0}{\partial A^m{}_\mu} = 2\frac{\partial\mathcal{L}_0}{\partial F^n{}_{\mu\nu}} c_m{}^n{}_q A^q{}_\nu + 2\frac{\partial\mathcal{L}_0}{\partial F^c{}_{\mu\nu}} I_m{}^c{}_a h^a{}_\nu. \tag{9.4c}$$

The second of the field equations (9.3) can be written down as follows

$$\partial_\nu \frac{\partial\mathcal{L}_0}{\partial F_{\mu\nu}} = \frac{1}{2}\sqrt{|\bar{g}|}(J^\mu_{(0)} + J^\mu), \tag{9.5a}$$

$$\sqrt{|\bar{g}|}J^\mu = -\frac{\partial\mathcal{L}_\psi}{\partial A_\mu} = \frac{\partial\mathcal{L}_\psi}{\partial D_\mu\psi^A} w\psi^A, \tag{9.5b}$$

$$\sqrt{|\bar{g}|}J^\mu_{(0)} = -\frac{\partial\mathcal{L}_0}{\partial A_\mu} = -2\frac{\partial\mathcal{L}_0}{\partial F^a{}_{\mu\nu}} h^a{}_\nu - 8\frac{\partial\mathcal{L}_0}{\partial Q_\mu}. \tag{9.5c}$$

The third of the field equations (9.3) can be represented as:

$$\partial_\nu \frac{\partial\mathcal{L}_0}{\partial F^a{}_{\mu\nu}} = -\frac{1}{2}\sqrt{|\bar{g}|}(t^\mu_{(0)a} + t^\mu_{(\psi)a}), \tag{9.6a}$$

$$\sqrt{|\bar{g}|}t^\mu_{(\psi)a} = \frac{\partial\mathcal{L}_\psi}{\partial h^a{}_\mu} = h^\mu{}_a \mathcal{L}_\psi - \frac{\partial\mathcal{L}_\psi}{\partial D_\mu\psi^A} D_a\psi^A, \tag{9.6b}$$

$$\sqrt{|\bar{g}|}t^\mu_{(0)a} = \frac{\partial\mathcal{L}_0}{\partial h^a{}_\mu} = h^\mu{}_a\mathcal{L}_0 - 2F^m{}_{a\nu}\frac{\partial\mathcal{L}_0}{\partial F^m{}_{\mu\nu}} - 2F_{a\nu}\frac{\partial\mathcal{L}_0}{\partial F_{\mu\nu}} - 2F^b{}_{a\nu}\frac{\partial\mathcal{L}_0}{\partial F^b{}_{\mu\nu}} + 2A^m{}_\nu I_m{}^b{}_a \frac{\partial\mathcal{L}_0}{\partial F^b{}_{\mu\nu}} - 2A_\nu\frac{\partial\mathcal{L}_0}{\partial F^a{}_{\mu\nu}} - Q_a\frac{\partial\mathcal{L}_0}{\partial Q_\mu}. \tag{9.6c}$$

The quantity (9.6b) represents the well-known expression for the canonical energy-momentum of an external field [9].

The quantities (9.4c), (9.5c), and (9.6c) represent an internal spin angular momentum, a proper dilatation current, and an energy-momentum of free gauge fields, respectively. The sums of these currents and the appropriate currents of an external field ψ^A are conserved. These conservation laws are just consequences of the gauge field equations (9.4a), (9.5a), and (9.6a):

$$\partial_\mu(\sqrt{|\bar{g}|}(S^\mu_{(0)m} + S^\mu{}_m)) = 0,$$

$$\partial_\mu(\sqrt{|\bar{g}|}(J^\mu_{(0)} + J^\mu)) = 0, \qquad \partial_\mu(\sqrt{|\bar{g}|}(t^\mu_{(0)a} + t^\mu_{(\psi)a})) = 0.$$

The field equations (9.4a), (9.5a), and (9.6a) can be represented in a geometrical form with the help of the gauge derivatives (5.7) that does not however act on the Greek coordinate indices:

$$D_\nu\left(\frac{\partial \mathcal{L}_0}{\partial F^m{}_{\mu\nu}}\right) = \frac{1}{2}\sqrt{|\bar{g}|}S^\mu{}_m + \frac{\partial \mathcal{L}_0}{\partial F^b{}_{\mu\nu}} I_m{}^b{}_a h^a{}_\nu, \tag{9.7a}$$

$$D_\nu\left(\frac{\partial \mathcal{L}_0}{\partial F_{\mu\nu}}\right) = \frac{1}{2}\sqrt{|\bar{g}|}J^\mu - \frac{\partial \mathcal{L}_0}{\partial F^a{}_{\mu\nu}} h^a{}_\nu - 4\frac{\partial \mathcal{L}_0}{\partial Q_\mu}, \tag{9.7b}$$

$$D_\nu\left(\frac{\partial \mathcal{L}_0}{\partial F^a{}_{\mu\nu}}\right) - F^m{}_{a\nu}\frac{\partial \mathcal{L}_0}{\partial F^m{}_{\mu\nu}} - F^b{}_{a\nu}\frac{\partial \mathcal{L}_0}{\partial F^b{}_{\mu\nu}} - F_{a\nu}\frac{\partial \mathcal{L}_0}{\partial F_{\mu\nu}} - \frac{1}{2}Q_a\frac{\partial \mathcal{L}_0}{\partial Q_\mu} + \frac{1}{2}h^\mu{}_a\mathcal{L}_0 = -\frac{1}{2}\sqrt{|\bar{g}|}t^\mu_{(\psi)a}. \tag{9.7c}$$

The above equations generalize the field equations for the Poincaré group [15,17], and the Eq. (9.7c) generalizes the Einstein equation to the arbitrary Lagrangian.

X. CONCLUSION

In the present paper, based upon the general principles, we constructed the gauge field theory for the Poincaré-Weyl group. The formal expression for the gauge derivative was obtained: $D_a = -A^R_a M_R$. It was shown that, contrary to [24,25], the tetrads were not true gauge fields, but represented some sufficiently complex functions of gauge fields: Lorentzian A^m_a, translational A^k_a, and dilatational A_a, the relation $A^k_a = D_a x^k$ being valid. It is possible to expect that the knowledge of the true gauge potentials of a gravitational field may become essential for constructing the quantum theory of gravity.

The gauge field equations were obtained. The geometrical interpretation of the theory was developed, and it was shown that, as a result of localization of the Poincaré-Weyl group, the space-time became the Weyl-Cartan space. Moreover, the geometrical interpretation of the Dirac's scalar field β [27]) (and thereby that of the dilaton field, as well as of the Utiyama measure scalar field [28]), as a component of the metric tensor of a tangent space in the Weyl-Cartan geometry was obtained. This field is essential for constructing the field theory in the Weyl-Cartan space [39,40].

We demonstrated that the gauge invariant Lagrangian of the proper gauge fields was an arbitrary scalar function of the gauge strengths of the theory containing derivatives of the gauge fields of the order not higher than the first:

$$\mathcal{L}_0 = \sqrt{|\bar{g}|}L_0(F^m{}_{ab}, F_{ab}, F^c{}_{ab}, Q_a, \beta(x)).$$

The most simple Lagrangian of this kind, allowing the gauge fields to be realized dynamically, can be constructed as

$$\begin{aligned}\mathcal{L}_0 &= 2hf_0\beta^4\Big(\Lambda + I_m{}^a{}_b F^m{}_a{}^b + \sum_k \rho_k \overset{(k)}{F}{}_{cab}\overset{(k)}{F}{}^{cab} + \sum_i f_i{}^{(i)}(I_{mab}F^m{}_{cd})^{(i)}(I_m{}^{ab}F^{mcd}) + \lambda F_{ab}F^{ab} + \xi Q_a Q^a + \zeta F^c{}_{ac}Q^a\Big) \\ &= 2hf_0\Big(\frac{1}{2}\beta^2\bar{R} + 2\sum_i f_i \overset{(i)}{\bar{R}}{}_{[ab]cd}\overset{(i)}{\bar{R}}{}^{[ab]cd} + \beta^2\Big(\sum_k \rho_k \overset{(k)}{T}{}_{cab}\overset{(k)}{T}{}^{cab} + 64\xi A_\mu A^\mu + 8\zeta T^\mu A_\mu\Big) \\ &\quad + 4\lambda(\partial_{[\mu}A_{\nu]})^2 - 8(\zeta T^\mu + 16\xi A^\mu)\beta\partial_\mu\beta + 64\xi g^{\mu\nu}(\partial_\mu\beta)(\partial_\nu\beta) + \Lambda\beta^4\Big).\end{aligned} \tag{10.1}$$

Here $\bar{R}^a{}_{b\mu\nu}$ is the Weyl-Cartan curvature tensor, $\bar{R}$ is the Weyl-Cartan curvature scalar, and the indexes (i), (k) numerate components of the irreducible decompositions (with respect to the Lorentz group) of the curvature and torsion tensors, respectively. Contraction of the Greek indices is performed with the Riemann-Cartan metric tensor $g_{\mu\nu}$.

The above given Lagrangian has some distinctive features. First, it reproduces the quadratic Lagrangian of the Poincaŕe-gauged theories of gravity [7,12,15]. Second, this Lagrangian, despite the gauge invariance, allows for the presence of a nonzero mass of the Weyl vector, and hence, of the dilatation gauge field, in contrast to [24,25]. This circumstance means that the gauge field, introduced by means of localization of the group of scale transformations, is not an electromagnetic field (contrary to the initial idea of Weyl and to [39]), but rather a field of a different nature, as it was pointed out in [36–38]. A nonzero mass of the Weyl field can play a positive role in interpretations of the modern observational data based upon the use of the post-Riemannian cosmological models [41–43], as well as for possible explanation of a smooth exit from the stage of inflation. Moreover, the last terms with the field $\beta(x)$ in this Lagrangian have the structure of the Higgs Lagrangian

[39], and hence they may play a decisive role in spontaneous violation of scale invariance and in formation of mass of particles [31].

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